Inventiones math. 36, 1 – 55 (1976)

Inventiones mathematicae

Harmonic Analysis on Real Reductive Groups. II

Wave-Packets in the Schwartz Space

Harish-Chandra (Princeton)

To Jean-Pierre Serre

§ 1. Introduction

The theory of the constant term, which has been developed in [1(e)] will now be applied to construct wave-packets in the Schwartz space of a reductive group G. Keeping to the notation of [1(e)], let A be the split component of a θ-stable Cartan subgroup of G. Fix a psgp $P_1 = MAN_1$ with the split component A and let τ be a unitary double representation of K on a finite-dimensional Hilbert space V. Then $L = {}^0\mathscr{C}(M, \tau_M)$ also has finite dimension [1(e), Theorem 27.3]. Put $\mathfrak{F} = \mathfrak{a}^*$ and consider the Eisenstein integral

$$\phi_\nu = E(P_1 : \psi : \nu) \qquad (\nu \in \mathfrak{F})$$

for a given $\psi \in L$. We compute the constant term ϕ_{ν, P_2} of ϕ_ν along a psgp $P_2 \in \mathscr{P}(A)$ (Theorem 18.1). The expression for ϕ_{ν, P_2} involves certain endomorphisms $c_{P_2|P_1}(s : \nu)$ $(s \in \mathfrak{w}(\mathfrak{a}))$ of L. We shall see later that these c-functions can be extended to meromorphic functions of ν on the whole complex space $\mathfrak{F}_c$.

Let $\mathfrak{F}'$ be the set of all regular elements in $\mathfrak{F}$. Fix $\alpha \in C_c^\infty(\mathfrak{F}')$ and put

$$\phi_\alpha = \int_{\mathfrak{F}} \alpha(\nu) \phi_\nu \, d\nu$$

where $d\nu$ is the Euclidean measure on $\mathfrak{F}$. Then $\phi_\alpha \in \mathscr{C}(G, \tau)$ (Theorem 13.1). Now fix $P_2 \in \mathscr{P}(A)$ and $m \in MA$ and consider the distribution

$$\alpha \to \phi_\alpha^{(P_2)}(m)$$

on $\mathfrak{F}'$. It turns out that this distribution is actually a function which can be written quite simply in terms of the c-functions (Theorem 19.2).

Theorems 13.1, 13.2 and 18.1 contain the main results of this paper. They may be regarded as generalizations of the corresponding results on spherical functions obtained in [1(a, b)]. In fact here we have combined the methods of [1(a, b)] with those of [1(d)] and our success depends in an essential way on the systematic use of the weak inequality.

As far as possible, we shall keep to the notation of [1(e)] and therefore any undefined symbols should be given the same meaning as in [1(e)].

Most of the work presented here was done some years ago and I have given lectures on it on various occasions.

§ 2. Recapitulation of Some Algebraic Results

Let $(P, A) \succ (P_0, A_0)$ be two p-pairs in G such that (P_0, A_0) is minimal. Then $P = MAN$, $P_0 = M_0 A_0 N_0$. Extend $\mathfrak{a}_0$ to a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}$. Then $\mathfrak{h}_0$ is θ-stable and $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}$. Put $W_0 = W(\mathfrak{g}/\mathfrak{h}_0)$ and let W_1 be the subgroup of those elements of W_0 which leave $\mathfrak{a}$ pointwise fixed. Put $S = \mathfrak{S}(\mathfrak{h}_{0c}) = S(\mathfrak{h}_{0c})$ and let J and J_1 denote the algebras of invariants of W_0 and W_1 respectively in S. Let $s_1, s_2, \ldots, s_q$ ($q = [W_0 : W_1]$) be a complete system of representatives for $W_1 \setminus W_0$ so that

$$W_0 = \bigcup_{1 \leqq i \leqq q} W_1 s_i.$$

Select homogeneous elements $u_1 = 1, u_2, \ldots, u_q$ in J_1 such that [1(a), Lemma 8]

$$J_1 = \sum_{1 \leqq i \leqq q} J u_i.$$

Fix a system of positive roots for $(\mathfrak{g}, \mathfrak{h}_0)$ and put

$$\varpi_0 = \varpi_{\mathfrak{g}/\mathfrak{h}_0}, \qquad \varpi_1 = \varpi_{\mathfrak{m}_1/\mathfrak{h}_0}, \qquad \varpi_{01} = \varpi_{\mathfrak{g}/\mathfrak{m}_1},$$

where $\mathfrak{m}_1 = \mathfrak{m} + \mathfrak{a}$. Then $\varpi_0 = \varpi_{01} \varpi_1$. Define $u^j \in C(J_1)$ by

$$\operatorname{tr}_{J_1/J}(u_i u^j) = \delta_i^j \qquad (1 \leqq i,\ j \leqq q)$$

and put $\tau^j = \varpi_{01} u^j$. Then [1(a), Lemma 12] $\tau^j \in J_1$.

Every element of S may be regarded as a polynomial function on $\mathfrak{h}_{0c}^*$. For $p \in J_1$ and $\Lambda \in \mathfrak{h}_{0c}^*$, define

$$f_\Lambda = \sum_{1 \leqq i \leqq q} \tau^i(\Lambda) u_i,$$

$$v^j(p:\Lambda) = \operatorname{tr}_{J_1/J}\{(p - p(\Lambda)) f_\Lambda u^j\} \qquad (1 \leqq j \leqq q).$$

Then it is clear that $v^j(p:\Lambda) \in J$ and, for p fixed, $\Lambda \mapsto v^j(p:\Lambda)$ is a polynomial mapping of $\mathfrak{h}_{0c}^*$ into J. Let S_Λ denote the set of all $p \in S$ such that $p(\Lambda) = 0$. Put $J_\Lambda = J \cap S_\Lambda$. Then it is obvious that $J_{s\Lambda} = J_\Lambda$ ($s \in W_0$).

Identify $\mathfrak{h}_0$ with its dual by means of the bilinear form B. We call an element $u \in J_1$ harmonic if $\partial(p) u = 0$ for all $p \in J \cap S_0$ in the notation of [1(c), § 3]. Then it is easy to conclude from [1(c), Lemma 4] that $u_1, \ldots, u_q$ may be so chosen as to span the space U of all harmonic elements in J_1. Moreover $J_1 = U + J_1 J_\Lambda$ where the sum is direct [1(a), p. 256]. The following lemma enables us to diagonalize the action of J_1 on $J_1/J_1 J_\Lambda \simeq U$.

Lemma 1. *Fix* $p \in J_1$, $\Lambda \in \mathfrak{h}_{0c}^*$ *and put* $\Lambda_i = s_i \Lambda$ $(1 \leqq i \leqq q)$. *Then*

1) $v^j(p:\Lambda_i) \in J_\Lambda$,
2) $(p - p(\Lambda_i)) f_{\Lambda_i} = \sum_{1 \leqq j \leqq q} v^j(p:\Lambda_i) u_j$,
3) $\sum_{1 \leqq k \leqq q} \varepsilon(s_k) \varpi_1(\Lambda_k) f_{\Lambda_k} = \varpi_0(\Lambda)$

for $1 \leqq i \leqq q$. *Here* $\varepsilon(s) = \pm 1$ *is defined as usual by* $\varpi_0^s = \varepsilon(s) \varpi_0$ $(s \in W_0)$.

We know from [1(a), Lemma 15] that

$$(p - p(\Lambda)) f_\Lambda \in SJ_\Lambda \cap J_1 = J_\Lambda J_1 .$$

Hence the first two statements are obvious. Both sides of 3) being polynomial in Λ, it is sufficient to consider the case when $\varpi_0(\Lambda) \neq 0$. Then the rational function u^j is defined at Λ_k and

$$\sum_k \varepsilon(s_k) \varpi_1(\Lambda_k) f_{\Lambda_k} = \varpi_0(\Lambda) \sum_{i,k} u^i(\Lambda_k) u_i .$$

But since

$$\sum_k (u^i)^{s_k^{-1}} = \mathrm{tr}_{J_1/J}\, u^i = \delta_1^i ,$$

we conclude that

$$\sum_{i,k} u^i(\Lambda_k) u_i = 1$$

and this proves 3).

§ 3. Further Algebraic Results

Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{h}_I + \mathfrak{h}_R$ as usual [1(e), § 8]. If $\lambda \in (\mathfrak{h}_I)_c^*$, $\nu \in (\mathfrak{h}_R)_c^*$, we extend them to linear functions on $\mathfrak{h}_c$ by defining $\lambda = 0$ on $\mathfrak{h}_R$ and $\nu = 0$ on $\mathfrak{h}_I$. In this way $\mathfrak{h}_c^*$ becomes the direct sum of $(\mathfrak{h}_I)_c^*$ and $(\mathfrak{h}_R)_c^*$.

An element $\lambda \in (\mathfrak{h}_I)_c^*$ is called singular if $\lambda(H_\alpha) = 0$ for some imaginary root α of $(\mathfrak{g}, \mathfrak{h})$. Otherwise we call it regular. Put $\mathfrak{F} = \mathfrak{h}_R^*$ and

$$\varpi = \varpi_{\mathfrak{g}/\mathfrak{h}} = \prod_{\alpha > 0} H_\alpha$$

where α runs over all positive roots of $(\mathfrak{g}, \mathfrak{h})$ (under some fixed order). Fix a regular element $\lambda \in (-1)^{1/2} \mathfrak{h}_I^*$ and let $\mathfrak{F}_c'(\lambda)$ denote the set of all $\nu \in \mathfrak{F}_c$ such that

$$\varpi(\lambda + (-1)^{1/2} \nu) \neq 0 .$$

Put $\mathfrak{F}'(\lambda) = \mathfrak{F} \cap \mathfrak{F}_c'(\lambda)$. Then $\mathfrak{F}'(\lambda)$ is an open and dense subset of $\mathfrak{F}$.

Now we use the notation of § 2. Fix $k_0 \in K$ such that $\mathfrak{h}_R^{k_0} \subset \mathfrak{a}_0$. Let $\mathfrak{z}$ denote the centralizer of $\mathfrak{h}_R^{k_0}$ in $\mathfrak{g}$. Then $\mathfrak{h}^{k_0}$ and $\mathfrak{h}_0$ are two Cartan subalgebras of $\mathfrak{z}$. Hence we can choose $y_0 \in G_c$ such that y_0 centralizes $\mathfrak{h}_R^{k_0}$ and $\mathfrak{h}_c^y = \mathfrak{h}_{0c}$ where $y = y_0 \mathrm{Ad}(k_0)$. Put $\Lambda_\nu = (\lambda + (-1)^{1/2} \nu)^y$ for $\nu \in \mathfrak{F}_c$. (Here we have identified $\mathfrak{h}_c$ with its dual by means of the restriction of the bilinear form B on $\mathfrak{h}_c$.) Then if $\nu \in \mathfrak{F}_c'(\lambda)$, it is clear that $\varpi_0(\Lambda_\nu) \neq 0$ and therefore the rational functions u^i are defined at Λ_ν.

Fix an element $\Lambda \in \mathfrak{h}_{0c}^*$ and let $W_0(\Lambda)$ be the subgroup of all $s \in W_0$ which leave Λ fixed. Let p_0 be the set of all positive roots of $(\mathfrak{g}, \mathfrak{h}_0)$ and $p_0(\Lambda)$ the set of those $\alpha \in p_0$ for which $\Lambda(H_\alpha) \neq 0$. Put

$$\varpi_{0,\Lambda} = \prod_{\alpha \in p_0(\Lambda)} H_\alpha.$$

Let $J(\Lambda)$ be the algebra of all invariants of $W_0(\Lambda)$ in S.

Lemma 1. *Let v be an element in $\mathbf{C}(S)$ such that $\operatorname{tr}_{S/J}(uv) \in J$ for all $u \in S$. Then $\varpi_{0,\Lambda} \operatorname{tr}_{S/J(\Lambda)}(v) \in S$.*

Put $v' = \operatorname{tr}_{S/J(\Lambda)} v$. Then if $u \in J(\Lambda)$, it is clear that

$$\operatorname{tr}_{J(\Lambda)/J}(v'u) = \operatorname{tr}_{S/J}(vu) \in J.$$

Hence we conclude from [1(a), Lemma 12] that $\varpi_{0,\Lambda} v' \in S$.

Now put

$$U = \sum_{1 \leq i \leq q} \mathbf{C} u_i$$

and $\varpi_{s,\lambda} = \varpi_{0,\Lambda}$ where $\Lambda = s\lambda^\nu$ $(s \in W_0)$. Define a rational mapping e_s $(s \in W_0)$ of $\mathfrak{F}_c$ into U by

$$e_s(\nu) = \sum_{1 \leq j \leq q} u^j(s\Lambda_\nu) u_j \qquad (\nu \in \mathfrak{F}_c'(\lambda)).$$

Since $u^j \in \mathbf{C}(J_1)$, it is clear that $e_{ts} = e_s$ $(t \in W_1)$.

Put $W_0(s, \lambda) = W_0(s\lambda^\nu)$.

Lemma 2. *Fix $s \in W_0$. Then the mapping*

$$\nu \mapsto \varpi_{s,\lambda}(s\Lambda_\nu) \sum_{t \in W_0(s,\lambda)} e_{ts}(\nu)$$

is a polynomial mapping of $\mathfrak{F}_c$ into U.

Let $u \in S$ and put $u' = \operatorname{tr}_{S/J_1} u$. Then $u' \in J_1$ and it is obvious that

$$\operatorname{tr}_{S/J}(u^j u) = \operatorname{tr}_{J_1/J}(u^j u') \in J \qquad (1 \leq j \leq q).$$

Hence we conclude from Lemma 1 that

$$\varpi_{0,\Lambda} \operatorname{tr}_{S/J(\Lambda)} u^j \in S$$

where $\Lambda = s\lambda^\nu$. Since $\mathbf{C}(J(\Lambda))$ is the fixed field of $W_0(\Lambda) = W_0(s,\lambda)$ in $\mathbf{C}(S)$, it follows that

$$\operatorname{tr}_{S/J(\Lambda)} u^j = \sum_{t \in W_0(s,\lambda)} (u^j)^t.$$

Hence the mapping

$$\nu \mapsto \varpi_{s,\lambda}(s\Lambda_\nu) \sum_{t \in W_0(s,\lambda)} \sum_j u^j(ts\Lambda_\nu) u_j$$
$$= \varpi_{s,\lambda}(s\Lambda_\nu) \sum_{t \in W_0(s,\lambda)} e_{ts}(\nu) \qquad (\nu \in \mathfrak{F}'_c(\lambda))$$

extends to a polynomial mapping of $\mathfrak{F}_c$ into U.

Let $p(\lambda)$ be the set of all positive roots α of $(\mathfrak{g}, \mathfrak{h})$ such that $\lambda(H_\alpha) \neq 0$. Put

$$\varpi_\lambda = \prod_{\alpha \in p(\lambda)} H_\alpha.$$

Lemma 3. *Fix $s \in W_0$ and $\nu \in \mathfrak{F}$. Then*

$$|\varpi_{s,\lambda}(s\Lambda_\nu)| \geqq |\varpi_{s,\lambda}(s\lambda^\nu)| = |\varpi_\lambda(\lambda)| > 0.$$

This is obvious from the definitions.

Now put $e_i = e_{s_i}$ and

$${}_ie = [W_1 \cap W_0(s_i,\lambda)]^{-1} \sum_{t \in W_0(s_i,\lambda)} e_{ts_i} \qquad (1 \leqq i \leqq q).$$

Let Q denote the set $\{1, 2, \ldots, q\}$. It is clear that ${}_ie = {}_je$ if $s_i\lambda^\nu = s_j\lambda^\nu$ $(i, j \in Q)$. Choose a maximal subset ${}^oQ$ of Q such that $s_i\lambda^\nu \neq s_j\lambda^\nu$ for $i \neq j$ in ${}^oQ$.

Lemma 4. *Fix $i \in Q$. Then ${}_ie$ is a rational mapping of $\mathfrak{F}_c$ into U which is everywhere defined on $\mathfrak{F}$. Moreover the mapping*

$$\nu \mapsto \varpi_{s_i,\lambda}(s_i\Lambda_\nu)\,{}_ie(\nu) \qquad (\nu \in \mathfrak{F}'_c(\lambda))$$

extends to a polynomial mapping from $\mathfrak{F}_c$ into U. Finally

$$\sum_{i \in {}^oQ} {}_ie = 1.$$

The first two statements follow from Lemmas 2 and 3. Moreover since $\mathrm{tr}_{J_1/J}\, u^j = \delta^j_1$, it is clear that

$$\sum_{1 \leqq i \leqq q} e_i = 1.$$

The third statement is an immediate consequence of this fact.

Put

$$v_{ij}(p:\nu) = \mathrm{tr}_{J_1/J}\{p - p(s_i\Lambda_\nu) u^j e_i(\nu)\} \qquad (\nu \in \mathfrak{F}'_c(\lambda),\ 1 \leqq i,\ j \leqq q)$$

for $p \in J_1$. Then $v_{ij}(p:\nu) \in J$.

Lemma 5. *Fix $\nu \in \mathfrak{F}'_c(\lambda)$ and $p \in J_1$. Then*

1) $v_{ij}(p:\nu) \in J_{\Lambda_\nu}$,
2) $(p - p(s_i\Lambda_\nu)) e_i(\nu) = \sum_{1 \leqq k \leqq q} v_{ik}(p:\nu) u_k$,
3) $\sum_{1 \leqq k \leqq q} e_k(\nu) = 1$,

for $1 \leqq i, j \leqq q$.

This follows immediately from Lemma 2.1.

We know from [1 (a), p. 256] that $J_1 = U + J_1 J_\mu$ for $\mu \in \mathfrak{h}^*_{0c}$, the sum being direct. Hence for any $\nu \in \mathfrak{F}_c$, we can define a representation Γ_ν of J_1 on U as follows. For $p \in J_1$, $\Gamma_\nu(p)$ is the linear transformation on U given by

$$\Gamma_\nu(p)u \equiv pu \bmod J_1 J_{\Lambda_\nu} \qquad (u \in U).$$

Corollary 1. *Fix* $\nu \in \mathfrak{F}'_c(\lambda)$. *Then*

$$\Gamma_\nu(p)\, e_i(\nu) = p(s_i \Lambda_\nu)\, e_i(\nu), \qquad \Gamma_\nu(e_i(\nu))\, e_j(\nu) = \delta_{ij}\, e_j(\nu)$$

for $p \in J_1$ *and* $1 \leqq i, j \leqq q$. *Moreover*

$$U = \sum_{1 \leqq i \leqq q} \mathbf{C}\, e_i(\nu).$$

This follows from Lemma 5 if we note that [1 (a), p. 259]

$$e_i(\nu : s_j \Lambda_\nu) = \sum_k u^k(s_i \Lambda_\nu)\, u_k(s_j \Lambda_\nu) = \delta_{ij}.$$

Corollary 2. $\Gamma_\nu(p\, e_i(\nu)) = p(s_i \Lambda_\nu)\, \Gamma_\nu(e_i(\nu))$ *and*

$$\Gamma_\nu(e_i(\nu)\, e_j(\nu)) = \delta_{ij}\, \Gamma(e_j(\nu))$$

for $1 \leqq i, j \leqq q$ *and* $\nu \in \mathfrak{F}'_c(\lambda)$.

This is obvious from Corollary 1 above.

Corollary 3. *For any* $p \in J_1$, $\nu \mapsto \Gamma_\nu(p)$ *is a polynomial mapping of* $\mathfrak{F}_c$ *into* End U.

Put $p_i^j = \mathrm{tr}_{J_1/J}(p\, u_i u^j) \in J$. It would be enough to verify that

$$\Gamma_\nu(p)\, u_i = \sum_j p_i^j(\Lambda_\nu)\, u_j \qquad (\nu \in \mathfrak{F}_c).$$

By Corollary 1 above, the left side is a rational function of ν. Hence it would be sufficient to prove this for $\nu \in \mathfrak{F}'_c(\lambda)$. Fix $\nu \in \mathfrak{F}'_c(\lambda)$. Then

$$\begin{aligned}\Gamma_\nu(p)\, u_i &= \Gamma_\nu(p\, u_i)\, 1 = \sum_k \Gamma_\nu(p\, u_i)\, e_k(\nu)\\ &= \sum_k p(s_k \Lambda_\nu)\, u_i(s_k \Lambda_\nu)\, e_k(\nu) \qquad \text{from Corollary 1,}\\ &= \sum_{k,j} p(s_k \Lambda_\nu)\, u_i(s_k \Lambda_\nu)\, u^j(s_k \Lambda_\nu)\, u_j.\end{aligned}$$

But [1 (a), p. 258]

$$\sum_k (p\, u_i u^j)^{s_k^{-1}} = \mathrm{tr}_{J_1/J}(p\, u_i u^j) = p_i^j$$

and therefore the required statement is obvious.

Corollary 4. *Let* $p \in J_1$. *Then*

$$\prod_{1 \leqq i \leqq q} \{\Gamma_\nu(p) - p(s_i \Lambda_\nu)\} = 0 \qquad (\nu \in \mathfrak{F}_c).$$

If $\nu \in \mathfrak{F}_c'(\lambda)$, then $e_i(\nu)$ $(1 \leqq i \leqq \nu)$ is a base for U and so our statement is obvious from Corollary 1. The rest follows from Corollary 3.

