DIFFRACTION OF LIGHT BY ULTRASONIC WAVES.

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Received July 21, 1938.
(Communicated by Sir C. V. Raman, Kt., F.R.S., N.L.)

1. *Introduction.*

A theory of the diffraction of light by high-frequency sound waves was developed by Sir C. V. Raman and Mr. N. S. Nagendra Nath jointly, and later also by the latter independently, and was published in a series of papers in these *Proceedings*. (These will be referred to below for brevity as R.-N. I, II, III, IV, V and N. I). In a paper published in the *Physica*, July 1937 (referred to below as V. C.), Van Cittert has treated the case of normal incidence; starting from the usual expression for a pencil of light and considering the light fluctuations at successive points in the medium and thus also the diffraction effects, he obtains a system of differential equations, and solves them in a series of Bessel-functions of which the first term agrees with the simplified theory contained in R.-N. I. In the present paper it will be shown that the results of the more general theory of Raman and Nath (R.-N. IV and N. I) for the case of normal incidence are completely identical with those of Van Cittert. The latter's method is then extended to the case of oblique incidence and the results thus obtained are proved to be also in agreement with those of Raman and Nath for this case contained in their papers (R.-N. V and N. I). The solution is developed *in extenso*, as a series of Bessel-functions. The method of parts as developed in Mr. Nagendra Nath's recent paper for two systems of sound-waves is extended, and the resulting generalised difference-differential equation solved.

Notation.

μ_0' = Refractive index of the medium in the undisturbed state.

μ = Maximum variation of the refractive index from μ_0'.

λ = Wave-length of incident light.

λ^* = Wave-length of sound wave.

ν^* = Frequency of the sound wave.

$$n = \mu(y, t) = \mu_0' - \mu \sin 2\pi \left(\nu^* t - \frac{y}{\lambda^*}\right)$$

= refractive index of the medium.

2. *Normal and Oblique Incidence.*

Considering the case of normal incidence, the results of R.-N. IV, and those of V. C. are proved to be the same as follows :—

(*a*) The restrictions under which the results are valid.

(*b*) The angles of inclination of the various orders of spectra to a fixed line.

(*c*) The expressions for the intensities, must be the same in both the theories.

The incident light will be diffracted at angles given by $\sin\theta = \pm\frac{n\lambda}{\lambda^*}$ for the nth order (R.-N. IV), where θ is the angle with the x axis. Van-Cittert's theory also gives the same expression ($\sin i_n = \frac{n\lambda}{\lambda^*}$ where i_n is the angle of the nth order with the axis of x).

The difference-differential equation as in V. C. is :

$$\frac{dS_p}{d\xi} = -\frac{i(\mu_p - \mu_0)}{\nu_0} S_p + \tfrac{1}{2}\{S_{p-1} - S_{p+1}\}, \tag{1}$$

where $\mu_p = \frac{2\pi\mu_0'}{\lambda}\cos r_p$, $\nu_p = \frac{2\pi\mu}{\lambda}\cos r_p$, and $\xi = \frac{2\pi\mu x}{\lambda}$.

$-\left(\frac{\mu_p - \mu_0}{\nu_0}\right) = \frac{\mu_0'}{\mu}(1 - \cos r_p)$, and from the relation $\sin r_p = \frac{p\,\lambda}{n\lambda^*}$,

$$= \tfrac{1}{2}p^2 \cdot \frac{\lambda^2\mu_0'}{\mu n^2\lambda^{*2}} = \tfrac{1}{2}p^2 \cdot \frac{\lambda^2}{\mu\mu_0'\lambda^{*2}}$$

(neglecting powers of μ higher than the first)

$$= \tfrac{1}{2}p^2\beta \text{ where } \beta = \frac{\lambda}{\mu\mu_0'\lambda^{*2}}.$$

$\therefore$ The equation takes the form

$$\frac{dS_p}{d\xi} = -\frac{ip^2\beta}{2} S_p + \tfrac{1}{2}\{S_{p-1} - S_{p+1}\}. \tag{2}$$

