GENERALIZATION OF NORMAL CURVATURE OF A CURVE IN A RIEMANNIAN V_n

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1. The object of the present paper is to define an invariant which is a generalization of the expression for the normal curvature of a curve in V_n, and to obtain generalizations of some known results.

2. Consider a Riemannian space V_n of co-ordinates x^i, $i = 1, 2, \ldots n$ and metric

$$ds^2 = g_{ij}dx^i dx^j \quad , \tag{2.1}$$

imbedded in a V_{n+1} of co-ordinates y^α, $\alpha = 1, 2, \ldots n+1$ and metric

$$ds^2 = a_{\alpha\beta}dy^\alpha dy^\beta. \tag{2.2}$$

We have the relation

$$g_{ij} = a_{\alpha\beta}y^\alpha{}_{,i}y^\beta{}_{,j} \quad , \tag{2.3}$$

where comma denotes covariant differentiation.

Let N^α denote the contravariant components of the unit normal to V_n, then

$$a_{\alpha\beta}N^\alpha N^\beta = 1, \tag{2.4}$$

and

$$a_{\alpha\beta}N^\alpha y^\beta{}_{;i} = 0, \; (i = 1, 2, \ldots n) \tag{2.5}$$

where (;) followed by an index denotes generalized covariant derivative (or tensor derivative) with respect to the x with that index.

We have

$$N^\alpha{}_{;i} = - \Omega_{ij}g^{jk}y^\alpha{}_{,k} \quad , \tag{2.6}$$

where Ω_{ij} are the components of a symmetric covariant tensor of the second order given by

$$\Omega_{ij} = y^\alpha{}_{;ij}a_{\alpha\beta}N^\beta. \tag{2.7}$$

From (2.6) we have

$$N^{a}{}_{;i}e^{i} = -e^{i}\Omega_{ij}g^{jk}y^{a}{}_{,k} \quad , \tag{2.8}$$

where e^{i} are the components of a unit vector in the hypersurface.

The resolved part of the derived vector $N^{a}{}_{;i}e^{i}$ in the direction of another unit vector **a** in the hypersurface is given by

$$(-e^{i}\Omega_{ij}g^{jk}y^{a}{}_{,k})\,a_{a\beta}\,(y^{\beta}{}_{,l}a^{l}) = -e^{i}\Omega_{ij}g^{jk}g_{kl}a^{l}$$

$$= -\Omega_{ij}e^{i}a^{j}, \tag{2.9}$$

and therefore the resolved part of the derived vector $N^{a}{}_{;i}e^{i}$ along

e^{i} is $-\Omega_{ij}e^{i}e^{j}$. (2.10)

3. Let λ^{a} be the components of a unit vector in the direction of a curve of a congruence such that one curve of the congruence passes through each point of V_{n}.

λ^{a} can be expressed in terms of $y^{a}{}_{,i}$ and N^{a} as

$$\lambda^{a} = y^{a}{}_{,i}t^{i} + rN^{a}. \tag{3.1}$$

Since λ^{a} are the components of a unit vector,

$$1 = a_{a\beta}\lambda^{a}\lambda^{\beta} = a_{a\beta}\,(y^{a}{}_{,i}t^{i} + rN^{a})\,(y^{\beta}{}_{,j}t^{j} + rN^{\beta})$$

$$= g_{ij}t^{i}t^{j} + r^{2}, \tag{3.2}$$

The tensor derivative of (3.1) with respect to x^{j} yields

$$\lambda^{a}{}_{;j} = y^{a}{}_{;ij}t^{i} + y^{a}{}_{;i}t^{i}{}_{;j} + rN^{a}{}_{;j} + r_{;j}N^{a}$$

$$= N^{a}\,[\Omega_{ij}t^{i} + r_{;j}] + y^{a}{}_{,k}\,[t^{k}{}_{,j} - r\Omega_{ij}g^{ik} \tag{3.3}$$