Corollary 5. *Fix* $i \in Q$. *Then*

$$\prod_{t \in W_0(s_i, \lambda)} (\Gamma_\nu(p) - p(t s_i \Lambda_\nu))\, \Gamma_\nu({}_i e(\nu)) = 0$$

for $\nu \in \mathfrak{F}$.

This is proved in the same way by taking Lemma 4 into account.

§ 4. Application to Differential Operators

We keep to the notation of §§ 2, 3. Put $\gamma_0 = \gamma_{\mathfrak{g}/\mathfrak{h}_0}$ and $\gamma_1 = \gamma_{\mathfrak{m}_1/\mathfrak{h}_0}$ (see [1(e), § 11]) where $\mathfrak{m}_1 = \mathfrak{m} + \mathfrak{a}$ as in § 2. Also define $\mathfrak{M}_1 = \mathfrak{M}\mathfrak{A}$ and $\mathfrak{Z}_1 = \mathfrak{Z}_M \mathfrak{A}$. (As usual $\mathfrak{Z}_M$ is the center of $\mathfrak{M}$.) Finally put

$$\eta_i(\nu) = \gamma_1^{-1}(f_{s_i \Lambda_\nu}) \in \mathfrak{Z}_1$$

$$z_{ij}(\zeta : \nu) = \gamma_0^{-1}(v^j(\gamma_1(\zeta) : s_i \Lambda_\nu)) \in \mathfrak{Z} \qquad (1 \leqq i,\ j \leqq q)$$

for $\zeta \in \mathfrak{Z}_1$ and $\nu \in \mathfrak{F}_c$ in the notation of Lemma 2.1. (Here $\Lambda_\nu = (\lambda + (-1)^{1/2} \nu)^y$ as in § 3.) Then for fixed i, j and ζ, $\nu \mapsto \eta_i(\nu)$ and $\nu \mapsto z_{ij}(\zeta : \nu)$ are polynomial mappings of $\mathfrak{F}_c$ into $\mathfrak{Z}_1$ and $\mathfrak{Z}$ respectively.

Put $\gamma = \gamma_{\mathfrak{g}/\mathfrak{h}}$ and $\mu = \gamma_{\mathfrak{g}/\mathfrak{m}_1}$ so that $\gamma_0 = \gamma_1 \circ \mu$ [1(e), § 11].

Lemma 1. *Define* $w_j = \gamma_1^{-1}(u_j) \in \mathfrak{Z}_1$. *Then*

1) $\gamma(z_{ij}(\zeta : \nu) : \lambda + (-1)^{1/2} \nu) = 0$,

2) $\zeta \eta_i(\nu) - \gamma_i(\zeta : s_i \Lambda_\nu) \eta_i(\nu) = \sum_{1 \leqq j \leqq q} \mu(z_{ij}(\zeta : \nu)) w_j$

for $\zeta \in \mathfrak{Z}_1$, $\nu \in \mathfrak{F}_c$ *and* $1 \leqq i,\ j \leqq q$.

This follows from 1) and 2) of Lemma 2.1.

Put $d(m) = d_P(m)$ [1(e), § 21] for $m \in M_1 = MA$ and define $v' = d^{-1} v \circ d$ $(v \in \mathfrak{M}_1)$ as usual [1(d), § 45]. Let

$$g_i(\zeta : \nu) = -\sum_{1 \leqq j \leqq q} \{z_{ij}(\zeta : \nu) - \mu(z_{ij}(\zeta : \nu))'\}\, w_j' \qquad (1 \leqq i \leqq q)$$

for $\zeta \in \mathfrak{Z}_1$ and $\nu \in \mathfrak{F}_c$.

Corollary. $g_i(\zeta : \nu) \in \theta(\mathfrak{n}) \mathfrak{G}\, \mathfrak{n}$ *and*

$$\zeta' \eta_i(\nu)' - \gamma_1(\zeta : s_i \Lambda_\nu) \eta_i(\nu)' = \sum_j z_{ij}(\zeta : \nu) w_j' + g_i(\zeta : \nu) \qquad (1 \leqq i \leqq q)$$

for $\zeta \in \mathfrak{Z}_1$ *and* $\nu \in \mathfrak{F}_c$. *Moreover for i and* ζ *fixed,* $\nu \mapsto g_i(\zeta : \nu)$ *is a polynomial mapping of* $\mathfrak{F}_c$ *into* $\theta(\mathfrak{n}) \mathfrak{G}\, \mathfrak{n}$.

This is obvious from the above lemma if we recall [1(d), p. 110] that

$$z - \mu(z)' \in \theta(\mathfrak{n}) \mathfrak{G}\, \mathfrak{n} \qquad (z \in \mathfrak{Z}).$$

§ 5. The Basic Differential Equations

Let V be a complete, locally convex, Hausdorff space and τ a differentiable double representation of K on V [1 (e), § 19]. Fix $\nu \in \mathfrak{F}_c$ and let ϕ be an element in $C^\infty(G, \tau)$ [1 (e), § 19] such that

$$z\phi = \gamma(z : \lambda + (-1)^{1/2}\nu)\phi \qquad (z \in \mathfrak{Z}).$$

Put

$$\phi_i(m) = d_P(m)\phi(m; \eta_i(\nu)') \qquad (m \in M_1).$$

Lemma 1. *Let* $m \in M_1$. *Then*

$$\varpi_0(\Lambda_\nu) d_P(m)\phi(m) = \sum_{1 \leqq i \leqq q} \varepsilon(s_i)\varpi_1(s_i\Lambda_\nu)\phi_i(m)$$

and

$$\phi_i(m; \zeta) = \gamma_1(\zeta : s_i\Lambda_\nu)\phi_i(m) + d_P(m)\phi(m; g_i(\zeta : \nu)) \qquad (1 \leqq i \leqq q)$$

for $\zeta \in \mathfrak{Z}_1$.

This follows from the corollary of Lemma 4.1.

Let α be a root of (P_0, A_0). Fix $X \in \mathfrak{n}_0$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}_0$.

Lemma 2. *Let* $g_1, g_2 \in \mathfrak{G}$ *and* $h \in A_0$. *Then*

$$\phi(g_1 ; h; \theta(X) g_2) = e^{-\alpha(\log h)}\phi(g_1\theta(X) ; h; g_2)$$

and

$$\phi(g_1 X ; h; g_2) = e^{-\alpha(\log h)}\phi(g_1 ; h; X g_2).$$

This is obvious.

Define

$$\psi_{i,\zeta}(m) = d_P(m)\phi(m; g_i(\zeta : \nu)) \qquad (1 \leqq i \leqq q,\ m \in M_1)$$

for $\zeta \in \mathfrak{Z}_1$. It is clear that $\psi_{i,\zeta}$ depends linearly on ζ. Since $\mathfrak{a} \subset \mathfrak{Z}_1$, the following result is an immediate consequence of Lemma 1.

Lemma 3.

$$\phi_i(m \exp TH) e^{-Ts_i\Lambda_\nu(H)} = \phi_i(m) + \int_0^T \psi_{i,H}(m \exp tH) e^{-ts_i\Lambda_\nu(H)} dt \qquad (1 \leqq i \leqq q)$$

for $m \in M_1$, $H \in \mathfrak{a}$ *and* $T \in \mathbf{R}$.

§ 6. Asymptotic Behavior of Eigenfunctions

For $\nu \in \mathfrak{F}$, let $\mathscr{A}(G, \tau, \lambda, \nu) = \mathscr{A}(\lambda, \nu) = \mathscr{A}(\nu)$ denote the space of all $\phi \in \mathscr{A}(G, \tau)$ [1 (e), § 21] such that

$$z\phi = \gamma(z : \lambda + (-1)^{1/2}\nu)\phi \qquad (z \in \mathfrak{Z}).$$

Fix $\nu \in \mathfrak{F}$, $\phi \in \mathscr{A}(\nu)$ and let us use the notation of § 5. Our object is to study the asymptotic behavior of ϕ_i. Put $M_1^+ = K_1 \cdot Cl(A_0^+) \cdot K_1$ as in [1 (e), § 22] where $K_1 =$

$K_M = K \cap M$. The following lemma is proved in the same way as [1(e), Lemma 22.1].

Lemma 1. *Fix* $\zeta \in \mathfrak{Z}_1$, v_1, $v_2 \in \mathfrak{M}_1$ *and* $\mathbf{s} \in \mathscr{S}(V)$. *Then we can choose numbers* c, $r \geqq 0$ *such that*

$$|\psi_{i,\zeta}(v_1; m \exp H; v_2)|_{\mathbf{s}} \leqq c\,\Xi_M(m)\,|(m, H)|^r e^{-\beta_P(H)}$$

for $m \in M_1^+$ *and* $H \in Cl\,\mathfrak{a}^+$.

Here the notation is the same as in [1(e), Lemma 22.3].

Let λ_i $(i \in Q)$ denote the restriction of $s_i \lambda^y$ on $\mathfrak{a}$. We decompose Q into three disjoint sets Q^+, Q^o and Q^- as follows. An element $i \in Q$ lies in Q^+ if $\lambda_i(H) > 0$ for some $H \in \mathfrak{a}^+$, $i \in Q^o$ if $\lambda_i = 0$ and $i \in Q^-$ if $\lambda_i(H) < 0$ for all $H \in \mathfrak{a}^+$. Define

$$\phi_{i\infty}(m) = \lim_{T \to +\infty} \phi_i(m \exp TH)\, e^{-T s_i \Lambda_v(H)} \qquad (m \in M_1)$$

for $i \in Q^o$ and $H \in \mathfrak{a}^+$. One proves as in [1(e), § 22] that this limit exists and is independent of the choice of H. Moreover $\phi_{i\infty} \in \mathscr{A}(M_1, \tau_M)$. Define $\phi_{i\infty} = 0$ for $i \in Q^+ \cup Q^-$.

Choose a number δ $(0 < \delta \leqq \frac{1}{2})$ such that

$$\lambda_i(H) \leqq -\delta\, \beta_P(H)$$

for all $i \in Q^-$ and $H \in \mathfrak{a}^+$. We have seen in [1(e), § 22] that this is possible.

Lemma 2. *Let* $i \in Q$. *Then* $\phi_{i\infty} = 0$ *unless* $i \in Q^o$. *Moreover* $\phi_{i\infty} \in \mathscr{A}(M_1, \tau_M)$ *and*

$$\zeta \phi_{i\infty} = \gamma_1(\zeta : s_i \Lambda_v)\, \phi_{i\infty} \qquad (\zeta \in \mathfrak{Z}_1).$$

Finally

$$|\phi_i(v_1; m \exp TH; v_2) - \phi_{i\infty}(v_1; m \exp TH; v_2)|_{\mathbf{s}}$$
$$\leqq e^{-T\delta\beta_P(H)} \left\{ |\phi_i(v_1; m; v_2)|_{\mathbf{s}} + \int_0^\infty |\psi_{i,H}(v_1; m \exp tH; v_2)|_{\mathbf{s}}\, e^{t\beta_P(H)/2} \right\}$$

for $v_1, v_2 \in \mathfrak{M}_1$, $m \in M_1$, $H \in \mathfrak{a}^+$, $T \geqq 0$ *and* $\mathbf{s} \in \mathscr{S}(V)$. *(In case* $P = G$, *the right side should be replaced by zero.)*

This is proved in the same way as [1(e), Theorem 22.1].

Lemma 3. *Fix* i $(1 \leqq i \leqq q)$ *and suppose* $v \in \mathfrak{F}'(\lambda)$. *Then* $\phi_{i\infty} = 0$ *unless* $s_i^{-1}\mathfrak{a} \subset \mathfrak{h}_R^{k_0}$.

Suppose $\phi_{i\infty} \neq 0$. Clearly $\mathfrak{h}_0$ is a θ-stable Cartan subalgebra of $\mathfrak{M}_1$. Hence by [1(e), Lemma 29.3] we can choose $s \in W(\mathfrak{m}_1/\mathfrak{h}_0)$ such that

$$s_i(\lambda - (-1)^{1/2} v)^y = s\theta s_i(\lambda + (-1)^{1/2} v)^y.$$

Choose $x \in G_c$ such that $x y^{-1} = s_i$ on $\mathfrak{h}_0$. Then

$$(\lambda - (-1)^{1/2} v)^x = s\theta(\lambda + (-1)^{1/2} v)^x$$

and $x \cdot \mathfrak{h}_c = \mathfrak{h}_{0c}$ since $y \cdot \mathfrak{h}_c = \mathfrak{h}_{0c}$ (see § 3). Fix $H_0 \in \mathfrak{a}^+$. Then we conclude from [1 (e), Lemma 33.1] that

$$\mathfrak{a} \subset x \cdot \mathfrak{h}_R = s_i \mathfrak{h}_R^y = s_i \mathfrak{h}_R^{k_0}.$$

This proves the lemma.

§ 7. The Functions $\phi_{P,s}$

Let $P = MAN$ be a psgp of G. Given $k \in K$, let s denote the restriction of $\mathrm{Ad}(k)$ on $\mathfrak{a}$. Then s determines the coset kK_M completely. Hence if H is any subgroup of G which is normalized by K_M, we can define $H^s = H^k = kHk^{-1}$. In particular $P^s = M^s A^s N^s$. For any $\phi \in \mathscr{A}(MA, \tau_M)$, we define $\phi^k = \phi^s \in \mathscr{A}((MA)^s, \tau_{M^s})$ by

$$\phi^s(m^k) = \tau(k)\phi(m)\tau(k^{-1}) \qquad (m \in MA).$$

It is easy to see that ϕ^s depends only on s. Similarly we define

$$\zeta^s = \zeta^k = \mathrm{Ad}(k)\zeta \quad (\zeta \in \mathfrak{Z}_M \mathfrak{A}), \qquad a^s = a^k \ (a \in A).$$

If $\mathfrak{h}$ is a Cartan subalgebra on $\mathfrak{g}$, sometimes it will be convenient to write $\gamma_{G/\mathfrak{h}}$ instead of $\gamma_{\mathfrak{g}/\mathfrak{h}}$.

Let $P' = M'A'N'$ be another psgp of G. Then we have [1 (e), § 5] the finite set $\mathfrak{w}(\mathfrak{a}'|\mathfrak{a})$ of linear injections of $\mathfrak{a}$ into $\mathfrak{a}'$. For every $s \in \mathfrak{w}(\mathfrak{a}'|\mathfrak{a})$ we can choose $k \in K$ such that $\mathrm{Ad}(k) = s$ on $\mathfrak{a}$ [1 (e), § 5]. Put $\mathfrak{w}(\mathfrak{a}) = \mathfrak{w}(\mathfrak{a}|\mathfrak{a})$. Then $\mathfrak{w}(\mathfrak{a})$ is a group of linear transformations in $\mathfrak{a}$.

Fix λ as in § 6.

Theorem 1. *Suppose* $\nu \in \mathfrak{F}'(\lambda)$ *and* $\phi \in \mathscr{A}(G, \tau, \lambda, \nu)$ *in the notation of* § 6. *Put* $\mathfrak{w} = \mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$. *Then there exist unique functions* $\phi_{P,s} \in \mathscr{A}(M_1, \tau_M)$ $(s \in \mathfrak{w})$ *with the following two properties.*

1) $\phi_P(m) = \sum_{s \in \mathfrak{w}} \phi_{P,s}(m) \qquad (m \in M_1)$,

2) $\zeta \phi_{P,s} = \gamma_{M_1^s/\mathfrak{h}}(\zeta^s : \lambda + (-1)^{1/2}\nu)\phi_{P,s} \qquad (\zeta \in \mathfrak{Z}_1,\ s \in \mathfrak{w})$.

Here ϕ_P is the constant term of ϕ along P [1 (e), § 21].

Corollary. $\phi_{P,s}(ma) = \phi_{P,s}(m) e^{(-1)^{1/2}\nu(\log a^s)} (m \in M_1, a \in A, s \in \mathfrak{w})$.

Since $\mathfrak{a} \subset \mathfrak{Z}_1$, the corollary is obvious from the second statement of the theorem. First we prove the following lemma.

Lemma 1. *Given* $s \in \mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$, *there exists a unique index* i $(1 \leq i \leq q)$ *such that* $sH = \mathrm{Ad}(k_0^{-1}) s_i^{-1} H$ *for all* $H \in \mathfrak{a}$.

Choose a representative $k \in K$ for s. (This means that $\mathrm{Ad}(k) = s$ on $\mathfrak{a}$.) Then

$$(\mathfrak{a})^{k_0 k} \subset \mathfrak{h}_R^{k_0} \subset \mathfrak{a}_0.$$

Hence we can choose $t \in W_0 = W(\mathfrak{g}/\mathfrak{h}_0)$ such that $\mathrm{Ad}(k_0 k) = t^{-1}$ on $\mathfrak{a}$. Clearly the coset $W_1 t$ is uniquely determined by this condition. Hence there exists a unique i such that $W_1 t = W_1 s_i$. This s_i satisfies our condition.

Lemma 2. *Let s and i be related as in Lemma* 1. *Then*

$$\gamma_1(\zeta: s_i \Lambda_\nu) = \gamma_{M_1^s/\mathfrak{h}}(\zeta^s: \lambda + (-1)^{1/2} \nu) \qquad (\zeta \in \mathfrak{Z}_1).$$

Choose $y_i \in G_c$ such that $y_i = s_i$ on $\mathfrak{h}_0$ and define k as in the proof of Lemma 1. Then it is clear that

$$m_1 = y_i \operatorname{Ad}(k_0 k) \in M_{1c}$$

where M_{1c} is the centralizer of $\mathfrak{a}$ in G_c. Now $s_i \Lambda_\nu = (\lambda + (-1)^{1/2} \nu)^{y_i y}$ and

$$y_i y = m_1 \operatorname{Ad}(k_0 k)^{-1} y.$$

Moreover $\operatorname{Ad}(k_0^{-1}) y$ centralizes $\mathfrak{h}_R$ (see § 3) and $\mathfrak{h}_R \supset s\,\mathfrak{a} = \mathfrak{a}^k$. Hence

$$m_2 = \operatorname{Ad}(k_0 k)^{-1} y \operatorname{Ad}(k) \in M_{1c}.$$

Put

$$m = m_1 m_2 = y_i y \operatorname{Ad}(k) \in M_{1c}$$

so that $y_i y = m \operatorname{Ad}(k^{-1})$. Since (§ 3)

$$(y_i y)^{-1} \mathfrak{h}_{0c} = y^{-1} \mathfrak{h}_{0c} = \mathfrak{h}_c$$

it follows that $m^{-1} \mathfrak{h}_{0c} = \mathfrak{h}_c^{k^{-1}}$. Therefore

$$\gamma_1(\zeta: s_i \Lambda_\nu) = \gamma_{M_1/\mathfrak{h}_0}(\zeta: (\lambda + (-1)^{1/2} \nu)^{y_i y})$$
$$= \gamma_{M_1/\mathfrak{h}^{k^{-1}}}(\zeta: (\lambda + (-1)^{1/2} \nu)^{k^{-1}}) = \gamma_{M_1^s/\mathfrak{h}}(\zeta^s: \lambda + (-1)^{1/2} \nu).$$

We now come to the proof of Theorem 1. Since $\varpi_0(\Lambda_\nu) \neq 0$, it is clear that $s\Lambda_\nu \neq t\Lambda_\nu$ for $s \neq t$ in W_0. Hence $s_i \Lambda_\nu$ and $s_j \Lambda_\nu$ cannot be conjugate under W_1 unless $i = j$. Put

$$\chi_s(\zeta) = \gamma_{M_1^s/\mathfrak{h}}(\zeta^s: \lambda + (-1)^{1/2} \nu) \qquad (s \in \mathfrak{w}, \zeta \in \mathfrak{Z}_1).$$

Then it follows from Lemma 2 that $\chi_s \neq \chi_t$ if $s \neq t$ in $\mathfrak{w}$. The uniqueness of $\phi_{P,s}$ is now obvious. On the other hand if s and i are related by Lemma 1 and we set

$$\phi_{P,s} = \varpi_{01}(s_i \Lambda_\nu)^{-1} \phi_{i\infty},$$

it follows from Lemmas 5.1 and 6.2 that all the conditions of Theorem 1 are fulfilled and this completes the proof.

We state the above result as a lemma for later reference.

Lemma 3. *Suppose s and i are related as in Lemma* 1. *Then*

$$\phi_{P,s} = \varpi_{01}(s_i \Lambda_\nu)^{-1} \phi_{i\infty}.$$

Let $(P', A') \prec (P, A)$ be another p-pair in G and put ${}^*P = P' \cap (MA)$. Then $({}^*P, A')$ is a p-pair in M_1. For any $s \in \mathfrak{w}(\mathfrak{h}_R | \mathfrak{a})$, let $\mathfrak{w}_s(\mathfrak{h}_R | \mathfrak{a}')$ be the set of all $t \in \mathfrak{w}(\mathfrak{h}_R | \mathfrak{a}')$ such that $t = s$ on $\mathfrak{a}$. (We note that $\mathfrak{a} \subset \mathfrak{a}'$.)