The solution of this equation is given as below (V. C.)

$$\left.\begin{aligned} S_0 &= J_0 - 2i\beta J_3 + 2\beta^2 J_4 + \cdots \\ S_1 &= J_1 + i\beta J_2 - \beta^2 J_3 + \cdots \\ S_2 &= J_2 + 5i\beta J_3 - 21\beta^2 J_4 + \cdots \\ S_3 &= J_3 + 14i\beta J_4 + \cdots \end{aligned}\right\} \text{where } J_n = J_n\left(\frac{2\pi\mu x}{\lambda}\right) \tag{2a}$$

Expressing these in powers of ξ, we have

$$S_0 = 1 - \frac{\xi^2}{4} - \frac{i\beta\xi^3}{24} + \frac{\xi^4}{64}\left(1 + \frac{\beta^2}{3}\right) - \cdots$$

$$S_1 = \frac{\xi}{2}\left\{1 + \frac{i\beta\xi}{4} - \left(1 + \frac{\beta^2}{3}\right)\frac{\xi^2}{8} + \cdots\right\} \tag{2b}$$

$$S_2 = \frac{\xi^2}{8}\left\{1 + \frac{5i\beta\xi}{6} - \frac{1}{4}\left(\frac{7\beta^2}{4} + \frac{1}{3}\right)\xi^2 + \cdots\right\}$$

$$S_3 = \frac{\xi^3}{48}\left\{1 + \frac{7i\beta\xi}{4} + \cdots\right\}$$

Intensity $= |S_p|^2$ for the pth order.

R.-N. IV and N. I.

$$\left\{\begin{aligned} \psi_0 &= 1 - \frac{\xi^2}{4} - \frac{i\rho\xi^3}{24} + \frac{1}{64}\left(1 + \frac{\rho^2}{3}\right)\xi^4 + \cdots \\ \psi_1 &= \frac{\xi}{2}\left\{1 + \frac{i\rho}{4}\xi - \frac{1}{24}(\rho^2 + 3)\xi^2 - \frac{i\rho}{192}(\rho^2 + 10)\xi^3 + \cdots\right\} \\ \psi_2 &= \frac{\xi^2}{8}\left\{1 + \frac{5i\rho}{6}\xi - \frac{1}{4}\left(\frac{7}{4}\rho^2 + \frac{1}{3}\right)\xi^2 + \cdots\right\} \\ \psi_3 &= \frac{\xi^3}{48}\left\{1 + \frac{7i\rho}{4}\xi + \cdots\right\} \end{aligned}\right. \tag{2c}$$

Intensity of the pth order $= |\psi_p|^2$.

$\rho = \frac{\lambda^2}{\mu\mu_0'\lambda^{*2}} = \beta$ and ξ signifies the same in both. Hence the expressions for the intensities are the same. The difference-differential equation is the same in both the cases.

$$i.e., \quad |\psi_p|^2 = |S_p|^2. \tag{3}$$

μ is small and of the order of 10^{-5}. β must be less than 1.

$$\beta = \frac{\lambda^2}{\mu\mu_0'\lambda^{*2}} = 0\,(10^{-1}) < 1. \quad \therefore \quad \rho < 1. \qquad \left\{\begin{aligned} \lambda &= 0\,(10^{-5}) \\ \lambda^* &= 0\,(10^{-2}) \\ \mu &= 0\,(10^{-5}) \\ \mu_0' &= 0\,(1). \end{aligned}\right.$$

Thus ρ also satisfies the condition.

$\therefore$ The results for the case of normal incidence from the general theory (R.-N. IV and N. I) and of V. C. are thus the same.