From (3.3) we have

$$a_{a\beta}\left(y^{\beta}{}_{,l}\frac{dx^{l}}{ds}\right)\left(\lambda^{a}{}_{;j}\frac{dx^{j}}{ds}\right)$$

$$= -\,[r\Omega_{lj} - t_{l;j}]\,\frac{dx^{l}}{ds}\frac{dx^{j}}{ds}. \tag{3.4}$$

If the congruence be one of normals to the hypersurface, then $t^{i} = 0$, $r = 1$, and the right-hand member of (3.4) reduces to (2.10). The expression

$$(r\Omega_{lj} - t_{l;j})\,\frac{dx^{l}}{ds}\frac{dx^{j}}{ds}$$

may therefore be considered as a generalization of the normal curvature of the curve whose unit tangent has components dx^i/ds. We shall call it the *generalized normal curvature of the curve relative to the congruence.* This quantity depends only on the direction of the curve C at the point considered, and is the same for all curves tangent to C at the point. We may therefore call it the *generalized normal curvature of the hypersurface relative to the congruence for the given direction.*

4. *Generalization of Dupin's Theorem on normal curvature.* If e^i are the components in the x's of a unit vector **e** at a point P in V_n, the generalized normal curvature of the hypersurface at P in the direction of **e** has the value

$$k_\lambda = (r\Omega_{lj} - t_{l;j})\, e^l e^j. \tag{4.1}$$

Consequently if $\mathbf{e}_{h|}$ $(h = 1, 2, \ldots . n)$ are the unit tangents at P to the curves of an orthogonal ennuple in V_n, the sum of the generalized normal curvatures of V_n for the directions of the ennuple is

$$\sum_h (r\Omega_{ij} - t_{i;j})\, e^i{}_{h|} e^j{}_{h|} = (r\Omega_{ij} - t_{i;j})\, g^{ij}. \tag{4.2}$$

Since the expression on the right is an invariant, we have the theorem: *The sum of the generalized normal curvatures of a hypersurface V_n relative to a given congruence, for* n *mutually orthogonal directions at a point is invariant and is equal to* $(r\Omega_{ij} - t_{i;j})g^{ij}$. This sum may be called *the mean generalized curvature of the hypersurface* at the point considered, relative to the given congruence. We shall denote it by M_λ. Thus

$$M_\lambda = (r\Omega_{ij} - t_{i;j})\, g^{ij} \tag{4.3}$$

If the congruence be one of normals to the hypersurface we have the known result:

The sum of the normal curvatures of a hypersurface V_n for n *mutually orthogonal directions at a point is invariant and equal to* $\Omega_{ij}g^{ij}$ (Weatherburn, 1950).

From the definition of the generalized normal curvature it also follows that *the generalized normal curvature of a hypersurface for any direction is the negative tendancy of the unit tangent to the curve of the congruence through the point in that direction.* But the sum of the tendancies of λ for n mutually orthogonal directions in V_n is equal to $div_n\lambda$.

Hence

$$M_{\lambda} = -\, div_{n}\lambda, \tag{4.4}$$

which may be stated as follows:

The mean generalized curvature of a hypersurface relative to a given congruence at any point is equal to the negative of the divergence of the unit tangent to the curve of the congruence through the point.

If the congruence be one of normals to the hypersurface, we have the known result:

The mean curvature of a hypersurface is equal to the negative of the divergence of the unit normal (Weatherburn, 1950).

5. *Principal generalized curvatures. Lines of generalized curvature.*

The generalized normal curvature of V_n for a variable direction **p** is given by

$$k_{\lambda} = \frac{(r\Omega_{ij} - t_{i;j})\, p^i p^j}{g_{ij} p^i p^j}. \tag{5.1}$$