Fix $s \in \mathfrak{w}(\mathfrak{h}_R | \mathfrak{a})$ and choose a representative $k \in K$ for s. Put

$$\psi = (\phi_{P,s})^s \in \mathscr{A}((MA)^s, \tau_{M^s}).$$

Then

$$\zeta\psi=\gamma_{M_1^s/\mathfrak{h}}(\zeta:\lambda+(-1)^{1/2}\nu)\psi \qquad (\zeta\in\mathfrak{Z}_1^s).$$

and $*P^k=(*P)^k$ is a psgp of M_1^s with split component $(A')^k$.

Lemma 4. *For any* $t\in\mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')$,

$$(\phi_{P',t})^t=(\psi_{*P^k,t\circ k^{-1}})^{t\circ k^{-1}}.$$

Here $t\circ k^{-1}$ denotes the mapping $H\mapsto t(\mathrm{Ad}(k^{-1})H)$ $(H\in(\mathfrak{a}')^k)$ of $\mathfrak{a}'^k$ into $\mathfrak{h}_R$. We know [1(e), Lemma 21.1] that

$$(\phi_{P,s})_{*P}=(\psi_{*P^k})^{k^{-1}}.$$

Let $\mathfrak{w}_0(\mathfrak{h}_R|\mathfrak{a}'^k)$ denote the set of all $t'\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a}'^k)$ such that $t'=\mathrm{Ad}(m^k)$ on $\mathfrak{a}'^k$ for some $m\in M_1$. Then it is easy to verify that $t\mapsto t\circ k^{-1}$ is a bijection of $\mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')$ on $\mathfrak{w}_0(\mathfrak{h}_R|\mathfrak{a}'^k)$. Therefore by applying Theorem 1 to (M_1^s,ψ) in place of (G,ϕ), we conclude that

$$\psi_{*P^k}=\sum_{t\in\mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')}\psi_{*P^k,t\circ k^{-1}}.$$

Now put $M_1'=M'A'$, $\mathfrak{Z}_1'=\mathfrak{Z}_{M'}\mathfrak{A}'$,

$$\chi_t(\eta)=\gamma_{(M_1')^t/\mathfrak{h}}(\eta^t:\lambda+(-1)^{1/2}\nu) \qquad (\eta\in\mathfrak{Z}_1')$$

and

$$\Psi(t)=(\psi_{*P^k,t\circ k^{-1}})^{k^{-1}}$$

for $t\in\mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')$. Then

$$\eta\Psi(t)=\chi_t(\eta)\Psi(t) \qquad (\eta\in\mathfrak{Z}_1').$$

On the other hand $\phi_{P'}=(\phi_P)_{*P}$ [1(e), Lemma 21.1]. Hence

$$\phi_{P'}=\sum_{s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})}(\phi_{P,s})_{*P}.$$

For every $s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$, choose a representative k_s in K and define

$$\Psi(s,t)=(\psi_{*P^k,t\circ k^{-1}})^{k^{-1}} \qquad (t\in\mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')),$$

with $\psi=(\phi_{P,s})^s$ and $k=k_s$. Then, by the above result,

$$(\phi_{P,s})_{*P}=\sum_{t\in\mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')}\Psi(s,t)$$

and

$$\eta\Psi(s,t)=\chi_t(\eta)\Psi(s,t) \qquad (\eta\in\mathfrak{Z}_1')$$

for $s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$ and $t\in\mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')$. Hence

$$\phi_{P'}=\sum_{s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})}(\phi_{P,s})_{*P}$$

$$=\sum_{s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})}\sum_{t\in\mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')}\Psi(s,t).$$

It is now obvious from Theorem 1 that

$$\phi_{P',t} = \Psi(s, t)$$

for $t \in \mathfrak{w}_s(\mathfrak{h}_R|\mathfrak{a}')$ and the statement of the lemma follows immediately.

We define the space ${}^o\mathscr{C}(M, \tau_M)$ as in [1(e), § 19].

Lemma 5. *Fix* $s \in \mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$ *and let* f *denote the restriction of* $\phi_{P,s}$ *on* M. *Then if* $\operatorname{prk} P = \dim \mathfrak{h}_R$, $f \in {}^o\mathscr{C}(M, \tau_M)$.

Let $^*P = {}^*M\, {}^*A\, {}^*N$ be a psgp of M with $\operatorname{prk} {}^*P \geqq 1$. Then by [1(e), Lemma 25.1], it is enough to verify that $f_{*P} = 0$. Let $P' = M'A'N'$ be the psgp of G corresponding to $^*P$ [1(e), Lemma 6.1] so that $(P', A') \prec (P, A)$. Then

$$\operatorname{prk} P' = \operatorname{prk} {}^*P + \operatorname{prk} P > \dim \mathfrak{h}_R$$

and therefore $\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a}')$ is empty. Fix a representative $k \in K$ for s and put $\psi = (\phi_{P,s})^s$, $Q = (P' \cap M_1)^k$. Then Q is a psgp of $M_1 = MA$ and it follows from the proof of Lemma 4 that $\psi_Q = 0$. Since [1(e), Lemma 21.1]

$$f_{*P} = (\psi_Q)^{k^{-1}}$$

on $^*M\, {}^*A$, we conclude that $f_{*P} = 0$.

§ 8. Functions of Type $II(\lambda)$

Now, instead of keeping ν fixed, we shall allow it to vary in $\mathfrak{F}$. Note that $\mathfrak{h}_R$, being a subspace of $\mathfrak{g}$, has a Euclidean norm. Hence, by duality, the same holds for $\mathfrak{F}$. Put

$$|(\nu, x)| = (1 + |\nu|)(1 + \sigma(x))$$

for $(\nu, x) \in \mathfrak{F} \times G$. Let $\mathfrak{D} = \mathfrak{D}(\mathfrak{F}_c)$ denote the algebra of polynomial differential operators on $\mathfrak{F}$ (or $\mathfrak{F}_c$) [1(c), § 3]. Put $\tilde{\mathfrak{G}} = \mathfrak{D} \otimes \mathfrak{G}^{(2)}$ [1(e), § 15]. Let ϕ be a C^∞ function from $\mathfrak{F} \times G$ to V. For $D \in \tilde{\mathfrak{G}}$, $\mathfrak{s} \in \mathscr{S}(V)$ and $r \geqq 0$, put

$$\mathfrak{s}_{D,r}(\phi) = \sup_{\mathfrak{F} \times G} |D\phi|_{\mathfrak{s}}\, \Xi^{-1}\, |(\nu, x)|^{-r}$$

in the notation of [1(e), § 15]. If F is a finite subset of $\tilde{\mathfrak{G}}$, we set

$$\mathfrak{s}_{F,r}(\phi) = \sum_{D \in F} \mathfrak{s}_{D,r}(\phi).$$

A function $\phi: \mathfrak{F} \times G \to V$ will be said to be of type $II(\lambda)$ if the following conditions hold.

1) ϕ is of class C^∞.

2) For any $\nu \in \mathfrak{F}$, the function $\phi_\nu = \phi(\nu)$ is a τ-spherical function on G and

$$z\phi_\nu = \gamma_{\mathfrak{g}/\mathfrak{h}}(z : \lambda + (-1)^{1/2}\nu)\,\phi_\nu \qquad (z \in \mathfrak{Z}).$$

3) For any $D \in \tilde{\mathfrak{G}}$ and $\mathfrak{s} \in \mathscr{S}(V)$, we can choose a number $r \geqq 0$ such that $\mathfrak{s}_{D,r}(\phi) < \infty$.

Fix a function ϕ of type $II(\lambda)$ and let us use the notation of § 5. Then ϕ_i and $\psi_{i,\zeta}$ ($\zeta \in \mathfrak{Z}_1$) are now functions on $\mathfrak{F} \times M_1$. Put

$$|(v, x, X)| = (1 + |v|)(1 + \sigma(x))(1 + \|X\|)$$

for $(v, x, X) \in \mathfrak{F} \times G \times \mathfrak{g}$.

Lemma 1. *Fix $\zeta \in \mathfrak{Z}_1$, $v_1, v_2 \in \mathfrak{M}_1$, $p \in S(\mathfrak{F}_c)$ and $\mathfrak{s} \in \mathscr{S}(V)$. Then we can choose c, $r \geqq 0$ such that*

$$|\psi_{i,\zeta}(v; \partial(p) \colon v_1; m \exp H; v_2)|_{\mathfrak{s}} \leqq c\, \Xi_M(m)\, |(v, m, H)|^r\, e^{-\beta_P(H)}$$

for $m \in M_1^+$, $H \in Cl\,\mathfrak{a}^+$, $v \in \mathfrak{F}$ and $1 \leqq i \leqq q$.

The proof is the same as for Lemma 6.1.

It follows without difficulty from the above estimates that $\phi_{i\infty}$, regarded as functions on $\mathfrak{F} \times M_1$, are of class C^∞. In fact we have the following analogue of Lemma 6.2.

Lemma 2. 1) $\phi_{i\infty}(v \colon m; \zeta) = \gamma_1(\zeta \colon s_i \Lambda_v)\, \phi_{i\infty}(v \colon m)$ ($\zeta \in \mathfrak{Z}_1$).

Given $v_1, v_2 \in \mathfrak{M}_1$, $p \in S(\mathfrak{F}_c)$ and $\mathfrak{s} \in \mathscr{S}(V)$, we can choose c, $r \geqq 0$ such that

2) $|\phi_{i\infty}(v; \partial(p) \colon v_1; m; v_2)|_s \leqq c\, \Xi_M(m)\, |(v, m)|^r$.

Finally

3) $|\phi_i(v \colon v_1; m \exp TH; v_2) - \phi_{i\infty}(v \colon v_1; m \exp TH; v_2)|_{\mathfrak{s}}$

$$\leqq e^{-T\delta\beta_P(H)} \left\{ |\phi_i(v \colon v_1; m; v_2|_{\mathfrak{s}} + \int_0^\infty |\psi_{i,H}(v \colon v_1; m \exp tH; v_2)|_{\mathfrak{s}}\, e^{t\beta_P(H)/2}\, dt \right\}$$

for $H \in \mathfrak{a}^+$, $T \geqq 0$.

Here $i \in Q$, $v \in \mathfrak{F}$, $m \in M_1$ and the right side in 3) *is to be replaced by zero in case $P = G$.*

We have only to comment on the proof of 2). Put

$$\phi_i^o(v \colon m \colon H) = \phi_i(v \colon m \exp H)\, e^{-s_i \Lambda_v(H)},$$

$$\psi_{i,\zeta}^o(v \colon m \colon H) = \psi_{i,\zeta}(v \colon m \exp H)\, e^{-s_i \Lambda_v(H)} \qquad (\zeta \in \mathfrak{Z}_1),$$

for $v \in \mathfrak{F}$, $m \in M_1$ and $H \in \mathfrak{a}$. Then

$$\phi_i^o(v \colon m \colon TH) = \phi_i(v \colon m) + \int_0^T \psi_{i,H}^o(v \colon m \colon tH)\, dt \qquad (T \in \mathbf{R})$$

from Lemma 5.3. Now if $i \in Q^o$, it follows from Lemma 1 that

$$\phi_{i\infty}(v; \partial(p) \colon v_1; m; v_2) = \phi_i(v; \partial(p) \colon v_1; m; v_2) + \int_0^\infty \psi_{i,H}^o(v; \partial(p) \colon v_1; m; v_2 \colon tH)\, dt$$

for $v_1, v_2 \in \mathfrak{M}_1$ and $p \in S(\mathfrak{F}_c)$. Now fix p. Then it is obvious that

$$\partial(p) \circ e^{-s_i \Lambda_v(H)} = e^{-s_i \Lambda_v(H)}\, \partial(p_H) \qquad (H \in \mathfrak{a})$$

where $H \mapsto p_H$ is a polynomial mapping of $\mathfrak{a}$ in $S(\mathfrak{F}_c)$. Hence 2) is an easy consequence of Lemma 1 and standard arguments [1(d), p. 69]. (We recall that by Lemma 6.2 $\phi_{i\infty}=0$ unless $i\in Q^o$.)

Put $\phi_P(\nu)=(\phi_\nu)_P$ $(\nu\in\mathfrak{F})$ and $\phi_{P,s}(\nu)=(\phi_\nu)_{P,s}$ for $\nu\in\mathfrak{F}'(\lambda)$ and $s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$.

Lemma 3. *Suppose* $\nu\in\mathfrak{F}'(\lambda)$. *Then*

$$\phi_P(\nu)=\sum_{i\in Q^o}\varpi_{01}(s_i\Lambda_\nu)^{-1}\phi_{i\infty}(\nu)=\sum_{s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})}\phi_{P,s}(\nu).$$

This is obvious from the results of § 7.

§ 9. Functions of Type $II'(\lambda)$

Let $\mathscr{P}$ be the set of all psgps of G. We keep to the notation of § 8.

Let ϕ be a function from $\mathfrak{F}\times G$ to V. We say that ϕ is of type $II'(\lambda)$, if it is of type $II(\lambda)$ and the following additional condition holds. Given $P=MAN$ in $\mathscr{P}$ and $s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$, the function $(\phi_{P,s})^s$ on $\mathfrak{F}'(\lambda)\times(MA)^s$ extends (uniquely) to a function of type $II(\lambda)$ on $\mathfrak{F}\times(MA)^s$.

Lemma 1. *Suppose ϕ is of type $II'(\lambda)$ on $\mathfrak{F}\times G$. Then for any $P=MAN$ in $\mathscr{P}$ and $s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$, $(\phi_{P,s})^s$ is of type $II'(\lambda)$ on $\mathfrak{F}\times(MA)^s$.*

This is an immediate consequence of Lemma 7.4.

Theorem 1. *Suppose ϕ is a function of type $II(\lambda)$ on $\mathfrak{F}\times G$. Define*

$$\psi(\nu:x)=\varpi(\lambda+(-1)^{1/2}\nu)\,\phi(\nu:x)\qquad(\nu\in\mathfrak{F},\ x\in G).$$

Then ψ is of type $II'(\lambda)$.

This is an immediate consequence of Lemmas 7.3 and 8.2.

§ 10. Continuity of ϕ_P

Fix a function ϕ of type $II(\lambda)$ on $\mathfrak{F}\times G$ and a psgp $P=MAN$ of G. We intend to show that ϕ_P is a continuous function on $\mathfrak{F}\times MA$. So we may assume that $P\neq G$.

We use the notation of § 3. Let U^* be the space dual to U and $(u_1^*,\dots,u_q^*)$ the base for U^* dual to $(u_1,\dots,u_q)$. For any $\nu\in\mathfrak{F}$, we have defined in § 3 a representation Γ_ν of J_1 on U. The corresponding (right-)representation Γ_ν^* on U^* is given by

$$\langle u^*\Gamma_\nu^*(p),u\rangle=\langle u^*,\Gamma_\nu(p)u\rangle\qquad(p\in J_1,\ u\in U,\ u^*\in U^*).$$

Define γ_1 as in § 4 and put $\eta_i=\gamma_1^{-1}(u_i)\in\mathfrak{Z}_1$ $(1\leqq i\leqq q)$.

We regard U^* as a Hilbert space with $(u_1^*,\dots,u_q^*)$ as an orthonormal base. Put $\mathbf{V}=V\otimes U^*$. Then by letting K act trivially on U^*, we get a double representation $\boldsymbol{\tau}$ of K on $\mathbf{V}$. Put

$$\boldsymbol{\Gamma}_\nu(\zeta)=1\otimes\Gamma_\nu^*(\gamma_1(\zeta))\qquad(\zeta\in\mathfrak{Z}_1).$$

Then $\boldsymbol{\Gamma}_\nu$ is a right-representation of $\mathfrak{Z}_1$ on $\mathbf{V}$ which commutes with $\boldsymbol{\tau}$.

We now proceed in the same way as in [1(e), § 22]. If $\mathbf{s} \in \mathscr{S}(V)$ and

$$\mathbf{v} = \sum_{1 \leq i \leq q} v_i \otimes u_i^* \qquad (v_i \in V),$$

we put

$$\mathbf{s}(\mathbf{v}) = |\mathbf{v}|_{\mathbf{s}} = (\sum_i |v_i|_{\mathbf{s}}^2)^{1/2}.$$

Let $\|T\|$ denote the Hilbert-Schmidt norm of a linear transformation T on U^*. (We write T on the right.) Then it is easy to verify that

$$\mathbf{s}(\mathbf{v} \cdot (1 \otimes T)) \leq \mathbf{s}(\mathbf{v}) \, \|T\| \qquad (\mathbf{s} \in \mathscr{S}(V), \mathbf{v} \in \mathbf{V}).$$

Now define a C^∞ function Φ from $\mathfrak{F} \times M_1$ to $\mathbf{V}$ by

$$\Phi(v : m) = d(m) \sum_{1 \leq i \leq q} \phi(v : m; \eta_i') \otimes u_i^* \qquad (v \in \mathfrak{F}, m \in M_1).$$

Here $M_1 = MA$, $d(m) = d_P(m)\,(m \in M_1)$ and $v' = d^{-1} v \circ d$ for $v \in \mathfrak{M}_1$ as in §4. Fix $\zeta \in \mathfrak{Z}_1$ and consider $\Phi(v : m; \zeta)$. Put $p = \gamma_1(\zeta) \in J_1$. Then

$$p u_i = \Gamma_v(p)\, u_i + \sum_{1 \leq j \leq q} v_{ij}(p : v)\, u_j \qquad (1 \leq i \leq q)$$

where

$$v_{ij}(p : v) = \operatorname{tr}_{J_1/J} \{u^j (p u_i - \Gamma_v(p)\, u_i)\} \in J_{\Lambda_v}$$

from the definition of $\Gamma_v(p)$. Define γ_0 and μ as in §4 and put

$$z_{ij}(\zeta : v) = \gamma_0^{-1}(v_{ij}(p : v)) \in \mathfrak{Z}.$$

Then it is clear that

$$\gamma(z_{ij}(\zeta : v) : \lambda + (-1)^{1/2} v) = 0$$

and [1(d), p. 110]

$$g_{ij}(\zeta : v) = z_{ij}(\zeta : v) - \mu(z_{ij}(\zeta : v))' \in \theta(\mathfrak{n})\, \mathfrak{G}\, \mathfrak{n}.$$

Put

$$g_i(\zeta : v) = - \sum_{1 \leq j \leq q} g_{ij}(\zeta : v)\, \eta_j'.$$

Then $g_i(\zeta : v)$ is linear in ζ and for fixed i and ζ, $v \mapsto g_i(\zeta : v)$ is a polynomial mapping of $\mathfrak{F}$ into $\theta(\mathfrak{n})\, \mathfrak{G}\, \mathfrak{n}$ by Corollary 3 of Lemma 3.5.

Lemma 1. *Fix* $\zeta \in \mathfrak{Z}_1$ *and put*

$$\Psi_\zeta(v : m) = d(m) \sum_{1 \leq i \leq q} \phi(v : m; g_i(\zeta : v)) \otimes u_i^* \qquad (v \in \mathfrak{F}, m \in M_1).$$

Then

$$\Phi(v : m; \zeta) = \Phi(v : m)\, \Gamma_v(\zeta) + \Psi_\zeta(v : m)$$

for $v \in \mathfrak{F}$ *and* $m \in M_1$.

Let $p=\gamma_1(\zeta)$. Then

$$\sum_i u_i \otimes u_i^* \Gamma_\nu^*(p)=\sum_i \Gamma_\nu(p)\, u_i \otimes u_i^*=\sum_i p\, u_i \otimes u_i^* - \sum_{i,j} v_{ij}(p:\nu)\, u_j \otimes u_i^*$$

in $J_1 \otimes U^*$. Therefore since $\gamma_1(\mu(z_{ij}(\zeta:\nu)))=v_{ij}(p:\nu)$ and $z_{ij}(\zeta:\nu)\,\phi(\nu)=0$, we conclude that

$$\Phi(\nu:m)\,\boldsymbol{\Gamma}_\nu(\zeta)=d(m)\sum_i \phi(\nu:m;\zeta'\eta_i')\otimes u_i^* - d(m)\sum_i \phi(\nu:m;g_i(\zeta:\nu))\otimes u_i^*$$

and this implies our assertion.

Lemma 2. *Let $H\in\mathfrak{a}$. Then*

$$\Phi(\nu:m\exp TH)\,e^{-T\boldsymbol{\Gamma}_\nu(H)}=\Phi(\nu:m)+\int_0^T \Psi_H(\nu:m\exp tH)\,e^{-t\boldsymbol{\Gamma}_\nu(H)}\,dt$$

for $\nu\in\mathfrak{F}$, $m\in M_1$ and $T\in\mathbf{R}$.

Since $\mathfrak{a}\subset\mathfrak{Z}_1$, this is an immediate consequence of Lemma 1.

Put

$$E_i(\nu)=\Gamma_\nu^*({}_ie(\nu)) \qquad (\nu\in\mathfrak{F},\ i\in Q)$$

in the notation of Lemma 3.4. Then it is clear that E_i is a C^∞ function from $\mathfrak{F}$ to End U^* and

$$\sum_{i\in{}^oQ} E_i(\nu)=1.$$

Moreover it is easy to verify from Corollary 2 of Lemma 3.5 that

$$E_i(\nu)\,E_j(\nu)=\delta_{ij}\,E_j(\nu) \qquad (i,j\in{}^oQ).$$

Put $\mathbf{E}_i(\nu)=1\otimes E_i(\nu)$. Since J_1 is an abelian algebra, it is obvious that $\mathbf{E}_i(\nu)$ commutes with $\boldsymbol{\Gamma}_\nu(\zeta)$ $(\zeta\in\mathfrak{Z}_1)$ and the operations of K on $\mathbf{V}$. Put

$$\Phi_i(\nu)=\Phi(\nu)\,\mathbf{E}_i(\nu) \qquad (\nu\in\mathfrak{F}).$$

Then the following result is immediate.