For the case of oblique incidence, we first have the same equation (1).

$$\frac{dS_p}{d\xi} = -\frac{i(\mu_p - \mu_0)}{\nu_0}S_p + \frac{1}{2}\{S_{p-1} - S_{p+1}\}. \tag{3a}$$

The angles of emergence for the various orders are given by

$$\sin r_{p\pm 1} - \sin r_p = \pm \frac{\lambda}{n\lambda^*}. \qquad \text{(V.C.)}$$

$$\sin r_p - \sin r_{p-1} = \frac{\lambda}{n\lambda^*}$$

$$\sin r_{p-1} - \sin r_{p-2} = \frac{\lambda}{n\lambda^*}$$

$$\sin r_1 - \sin r_0 = \frac{\lambda}{n\lambda^*}$$

$$\therefore \sin r_p - \sin r_0 = p \cdot \frac{\lambda}{n\lambda^*}.$$

Let the angle of incidence be ϕ ($i_0 = \phi$). We have the relation $\dfrac{\sin i_p}{\sin r_p} = n$ (refractive index of the medium).

$$\boxed{\therefore \sin i_p = \sin\phi + p\frac{\lambda}{\lambda^*}}. \qquad (3b)$$

This equation is the same as in R.-N. II (oblique incidence).

$$-\left(\frac{\mu_p - \mu_0}{\nu_0}\right) = \frac{\mu_0'}{\mu}\left\{\frac{\cos r_0 - \cos r_p}{\cos r_0}\right\}$$

$$\cos r_0 = \sqrt{1 - \frac{\sin^2\phi}{n^2}} = 1$$ (neglecting powers of ϕ higher than the first).

$$\cos r_p = \sqrt{1 - \left(\frac{\sin\phi}{n} + p\frac{\lambda}{n\lambda^*}\right)^2}$$

$$= 1 - \frac{1}{2}\left\{\frac{p^2\lambda^2}{n^2\lambda^{*2}} + 2p\frac{\sin\phi\lambda}{n^2\lambda^*}\right\}$$ (neglecting higher terms).

$$\therefore -\left(\frac{\mu_p - \mu_0}{\nu_0}\right) = \frac{\mu_0'}{2\mu}\left\{p^2\frac{\lambda^2}{n^2\lambda^{*2}} + 2p\frac{\lambda\sin\phi}{n^2\lambda^*}\right\}$$

$$= \tfrac{1}{2}\left\{p^2\frac{\lambda^2}{\mu\mu_0'\lambda^{*2}} + 2p\frac{\lambda\sin\phi}{\mu\mu_0'\lambda^*}\right\}$$

Setting $\beta = \dfrac{\lambda^2}{\mu\mu_0'\lambda^{*2}}$ and $a = \dfrac{2\lambda\sin\phi}{\mu\mu_0'\lambda^*} = \alpha\beta$ the equation becomes

$$\boxed{2\frac{dS_p}{d\xi} = i\beta(p^2 + \alpha p) + (S_{p-1} - S_{p+1})}. \qquad (4)$$

[In deriving these equations, powers of ϕ higher than the first have been neglected : μ^2 is neglected. ϕ in this paper corresponds to $\pi + \phi$ in R.-N. V and N. I]. It can be easily seen that the diffraction pattern is asymmetric (R.-N. V).

3. *The Difference-Differential Equation.*

$$2\,\frac{dS_p}{d\xi} = C_p S_p + \{S_{p-1} - S_{p+1}\}, \tag{5}$$

where $C_p = i\beta\,((p^2 + \alpha p)$.

Substituting $S_p = J_p + \sum_1^{\infty} a_{p,\,n+p}\, J_{n+p}$ in equation (5), we get the following recurrence relations.

$$a_{0,1} = 0$$

$$a_{0,2} = 0$$

$$a_{0,\,n+1} - a_{0,\,n-1} = -\{a_{1,\,n} - a_{-1,\,n}\}.$$

$$a_{p,\,p+1} - a_{p-1,\,p} = C_p$$

$$a_{p,\,n+p+1} - a_{p-1,\,p+n} + a_{p+1,\,p+n} - a_{p,\,p+n-1} = C_p\, a_{p,\,p+n}$$

$$\therefore\; a_{p-1,\,n+p} - a_{p-2,\,p+n-1} + a_{p,\,p+n-1} - a_{p-1,\,p+n-2} = C_{p-1}\, a_{p-1,\,p+n-1}$$

. . . .