The stationary values of k_{λ} are those determined by the directions for which

$$\frac{dk_{\lambda}}{dp^j} = 0\ (j = 1, 2, \ldots. n).$$

Writing (5.1) in the form

$$(r\Omega_{ij} - t_{i;j})\, p^i p^j - k_{\lambda} g_{ij} p^i p^j = 0$$

and differentiating with respect to p^j, we see that the directions corresponding to the maximum and minimum values of k_{λ} are those which satisfy the equations

$$[r\Omega_{ij} - \tfrac{1}{2}(t_{i;j} + t_{j;i}) - k_{\lambda} g_{ij}]\, p^i = 0 \tag{5.2}$$

and the maximum and minimum values of k_{λ} are the roots of the equation

$$|\, r\Omega_{ij} - \tfrac{1}{2}(t_{i;j} + t_{j;i}) - k_{\lambda} g_{ij}\,| = 0 \tag{5.3}$$

The directions satisfying (5.2) are the principal directions in V_n determined by the symmetric part of the covariant tensor $(r\Omega_{ij} - t_{i;j})$. We shall therefore call the directions determined by (5.2) as the *principal generalized directions*, and the maximum and minimum values of (5.1) [which are the generalized curvatures of V_n in the principal generalized directions and are the roots of the equation (5.3)] as the *principal generalized curvatures*. A curve whose direction at each point is a principal generalized direction is a *line of generalized curvature* in V_n. There are thus n congruences of lines of generalized curvature in V_n corresponding to a given congruence λ in V_n. If the roots of (5.3) are all simple, the principal generalized directions relative to a given congruence are uniquely determined and are mutually orthogonal. Corresponding to a multiple root of order m it is possible in a multiply-infinite number of ways, to choose m mutually orthogonal principal generalized directions. These are orthogonal to the principal generalized directions determined by the other roots of (5.3). Also the principal generalized directions satisfy the equations

$$[r\Omega_{ij} - \tfrac{1}{2}(t_{i;j} + t_{j;i})]\, p^i_{h|} p^j_{k|} = 0 \; (h \neq k) \tag{5.4}$$

Let $\mathbf{e}_{h|}$ be the unit tangents to the n congruences of lines of generalized curvature. Then the principal generalized curvatures are given by

$$k_{\lambda|h} = (r\Omega_{ij} - t_{i;j})\, e^i_{h|} e^j_{h|}, \; (h = 1, 2, \ldots . n), \tag{5.5}$$

Any other unit vector $\mathbf{a}$ in V_n is expressible in the form

$$\mathbf{a} = \sum_h \mathbf{e}_{h|} \cos \alpha_h \tag{5.6}$$

where

$$\cos \alpha_h = \mathbf{a} \cdot \mathbf{e}_{h|} = a^i e_{h|i} \quad ,$$

a^i being the contravariant components of $\mathbf{a}$ and α_h being the inclination of $\mathbf{a}$ to $\mathbf{e}_{h|}$. The generalized normal curvature of V_n for the direction of $\mathbf{a}$ relative to the congruence is given by

$$k_\lambda = (r\Omega_{ij} - t_{i;j})\, a^i a^j \quad ,$$

Substituting in the second member the values of a^i corresponding to (5.6) we obtain,

$$k_\lambda = \sum_h k_{\lambda|h} \cos^2 \alpha_h . \tag{5.7}$$

The sum of the principal generalized curvatures, being the sum of the generalized normal curvatures for n mutually orthogonal directions in V_n, is equal to the mean generalized curvature.

Thus

$$\sum_h k_{\lambda|h} = M_\lambda = - div_n \lambda . \tag{5.8}$$

For the congruence of normals to V_n, the principal generalized directions, the principal generalized curvatures, and the lines of generalized curvature are the principal directions, the principal curvatures, and the lines of curvature respectively. In this case (5.7) reduces to Euler's formula for the normal curvature of a hypersurface V_n imbedded in V_{n+1} (Weatherburn, 1950).