Lemma 3. *Let $H\in\mathfrak{a}$. Then*

$$\Phi_i(\nu:m\exp TH)\,e^{-T\boldsymbol{\Gamma}_\nu(H)}=\Phi_i(\nu:m)+\int_0^T \Psi_H(\nu:m\exp tH)\,\mathbf{E}_i(\nu)\,e^{-t\boldsymbol{\Gamma}_\nu(H)}\,dt$$

for $\nu\in\mathfrak{F}$, $m\in M_1$, $T\in\mathbf{R}$ and $i\in Q$.

Let λ_i $(i\in Q)$ denote the restriction of $s_i\lambda^y$ on $\mathfrak{a}$ as in § 6.

Lemma 4. *Put*

$$\Gamma_i(\nu:H)=E_i(\nu)\,e^{\Gamma_\nu^*(H)-\lambda_i(H)}$$

for $i\in Q$, $\nu\in\mathfrak{F}$, $H\in\mathfrak{a}$. Then we can choose c_0, $r_0\geqq 0$ such that

$$\|\Gamma_i(\nu:H)\|\leqq c_0(1+\|H\|)^{r_0}(1+|\nu|)^{r_0} \qquad (i\in Q,\ H\in\mathfrak{a},\ \nu\in\mathfrak{F}).$$

Put

$$F_i(\nu:H)=E_i(\nu)(\Gamma_\nu^*(H)-\lambda_i(H))$$

and fix $H\in\mathfrak{a}$. We claim that all eigenvalues of $F_i(\nu:H)$ are purely imaginary. Since $F_i(\nu:H)$ is a continuous function of $\nu\in\mathfrak{F}$, it would be enough to verify this for $\nu\in\mathfrak{F}'(\lambda)$. But this follows from Corollary 1 of Lemma 3.5 since

$$\Gamma_\nu(H)\, e_j(\nu)=s_j\, \Lambda_\nu(H)\, e_j(\nu) \qquad (1\leqq j\leqq q).$$

Now

$$\Gamma_i(\nu:H)=E_i(\nu)\, e^{F_i(\nu:H)}.$$

Since

$$\nu\mapsto \varpi_{s_i,\lambda}(s_i\Lambda_\nu)\, E_i(\nu)$$

is a polynomial mapping (Lemma 3.4) and

$$|\varpi_{s_i,\lambda}(s_i\Lambda_\nu)|\geqq|\varpi_\lambda(\lambda)|>0$$

(Lemma 3.3), the required result follows from [1(a), Lemma 60].

Lemma 5. *Fix* $\zeta\in\mathfrak{Z}_1$, v_1, $v_2\in\mathfrak{M}_1$ *and* $\mathbf{s}\in\mathcal{S}(V)$. *Then we can choose* c, $r\geqq 0$ *such that*

$$|\Psi_\zeta(\nu: v_1; m\exp H; v_2)|_{\mathbf{s}}\leqq c\, \Xi_M(m)\, |(\nu, m, H)|^r\, e^{-\beta_P(H)}$$

for $m\in M_1^+$, $H\in Cl\,\mathfrak{a}^+$ *and* $\nu\in\mathfrak{F}$.

We recall that $\nu\mapsto g_j(\zeta:\nu)$ $(1\leqq j\leqq q)$ are polynomial mappings of $\mathfrak{F}$ into $\theta(\mathfrak{n})\mathfrak{G}\,\mathfrak{n}$. Therefore our assertion follows without difficulty from Lemma 5.2.

Define Q^+, Q^o and Q^- as in §6. Then (see §6) we can choose δ $(0<\delta\leqq\frac{1}{2})$ such that

$$\lambda_i(H)\leqq -\delta\,\beta_P(H)$$

for all $i\in Q^-$ and $H\in\mathfrak{a}^+$.

Fix $i\in Q^o$, v_1, $v_2\in\mathfrak{M}_1$, $\mathbf{s}\in\mathcal{S}(V)$ and $H\in\mathfrak{a}^+$. Then it follows from Lemmas 4 and 5 that the integral

$$\int_0^\infty |\Psi_H(\nu: v_1; m\exp tH; v_2)|_{\mathbf{s}}\, \|\Gamma_i(\nu: -tH)\|\, dt$$

converges uniformly as ν and m vary within compact subsets of $\mathfrak{F}$ and M_1 respectively. Put

$$\Phi_{i\infty}(\nu:m)=\lim_{t\to+\infty}\Phi_i(\nu:m\exp tH)\, e^{-t\Gamma_\nu(H)} \qquad (\nu\in\mathfrak{F}, m\in M_1).$$

Then, from Lemma 3, this limit exists and we prove as in [1(e), §22] that it is independent of $H\in\mathfrak{a}^+$. Moreover $\Phi_{i\infty}$ is a continuous function from $\mathfrak{F}\times M_1$ to $\mathbf{V}$ which is differentiable in $m\in M_1$. In fact

$$\Phi_{i\infty}(\nu: v_1; m; v_2)=\lim_{t\to+\infty}\Phi_i(\nu: v_1; m\exp tH; v_2)\, e^{-t\Gamma_\nu(H)}$$

for v_1, $v_2\in\mathfrak{M}_1$ and $H\in\mathfrak{a}^+$.

Define $\Phi_{i\infty}=0$ for $i\in Q^+\cup Q^-$.

Lemma 6. *Fix $i \in Q$. Then*

$$\Phi_{i\infty}(\nu:m;\zeta)=\Phi_{i\infty}(\nu:m)\Gamma_\nu(\zeta) \qquad (\zeta\in\mathfrak{Z}_1)$$

and

$$|\Phi_i(\nu:v_1:m\exp TH;v_2)-\Phi_{i\infty}(\nu:v_1:m\exp TH;v_2)|_s$$
$$\leqq e^{-\delta T\beta_P(H)}\left\{|\Phi(\nu:v_1:m;v_2)|_s\,\|\Gamma_i(\nu:TH)\| + \int_0^\infty |\Psi_H(\nu:v_1:m\exp tH;v_2)|_s\,\|\Gamma_i(\nu:(T-t)H\|\, e^{t\beta_P(H)/2}\,dt\right\}$$

for $v_1, v_2\in\mathfrak{M}_1$, $m\in M_1$, $H\in\mathfrak{a}^+$, $\nu\in\mathfrak{F}$, $s\in\mathcal{S}(V)$ and $T\geqq 0$.

This is proved in the same way as [1(e), Theorem 22.1].

Put ${}^oQ^o={}^oQ\cap Q^o$. Since

$$\sum_{i\in {}^oQ} E_i(\nu)=1,$$

we get the following corollary.

Corollary.

$$|\Phi(\nu:v_1:m\exp TH;v_2)-\sum_{i\in {}^oQ^o}\Phi_{i\infty}(\nu:v_1:m\exp TH;v_2)|_s$$
$$\leqq e^{-\delta T\beta_P(H)}\sum_{i\in {}^oQ}\left\{|\Phi(\nu:v_1:m;v_2)|_s\,\|\Gamma_i(\nu:TH)\| + \int_0^\infty |\Psi_H(\nu:v_1:m\exp tH;v_2)|_s\,\|\Gamma_i(\nu:(T-t)H\|\, e^{t\beta_P(H)/2}\,dt\right\}.$$

Define functions ψ_i $(i\in Q)$ from $\mathfrak{F}\times M_1$ to V by the formula

$$\sum_{i\in {}^oQ^o}\Phi_{i\infty}=\sum_{i\in Q}\psi_i\otimes u_i^*.$$

Since $u_1=1$, it is clear from the above results and the definition of ϕ_P [1(e), Theorem 21.1] that $\psi_1=\phi_P$. The following result is now obvious from Lemma 6.

Lemma 7. *Fix $v_1, v_2\in\mathfrak{M}_1$. Then the function $(\nu, m)\mapsto\phi_P(\nu:v_1:m;v_2)$ is continuous on $\mathfrak{F}\times M_1$. Moreover for each $s\in\mathcal{S}(V)$, we can choose $c, r\geqq 0$ such that*

$$|\phi_P(\nu:v_1:m;v_2)|_s\leqq c\,\Xi_M(m)\,|(\nu,m)|^r$$

for $\nu\in\mathfrak{F}$ and $m\in M_1$.

Corollary. *Suppose ϕ is of type $II'(\lambda)$. Then*

$$\phi_P=\sum_{s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})}\phi_{P,s}$$

on $\mathfrak{F}\times M_1$.

By Lemma 8.3 the equality holds on $\mathfrak{F}'(\lambda)\times M_1$. But since both sides are continuous, it must hold on $\mathfrak{F}\times M_1$.

We recall that $P \neq G$. Fix a compact subset Ω of $\mathfrak{a}^+$ and choose $\varepsilon_0 > 0$ such that $\beta_P(H) \geqq 2\varepsilon_0$ for all $H \in \Omega$. Put $\varepsilon = \delta\varepsilon_0$. Then the following result is an easy consequence of the corollary of Lemma 6.

Lemma 8. *Given $v_1, v_2 \in \mathfrak{M}_1$ and $\mathbf{s} \in \mathscr{S}(V)$, we can choose $c, r \geqq 0$ such that*

$$|d_P(m \exp TH)\, \phi(\nu : v_1' ; m \exp TH ; v_2') - \phi_P(\nu : v_1 ; m \exp TH ; v_2)|_{\mathbf{s}}$$
$$\leqq c\, e^{-\varepsilon T}\, \Xi_M(m)\, |(\nu, m)|^r$$

for $\nu \in \mathfrak{F}$, $m \in M_1^+$, $H \in \Omega$ and $T \geqq 0$.

§ 11. A Criterion for a Function to be of Type $II'(\lambda)$

We assume in this section that τ is a unitary [1(e), § 20]. Let $\mathscr{P}(\mathfrak{h}_R)$ denote the set of all psgps $P = MAN$ of G such that $\mathfrak{a} = \mathfrak{h}_R$. Clearly M is independent of $P \in \mathscr{P}(\mathfrak{h}_R)$.

Let ϕ be a function on $\mathfrak{F} \times G$ of type $II(\lambda)$. Put $\mathfrak{a} = \mathfrak{h}_R$ and fix $P \in \mathscr{P}(\mathfrak{a})$, $s \in \mathfrak{w}(\mathfrak{a})$ and $\nu \in \mathfrak{F}'(\lambda)$ $(P = MAN)$. Then the function $m \mapsto \phi_{P,s}(\nu : m)$ $(m \in M)$ lies in ${}^0\mathscr{C}(M, \tau_M)$ (Lemma 7.5). We observe that ${}^0\mathscr{C}(M, \tau_M)$, being a closed subspace of $\mathscr{C}(M, \tau_M)$ [1(e), § 18], is a locally convex space.

Lemma 1. *Let ϕ be a function on $\mathfrak{F} \times G$ of type $II(\lambda)$ and $P' = M'A'N'$ a psgp of G. Then $\phi_{P'}(\nu) \sim 0$ $(\nu \in \mathfrak{F})$ unless $\mathfrak{a}'$ is a conjugate to $\mathfrak{a}$ under K.*

Fix $a' \in A'$, $f \in {}^0\mathscr{C}(M', \tau_{M'})$ and assume that $\mathfrak{a}'$ is not conjugate to under K. Then it follows from [1(e), Theorem 29.1] that

$$\int_{M'} (f(m'),\ \phi_{P'}(\nu : m' a'))\, dm' = 0$$

for $\nu \in \mathfrak{F}'(\lambda)$. On the other hand, it is obvious from Lemma 10.7 that the left side is a continuous function of $\nu \in \mathfrak{F}$. Hence $\phi_{P'}(\nu) \sim 0$ for all $\nu \in \mathfrak{F}$.

Corollary. *Fix $\nu \in \mathfrak{F}$ and suppose $\phi_P(\nu) = 0$ for all $P \in \mathscr{P}(\mathfrak{a})$. Then $\phi(\nu) = 0$.*

This is an immediate consequence of [1(e), Lemma 25.2] and the above result.

Theorem 1. *Let ϕ be as above and S a collection of continuous seminorms on ${}^0\mathscr{C}(M, \tau_M)$. We assume that $f \in {}^0\mathscr{C}(M, \tau_M)$ and $\mathbf{s}(f) = 0$ for all $\mathbf{s} \in S$, implies that $f = 0$. Then, in order that ϕ be of type $II'(\lambda)$, it is necessary and sufficient that the following condition holds. For $P \in \mathscr{P}(\mathfrak{a})$ and $s \in \mathfrak{w}(\mathfrak{a})$, let $f_{P,s}(\nu)$ denote the restriction of $\phi_{P,s}(\nu)$ on M $(\nu \in \mathfrak{F}'(\lambda))$. Then $\mathbf{s}(f_{P,s}(\nu))$ should remain locally bounded on $\mathfrak{F}$ for every $P \in \mathscr{P}(\mathfrak{a})$, $s \in \mathfrak{w}(\mathfrak{a})$ and $\mathbf{s} \in S$.*

For example we can take S to consist of the single element $\mathbf{s}$ given by

$$\mathbf{s}(f) = \|f\|_M \qquad (f \in {}^0\mathscr{C}(M, \tau_M)),$$

where

$$\|f\|_M^2 = \int_M |f(m)|^2\, dm.$$

We first need a simple result.

Lemma 2. *Let $H_0 \neq 0$ be a point in $\mathfrak{a}$ and ϕ a function of type $II(\lambda)$ on $\mathfrak{F} \times G$ such that $\phi(v)=0$ whenever $v(H_0)=0$ ($v \in \mathfrak{F}$). Then the function*

$$\psi(v:x)=v(H_0)^{-1}\,\phi(v:x) \qquad (v \in \mathfrak{F}, x \in G)$$

is also of type $II(\lambda)$.

This follows from Lemma 22.1.

Now we come to the proof of Theorem 1. If ϕ is of type $II'(\lambda)$, then for fixed $P \in \mathscr{P}(\mathfrak{a})$ and $s \in \mathfrak{w}(\mathfrak{a})$, $f_{P,s}$ defines a C^∞ mapping from $\mathfrak{F}$ to ${}^o\mathscr{C}(M, \tau_M)$ (see Lemma 12.1 below). Hence our condition is certainly necessary. So it remains to verify that it is sufficient.

Put

$$\psi(v:x)=\varpi(\lambda+(-1)^{1/2}\,v)\,\phi(v:x) \qquad (v \in \mathfrak{F}, x \in G).$$

Then by Theorem 9.1, ψ is of type $II'(\lambda)$. Let p be the set of all positive roots of $(\mathfrak{g}, \mathfrak{h})$, $p(\lambda)$ the subset of those $\alpha \in p$ for which $\lambda(H_\alpha) \neq 0$ and $p'(\lambda)$ the complement of $p(\lambda)$ in p. Put

$$\varpi_\lambda = \prod_{\alpha \in p(\lambda)} H_\alpha, \qquad \varpi'_\lambda = \prod_{\alpha \in p'(\lambda)} H_\alpha.$$

Then $\varpi = \varpi_\lambda \cdot \varpi'_\lambda$ and

$$|\varpi_\lambda(\lambda+(-1)^{1/2}\,v)| \geqq |\varpi_\lambda(\lambda)| > 0 \qquad (v \in \mathfrak{F}).$$

Hence it follows without difficulty that

$$\begin{aligned}\psi'(v:x) &= \varpi_\lambda(\lambda+(-1)^{1/2}\,v)^{-1}\,\psi(v:x)\\ &= \varpi'_\lambda(\lambda+(-1)^{1/2}\,v)\,\phi(v:x)\end{aligned}$$

is a function of type $II'(\lambda)$. Since λ is a regular element in $(-1)^{1/2}\,\mathfrak{h}_I^*$ (see § 3), it is clear that we can choose elements $H_i \neq 0$ $(1 \leqq i \leqq r)$ in $\mathfrak{a}$ and a complex number $c \neq 0$ such that

$$\varpi'_\lambda(\lambda+(-1)^{1/2}\,v) = c \prod_{1 \leqq i \leqq r} v(H_i) \qquad (v \in \mathfrak{F}).$$

Hence it is enough to prove the following result.

Lemma 3. *Put*

$$Q(v) = \prod_{1 \leqq i \leqq r} v(H_i) \qquad (v \in \mathfrak{F})$$

where $H_i \neq 0$ are elements in $\mathfrak{a}$. Suppose ϕ satisfies the condition of Theorem 1 and

$$\psi(v:x) = Q(v)\,\phi(v:x) \qquad (v \in \mathfrak{F},\; x \in G)$$

is a function of type $II'(\lambda)$. Then ϕ is also of type $II'(\lambda)$.

By induction we are reduced to the case $r=1$. Fix a psgp $P'=M'A'N'$ and $t \in \mathfrak{w}(\mathfrak{a}|\mathfrak{a}')$. Then

$$\psi_{P',t}(v) = v(H_1)\,\phi_{P',t}(v) \qquad (v \in \mathfrak{F}'(\lambda)).$$

We have to verify that $(\phi_{P',t})^{t}$ is of type $II(\lambda)$. Since ψ is of type $II'(\lambda)$, we know from Lemma 9.1 that $(\psi_{P',t})^{t}$ is also of type $II'(\lambda)$. Hence in view of Lemma 2, it would be enough to verify that $\psi_{P',t}(v)=0$ whenever $v(H_1)=0$.

Now fix $P\in\mathscr{P}(\mathfrak{a})$ and $s\in\mathfrak{w}(\mathfrak{a})$. Then

$$\psi_{P,s}(v)=v(H_1)\,\phi_{P,s}(v) \qquad (v\in\mathfrak{F}'(\lambda)).$$

Let $g(v)$ denote the restriction of $\psi_{P,s}(v)$ on M. Then we conclude from Lemma 12.1 below that $v\mapsto g(v)$ is a continuous mapping from $\mathfrak{F}$ into ${}^{0}\mathscr{C}(M,\tau_M)$.

Fix a point $v_0\in\mathfrak{F}$ such that $v_0(H_1)=0$. Let v be a variable point in $\mathfrak{F}'(\lambda)$ which tends to v_0. Then if $\mathrm{s}\in S$,

$$\mathrm{s}(g(v_0))=\lim_{v}\mathrm{s}(g(v))=\lim_{v}|v(H_1)|\,\mathrm{s}(f_{P,s}(v))=0$$

by our assumption on ϕ. Hence $g(v_0)=0$ and this implies (Corollary of Theorem 7.1) that $\psi_{P,s}(v_0)=0$. But then we conclude from Lemma 7.4 and the corollary of Lemma 1 that $\psi_{P',t}(v_0)=0$.

This proves Lemma 3 and therefore also Theorem 1.

§ 12. An Auxiliary Result

Let $G=MA$ be the Langlands decomposition of G and assume $\mathfrak{a}=\mathfrak{h}_R$. Let ϕ be a function of type $II(\lambda)$ on $\mathfrak{F}\times G$ and ψ its restriction on $\mathfrak{F}\times M$. Then we know from Lemma 7.5 that $\psi(v)\in{}^{0}\mathscr{C}(M,\tau_M)$ for $v\in\mathfrak{F}$. (We note that $\mathfrak{F}'(\lambda)=\mathfrak{F}$ and $\mathfrak{w}(\mathfrak{a})=\{1\}$ in this case.)

Lemma 1. *$v\mapsto\psi(v)$ is a C^∞ mapping of $\mathfrak{F}$ into ${}^{0}\mathscr{C}(M,\tau_M)$.*

This is an immediate consequence of the following lemma.

Lemma 2. *Suppose $\mathfrak{h}_R=\{0\}$. Fix $g_1,g_2\in\mathfrak{G}$ and $r_0\geqq 0$. Then we can choose a finite subset F of $\mathfrak{G}^{(2)}$ with the following property. Given $r\geqq 0$ and $\mathrm{s}\in\mathscr{S}(V)$, we can choose a number $c>0$ such that*

$$|\phi(g_1;x;g_2)|_{\mathrm{s}}\,\Xi(x)^{-1}(1+\sigma(x))^{r_0}\leqq c\,\mathrm{s}_{F,r}(\phi) \qquad (x\in G)$$

for all functions ϕ on G of type $II(\lambda)$.

Let $P=MAN$ be a psgp of G $(P\neq G)$. Since $\mathfrak{h}_R=\{0\}$, $\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})=\emptyset$ and $\mathfrak{F}'(\lambda)=\mathfrak{F}=\{0\}$. Therefore $\phi_P=0$ by Lemma 8.3. Moreover

$$z\phi=\gamma(z:\lambda)\phi \qquad (z\in\mathfrak{Z}).$$

Therefore Lemma 23.4 of [1(e)] is applicable. Fix a minimal p-pair (P_0,A_0) in G. Then $G=K\cdot Cl A_0^{+}\cdot K$ and there are only a finite number of p-pairs $(P,A)\succ(P_0,A_0)$. Our assertion is an easy consequence of these facts.