$$a_{1,\,n+2} - a_{0,\,n+1} + a_{2,\,n+1} - a_{1,\,n} = C_1\, a_{1,\,n+1}$$

On adding, we have

$$\boxed{a_{p,\,p+n+1} = a_{0,\,n+1} - a_{p+1,\,p+n} + a_{1,\,n} + \sum_1^p C_p\, a_{p,\,p+n}} \tag{6}$$

which can also be written as

$$\boxed{a_{p,\,p+n+1} = a_{0,\,n-1} = a_{p+1,\,p+n} + a_{-1,\,n} + \sum_1^p C_p\, a_{p,\,p+n}.} \tag{6a}$$

$$a_{p,\,p+1} - a_{p-1,\,p} = C_p$$

$$\therefore\quad \boxed{a_{p,\,p+1} = \sum_1^p C_p.} \tag{7}$$

$$a_{p,\,p+2} = \sum_1^p C_p\, a_{p,\,p+1}$$

$$= \sum_1^p C_p\,(C_1 + C_2 + \cdots + C_p) = \tfrac{1}{2} \sum_{r=1}^{p} \sum_{s=1}^{p} C_r\, C_s.$$

$$= \tfrac{1}{2}\left\{\left(\sum_1^p C_r\right)^2 + \sum_1^p C_r^2\right\}\cdot$$

$$\boxed{a_{p,\,p+2} = \tfrac{1}{2}\left\{\left(\sum_1^p C_r\right)^2 + \sum_1^p C_r^2\right\}} \tag{7a}$$

$$a_{p,\,p+3} = a_{0,\,3} - \left\{C_2 + C_3 + \cdots + C_{p+1}\right\} + \sum_1^p C_p\, a_{p,\,p+2}.$$

$$a_{p,\,p+3} = -\left\{C_{-1} + \sum_1^{p+1} C_r\right\} + \sum_1^p C_r C_s C_t \rightarrow \text{(all like terms being counted once only)} \tag{7b}$$

$$C_p = i\beta\,(p^2 + \alpha p).$$

$$\left\{\begin{aligned} a_{p,\,p+1} &= i\beta\left\{\frac{p\,(p+1)\,(2p+1)}{6} + \alpha\cdot\frac{p\,(p+1)}{2}\right\} \\ a_{p,\,p+2} &= -\frac{\beta^2}{2}\left\{\frac{p\,(p+1)\,(2p+1)\,(10p^3+33p^2+23p-6)}{180}\right. \\ &\qquad\left. + \frac{\alpha}{3}p^2\,(p+1)^2\,(p+2) + \frac{\alpha^2}{12}\,p\,(p+1)\,(p+2)\,(3p+1).\right\} \\ a_{p,\,p+3} &= -i\beta\left\{\frac{(p+1)\,(p+2)\,(2p+3)}{6} + 1 + \frac{\alpha}{2}p\,(p+3)\right\} + \sum_1^p C_p\, a_{p,\,p+2} \end{aligned}\right.$$

Expressions for the amplitudes $S_0, S_1, \cdots$ are as below:

$$\begin{aligned} S_0 = J_0 &- 2i\beta J_3 + 2\beta^2(1+\alpha^2)\,J_4 + 2i\beta\,\{5 + \beta^2(1+3\alpha^2)\}\,J_5 \\ &- 2\beta^2\{(27+9\alpha^2) + \beta^2(1+6\alpha^2+\alpha^4)\}\,J_6 \\ &- 2i\beta\,\{18 + \beta^2(113+119\,\alpha^2) + \beta^4(1+10\alpha^2+5\alpha^4)\}\,J_7 \\ &+ 2\beta^2\{(246+22\alpha^2) + \beta^2(459+922\alpha^2+55\alpha^4) + \beta^4(1+15\alpha^2 \\ &+ 15\alpha^4+\alpha^6)\,J_8 + \cdots \end{aligned}$$