6. *Generalized conjugate directions and generalized asymptotic directions.* The directions of two vectors, **a** and **b**, at a point in V_n, may be said to be *generalized conjugate* relative to a congruence of curves in V_{n+1} if

$$[r\Omega_{ij} - \tfrac{1}{2}(t_{i;j} + t_{j;i})]\, a^i b^j = 0 \tag{6.1}$$

and two congruences of curves in the hypersurface may be said to be generalized conjugate relative to a congruence of curves in V_{n+1} if the directions of the two curves through any point are generalized conjugate. It follows from (5.4) that *the principal generalized directions at a point of a hypersurface are generalized conjugate.*

Also we have the following more general theorem:

A vector at a point of a hypersurface V_n, whose components are linear combinations of the components of p *vectors tangent to lines of generalized curvature relative to a given congruence in V_n, is generalized conjugate to the vector whose components are linear combinations of the remaining* n–p *vectors tangent to lines of generalized curvature.*

For the congruence of normals to the hypersurface these reduce to the following known results:

1. *The directions of two lines of curvature at a point of a hypersurface are conjugate.*

2. *A vector at a point of a hypersurface whose components are linear combinations of the components of* p *vectors tangent to the lines of curvature is conjugate to the vector whose components are linear combinations of the remaining* n–p *vectors tangent to lines of curvature* (Eisenhart, 1949).

A direction in V_n, which is generalized self-conjugate, is said to be *generalized asymptotic.* Thus the condition that the direction of the vector **a** at a point of V_n be generalized asymptotic is

$$[r\Omega_{ij} - \tfrac{1}{2}(t_{i;j} + t_{j;i})]\, a^i a^j = 0,$$

which may be put in the form

$$(r\Omega_{ij} - t_{i;j})\, a^i a^j = 0. \tag{6.2}$$

A generalized asymptotic line in a hypersurface relative to a given congruence in V_{n+1} is a curve, whose direction at every point is generalized asymptotic. The differential equation of such a line is

$$(r\Omega_{ij} - t_{i;j})\, dx^i dx^j = 0. \tag{6.3}$$

From (6.2) and (4.1) it follows that *the generalized normal curvature of a hypersurface in a generalized asymptotic direction is zero.* Relative to the congruence of normals to the hypersurface, the generalized conjugate directions are the conjugate directions and the generalized asymptotic directions reduce to asymptotic directions.

From (3.3) we see that if **a** and **b** be unit vectors in V_n, the resolved part along **b** of the derived vector of $\boldsymbol{\lambda}$ along **a** is

$$-[r\Omega_{ij} - t_{i;j}]\, a^i b^j \tag{6.4}$$

Using (6.1) and (6.4) we find that a pair of generalized conjugate directions in V_n relative to a congruence in V_{n+1} is such that *the resolved part in one direction of the derived vector of the unit tangent to the congruence in the other direction is the negative of the resolved part in the second direction, of the derived vector of the unit tangent to the congruence in the first direction.* For the particular case in which the direction is generalized self-conjugate, we may state the theorem as follows:

The derived vector of the unit tangent to a congruence in V_{n+1} along a curve of the hypersurface will be orthogonal to the curve provided the curve be a generalized asymptotic line in the hypersurface, relative to the congruence.

If the congruence be one of normals to V_n, these reduce to the following known results:

1. *Conjugate directions in a hypersurface are such that the derived vector of the unit normal in either direction is orthogonal to the other direction* (Weatherburn, 1950).

2. *The derived vector of the unit normal along a curve of the hypersurface will be orthogonal to the curve provided the curve be an asymptotic line in the hypersurface* (Weatherburn, 1950).

Summary

In the present paper a congruence of curves through points of a hypersurface V_n imbedded in a Riemannian V_{n+1} has been considered. In analogy with the normal curvature of a curve C in V_n, the generalized normal curvature of C at any point of it, relative to the curve of the congruence through

that point, has been defined as the negative of the resolved part along C, of the derived vector of the unit tangent to the curve of the congruence through the point along C. The concepts of normal curvature of a hypersurface, principal directions, principal curvatures, lines of curvature, conjugate directions, asymptotic directions and asymptotic lines have been generalized and generalizations of several known theorems on the curvature of a hypersurface V_n in V_{n+1} have been obtained.

REFERENCES

Eisenhart, L. P. .. *Riemannian Geometry*, 1949, 156.

Weatherburn, C. E. .. *An Introduction to Riemannian Geometry and the Tensor Calculus*, 1950, 121–39.