§ 13. Statement of the Two Main Theorems

We keep to the notation of § 8. For $D\in\tilde{\mathfrak{G}}$, $\mathbf{s}\in\mathscr{S}(V)$ and $r\geqq 0$, define

$$^{o}\mathbf{s}_{D,r}(f)=\sup_{\mathfrak{F}\times G}|Df|_{\mathbf{s}}\Xi^{-1}(1+\sigma)^{-r}\qquad (f\in C^{\infty}(\mathfrak{F}\times G, V)).$$

Similarly if F is any finite subset of $\tilde{\mathfrak{G}}$, we write

$$^{o}\mathbf{s}_{F,r}(f)=\sum_{D\in F}{}^{o}\mathbf{s}_{D,r}(f).$$

A function ϕ: $\mathfrak{F}\times G\to V$ will be said to be of type $I(\lambda)$ if:

1) ϕ is of type $II(\lambda)$.

2) For any $D\in\tilde{\mathfrak{G}}$ and $\mathbf{s}\in\mathscr{S}(V)$, we can choose $r\geqq 0$ such that $^{o}\mathbf{s}_{D,r}(\phi)<\infty$.

Moreover we say that ϕ is of type $I'(\lambda)$ if it is both of type $I(\lambda)$ and type $II'(\lambda)$.

Let $\mathscr{E}(I'(\lambda))$ denote the space of all functions of type $I'(\lambda)$ and $d\nu$ the Euclidean measure on $\mathfrak{F}$.

Theorem 1. *For $\phi\in\mathscr{E}(I'(\lambda))$, define*

$$j_{\phi}(x)=\int_{\mathfrak{F}}\phi(\nu:x)\,d\nu\qquad (x\in G).$$

Then $j_{\phi}\in\mathscr{C}(G,\tau)$. Fix $g_1, g_2\in\mathfrak{G}$ and $r_0\geqq 0$. Then we can choose a finite subset F of $\tilde{\mathfrak{G}}$ with the following property. Given $r\geqq 0$ and $\mathbf{s}\in\mathscr{S}(V)$, there exists a number $c>0$ such that

$$|j_{\phi}(g_1;x;g_2)|_{s}\leqq c\,{}^{o}\mathbf{s}_{F,r}(\phi)\,\Xi(x)(1+\sigma(x))^{-r_0}\qquad (x\in G)$$

for all $\phi\in\mathscr{E}(I'(\lambda))$.

Define the Schwartz space $\mathscr{C}(\mathfrak{F})$ as usual.

Corollary. *Fix a function ϕ on $\mathfrak{F}\times G$ of type $II'(\lambda)$ and define*

$$\phi_{\alpha}(x)=\int_{\mathfrak{F}}\alpha(\nu)\phi(\nu:x)\,d\nu\qquad (x\in G)$$

for $\alpha\in\mathscr{C}(\mathfrak{F})$. Then $\alpha\mapsto\phi_{\alpha}$ is a continuous mapping of $\mathscr{C}(\mathfrak{F})$ into $\mathscr{C}(G,\tau)$ and

$$\phi_{\alpha}(g_1;x;g_2)=\int_{\mathfrak{F}}\alpha(\nu)\phi(\nu:g_1;x;g_2)\,d\nu\qquad (x\in G)$$

for $g_1, g_2\in\mathfrak{G}$ and $\alpha\in\mathscr{C}(\mathfrak{F})$.

This is an immediate consequence of Theorem 1.

Fix ϕ as in the above corollary. Then if $P=MAN$ is a psgp of G and $\alpha\in\mathscr{C}(\mathfrak{F})$, it follows from Lemma 9.1 and the corollary of Lemma 10.7 that the function

$$\phi_{P,\alpha}(m)=\int_{\mathfrak{F}}\alpha(\nu)\phi_P(\nu:m)\,d\nu\qquad (m\in MA)$$

lies in $\mathscr{C}(MA,\tau_M)$. Extend it to a function on G by setting

$$\phi_{P,\alpha}(kmn)=\tau(k)\phi_{P,\alpha}(m)\qquad (k\in K,\ m\in MA,\ n\in N).$$

Put $\bar{P}=\theta(P)$, $\bar{N}=\theta(N)$, $\rho=\rho_P$, $H(x)=H_P(x)$ $(x\in G)$ and define $\phi_\alpha^{(P)}$ as in [1(e), Lemma 16.1].

Theorem 2. *Let $d\bar{n}$ denote the Haar measure on $\bar{N}$. Then*

$$\phi_\alpha^{(P)}(m)=d_P(m)\int_{\bar{N}}\phi_\alpha(\bar{n}m)\,d\bar{n}=\int_{\bar{N}}e^{-\rho(H(\bar{n}))}\phi_{P,\alpha}(\bar{n}m)\,d\bar{n}$$

for $m\in MA$ and $\alpha\in\mathscr{C}(\mathfrak{F})$.

This is a generalization of [1(b), Theorem 4, p. 610]. (It is part of the assertion of the theorem that the above integrals are well defined.)

In view of the corollary of Lemma 10.7, the following result is obvious.

Corollary. *$\phi_\alpha^{(P)}=0$ unless $\mathfrak{a}^k\subset\mathfrak{h}_R$ for some $k\in K$.*

The above two theorems contain the main results of this paper. The significance of Theorem 2 may be explained as follows. Extend d_P and $\phi_P(v)$ to functions on G as in [1(e), § 24]. Then Theorem 2 asserts that

$$\int_{\bar{N}}d_P(\bar{n})^{-1}d\bar{n}\int_{\mathfrak{F}}\alpha(v)\,d_P(\bar{n}m)\,\phi(v:\bar{n}m)\,dv$$
$$=\int_{\bar{N}}d_P(\bar{n})^{-1}d\bar{n}\int_{\mathfrak{F}}\alpha(v)\,\phi_P(v:\bar{n}m)\,dv\qquad(m\in MA)$$

for $\alpha\in\mathscr{C}(\mathfrak{F})$. This shows that the integral on the left remains unchanged when we replace $d_P\phi(v)$ by its asymptotic value $\phi_P(v)$ [1(e), Lemma 24.1].

§ 14. Some Preparation

Put $\mathfrak{D}=\mathfrak{D}(\mathfrak{F}_c)$, $\mathscr{E}'=\mathscr{E}(I'(\lambda))$ and let $\mathscr{E}=\mathscr{E}(I(\lambda))$ denote the space of all functions on $\mathfrak{F}\times G$ of type $I(\lambda)$. It is obviously enough to prove the statement of Theorem 13.1 for $\mathbf{s}\in\mathscr{S}^o(V)$ [1(e), § 22].

Let $\mathbf{R}_+$ denote the set of all real numbers $r\geqq 0$. In order to avoid tedious repetitions, we agree to the following conventions. The variables r, $\mathbf{s}$ and v shall range freely over $\mathbf{R}_+$, $\mathscr{S}^o(V)$ and $\mathfrak{F}$ respectively unless explicitly mentioned otherwise. Let Y be any set and f, g two functions from $\mathbf{R}_+\times\mathscr{S}^o(V)\times\mathfrak{F}\times Y$ to $\mathbf{R}_+\cup\{\infty\}$. Then we write

$$f(r,\mathbf{s},v,y)\prec g(r,\mathbf{s},v,y)\qquad(y\in Y),$$

if for any given r and $\mathbf{s}$ we can choose a real number $c(r,\mathbf{s})>0$ such that

$$f(r,\mathbf{s},v,y)\leqq c(r,\mathbf{s})\,g(r,\mathbf{s},v,y)$$

for all $v\in\mathfrak{F}$ and $y\in Y$. Finally the letter F will always stand for a finite set. Thus $F\subset Y$ means that F is a finite subset of Y.

We now use the notation of § 5 and fix numbers c_0, $d_0\geqq 0$ such that

$$d_P(m)\,\Xi(m)\leqq c_0\,\Xi_M(m)(1+\sigma(m))^{d_0}\qquad(m\in M_1^+).$$

Lemma 1. *Fix $\zeta\in\mathfrak{Z}_1$, $v_1, v_2\in\mathfrak{M}_1$ and $D\in\mathfrak{D}$. Then we can choose $F\subset\mathfrak{G}$ such that*

$$|\psi_{i,\zeta}(v;D:v_1;m;v_2)|_s\leqq{}^o\mathrm{s}_{F,r}(\phi)\Xi_M(m)|(m,H)|^{d_0+r}e^{-\beta_P(H)}$$

for $\phi\in\mathscr{E}$, $m\in M_1^+$, $H\in Cl\,\mathfrak{a}^+$ and $1\leqq i\leqq q$.

Here $\psi_{i,\zeta}$ is the function defined in § 5 corresponding to ϕ. This lemma is proved in the same way as [1(e), Lemma 22.3].

Lemma 2. *Given $D\in\mathfrak{D}$ and $v_1, v_2\in\mathfrak{M}_1$, we can choose $F\subset\mathfrak{G}$ such that*

$$|\phi_{i\infty}(v;D:v_1;m;v_2)|_s\prec{}^o\mathrm{s}_{F,r}(\phi)\Xi_M(m)(1+\sigma(m))^{d_0+r}\qquad(i\in Q^o)$$

for $m\in M_1$ and *$\phi\in\mathscr{E}$.*

We use the notation of the proof of Lemma 8.2. Fix $H\in\mathfrak{a}^+$. Then

$$\begin{aligned}&\phi_{i\infty}(v;D:v_1;m;v_2)\\&\quad=\phi_i(v;D:v_1;m;v_2)+\int_0^\infty\psi_{i,H}{}^o(v;D:v_1;m;v_2:tH)\,dt\end{aligned}$$

and our assertion follows without difficulty.

Now suppose $\phi\in\mathscr{E}'$. Then for any $\mathrm{s}\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$, $\phi_{P,s}$ extends to a C^∞ function on $\mathfrak{F}\times M_1$.

Lemma 3. *Given $D\in\mathfrak{D}$ and $v_1, v_2\in\mathfrak{M}_1$, we can choose $F\subset\mathfrak{G}$ such that*

$$|\phi_{P,s}(v;D:v_1;m;v_2)|_s\prec{}^o\mathrm{s}_{F,r}(\phi)\Xi_M(m)(1+\sigma(m))^{d_0+r}$$

for $m\in M_1$, $\phi\in\mathscr{E}'$ and $s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$.

In view of Lemmas 3.3 and 7.3, this is an immediate consequence of Lemmas 2 and 22.2.

Corollary. *If $\phi\in\mathscr{E}'$, then for any $s\in\mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$, $(\phi_{P,s})^s$ is a function of type $I'(\lambda)$ on $\mathfrak{F}\times M_1^s$.*

This follows from Lemmas 3 and 9.1.

Now let us use the notation of Lemma 10.5.

Lemma 4. *Fix $\zeta\in\mathfrak{Z}_1$, $v_1, v_2\in\mathfrak{M}_1$ and $r_1\geqq 0$. Then we can choose $F\subset\mathfrak{G}$ such that*

$$|\Psi_\zeta(v:v_1;m\exp H;v_2)|_s(1+|v|)^{r_1}\leqq{}^o\mathrm{s}_{F,r}(\phi)\Xi_M(m)|(m,H)|^{d_0+r}e^{-\beta_P(H)}$$

for $m\in M_1^+$, $H\in Cl\,\mathfrak{a}^+$, $\phi\in\mathscr{E}$.

As before this follows from Lemma 5.2.

Now assume that $P\neq G$. Fix a compact set Ω in $\mathfrak{a}^+$ and choose $\varepsilon_0>0$ such that $\beta_P(H)\geqq 2\varepsilon_0$ for all $H\in\Omega$. Select δ $(0<\delta\leqq\frac{1}{2})$ as in § 10 and put $\varepsilon=\delta\varepsilon_0$.

Lemma 5. *Given $v_1, v_2\in\mathfrak{M}_1$ and $r_1\geqq 0$, we can choose $F\subset\mathfrak{G}$ such that*

$$\begin{aligned}&|d_P(m\exp TH)\phi(v:v_1';m\exp TH;v_2')-\phi_P(v:v_1;m\exp TH;v_2)|_s(1+|v|)^{r_1}\\&\qquad\prec{}^o\mathrm{s}_{F,r}(\phi)\Xi_M(m)(1+\sigma(m))^{d_0+r}e^{-\varepsilon T}\end{aligned}$$

for $m\in M_1^+$, $H\in\Omega$, $T\geqq 0$ and $\phi\in\mathscr{E}$.

This is proved in the same way as Lemma 10.8.

Now fix $r_1 \geqq 0$ such that

$$\int_{\mathfrak{F}} (1+|v|)^{-r_1} dv < \infty .$$

If $\phi \in \mathscr{E}'$, we know (Corollary of Lemma 10.7) that

$$\phi_P = \sum_{s \in \mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})} \phi_{P,s} .$$

Put $j(\phi : x) = j_\phi(x)$ $(x \in G)$ and

$$j(\phi_{P,s} : m) = \int_{\mathfrak{F}} \phi_{P,s}(v : m) dv \qquad (m \in M_1)$$

for $s \in \mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$ and $\phi \in \mathscr{E}'$.

Corollary.

$$|d_P(m \exp TH) j(\phi : v_1'; m \exp TH; v_2') - \sum_{s \in \mathfrak{w}(\mathfrak{h}_R|\alpha)} j(\phi_{P,s} : v_1; m \exp TH; v_2)|_s$$
$$< {}^0\mathbf{s}_{F,r}(\phi) \Xi_M(m)(1+\sigma(m))^{d_0+r} e^{-\varepsilon T}$$

for $m \in M_1^+$, $H \in \Omega$, $T \geqq 0$ *and* $\phi \in \mathscr{E}'$.

This follows immediately from Lemma 5.

§ 15. Proof of Theorem 13.1

We now come to the proof of Theorem 13.1. It is clearly enough to prove the second part of the theorem.

We proceed by induction on dim G. First assume that prk $G > 0$ and let $G = MA$ be the Langlands decomposition of G. Then $\mathfrak{h}_R = \mathfrak{m} \cap \mathfrak{h}_R + \mathfrak{a}$ where the sum is direct. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be the subspace consisting of all $v \in \mathfrak{F}$ which vanish identically on $\mathfrak{m} \cap \mathfrak{h}_R$ and $\mathfrak{a}$ respectively. Then $\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2$ where the sum is direct. We note that $\mathfrak{D}_i = \mathfrak{D}(\mathfrak{F}_{ic}) \subset \mathfrak{D} = \mathfrak{D}(\mathfrak{F}_c)$ [1(c), p. 540]. Let dv_i denote the Euclidean measure on $\mathfrak{F}_i$ so normalized that $dv = dv_1 dv_2$ $(v = v_1 + v_2,\ v_i \in \mathfrak{F}_i,\ i = 1, 2)$. Since $\mathfrak{a} \subset \mathfrak{Z}$, it follows from our assumptions that

$$\phi(v_1 + v_2 : ma) = \phi(v_1 + v_2 : m) e^{(-1)^{1/2} v_1(\log a)} \qquad (m \in M,\ a \in A)$$

for $\phi \in \mathscr{E}'$ and $v_i \in \mathfrak{F}_i$. Fix $v_1, v_2 \in \mathfrak{M}$ and $u \in \mathfrak{A}$. Then

$$j_\phi(v_1; ma; v_2 u) = \int \phi(v_1 + v_2 : v_1; m; v_2) u((-1)^{1/2} v_1) e^{(-1)^{1/2} v_1(\log a)} dv_1 dv_2 .$$

(We regard u as a polynomial function on $\mathfrak{F}_{1c}$ in the right side.) Now fix $r_0 \geqq 0$. Then we can choose $p \in S(\mathfrak{F}_{1c})$ such that

$$p((-1)^{1/2} H) \geqq (1 + \|H\|)^{r_0} \qquad (H \in \mathfrak{a}).$$

Also we can select a polynomial function p_1 on $\mathfrak{F}_1$ such that $p_1 \geqq 1$ on $\mathfrak{F}_1$ and

$$\int_{\mathfrak{F}_1} p_1^{-1} dv_1 < \infty .$$

Hence it is obvious that there exists an element $D_1 \in \mathfrak{D}_1$ such that

$$|j_\phi(v_1; ma; v_2 u)|_s (1+\sigma(a))^{r_0} \leqq \sup_{v_1 \in \mathfrak{F}_1} \Big| \int_{\mathfrak{F}_2} \phi(v_1+v_2; D_1 : v_1; m; v_2)\, dv_2 \Big|_s$$

for $m \in M$, $a \in A$ and $\phi \in \mathscr{E}'$.

On the other hand $\dim M < \dim G$ and so the induction hypothesis is applicable to M. Let $\mathscr{E}'_M$ be the space of all functions ψ on $\mathfrak{F}_2 \times M$ of type $I'(\lambda)$. Then we can choose a finite subset F_2 of $\mathfrak{M} = \mathfrak{D}_2 \otimes \mathfrak{M}^{(2)}$ such that

$$\Big| \int_{\mathfrak{F}_2} \psi(v_2 : v_1; m; v_2)\, dv_2 \Big|_s \prec {}^0\mathbf{s}_{F_2, r}(\psi)\, \Xi(m)(1+\sigma(m))^{-r_0}$$

for $m \in M$ and $\psi \in \mathscr{E}'_M$.

We regard $\mathfrak{M}$ as a subalgebra of $\mathfrak{G} = \mathfrak{D} \otimes \mathfrak{G}^{(2)}$. Let F denote the subset of $\mathfrak{G}$ consisting of all elements of the form $D_2 D_1$ ($D_2 \in F_2$). Fix $\phi \in \mathscr{E}'$, $v_1 \in \mathfrak{F}_1$ and put

$$\psi(v_2 : m) = \phi(v_1+v_2; D_1 : m) \qquad (v_2 \in \mathfrak{F}_2,\ m \in M).$$

Then $\psi \in \mathscr{E}'_M$ and so we conclude from the above result that

$$|j_\phi(v_1; ma; v_2 u)|_s \prec {}^0\mathbf{s}_{F,r}(\phi)\, \Xi(m)(1+\sigma(m))^{-r_0}(1+\sigma(a))^{-r_0}$$

for $m \in M$, $a \in A$ and $\phi \in \mathscr{E}'$. This obviously implies Theorem 13.1 in this case.

So now suppose $\operatorname{prk} G = 0$. The case $G = K$ being trivial, we may assume that G is not compact. Fix a minimal p-pair (P_0, A_0) in G and let S^+ be the set of all $H \in Cl\,\mathfrak{a}_0^+$ with $\|H\| = 1$. Fix $H_0 \in S^+$ and let F_0 be the set of all simple roots of (P_0, A_0) which vanish at H_0. Put $(P, A) = (P_0, A_0)_{F_0}$. Then $H_0 \in \mathfrak{a}^+$. Fix a compact neighborhood Ω_0 of H_0 in S^+ such that

$$\alpha(H) \geqq \alpha(H_0)/2 \qquad (H \in \Omega_0)$$

for every root α of (P_0, A_0). Put $\varepsilon_0 = \beta_P(H_0)/4$ and $\varepsilon = \delta \varepsilon_0$ where δ is defined as in § 10. Since

$$\exp tH = m_t \exp(tH_0/2) \qquad (H \in \Omega_0,\ t \geqq 0),$$

where $m_t = \exp t(H - \frac{1}{2}H_0) \in Cl\,A_0^+ \subset M_1^+$, we get the following result from the corollary of Lemma 14.5.

Lemma 1. *Given $v_1, v_2 \in \mathfrak{M}_1$, we can choose $F \subset \mathfrak{G}$ such that*

$$\Big| d_P(\exp tH) j(\phi : v_1'; \exp tH; v_2') - \sum_{s \in \mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})} j(\phi_{P,s} : v_1; \exp tH; v_2) \Big|_s$$
$$\prec {}^0\mathbf{s}_{F,r}(\phi)\, \Xi_M(\exp tH)(1+t)^{d_0+r} e^{-\varepsilon t}$$

for $H \in \Omega_0$, $t \geqq 0$ and $\phi \in \mathscr{E}'$.

On the other hand since $\operatorname{prk} G = 0$ and $H_0 \neq 0$, it is clear that $\dim M_1 < \dim G$. Moreover for $s \in \mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$ and $\phi \in \mathscr{E}'$, $(\phi_{P,s})^s$ is a function of type $I'(\lambda)$ on $\mathfrak{F} \times M_1^s$ (Corollary of Lemma 14.3). Hence if we take into account Lemma 14.3 and apply the induction hypothesis to M_1^s, we get the following result immediately.

Lemma 2. *Fix* $v_1, v_2 \in \mathfrak{M}_1$ *and* $r_0 \geqq 0$. *Then we can choose* $F \subset \tilde{\mathfrak{G}}$ *such that*

$$|j(\phi_{P,s}: v_1; m; v_2)|_s \prec {}^o s_{F,r}(\phi) \Xi_M(m)(1+\sigma(m))^{-r_0}$$

for $s \in \mathfrak{w}(\mathfrak{h}_R|\mathfrak{a})$, $m \in M_1$ *and* $\phi \in \mathscr{E}'$.

Combining this with Lemma 1 and standard inequalities relating Ξ and Ξ_M, we get the following result.