$$\begin{aligned} S_1 = J_1 &+ i\beta\,(1+\alpha)\,J_2 - \beta^2(1+\alpha)^2\,J_3 - i\beta\,\{(6+2\alpha) + \beta^2(1+\alpha)^3\}J \\ &+ \beta^2\{(28+30\alpha+10\alpha^2) + \beta^2(1+\alpha)^4\}\,J_5 + i\beta\,\{(25+3\alpha) \\ &+ \beta^2(114+196\alpha+122\alpha^2+24\alpha^3) + \beta^4(1+\alpha)^5\}\,J_6 \\ &- \beta^2\{(273+190\alpha+31\alpha^2) + \beta^2(460+1044\alpha+928\alpha^2+364\alpha^3 \\ &+ 56\alpha^4) + \beta^4(1+\alpha)^6)\}\,J_7 + \cdots \end{aligned}$$

$$\begin{aligned} S_2 = J_2 &+ i\beta\,(5+3\alpha)\,J_3 - \beta^2(21+24\alpha+7\alpha^2)\,J_4 - i\beta\,\{(15+5\alpha) \\ &+ \beta^2(85+141\alpha+79\alpha^2+15\alpha^3)\}\,J_5 \\ &+ \beta^2\{(214+176\alpha+38\alpha^2) + \beta^2(341+738\alpha+604\alpha^2+222\alpha^3 \\ &+ 31\alpha^4)\}\,J_6 + i\beta\,\{(39+16\alpha) + \beta^2(2291+2849\alpha+1205\alpha^2 \\ &+ 175\alpha^3) + \beta^4(1364+3634\alpha+3892\alpha^2+2096\alpha^3+568\alpha^4+ \\ &62\alpha^5)\}\,J_7 + \cdots \end{aligned}$$

$$S_3 = J_3 + i\beta(14 + 6\alpha) J_4 - \beta^2(147 + 120\alpha + 25\alpha^2) J_5 - i\beta\{(31 + 9\alpha) + \beta^2(1408 + 1662\alpha + 664\alpha^2 + 90\alpha^3)\} J_6 + \beta^2\{(973 + 618\alpha + 132\alpha^2) + \beta^2(13013 + 19910\alpha + 11566\alpha^2 + 3024\alpha^3 + 270\alpha^4)\} J_7 + \cdots$$

$$S_4 = J_4 + i\beta(30 + 10\alpha) J_5 - \beta^2(627 + 400\alpha + 65\alpha^2) J_6 - i\beta\{(56 + 14\alpha) + \beta^2(11440 + 10570\alpha + 3304\alpha^2 + 350\alpha^3)\} J_7 + \beta^2\{(3872 + 2114\alpha + 329\alpha^2) + \beta^2(653653 + 771990\alpha + 344570\alpha^2 + 68880\alpha^3 + 3770\alpha^4)\} J_8 + \cdots$$

$$S_5 = J_5 + i\beta(55 + 15\alpha) J_6 - \beta^2(2002 + 1050\alpha + 140\alpha^2) J_7 - i\beta\{(92 + 20\alpha) + \beta^2(61490 + 36830\alpha + 12054\alpha^2 + 1050\alpha^3)\} J_8 + \cdots$$

$$S_6 = J_6 + i\beta(91 + 21\alpha) J_7 - \beta^2(5378 + 2402\alpha + 266\alpha^2) J_8 - i\beta\{(141 + 21\alpha) + \beta^2(255098 + 155570\alpha + 36042\alpha^2 + 2646\alpha^3)\} J_9 + \cdots$$

$$S_7 = J_7 + i\beta\{(140 + 28\alpha)\} J_8 - \beta^2(12138 + 4704\alpha + 462\alpha^2) J_9 + \cdots$$

. . . .

$$S_p = J_p + i\beta\left\{\frac{p(p+1)(2p+1)}{6} + \frac{\alpha}{2}p(p+1)\right\} J_{p+1} - \frac{\beta^2}{2}\left\{\frac{p(p+1)(2p+1)(10p^3 + 33p^2 + 23p - 6)}{180} + \frac{\alpha}{3}p^2(p+1)^2(p+2) + \frac{\alpha^2}{4}p(p+1)(p+2)(3p+1)\right\} J_{p+2} + \cdots + a_{p,\,p+n} J_{p+n} + \cdots$$

$$\text{where} \quad J_n = J_n\left(\frac{2\pi\mu x}{\lambda}\right).$$

4. *Systems of Parallel Sound Waves.*

Let us consider the case of three systems of sound waves with different frequencies, amplitudes and wave-lengths. The refractive index of the medium is given by $\mu(y, t) = \mu_0 + \sum_{r=1}^{3} \mu_r \sin 2\pi\left(\nu_r^* t - \frac{y}{\lambda_r^*}\right)$. It can easily be proved that combinational orders are present and that they occur at angles given by $\left(\sin\theta = r\frac{\lambda}{\lambda_1^*} + s\frac{\lambda}{\lambda_2^*} + t\frac{\lambda}{\lambda_3^*}\right)$ for the (r, s, t)th order. Extending the method of parts as in Mr. Nath's recent paper, it can easily be seen that the equation giving the amplitudes of successive orders is

$$2\frac{d}{dx}\phi_{r,s,t}(x) = \sigma_1\left\{\phi_{r-1,s,t} - \phi_{r+1,s,t}\right\} + \sigma_2\left\{\phi_{r,s-1,t} - \phi_{r,s+1,t}\right\} + \sigma_3\left\{\phi_{r,s,t-1} - \phi_{r,s,t+1}\right\} \text{ where } \sigma_r = \frac{2\pi\mu_r}{\lambda}. \quad (8)$$

The boundary conditions are $\phi_{r,s,t}(0) = 0$, and $\phi_{0,0,0}(0) = 1$.

This can easily be generalised to the case of N systems of sound waves.

To solve the equation

$$2\frac{d}{dz}\phi_{r,s,t}(z) = \sigma_1\left\{\phi_{r-1,s,t} - \phi_{r+1,s,t}\right\} + \sigma_2\left\{\phi_{r,s-1,t} - \phi_{r,s+1,t}\right\} + \sigma_3\left\{\phi_{r,s,t-1} - \phi_{r\ s,t+1}\right\} \tag{9}$$

with the conditions $\begin{cases}\phi_{r,s,t}(0) = 0\\ \phi_{0,0,0}(0) = 1.\end{cases}$

Let $\phi_{r,s,t}\ (z) = \phi\ \dfrac{e^{\rho z}}{\xi_1^{\ r+1}\ \xi_2^{\ s+1}\ \xi_3^{\ t+1}}$. This is a solution of the above equation if $2\rho = \sum\limits_{r=1}^{3}\ \sigma_r\left(\xi_r - \dfrac{1}{\xi_r}\right)$. The most general solution is obtained by summing up similar solutions.

i.e.,

$$\phi_{r,s,t}(z) = \left(\frac{1}{2\pi i}\right)^3 \int\!\!\int\!\!\int \phi(\xi_1, \xi_2, \xi_3)\, e^{\frac{z}{2}\left\{\sum\limits_{r=1}^{3}\sigma_r\left(\xi_r - \frac{1}{\xi_r}\right)\right\}}\ \frac{d\xi_1\cdot d\xi_2\cdot d\xi_3}{\xi_1^{\ r+1}\xi_2^{\ s+1}\xi_3^{\ t+1}}. \tag{9a}$$

where ϕ is any function of ξ_1, ξ_2, ξ_3. To obtain the solution satisfying the given boundary conditions, we can regard the quantities (ξ_r) as complex variables and take each integral round a closed contour which encircles the orgin once.