Lemma 3. *Given* $v_1, v_2 \in \mathfrak{M}_1$ *and* $r_0 \geqq 0$, *we can choose* $F \subset \tilde{\mathfrak{G}}$ *such that*

$$|j(\phi: v_1; \exp tH; v_2)|_s \prec {}^o s_{F,r}(\phi) \Xi(\exp tH)(1+t)^{-r_0}$$

for $H \in \Omega_0$, $t \geqq 0$ *and* $\phi \in \mathscr{E}'$.

On the other hand the following result is an immediate consequence of Lemma 5.2.

Lemma 4. *Fix* $D \in \mathfrak{D}$, $g_i \in \mathfrak{G}$ $(1 \leqq i \leqq 4)$ *such that* $g_1 \in \mathfrak{G}\mathfrak{n}$ *and* $g_4 \in \theta(\mathfrak{n})\mathfrak{G}$. *Then we can choose* $F \subset \tilde{\mathfrak{G}}$ *such that*

$$\sum_{i=1,3} |\phi(v; D: g_i; \exp tH; g_{i+1})|_s \leqq {}^o s_{F,r}(\phi) \Xi(\exp tH)(1+t)^r e^{-2\varepsilon_0 t}$$

for $\phi \in \mathscr{E}$, $H \in \Omega_0$ *and* $t \geqq 0$.

Now fix a polynomial function p on $\mathfrak{F}$ such that $p \geqq 1$ on $\mathfrak{F}$ and

$$\int_{\mathfrak{F}} p^{-1}\, dv < \infty.$$

Then taking $D = p$ in the above lemma, we get the following corollary.

Corollary. *Let* g_i $(1 \leqq i \leqq 4)$ *be as above. Then we can choose* $F \subset \tilde{\mathfrak{G}}$ *such that*

$$\sum_{i=1,3} |j(\phi: g_i; \exp tH; g_{i+1})|_s \leqq {}^o s_{F,r}(\phi) \Xi(\exp tH)(1+t)^r e^{-2\varepsilon_0 t}$$

for $\phi \in \mathscr{E}$, $H \in \Omega_0$ *and* $t \geqq 0$.

Now fix $g_1, g_2 \in \mathfrak{G}$. Since $G = K \cdot ClA_0^+ \cdot K$, S^+ is compact and

$$j_\phi(g_1; k_1^{-1} a k_2; g_2) = \tau(k_1^{-1}) j_\phi(g_1^{k_1}; a; g_2^{k_2}) \tau(k_2)$$

$(k_1, k_2 \in K, a \in A_0, \phi \in \mathscr{E}')$, in order to prove Theorem 13.1, it would be enough to verify the following lemma.

Lemma 5. *Fix* $g_1, g_2 \in \mathfrak{G}$ *and* $H_0 \in S^+$. *Then we can choose a neighborhood* Ω_0 *of* H_0 *in* S^+ *satisfying the following condition. Given* $r_0 \geqq 0$, *there exists* $F \subset \tilde{\mathfrak{G}}$ *such that*

$$|j(\phi: h_1; \exp tH; g_2)|_s \prec {}^o s_{F,r}(\phi) \Xi(\exp tH)(1+t)^{-r_0}$$

for $\phi \in \mathscr{E}'$, $H \in \Omega_0$ *and* $t \geqq 0$.

Since

$$\mathfrak{G} = \mathfrak{K}\mathfrak{M}_1 \mathfrak{N} = \theta(\mathfrak{N}) \mathfrak{M}_1 \mathfrak{K}$$

and τ is differentiable, we may without loss of generality assume that $g_1 \in \mathfrak{M}_1 \mathfrak{N}$ and $g_2 \in \theta(\mathfrak{N}) \mathfrak{M}_1$. Then we can choose $v_i \in \mathfrak{M}_1$ $(i=1,2)$ such that

$$g_1 - v_1 \in \mathfrak{G}\mathfrak{n}, \quad g_2 - v_2 \in \theta(\mathfrak{n})\,\mathfrak{G}.$$

Our assertion now follows immediately from Lemma 3 and the corollary of Lemma 4.

This completes the proof of Theorem 13.1.

§ 16. Proof of Theorem 13.2

We shall now begin preparation for the proof of Theorem 13.2. Fix a function ϕ on $\mathfrak{F} \times G$ of type $II'(\lambda)$. We use the notation of § 10 and assume, as we may, that $P \neq G$. We also agree to the convention that the variables v, $\bar{n}$ and m shall range freely over $\mathfrak{F}$, $\bar{N}$ and M_1 respectively unless explicitly stated otherwise. Put

$$\Phi(v:\bar{n}:m) = d_P(m) \sum_{1 \leq i \leq q} \phi(v:\bar{n}m;\eta_i') \otimes u_i^*$$

and consider the obvious pairing [1(e), § 21] of $V \otimes U^*$, U into V given by

$$\langle v \otimes u^*, u \rangle = \langle u^*, u \rangle\, v \qquad (v \in V,\ u^* \in U^*,\ u \in U).$$

For any $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U) = \mathscr{C}(\mathfrak{F}) \otimes U$, define

$$\Phi(\mathbf{b}:\bar{n}:m) = \int_{\mathfrak{F}} \langle \Phi(v:\bar{n}:m), \mathbf{b}(v) \rangle\, dv$$

and put

$$F_{\mathbf{b}}(m) = F(\mathbf{b}:m) = \int_{\bar{N}} \Phi(\mathbf{b}:\bar{n}:m)\, d\bar{n}.$$

It follows from the corollary of Theorem 13.1 and [1(e), § 16] that this integral is well defined and in fact we have the following result.

Lemma 1. $\mathbf{b} \to F_{\mathbf{b}}$ *is a continuous mapping of* $\mathscr{C}(\mathfrak{F}, U)$ *into* $\mathscr{C}(M_1, \tau_M)$.

For $\zeta \in \mathfrak{Z}_1$, define $g_i(\zeta:v)$ $(1 \leq i \leq q)$ as in § 10 and put

$$\Psi_\zeta(v:\bar{n}:m) = d_P(m) \sum_{1 \leq i \leq q} \phi(v:\bar{n}m; g_i(\zeta:v)) \otimes u_i^*.$$

Lemma 2. *Let* $\zeta \in \mathfrak{Z}_1$. *Then*

$$\Phi(v:\bar{n}:m;\zeta) = \Phi(v:\bar{n}:m)\, \Gamma_v(\zeta) + \Psi_\zeta(v:\bar{n}:m).$$

This is proved in the same way as Lemma 10.1.

Now put

$$\Psi_\zeta(\mathbf{b}:\bar{n}:m) = \int \langle \Psi_\zeta(v:\bar{n}:m), \mathbf{b}(v) \rangle\, dv$$

for $\zeta \in \mathfrak{Z}_1$ and $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)$.

Lemma 3. *Let* $\zeta \in \mathfrak{Z}_1$ *and* $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)$. *Then*

$$\int_{\bar{N}} \Psi_\zeta(\mathbf{b}:\bar{n}:m)\, d\bar{n} = 0.$$

We know (see § 10) that $\nu \mapsto g_i(\zeta : \nu)$ is a polynomial mapping of $\mathfrak{F}$ into $\bar{\mathfrak{n}}\mathfrak{G}$. Therefore (Corollary of Theorem 13.1) the above integral is defined and it would be enough to verify the following result.

Lemma 4. *Fix $X \in \bar{\mathfrak{n}}$, $g \in \mathfrak{G}$ and $b \in \mathscr{C}(\mathfrak{F})$. Then*

$$\int_{\bar{N}} d\bar{n} \int_{\mathfrak{F}} b(\nu)\, \phi(\nu : \bar{n}m;\, X g)\, d\nu = 0.$$

Put

$$\psi(x) = \int_{\mathfrak{F}} b(\nu)\, \phi(\nu : x;\, g)\, d\nu \qquad (x \in G).$$

Then $\psi \in \mathscr{C}(G, V)$ (Corollary of Theorem 13.1). Let

$$f(x) = \int_{\bar{N}} \psi(\bar{n}x)\, d\bar{n} \qquad (x \in G).$$

Then [1(e), § 16] $f \in C^\infty(G, V)$ and

$$f(x;\, X) = \int_{\bar{N}} \psi(\bar{n}x;\, X)\, d\bar{n}.$$

Therefore since $f(\bar{n}x) = f(x)$ and $X^m \in \bar{\mathfrak{n}}$, we conclude that

$$f(m;\, X) = f(X^m;\, m) = 0.$$

This proves the lemma.

For any $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)$ and $\zeta \in \mathfrak{Z}_1$, let $\Gamma(\zeta)\mathbf{b}$ denote the function $\nu \mapsto \Gamma_\nu(\gamma_1(\zeta))\, \mathbf{b}(\nu)$ from $\mathfrak{F}$ to U in the notation of § 10. It is clear from Corollary 3 of Lemma 3.5 that for a fixed ζ, $\mathbf{b} \mapsto \Gamma(\zeta) b$ is a continuous endomorphism of $\mathscr{C}(\mathfrak{F}, U)$.

Lemma 5. *Let $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)$ and $\zeta \in \mathfrak{Z}_1$. Then*

$$F(\mathbf{b} : m;\, \zeta) = F(\Gamma(\zeta)\, \mathbf{b} : m).$$

This is an immediate consequence of Lemmas 2 and 3.

Define ${}^oQ$ and ${}_ie\,(i \in {}^oQ)$ as in Lemma 3.4 and put

$${}_i\mathbf{b}(\nu) = \Gamma_\nu({}_ie(\nu))\, \mathbf{b}(\nu)$$

for $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)$. Then it is clear from Lemmas 3.3, 3.4 and Corollary 3 of Lemma 3.5 that $\mathbf{b} \mapsto {}_i\mathbf{b}$ is a continuous endomorphism of $\mathscr{C}(\mathfrak{F}, U)$ and

$$\mathbf{b} = \sum_{i \in {}^oQ} {}_i\mathbf{b}.$$

Put ${}^oQ^o = {}^oQ \cap Q^o$ as in § 10 and define

$$e^o(\nu) = \sum_{i \in {}^oQ^o} {}_ie(\nu),$$

$$\mathbf{b}^o = \sum_{i \in {}^oQ^o} {}_i\mathbf{b} \qquad (\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)).$$

Lemma 6. *Let $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)$ and $i \in {}^oQ$. Then $F({}_i\mathbf{b} : m) = 0$ unless $i \in {}^oQ^o$. Hence $F(\mathbf{b} : m) = F(\mathbf{b}^o : m)$.*

Put

$$F(\mathbf{b} : m : \mu) = \int_{\mathfrak{a}} F(\mathbf{b} : m \exp H)\, e^{-(-1)^{1/2}\mu(H)}\, dH$$

for $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)$ and $\mu \in \mathfrak{a}^*$. (Here dH denotes the Euclidean measure on $\mathfrak{a}$ and $\mathfrak{a}^*$ the dual of $\mathfrak{a}$.) It follows from Lemma 1 that for (m, μ) fixed,

$$\mathbf{b} \to F(\mathbf{b}: m: \mu)$$

is a continuous mapping of $\mathscr{C}(\mathfrak{F}, U)$ into V. Moreover we conclude from Lemma 5 that

$$F(\Gamma(H)\,\mathbf{b}: m: \mu) = (-1)^{1/2}\,\mu(H)\,F(\mathbf{b}: m: \mu)$$

for $H \in \mathfrak{a}$.

Now fix $m_0 \in M_1$, $\mu \in \mathfrak{a}^*$, $i \in {}^oQ$ and put

$$T(\mathbf{b}) = F({}_i\mathbf{b}: m_0: \mu) \qquad (\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)).$$

Since $\dim U < \infty$, T may be regarded as a tempered distribution on $\mathfrak{F}$ with values in $V \otimes U^*$ (i.e. a continuous linear mapping of $\mathscr{C}(\mathfrak{F})$ into $V \otimes U^*$).

Lemma 7. *Fix $H \in \mathfrak{a}$, $\mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)$ and put*

$$\mathbf{b}'(v) = \prod_{t \in W_0(s_i, \lambda)} \{(-1)^{1/2}\,\mu(H) - t\,s_i\,\Lambda_v(H)\} \cdot {}_i\mathbf{b}(v)$$

in the notation of § 3. *Then $T(\mathbf{b}') = 0$.*

It follows from what we have seen above that

$$T(\Gamma(H)\,\mathbf{b}) = (-1)^{1/2}\,\mu(H)\,T(\mathbf{b}) \qquad (H \in \mathfrak{a}, \mathbf{b} \in \mathscr{C}(\mathfrak{F}, U)).$$

Hence our assertion is an immediate consequence of Corollary 5 of Lemma 3.5.

Now suppose $T \neq 0$. Then if $v_0 \in \operatorname{Supp} T$ ($v_0 \in \mathfrak{F}$), it follows from Lemma 7 that

$$\prod_{t \in W_0(s_i, \lambda)} \{(-1)^{1/2}\,\mu(H) - t\,s_i\,\Lambda_{v_0}(H)\} = 0$$

for all $H \in \mathfrak{a}$. Since $\Re\, t\,s_i\,\Lambda_{v_0}(H) = s_i\,\lambda^y(H)$, this implies that $i \in {}^oQ^o$. Therefore if $i \notin {}^oQ^o$, we conclude that $F({}_i\mathbf{b}: m: \mu) = 0$ for all $m \in M_1$ and $\mu \in \mathfrak{a}^*$. The statement of Lemma 6 now follows immediately by Fourier transform.

Now introduce the structure of a Hilbert space on U so that $(u_1, \ldots, u_q)$ becomes an orthonormal base. Moreover for any $E \in \operatorname{End} U$, let $\|E\|$ denote the Hilbert-Schmidt norm of E.

Lemma 8. *Put*

$$E(H: v) = e^{\Gamma_v(H)}\,\Gamma_v(e^o(v)) \qquad (H \in \mathfrak{a}).$$

Then for a given $D \in \mathfrak{D}(\mathfrak{F}_c)$, we can choose $c, r \geqq 0$ such that

$$\|E(H: v; D)\| \leqq c(1 + |v|)^r (1 + \|H\|)^r$$

for all $H \in \mathfrak{a}$.

Set

$$p(v) = \prod_{i \in {}^oQ^o} \varpi_{s_i, \lambda}(s_i\,\Lambda_v)$$

in the notation of § 3. Then p is a polynomial function on $\mathfrak{F}$ and by Lemma 3.3,

$$|p(v)| \geqq |\varpi_\lambda(\lambda)|^{q_0} > 0$$

where q_0 is the number of elements in ${}^oQ^o$. Put $E^o(\nu)=\Gamma_\nu(e^o(\nu))$. Then for a fixed $H\in\mathfrak{a}$, $\nu\mapsto\Gamma_\nu(H)$ and $\nu\mapsto p(\nu)\,E^o(\nu)$ are polynomial mappings of $\mathfrak{F}$ into End U (see § 3). Moreover

$$E(H:\nu)=e^{\Gamma_\nu(H)\,E^o(\nu)}\cdot E^o(\nu)$$

and all eigenvalues of $\Gamma_\nu(H)\,E^o(\nu)$ are pure imaginary (Corollaries 2 and 5 of Lemma 3.5). Hence our assertion follows without difficulty from [1(a), Lemma 60].

Now fix $H_0\in\mathfrak{a}$, $\mathbf{a}\in\mathscr{C}(\mathfrak{F},U)$ and for any $t\in\mathbf{R}$, put

$$\mathbf{a}_t(\nu)=E(-tH_0:\nu)\,\mathbf{a}(\nu).$$

Then it follows from Lemma 8 that $t\mapsto\mathbf{a}_t$ is a C^∞ function from $\mathbf{R}$ to $\mathscr{C}(\mathfrak{F},U)$.

Lemma 9. *Fix $m\in M_1$ and $\mu\in\mathfrak{a}^*$. Then*

$$F(\mathbf{a}:m:\mu)=F(\mathbf{a}^o:m:\mu)=e^{(-1)^{1/2}t\mu(H_0)}F(\mathbf{a}_t:m:\mu)$$

for $t\in\mathbf{R}$.

Put

$$T(\mathbf{b})=F(\mathbf{b}:m:\mu)\qquad(\mathbf{b}\in\mathscr{C}(\mathfrak{F},U)).$$

Then, as we have seen above, T is a continuous linear mapping of $\mathscr{C}(\mathfrak{F},U)$ into V and

$$T(\Gamma(H)\,\mathbf{b})=(-1)^{1/2}\mu(H)\,T(\mathbf{b})\qquad(H\in\mathfrak{a}).$$

Now let

$$f(t)=T(\mathbf{a}_t)\qquad(t\in\mathbf{R}).$$

It follows from the definition of $\mathbf{a}_t$ that

$$d\mathbf{a}_t/dt=-\Gamma(H_0)\,\mathbf{a}_t$$

and therefore

$$df/dt=-(-1)^{1/2}\mu(H_0)f.$$

This implies that

$$f(t)=e^{-(-1)^{1/2}t\mu(H_0)}f(0),$$

which is equivalent to the required result, if we take Lemma 6 into account.

Now assume that $H_0\in\mathfrak{a}^+$. Then it is clear from Lemma 2 that

$$\begin{aligned} &d\Phi(\mathbf{a}_t:\bar n:m\exp tH_0)/dt\\ &\quad=-\Phi(\Gamma(H_0)\,\mathbf{a}_t:\bar n:m\exp tH_0)+\Phi(\mathbf{a}_t:\bar n:m\exp tH_0;H_0)\\ &\quad=\Psi_{H_0}(\mathbf{a}_t:\bar n:m\exp tH_0)\qquad(t\in\mathbf{R}).\end{aligned}$$

Put $\Psi=\Psi_{H_0}$ and for any $\mathbf{b}\in\mathscr{C}(\mathfrak{F},U)$ and $\alpha\in C_c^\infty(\bar N)$, define

$$\Phi(\mathbf{b}:\alpha:\bar n_0:m)=\int_{\bar N}\alpha(\bar n)\,\Phi(\mathbf{b}:\bar n_0\,\bar n:m)\,d\bar n,$$

$$\Psi(\mathbf{b}:\alpha:\bar n_0:m)=\int_{\bar N}\alpha(\bar n)\,\Psi(\mathbf{b}:\bar n_0\,n:m)\,d\bar n$$

for $\bar n_0\in\bar N$. Then the following result is obvious.

Lemma 10. *$d\Phi(\mathbf{a}_t:\alpha:\bar{n}:m\ \exp tH_0)/dt=\Psi(\mathbf{a}_t:\alpha:\bar{n}:m\exp tH_0)$ for $\alpha\in C_c^\infty(\bar{N})$ and $t\in\mathbf{R}$.*

Let us now put

$$\phi_b(\bar{n}:x)=\phi_b(\bar{n}x)\qquad(x\in G)$$

for $b\in\mathscr{C}(\mathfrak{Z})$, and define

$$\phi_b(\alpha:\bar{n}_0:x)=\int_{\bar{N}}\alpha(\bar{n})\,\phi_b(\bar{n}_0\bar{n}:x)\,d\bar{n}\qquad(\bar{n}_0\in\bar{N},\ x\in G)$$

for $\alpha\in C_c^\infty(\bar{N})$. Then if $X\in\bar{\mathfrak{n}}$ and $g\in\mathfrak{G}$, it is clear that

$$\phi_b(\bar{n}m;Xg)=\phi_b(\bar{n};X^m:m;g).$$

Since $X^m\in\bar{\mathfrak{n}}$, it follows that

$$\phi_b(\alpha:\bar{n}:m;Xg)=-\phi_b(X^m\alpha:\bar{n}:m;g).$$

Put $\beta_P(H_0)=2\varepsilon$ so that $\varepsilon>0$.

Lemma 11. *Fix $m_0\in M_1$, $\alpha\in C_c^\infty(\bar{N})$, $X\in\bar{\mathfrak{n}}$, $g\in\mathfrak{G}$, $\mathbf{s}\in\mathscr{S}(V)$ and $r_0\geqq 0$. Then we can choose a continuous seminorm $\mathbf{t}$ on $\mathscr{C}(\mathfrak{Z})$ such that*

$$|\phi_b(\alpha:\bar{n}:m_t:Xg)|_{\mathbf{s}}\leqq\mathbf{t}(b)\,e^{-2\varepsilon t}\int_{\omega}\Xi_{r_0}(\bar{n}\bar{n}_0m_t)\,d\bar{n}_0$$

for $t\geqq 0$ and $b\in\mathscr{C}(\mathfrak{Z})$. Here $m_t=m_0\exp tH_0$,

$$\Xi_{r_0}(x)=\Xi(x)(1+\sigma(x))^{-r_0}\qquad(x\in G)$$

and $\omega=\operatorname{Supp}\alpha$.

This follows from the corollary of Theorem 13.1 and the above remarks.

Corollary. *We can choose $c\geqq 0$ such that*

$$|\Psi(\mathbf{a}_t:\alpha:\bar{n}:m_t)|_{\mathbf{s}}\leqq c\,e^{-\varepsilon t}\,d_P(m_t)\int_{\omega}\Xi_{r_0}(\bar{n}\bar{n}_0m_t)\,d\bar{n}_0$$

for $t\geqq 0$.

This follows from the corollary of Theorem 13.1 and the above remarks.