The boundary conditions are

$$\begin{cases}(1)\ \phi_{r,s,t}\ (0) = 0 \qquad r \neq s,\ t \neq 0\\ (2)\ \phi_{0,0,0}\ (0) = 1\end{cases}$$

$$\therefore\quad (1) \text{ gives } \left(\frac{1}{2\pi i}\right)^3 \oint\!\!\oint\!\!\oint\ \phi(\xi_1, \xi_2, \xi_3)\ \frac{d\xi_1\, d\xi_2\, d\xi_3}{\xi_1^{\ r+1}\ \xi_2^{\ s+1}\ \xi_5^{\ t+1}} = 0.$$

This implies that $\phi(\xi_1, \xi_2, \xi_3)$ is a constant $= \alpha = 1$ by (2).

$$\therefore\quad \phi_{r,s,t}(z) = \left(\frac{1}{2\pi i}\right)^3\ \oint\!\!\oint\!\!\oint\ e^{\frac{z}{2}\left\{\sum\limits_{1}^{3}\sigma_r\left(\xi_r - \frac{1}{\xi_r}\right)\right\}}\ \frac{d\xi_1\cdot d\xi_2\cdot d\xi_3}{\xi^{r}_{\ 1}{}^{+1}\ \xi^{s}_{\ 2}{}^{+1}\ \xi^{t}_{\ 3}{}^{+1}}.$$

We know that $\dfrac{1}{2\pi i}\oint \dfrac{e^{\frac{\sigma z}{2}\left(\xi - \frac{1}{\xi}\right)}}{\xi^{n+1}}\ d\xi = J_n(\sigma z)$. This can be proved as follows :

$$\oint \frac{1}{2\pi i} e^{\frac{z}{2}\left(\xi - \frac{1}{\xi}\right)} \frac{d\xi}{\xi^{n+1}} = \text{Residue at the origin.}$$

$$e^{\frac{z}{2}\xi} \cdot e^{-\frac{z}{2\xi}} = \sum_0^\infty \frac{\left(\frac{z}{2}\xi\right)^n}{\lfloor n} \cdot \sum_0^\infty \frac{\left(-\frac{z}{2\xi}\right)^n}{\lfloor n}.$$

Coefficient of ξ^n is given by

$$\left(\frac{z}{2}\right)^n \frac{1}{\lfloor n} - \frac{\left(\frac{z}{2}\right)^{n+2}}{\lfloor 1 \lfloor n+1} + \left(\frac{z}{2}\right)^{n+4} \frac{1}{\lfloor 2 \lfloor n+2} \cdots$$

$$= \left(\frac{z}{2}\right)^n \left\{\frac{1}{\lfloor n} - \frac{\left(\frac{z}{2}\right)^2}{\lfloor 1 \lfloor n+1} + \cdots\right\}$$

$$= \left(\frac{z}{2}\right)^n \left\{\sum_0^\infty (-1)^r \left(\frac{z}{2}\right)^{2r} \frac{1}{\lfloor r \lfloor n+r}\right\}$$

$$= J_n(z).$$

$$\therefore \frac{1}{2\pi i} \oint e^{\frac{z}{2}\left(\xi - \frac{1}{\xi}\right)} \frac{d\xi}{\xi^{n+1}} = J_n(z).$$

$$\therefore \phi_{r,s,t}(z) = \frac{1}{2\pi i} \oint e^{\frac{\sigma_1 z}{2}\left(\xi_1 - \frac{1}{\xi_1}\right)} \frac{d\xi_1}{\xi_1^{r+1}} \cdot \frac{1}{2\pi i} \oint e^{\frac{\sigma_2 z}{2}\left(\xi_2 - \frac{1}{\xi_2}\right)} \frac{d\xi_2}{\xi_2^{s+1}} \times$$

$$\times \frac{1}{2\pi i} \cdot \oint e^{\frac{\sigma_3 z}{2}\left(\xi_3 - \frac{1}{\xi_3}\right)} \frac{d\xi_3}{\xi_3^{t+1}}.$$

$$\therefore \boxed{\phi_{r,s,t}(z) = J_r(\sigma_1 z)\, J_s(\sigma_2 z)\, J_t(\sigma_3 z).} \qquad (10)$$

The generalised form of the above equation can be written as

$$2 \frac{d}{dz} \phi_{n_1, n_2, \cdots n_N}(z) = \sigma_1 \left\{\phi_{n_1-1, n_2 \cdots n_N} - \phi_{n_1+1, n_2, \cdots n_N}\right\}$$

$$+ \sigma_2 \left\{\phi_{n_1, n_2-1, \cdots n_N} - \phi_{n_1, n_2+1, \cdots n_N}\right\} + \cdots$$

$$+ \sigma_N \left\{\phi_{n_1, n_2, \cdots n_N - 1} - \phi_{n_1, n_2, \cdots n_N + 1}\right\}. \qquad (11)$$

where $n_1, n_2, \cdots n_N$ are integers and $\{\sigma_r\}$ are constants. $\left(\sigma_r = \frac{2\pi\mu_r}{\lambda}\right)$.