Now fix $\alpha\in C_c^\infty(\bar{N})$. Then it follows from the above corollary and [1(e), §10] that

$$\int_0^\infty|\Psi(\mathbf{a}_t:\alpha:\bar{n}:m\exp tH_0)|_{\mathbf{s}}\,dt<\infty$$

for $\mathbf{s}\in\mathscr{S}(V)$. Put

$$\Phi_\infty(\mathbf{a}:\alpha:\bar{n}:m)=\Phi(\mathbf{a}^o:\alpha:\bar{n}:m)+\int_0^\infty\Psi(\mathbf{a}_t:\alpha:\bar{n}:m\exp tH_0)\,dt.$$

Then it follows from Lemma 10 that

$$\Phi_\infty(\mathbf{a}:\alpha:\bar{n}:m)=\lim_{t\to+\infty}\Phi(\mathbf{a}_t:\alpha:\bar{n}:m\exp tH_0).$$

Lemma 12. *Fix $\alpha\in C_c^\infty(\bar{N})$ such that $\int_{\bar{N}}\alpha(\bar{n})\,d\bar{n}=1$. Then*

$$F(\mathbf{a}:m)=\int_{\bar{N}}\Phi_\infty(\mathbf{a}:\alpha:\bar{n}:m)\,d\bar{n}.$$

It follows from [1(e), §10] and the corollary of Lemma 11 that

$$\int_{\bar{N}} d\bar{n} \int_0^\infty |\Psi(\mathbf{a}_t: \alpha: \bar{n}: m \exp tH_0)|_{\mathbf{s}}\, dt < \infty$$

for $\mathbf{s} \in \mathscr{S}(V)$. Therefore we conclude from the corollary of Theorem 13.1 that

$$\int_{\bar{N}} |\Phi_\infty(\mathbf{a}: \alpha: \bar{n}: m)|_{\mathbf{s}}\, d\bar{n} < \infty.$$

On the other hand it is clear from Lemma 3 that

$$\int_{\bar{N}} \Psi(\mathbf{a}_t: \alpha: \bar{n}: m \exp tH_0)\, d\bar{n} = 0.$$

Therefore by Fubini's theorem we obtain

$$\int_{\bar{N}} \Phi_\infty(\mathbf{a}: \alpha: \bar{n}: m)\, d\bar{n} = \int_{\bar{N}} \Phi(\mathbf{a}^o: \alpha: \bar{n}: m)\, d\bar{n} = F(\mathbf{a}^o: m) = F(\mathbf{a}: m)$$

from Lemma 6.

Now put

$$\Phi^o(v: \bar{n}: m) = \Phi(v: \bar{n}: m)\, E^{o*}(v),$$

$$\Psi^o(v: \bar{n}: m) = \Psi(v: \bar{n}: m)\, E^{o*}(v)$$

where $E^{o*}(v) = 1 \otimes \Gamma_v^*(e^o(v))$. Then

$$\Phi(\mathbf{a}_t: \bar{n}: m \exp tH_0) = \int_{\mathfrak{F}} \langle \Phi(v: \bar{n}: m \exp tH_0), \mathbf{a}_t(v)\rangle\, dv = \int_{\mathfrak{F}} \langle \Phi^o(v: \bar{n}: m \exp tH_0)\, e^{-t\Gamma_v(H_0)}, \mathbf{a}(v)\rangle\, dv.$$

Lemma 13. *Fix $x \in G$, $X \in \bar{\mathfrak{n}}$, $g \in \mathfrak{G}$ and $\mathbf{s} \in \mathscr{S}(V)$. Then we can choose $c, r \geqq 0$ such that*

$$|\phi(v: x \exp tH_0; Xg)|_{\mathbf{s}} \leqq c(1+|v|)^r e^{-2\varepsilon t}\, \Xi(x \exp tH_0)(1+t)^r$$

for $t \geqq 0$.

This follows immediately from the fact that

$$\phi(v: x_t; Xg) = \phi(v: \mathrm{Ad}(x_t) X; x_t; g)$$

where $x_t = x \exp tH_0$.

Corollary. *Fix $\bar{n} \in \bar{N}$, $m \in M_1$ and $\mathbf{s} \in \mathscr{S}(V)$. Then we can choose $c, r \geqq 0$ such that*

$$|\Psi^o(v: \bar{n}: m \exp tH_0)\, e^{-t\Gamma_v(H_0)}|_{\mathbf{s}} \leqq c\, e^{-\varepsilon t}(1+|v|)^r$$

for $t \geqq 0$.

This is an immediate consequence of Lemmas 13 and 8.

On the other hand it follows from Lemma 2 that

$$\Phi^o(v: \bar{n}: m \exp TH_0)\, e^{-T\Gamma_v(H_0)} = \Phi^o(v: \bar{n}: m) + \int_0^T \Psi^o(v: \bar{n}: m \exp tH_0)\, e^{-t\Gamma_v(H_0)}\, dt.$$

Moreover we conclude from the above corollary that

$$\int_0^\infty |\Psi^o(\nu:\bar{n}:m\exp tH_0)\,e^{-t\Gamma_\nu(H_0)}|_{\mathbf{s}}\,dt<\infty$$

for $\mathbf{s}\in\mathscr{S}(V)$. Therefore if we put

$$\Phi_\infty(\nu:\bar{n}:m)=\Phi^o(\nu:\bar{n}:m)+\int_0^\infty \Psi^o(\nu:\bar{n}:m\exp tH_0)\,e^{-t\Gamma_\nu(H_0)}\,dt,$$

it follows that

$$|\Phi_\infty(\nu:\bar{n}:m)-\Phi^o(\nu:\bar{n}:m\exp TH_0)\,e^{-T\Gamma_\nu(H_0)}|_{\mathbf{s}} \leqq \int_T^\infty |\Psi^o(\nu:\bar{n}:m\exp tH_0)\,e^{-t\Gamma_\nu(H_0)}|_{\mathbf{s}}\,dt$$

for $T\geqq 0$ and $\mathbf{s}\in\mathscr{S}(V)$. Hence we get the following result from the corollary of Lemma 13.

Lemma 14. *Fix $\bar{n}\in\bar{N}$, $m\in M$ and put*

$$\Phi_\infty(\nu:\bar{n}:m)=\lim_{t\to+\infty}\Phi^o(\nu:\bar{n}:m\exp tH_0)\,e^{-t\Gamma_\nu(H_0)}.$$

Then for any $\mathbf{s}\in\mathscr{S}(V)$, we can choose $c, r\geqq 0$ such that

$$|\Phi_\infty(\nu:\bar{n}:m)-\Phi^o(\nu:\bar{n}:m\exp TH_0)\,e^{-T\Gamma_\nu(H_0)}|_{\mathbf{s}}\leqq c\,e^{-\varepsilon T}(1+|\nu|)^r$$

for $T\geqq 0$.

Now define

$$\Phi_\infty(\mathbf{a}:\bar{n}:m)=\lim_{t\to+\infty}\Phi(\mathbf{a}_t:\bar{n}:m\exp tH_0).$$

It is clear that this limit exists and in fact

$$\Phi_\infty(\mathbf{a}:\bar{n}:m)=\lim_{t\to+\infty}\int_{\mathfrak{F}}\langle\Phi^o(\nu:\bar{n}:m\exp tH_0)\,e^{-t\Gamma_\nu(H_0)},\mathbf{a}(\nu)\rangle\,d\nu = \int_{\mathfrak{F}}\langle\Phi_\infty(\nu:\bar{n}:m),\mathbf{a}(\nu)\rangle\,d\nu.$$

On the other hand, let us put

$$\Phi_\infty(\nu:m)=\Phi_\infty(\nu:1:m) = \lim_{t\to+\infty}\Phi(\nu:m\exp tH_0)\,E^{o*}(\nu)\,e^{-t\Gamma_\nu(H_0)}.$$

Extend this to a function on $\mathfrak{F}\times G$ by setting

$$\Phi_\infty(\nu:kmn)=\tau(k)\,\Phi_\infty(\nu:m)\qquad(k\in K,\ m\in M_1,\ n\in N).$$

Lemma 15. $\Phi_\infty(\nu:\bar{n}:m)=e^{-\rho(H(\bar{n}))}\,\Phi_\infty(\nu:\bar{n}m)$.

It is obvious from Lemma 14 that for fixed $\bar{n}$ and m, $\Phi(\nu:\bar{n}:m)$ is a continuous function of ν. Therefore, in view of its definition, the same holds for $\Phi_\infty(\nu:\bar{n}m)$. Hence it would be enough to verify the above relation for $\nu\in\mathfrak{F}'(\lambda)$.

Fix $\nu\in\mathfrak{F}'(\lambda)$ and let $e_i^*(\nu)$ $(1\leqq i\leqq q)$ be the base of U^* dual to $e_i(\nu)$ $(1\leqq i\leqq q)$. Then

$$\sum_i u_i\otimes u_i^*=\sum_i e_i(\nu)\otimes e_i^*(\nu).$$

Hence

$$\sum_i u_i \otimes u_i^* \Gamma_\nu^*(e^o(\nu))\, e^{-t\Gamma_\nu^*(H_0)} = \sum_i e^{-t\Gamma_\nu(H_0)} \Gamma_\nu(e^o(\nu))\, e_i(\nu) \otimes e_i^*(\nu)$$
$$= \sum_{i \in Q^o} e^{-ts_i \Lambda_\nu(H_0)}\, e_i(\nu) \otimes e_i^*(\nu)$$

from Corollary 1 of Lemma 3.5. Hence

$$\Phi^o(\nu: \bar{n}: m_t)\, e^{-t\Gamma_\nu(H_0)}$$
$$= d_P(m_t) \sum_{i \in Q^o} \varpi_{01}(s_i \Lambda_\nu)^{-1}\, e^{-ts_i \Lambda_\nu(H_0)}\, \phi(\nu: \bar{n} m_t; \eta_i(\nu)') \otimes e_i^*(\nu),$$

where $m_t = m \exp tH_0$ and ϖ_{01}, $\eta_i(\nu)$ have the same meaning as in §2 and §4 respectively.

Now fix $i \in Q^o$. Then (see §6)

$$\phi_{i\infty}(\nu: m) = \lim_{t \to +\infty} d_P(m_t)\, \phi(\nu: m_t; \eta_i(\nu)')\, e^{-ts_i \Lambda_\nu(H_0)}$$

and

$$\phi_{i\infty}(\nu: m \exp H) = \phi_{i\infty}(\nu: m)\, e^{s_i \Lambda(H)} \qquad (H \in \mathfrak{a})$$

from Lemma 6.2. Therefore

$$\lim_{t \to +\infty} \{d_P(m_t)\, \phi(\nu: m_t; \eta_i(\nu)') - \phi_{i\infty}(\nu: m_t)\} = 0$$

and we conclude from [1(e), Lemmas 21.3 and 24.1] that

$$\phi_P(\nu: m; \eta_i(\nu)) = \phi_{i\infty}(\nu: m).$$

Extend $\phi_{i\infty}(\nu)$ to a function on G by setting

$$\phi_{i\infty}(\nu: kmn) = \tau(k)\, \phi_{i\infty}(\nu: m) \qquad (k \in K, m \in M_1, n \in N).$$

Then we conclude from [1(e), Lemma 24.1] that

$$\lim_{t \to +\infty} \{d_P(x_t)\, \phi(\nu: x_t; \eta_i(\nu)') - \phi_{i\infty}(\nu: x_t)\} = 0.$$

Here x is a fixed element in G and $x_t = x \exp tH_0$. But this implies that

$$\lim_{t \to +\infty} d_P(x_t)\, \phi(\nu: x_t; \eta_i(\nu)')\, e^{-ts_i \Lambda_\nu(H_0)} = \phi_{i\infty}(\nu: x)$$

and therefore

$$\Phi_\infty(\nu: \bar{n}: m) = \lim_{t \to +\infty} \Phi^o(\nu: \bar{n}: m_t)\, e^{-t\Gamma_\nu(H_0)}$$
$$= e^{-\rho(H(\bar{n}))} \sum_{i \in Q^o} \varpi_{01}(s_i \Lambda_\nu)^{-1}\, \phi_{i\infty}(\nu: \bar{n} m) \otimes e_i^*(\nu).$$

The assertion of the lemma is now obvious from the definition of $\Phi_\infty(\nu: m)$.

Lemma 16. *$e_j^*(\nu) = \sum_{1 \leq i \leq q} u_i(s_j \Lambda_\nu)\, u_i^*$ $(1 \leq j \leq q)$ for $\nu \in \mathfrak{F}_c'(\lambda)$.*

By Corollary 1 of Lemma 3.5

$$u_i = \Gamma_\nu(u_i)\, 1 = \sum_{1 \leq j \leq q} \Gamma_\nu(u_i)\, e_j(\nu)$$
$$= \sum_j u_i(s_i \Lambda_\nu)\, e_j(\nu).$$

Therefore

$$\sum_i u_i \otimes u_i^* = \sum_{i,j} u_i(s_j \Lambda_\nu)\, e_j(\nu) \otimes u_i^*.$$

But since

$$\sum_i u_i \otimes u_i^* = \sum_j e_j(\nu) \otimes e_j^*(\nu),$$

our assertion is now obvious.

Corollary. *Let* $\nu \in \mathfrak{F}'(\lambda)$. *Then*

$$\Phi_\infty(\nu : m) = \sum_{i \in Q^o} \sum_{1 \leq j \leq q} \varpi_{01}(s_i \Lambda_\nu)^{-1} \phi_{i\infty}(\nu : m) \otimes u_j(s_i \Lambda_\nu)\, u_j^*.$$

This follows immediately from Lemma 16 and what we have seen above. Now put

$$\Phi_\infty(\mathbf{a} : m) = \int_{\mathfrak{F}} \langle \Phi_\infty(\nu : m), \mathbf{a}(\nu) \rangle \, d\nu.$$

Then it follows from Lemmas 7.3, 8.2 and the corollary of Theorem 13.1 that $\Phi_\infty(\mathbf{a}) \in \mathscr{C}(M_1, \tau_M)$. Hence we conclude from [1(e), Lemma 32.1] that

$$\int_{\bar{N}} e^{-\rho(H(\bar{n}))} |\Phi_\infty(\mathbf{a} : \bar{n} m)|_{\mathbf{s}} \, d\bar{n} < \infty \qquad (\mathbf{s} \in \mathscr{S}(V)),$$

provided $\Phi_\infty(\mathbf{a})$ is extended to a function on G in the usual way so that

$$\Phi_\infty(\mathbf{a} : kmn) = \tau(k)\, \Phi_\infty(\mathbf{a} : m) \qquad (k \in K,\ m \in M_1,\ n \in N).$$

Now put, as before,

$$\Phi_\infty(\mathbf{a} : \bar{n} : m) = \int_{\mathfrak{F}} \langle \Phi_\infty(\nu : \bar{n} : m), \mathbf{a}(\nu) \rangle \, d\nu.$$

Then it follows from Lemma 15 that

$$\Phi_\infty(\mathbf{a} : \bar{n} : m) = e^{-\rho(H(\bar{n}))}\, \Phi_\infty(\mathbf{a} : \bar{n} m)$$

and therefore from Lemma 12 that

$$F(\mathbf{a} : m) = \int_{\bar{N}} e^{-\rho(H(\bar{n}))}\, \Phi_\infty(\mathbf{a} : \bar{n} m)\, d\bar{n}.$$

Substituting the definition of $F(\mathbf{a})$ we obtain the following result.

Lemma 17. *Let* $\mathbf{a} \in \mathscr{C}(\mathfrak{F}, U)$. *Then*

$$\int_{\bar{N}} \Phi(\mathbf{a} : \bar{n} : m)\, d\bar{n} = \int_{\bar{N}} e^{-\rho(H(\bar{n}))}\, \Phi_\infty(\mathbf{a} : \bar{n} m)\, d\bar{n}.$$

In order to prove Theorem 13.2 we take $\mathbf{a}(\nu) = \alpha(\nu)\, u_1$. Then we claim that

$$\langle \Phi_\infty(\nu : m), \mathbf{a}(\nu) \rangle = \alpha(\nu)\, \phi_P(\nu : m).$$

Since both sides are continuous in ν, it is sufficient to verify this for $\nu \in \mathfrak{F}'(\lambda)$. But $u_1 = 1$ and so this is an immediate consequence of Lemma 7.3 and the corollary of Lemma 16. The statement of Theorem 13.2 is now obvious from Lemma 17.

§ 17. Application to Eisenstein Integrals

Let U be an open subset of $\mathfrak{F}_c$. A function $f\colon U\times G\to V$ will be said to be of type $H\times C^\infty$ if 1) it is of class C^∞ on $U\times G$ and 2) for all $x\in G$ the function $\nu\to f(\nu\colon x)$ from U to V is holomorphic.

Fix a psgp $P_1=MAN_1$ in $\mathscr{P}(\mathfrak{h}_R)$. Then for any $\psi\in C^\infty(M,\tau_M)$, we consider the Eisenstein integral $E(P_1\colon\psi)$ [1(e), §9]. Clearly it is a function of type $H\times C^\infty$ on $\mathfrak{F}_c\times G$.

Put

$$\mathbf{s}_\delta(\psi)=\sup_M|\delta\psi|_{\mathbf{s}}\,\Xi_M^{-1}$$

for $\mathbf{s}\in\mathscr{S}(V)$, $\delta\in\mathfrak{M}^{(2)}=\mathfrak{M}\otimes\mathfrak{M}$ and $\psi\in C^\infty(M,V)$. Moreover let

$$\mathbf{s}_F(\psi)=\sum_{\delta\in F}\mathbf{s}_\delta(\psi)$$

for any finite subset F of $\mathfrak{M}^{(2)}$. If $\nu\in\mathfrak{F}_c$, define ν_R and ν_I in $\mathfrak{F}$ by $\nu=\nu_R+(-1)^{1/2}\nu_I$. Then it is easy to see that we can choose $c_0\geqq 0$ such that

$$|\mathfrak{R}(-1)^{1/2}\nu(H_{P_1}(x))|\leqq c_0|\nu_I|\,\sigma(x)\qquad(\nu\in\mathfrak{F}_c,\ x\in G).$$

Extend the norm on $\mathfrak{F}_c$ by setting

$$|\nu|^2=|\nu_R|^2+|\nu_I|^2$$

and put

$$|(\nu,x)|=(1+|\nu|)(1+\sigma(x))\qquad(\nu\in\mathfrak{F}_c,\ x\in G).$$

Lemma 1. *Fix $g_1,g_2\in\mathfrak{G}$ and $D\in\mathfrak{D}(\mathfrak{F}_c)$. Then we can choose $r\geqq 0$ and a finite subset F of $\mathfrak{M}^{(2)}$ with the following property. For any $\mathbf{s}\in\mathscr{S}(V)$, there exists a number $c>0$ such that*

$$|E(P_1\colon\psi\colon\nu;\,D\colon g_1;x;g_2)|_{\mathbf{s}}\leqq c\,\mathbf{s}_F(\psi)\,\Xi(x)\,|(\nu,x)|^r\exp\{c_0|\nu_I|\,\sigma(x)\}$$

for all $\psi\in C^\infty(M,\tau_M)$, $\nu\in\mathfrak{F}_c$ and $x\in G$.

It is enough to consider the case $D=1$. The general result would follow from this if we fix x, consider the complex polycylinder with center ν and radius $(1+\sigma(x))^{-1}$ and apply the Cauchy integral formula.

We drop the subscript and write $P=P_1$, $N=N_1$. Put

$$\psi_\nu(x)=\psi(x)\exp\{((-1)^{1/2}\nu-\rho)(H(x))\}\qquad(x\in G)$$

in the usual notation [1(e), § 19] where $\rho=\rho_P$ and $H(x)=H_P(x)$. Then it is obvious that

$$|E(P\colon\psi\colon\nu\colon g_1;x;g_2)|_{\mathbf{s}}\leqq\int_K|\psi_\nu(g_1;xk;g_2^k)|_{\mathbf{s}}\,dk$$

for $\mathbf{s}\in\mathscr{S}^o(V)$ [1(e), §22]. But if $x=kman$ ($k\in K$, $m\in M$, $a\in A$, $n\in N$), it is clear that

$$\psi_\nu(g_1;kman;g_2)=\tau(k)\,\psi_\nu(g_1^{k^{-1}};man;g_2).$$

Moreover

$$\mathfrak{G}=\mathfrak{G}\mathfrak{n}+\mathfrak{K}\mathfrak{M}_1=\mathfrak{G}\mathfrak{n}+\mathfrak{M}_1\mathfrak{K}.$$

Therefore for given $g_1, g_2 \in \mathfrak{G}$, we can choose $r \geqq 0$ and $u_i, v_i \in \mathfrak{M}$ $(1 \leqq i \leqq p)$ such that

$$|\psi_\nu(g_1; kman; g_2)|_{\mathbf{s}} \leqq \sum_{1 \leqq i \leqq p} |\psi(u_i \quad m; v_i)|_{\mathbf{s}} (1+|\nu|)^r e^{-(\nu_I+\rho)(\log a)}$$

for all $\nu \in \mathfrak{F}_c$, $\mathbf{s} \in \mathscr{S}^o(V)$ and $(k, m, a, n) \in K \times M \times A \times N$. The required result now follows immediately from [1(e), Corollary of Lemma 30.1].