Proceeding as before, we have the general solution given by

$$\phi_{n_1, n_2, \cdots n_N}(z) = \left(\frac{1}{2\pi i}\right)^N \underbrace{\iint \cdots \iint}_{N} \phi(\xi_1, \xi_2 \cdots \xi_n) \cdot e^{\frac{z}{2}\left\{\sum_{r=1}^{N} \sigma_r \left(\xi_r - \frac{1}{\xi_r}\right)\right\}}$$

$$\times \frac{d\xi_1 \cdot d\xi_2 \cdots d\xi_N}{\xi_1^{n_1+1}\, \xi_2^{n_2+1} \cdots \xi_N^{n_N+1}}. \tag{11a}$$

To obtain the solution satisfying the boundary conditions

$$\left.\begin{aligned} \phi_{n_1, n_2, \cdots n_N}(0) &= 0 \\ \phi_{0,0,\cdots 0}(0) &= 1 \end{aligned}\right\}$$

we regard the quantities (ξ_r) as complex variables and integrate them round contours enclosing the origin once. We therefore get,

$$\phi_{n_1, n_2, \cdots n_N}(z) = \left(\frac{1}{2\pi i}\right)^N \underbrace{\oint\oint \cdots \oint\oint}_{N} e^{\frac{z}{2}\left\{\sum_1^N \sigma_r \left(\xi_r - \frac{1}{\xi_r}\right)\right\}} \cdot \frac{d\xi_1 \cdot d\xi_2 \cdots d\xi_N}{\xi_1^{n_1+1}\, \xi_2^{n_2+1} \cdot \xi_N^{n_N+1}}.$$

$$= \prod_{r=1}^{N} \left\{\frac{1}{2\pi i} \oint e^{\frac{\sigma_r z}{2}\left(\xi_r - \frac{1}{\xi_r}\right)} \frac{d\xi_r}{\xi_r^{n_r+1}}\right\}.$$

$$= J_{n_1}(\sigma_1 z) \cdot J_{n_2}(\sigma_2 z) \cdots J_{n_N}(\sigma_N z).$$

$$\boxed{\phi_{n_1, n_2, \cdots n_N}(z) = \prod_{r=1}^{N} J_{nr}(\sigma_r z).} \tag{12}$$

$$\text{Intensity} = |\phi_{n_1, n_2, \cdots n_N}(z)|^2 = \prod_{r=1}^{N} J_n^2(\sigma_r z). \tag{13}$$

This same expression has been obtained by E. Fues by applying the simplified theory of Raman-Nath (R.-N. I). In conclusion it is my greatest pleasure to record my respectful thanks to Professor Sir C. V. Raman, for suggesting the present investigation and for much valuable guidance and criticism in the course of the work.

5. *Summary.*

The results of Raman and Nath in their general theory of the diffraction light by ultrasonic waves at normal and oblique incidences are shown to be in complete agreement with those of Van Cittert for the case of normal incidence and also with those obtained by extending Van Cittert's method to the case of oblique incidence. The amplitude function for the latter case is developed *in extenso* in a series of Bessel functions. The expression for the

intensity with normal incidence in the case of N systems of sound waves is $| J_{n_1} (\sigma_1 z) J_{n_2} (\sigma_2 z) \cdots J_{n_N} (\sigma_N z) |^2$, where $n_1, n_2, \cdots n_N$ are the orders excited. This result is the same as that which has been obtained by E. Fues by applying the simplified Raman-Nath method to the case considered.

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