Let $\psi \neq 0$ be an eigenfunction of $\mathfrak{Z}_M$ in $\mathscr{C}(M, \tau_M)$. Then [1(e), Theorem 18.3] there exists a regular element $\lambda \in (-1)^{1/2}\, \mathfrak{h}_I^*$ such that

$$\zeta\psi = \gamma_{\mathfrak{m}/\mathfrak{h}_I}(\zeta : \lambda)\psi \qquad (\zeta \in \mathfrak{Z}_M).$$

Put $\phi = E(P_1 : \psi)$. Then it is obvious from Lemma 1 and [1(e), Lemma 19.1] that ϕ defines a function of type $II(\lambda)$ (see § 8) on $\mathfrak{F} \times G$.

Let $P = MAN$ be another psgp in $\mathscr{P}(\mathfrak{h}_R)$. We shall now investigate the behavior of

$$d_P(a)\, \phi(\nu : ma) \qquad (m \in M, a \in A)$$

as $a \xrightarrow[P]{} \infty$. The case $P = G$ being trivial, we assume that $\mathfrak{a} = \mathfrak{h}_R$ does not lie in the center of $\mathfrak{g}$.

We now use the notation of § 5 and put

$$|(\nu, x, H)| = |(\nu, x)|\, (1 + \|H\|)$$

for $\nu \in \mathfrak{F}_c$, $x \in G$ and $H \in \mathfrak{a}$. Note that $\mathfrak{a} = \mathfrak{h}_R \subset \mathfrak{a}_0$ and therefore we may assume that $k_0 = 1$ (see § 3). Then y centralizes $\mathfrak{a}$ and therefore $\Lambda_\nu = \lambda^y + (-1)^{1/2}\, \nu$.

Lemma 2. *Fix $\zeta \in \mathfrak{Z}_1$ and $v_1, v_2 \in \mathfrak{M}$. Then we can choose $r \geqq 0$ and for each $\mathbf{s} \in \mathscr{S}(V)$ a number $c(\mathbf{s}) \geqq 0$ such that*

$$|\psi_{i,\zeta}(\nu : v_1; m \exp H; v_2)|_{\mathbf{s}}$$
$$\leqq c(\mathbf{s})\, \Xi_M(m)\, |(\nu, m, H)|^r e^{-\beta_P(H)} \exp\{c_0 |\nu_I| (\sigma(m) + \|H\|)\}$$

for $\nu \in \mathfrak{F}_c$, $m \in M_1^+$, $H \in Cl\, \mathfrak{a}^+$ and $1 \leqq i \leqq q$.

This is proved in the same way as Lemma 6.1.

Define λ_i $(i \in Q)$ and Q^o as in § 6. Fix two positive numbers ε, δ and an element $H_0 \in \mathfrak{a}^+$ with $\|H_0\| = 1$. Let $\mathfrak{F}_c(\delta)$ denote the set of all $\nu \in \mathfrak{F}_c$ with $|\nu_I| < \delta$. By choosing ε, δ sufficiently small, we can assume that:

1) $\beta_P(H_0) \geqq 4\varepsilon$,

2) $|\lambda_i(H_0)| \geqq 3\varepsilon$ if $\lambda_i(H_0) \neq 0$,

3) $|s_i \nu_I(H_0)| + c_0 |\nu_I| \leqq \varepsilon$

for $i \in Q$ and $\nu \in \mathfrak{F}_c(\delta)$. Put

$$\psi_i^o(\nu : m : t) = \psi_{i, H_0}(\nu : m \exp tH_0)\, e^{-t s_i \Lambda_\nu(H_0)}.$$

Fix $\mathbf{s} \in \mathscr{S}(V)$ and $v \in \mathfrak{M}_1$. Then it follows from Lemma 2 that if $\lambda_i(H_0) \geqq 0$, the integral

$$\int_0^\infty |\psi_i^o(\nu : m; v : t)|_{\mathbf{s}}\, dt$$

converges uniformly as (ν, m) varies within a compact subset of $\mathfrak{F}_c(\delta)\times M_1$. Hence by Lemma 5.3, we can define

$$\phi_{i\infty}(\nu: m)=\lim_{t\to+\infty}\phi_i(\nu: m\exp tH_0)\,e^{-ts_i\Lambda_\nu(H_0)}$$

for $(\nu, m)\in\mathfrak{F}_c(\delta)\times M_1$. Then $\phi_{i\infty}$ is a function of type $H\times C^\infty$.

Lemma 3. *Fix i such that $\lambda_i(H_0)\geqq 0$. Then*

$$\phi_{i\infty}(\nu: m;\zeta)=\gamma_1(\zeta: s_i\Lambda_\nu)\,\phi_{i\infty}(\nu: m)\qquad(\zeta\in\mathfrak{Z}_1)$$

for $\nu\in\mathfrak{F}_c(\delta)$ and $m\in M_1$. Moreover $\phi_{i\infty}=0$ unless $i\in Q^o$ and $s_i^{-1}\mathfrak{a}=\mathfrak{a}$.

If $\lambda_i(H_0)>0$, it is clear that

$$\Re\, s_i\Lambda_\nu(H_0)-c_0|\nu_I|>0$$

for $\nu\in\mathfrak{F}_c(\delta)$ and therefore $\phi_{i\infty}=0$ from Lemma 1. So now assume that $\lambda_i(H_0)=0$. Then it follows easily from Lemmas 6.2 and 6.3 that our statement is true if $\nu\in\mathfrak{F}$. The rest is obvious by holomorphy.

Corollary. $\phi_{i\infty}(\nu: m\exp H)=\phi_{i\infty}(\nu: m)\,e^{s_i\Lambda_\nu(H)}$ *for $m\in M_1$, $H\in\mathfrak{a}$ and $\nu\in\mathfrak{F}_c(\delta)$.*

This is obvious from Lemma 3.
Define $\phi_{i\infty}=0$ if $\lambda_i(H_0)<0$.

Lemma 4. *Fix $\nu\in\mathfrak{F}_c(\delta)$, $m\in M_1$. Then*

$$|\phi_i(\nu: m\exp TH_0)-\phi_{i\infty}(\nu: m\exp TH_0)|_{\mathbf{s}}$$
$$\leqq e^{-2\varepsilon T}\left\{|\phi_i(\nu: m)|_{\mathbf{s}}+\int_0^\infty|\psi_{i,H_0}(\nu: m\exp tH_0)|_{\mathbf{s}}\,e^{2\varepsilon t}\,dt\right\}$$

for $\mathbf{s}\in\mathscr{S}(V)$, $T\geqq 0$ and $i\in Q$.

Put $m_t=m\exp tH_0$ $(t\in\mathbf{R})$ and first suppose $\lambda_i(H_0)\geqq 0$. Then

$$\phi_{i\infty}(\nu: m_T)=\phi_i(\nu: m_T)+\int_T^\infty\psi_{i,H_0}(\nu: m_t)\,e^{-(t-T)s_i\Lambda_\nu(H_0)}\,dt$$

from Lemma 5.3. Moreover

$$\Re\, s_i\Lambda_\nu(H_0)=\lambda_i(H_0)-s_i\nu_I(H_0)\geqq-\varepsilon.$$

Hence

$$|\phi_{i\infty}(\nu: m_T)-\phi_i(\nu: m_T)|_{\mathbf{s}}\leqq\int_T^\infty|\psi_{i,H_0}(\nu: m_t)|\,e^{\varepsilon(t-T)}\,dt$$

and this implies the required inequality.
Now suppose $\lambda_i(H_0)<0$. Then $\phi_{i\infty}=0$ and

$$\phi_i(\nu: m_T)=\phi_i(\nu: m)\,e^{Ts_i\Lambda_\nu(H_0)}+\int_0^T\psi_{i,H_0}(\nu: m_t)\,e^{(T-t)s_i\Lambda_\nu(H_0)}\,dt$$

from Lemma 5.3. But

$$\Re\, s_i\Lambda_\nu(H_0)=\lambda_i(H_0)-s_i\nu_I(H_0)\leqq-2\varepsilon$$

and therefore

$$|\phi_i(\nu: m \exp TH_0)|_{\mathbf{s}} \leqq e^{-2\varepsilon T}\left\{|\phi_i(\nu: m)|_{\mathbf{s}} + \int_0^\infty |\psi_{i,H_0}(\nu: m \exp tH_0)|_{\mathbf{s}}\, e^{2\varepsilon t}\, dt\right\}.$$

This proves the lemma.

Let $\mathfrak{F}'_c(\delta, \lambda)$ denote the set of all $\nu \in \mathfrak{F}_c(\delta)$ where $\varpi(\lambda + (-1)^{1/2}\nu) \neq 0$, so that (see § 3)

$$\mathfrak{F}'_c(\delta, \lambda) = \mathfrak{F}_c(\delta) \cap \mathfrak{F}'_c(\lambda).$$

For any $s \in \mathfrak{w} = \mathfrak{w}(\mathfrak{a})$, there exists a unique index $i \in Q$ such that $s = s_i^{-1}$ on $\mathfrak{a}$ (Lemma 7.1). Define

$$\phi_{P,s}(\nu: m) = \varpi_{01}(s_i \Lambda_\nu)^{-1}\, \phi_{i\infty}(\nu: m)$$

for $\nu \in \mathfrak{F}'_c(\delta, \lambda)$, $m \in M_1$. Note that

$$\begin{aligned} s_i \Lambda_\nu(H) = \Lambda_\nu(sH) &= \lambda^y(sH) + (-1)^{1/2}\nu(sH) \\ &= (-1)^{1/2}\nu(sH) \qquad (H \in \mathfrak{a}) \end{aligned}$$

since y centralizes $\mathfrak{a}$ and $\lambda = 0$ on $\mathfrak{a} = \mathfrak{h}_R$. This shows that $i \in Q^o$. Therefore the following result is obvious from Lemmas 2 and 4, Corollary of Lemma 3 and Lemma 5.1.

Lemma 5. *Let $\nu \in \mathfrak{F}'_c(\delta, \lambda)$, $m \in M_1$ and $\mathbf{s} \in \mathscr{S}(V)$. Then*

$$\lim_{t \to +\infty} e^{\varepsilon t}\left|d_P(m_t)\,\phi(\nu: m_t) - \sum_{s \in \mathfrak{w}} \phi_{P,s}(\nu: m)\, e^{(-1)^{1/2} t\nu(sH_0)}\right|_{\mathbf{s}} = 0$$

where $m_t = m \exp tH_0$.

Corollary. *Fix $\nu \in \mathfrak{F}'_c(\delta, \lambda)$ and $s_0 \in \mathfrak{w}$ and suppose $\nu_I(s_0 H_0) < \nu_I(sH_0)$ for every $s \neq s_0$ in $\mathfrak{w}$. Then*

$$\lim_{t \to +\infty} d_P(m_t)\,\phi(\nu: m_t)\, e^{-(-1)^{1/2} t\nu(s_0 H_0)} = \phi_{P,s_0}(\nu: m)$$

for $m \in M_1$.

Since $|\nu_I(s_0 H_0)| \leqq \varepsilon$, this follows from Lemma 5 if we observe that

$$\Re(-1)^{1/2}\{\nu(sH_0) - \nu(s_0 H_0)\} = \nu_I(s_0 H_0) - \nu_I(sH_0) < 0$$

for $s \neq s_0$.

§ 18. The c-Functions

Now assume that $\dim \tau < \infty$ and τ is unitary. Put

$$L = {}^o\mathscr{C}(M, \tau_M).$$

Then by [1(e), Theorem 27.9], $\dim L < \infty$. Let $|\cdot|$ denote the norm in the finite-dimensional Hilbert space V. Put

$$\|\psi\|^2 = \int_M |\psi(m)|^2\, dm$$

for $\psi \in L$. This defines the structure of a Hilbert space on L.

Let $\mathscr{E}_2(M)$ be the discrete series of M (i.e. the set of all equivalence classes of irreducible, square-integrable representations of M). For $\omega \in \mathscr{E}_2(M)$, put

$$L(\omega) = L \cap (\mathfrak{H}_\omega \otimes V)$$

where $\mathfrak{H}_\omega$ is the smallest closed subspace of $L_2(M)$ containing all the matrix coefficients of ω. Then

$$L = \sum_\omega L(\omega)$$

where the sum is orthogonal.

We keep to the notation of § 17 and put $\mathfrak{a} = \mathfrak{h}_R$ and $\mathfrak{w} = \mathfrak{w}(\mathfrak{a})$. Fix $P \in \mathscr{P}(\mathfrak{a})$ and define

$$\pi(v) = \prod_{1 \leqq i \leqq r} \langle \alpha_i, v \rangle^{m_i} \qquad (v \in \mathfrak{F}_c)$$

where $\alpha_1, \ldots, \alpha_r$ are all the distinct roots of (P, A) and m_i the multiplicity of α_i. As usual $\langle \alpha_1, v \rangle = \alpha_i(H_v)$. Let $\mathfrak{F}_c'$ be the set of all $v \in \mathfrak{F}_c$ where $\pi(v) \neq 0$. Clearly $\mathfrak{F}_c'$ is independent of the choice of P in $\mathscr{P}(\mathfrak{a})$. Put $\mathfrak{F}' = \mathfrak{F} \cap \mathfrak{F}_c'$ and $\mathfrak{F}_c'(\delta) = \mathfrak{F}_c(\delta) \cap \mathfrak{F}_c'$ for $\delta > 0$.

Theorem 1. *Fix $v \in \mathfrak{F}'$ and $P_1, P_2 \in \mathscr{P}(\mathfrak{a})$. Then there exist unique elements $c_{P_2|P_1}(s: v) \in \operatorname{End} L$ $(s \in \mathfrak{w})$ such that*

$$E_{P_2}(P_1: \psi: v: ma) = \sum_{s \in \mathfrak{w}} (c_{P_2|P_1}(s: v)\psi)(m)\, e^{(-1)^{1/2} sv(\log a)}$$

for $\psi \in L$, $m \in M$ and $a \in A$. Moreover we can choose $\delta > 0$ such that for every $s \in \mathfrak{w}$, $\pi(v)\, c_{P_2|P_1}(s: v)$ extends to a holomorphic function of v on $\mathfrak{F}_c(\delta)$.

Fix $v \in \mathfrak{F}'$. Then $sv \neq v$ for $s \neq 1$ in $\mathfrak{w}$ (Lemma 22.3). Hence the uniqueness is obvious. So now we have to prove existence. Fix $\omega \in \mathscr{E}_2(M)$ such that $L(\omega) \neq \{0\}$. It is enough to define $c_{P_2|P_1}(s: v)$ on $L(\omega)$. By [1 (e), Theorem 18.3] there exists a regular element $\lambda \in (-1)^{1/2}\, \mathfrak{h}_I^*$ such that

$$\zeta\psi = \gamma_{\mathfrak{m}/\mathfrak{h}_I}(\zeta: \lambda)\psi \qquad (\zeta \in \mathfrak{Z}_M)$$

for all $\psi \in L(\omega)$. Now fix $\psi \in L(\omega)$ and put $\phi = E(P_1 : \psi : v)$. It is easy to verify that $\mathfrak{F}'(\lambda) \subset \mathfrak{F}'$ and therefore by Theorem 7.1

$$\phi_{P_2} = \sum_{s \in \mathfrak{w}} \phi_{P_2, s}.$$

Moreover by Lemma 7.5 the functions

$$m \mapsto \phi_{P_2, s}(m) \qquad (m \in M)$$

are in L. Now define

$$c_{P_2|P_1}(s^{-1}: v)\psi = \phi_{P_2, s} \qquad (s \in \mathfrak{w}).$$

Then the first statement of the theorem follows from Theorem 7.1 and its corollary.

For any linear function μ on V and $m \in M$, put

$$\mu_m(\psi) = \mu(\psi(m)) \qquad (\psi \in L).$$

Then μ_m is a linear function on L. For a given $\psi \in L$, the condition $\mu_m(\psi)=0$ for all μ and m, implies that $\psi=0$. Hence we can choose a base $(\Lambda_1, \ldots, \Lambda_n)$ for the space dual to L, consisting of linear functions of the form μ_m. Let $(\psi_1, \ldots, \psi_n)$ be the dual base for L. For each i, choose $m_i \in M$ and a linear function μ_i on V such that $\Lambda_i(\psi)=\mu_i(\psi(m_i))$ for $\psi \in L$. Then

$$\psi=\sum_i \Lambda_i(\psi)\,\psi_i=\sum_i \mu_i(\psi(m_i))\,\psi_i \qquad (\psi \in L).$$

Now fix ω and $\psi \in L(\omega)$ as above and put

$$\psi_s(v)=\varpi(\lambda+(-1)^{1/2}\,v)\,\phi_{P_2,s}(v)$$

for $v \in \mathfrak{F}_c(\delta)$ in the notation of § 17 where

$$\phi(v)=E(P_1:\psi:v).$$

Then for a fixed $s \in \mathfrak{w}$, the function

$$(v, m) \mapsto \psi_s(v:m)$$

on $\mathfrak{F}_c(\delta)\times M$ is of class $H\times C^\infty$. Moreover $\psi_s(v) \in L$ for $v \in \mathfrak{F}'$. Hence

$$\psi_s(v:m)=\sum_i \mu_i(\psi_s(v:m_i))\,\psi_i(m) \qquad (m \in M)$$

for $v \in \mathfrak{F}'$. Therefore by holomorphy this relation holds for all $v \in \mathfrak{F}_c(\delta)$. This shows that $\psi_s(v) \in L$ and $v \to \psi_s(v)$ is a holomorphic mapping from $\mathfrak{F}_c(\delta)$ to L. The second statement of Theorem 1 is now obvious.

We observe that $\mathfrak{w}$ operates on L. For if $s \in \mathfrak{w}$ and $\psi \in L$, then $s\psi=\psi^s$ (see § 7) is also in L. Clearly the sets $\mathfrak{F}_c(\delta)$ and $\mathfrak{F}'$ are also stable under $\mathfrak{w}$.

Lemma 1. *Let* $P_1, P_2 \in \mathscr{P}(\mathfrak{a})$ *and* $s, t \in \mathfrak{w}$. *Then*

$$s\,c_{P_2|P_1}(t:v)=c_{P_2^s|P_1}(st:v)$$

$$c_{P_2|P_1}(t:v)\,s^{-1}=c_{P_2|P_1^s}(t s^{-1}:sv)$$

for $v \in \mathfrak{F}_c(\delta)$.

It is enough to prove this for v in $\mathfrak{F}'$. Fix $\psi \in L$, $v \in \mathfrak{F}'$ and put $\phi=E(P_1:\psi:v)$. Then it follows from [1 (e), Lemma 21.1] that

$$(\phi_{P_2})^s=\phi_{P_2^s}$$

and the first assertion is an immediate consequence of this fact.

Similarly the second statement is an easy consequence of the following lemma.

Lemma 2. *Fix* $P \in \mathscr{P}(\mathfrak{a})$ *and* $s \in \mathfrak{w}$. *Then*

$$E(P:\psi:v)=E(\mathrm{P}^s:s\psi:sv)$$

for $\psi \in L$ *and* $v \in \mathfrak{F}_c$.

For $f, g \in C^\infty(G, \tau)$ and $\alpha, \beta \in C^\infty(M, \tau_M)$, put

$$(f, g)_G=\int_G (f(x), g(x))\,dx,$$

$$(\alpha, \beta)_M=\int_M (\alpha(m), \beta(m))\,dm,$$

provided the integrals are absolutely convergent. Moreover for $f\in C_c^\infty(G,\tau)$, define $f_\nu^{(P)}\in C_c^\infty(M,\tau_M)$ $(\nu\in\mathfrak{F}_c)$ by

$$f_\nu^{(P)}(m)=\int_A f^{(P)}(ma)\,e^{-(-1)^{1/2}\nu(\log a)}\,da \qquad (m\in M)$$

in the notation of [1 (e), § 16]. Then it is clear that

$$(E(P:\psi:\nu),f)_G=(\psi,f_\nu^{(P)})_M$$

for $\psi\in L$ and $\nu\in\mathfrak{F}$. Similarly

$$(E(P^s:s\psi:s\nu),f)_G=(s\psi,f_{s\nu}^{(P^s)})_M.$$

However it is easy to verify that

$$f_{s\nu}^{(P^s)}=s(f_\nu^{(P)})$$

and therefore

$$(E(P:\psi:\nu),f)_G=(E(P^s:s\psi:s\nu),f)_G$$

for all $f\in C_c^\infty(G,\tau)$. The statement of Lemma 2 is now obvious.

Lemma 1 shows that it is sufficient to investigate the functions $c_{P_2|P_1}(1:\nu)$ for $P_1,P_2\in\mathscr{P}(\mathfrak{a})$.

Lemma 3. *Fix $P\in\mathscr{P}(\mathfrak{a})$, $\psi\in L$, $\nu\in\mathfrak{F}$ and let $P'=M'A'N'$ be a psgp of G. Then*

$$E_{P'}(P:\psi:\nu)\sim 0$$

unless A' is conjugate to A under K.

We may assume, without loss of generality, that $\psi\in L(\omega)$ for some $\omega\in\mathscr{E}_2(M)$. Then our assertion follows from Lemmas 11.1. and 17.1.

§ 19. Some Integral Formulas

Fix $P\in\mathscr{P}(\mathfrak{a})$ and let $\mathfrak{F}_c(P)$ denote the set of all $\nu\in\mathfrak{F}_c$ such that $\langle\alpha,\nu_I\rangle>0$ for every root α of (P,A). Put $\rho=\rho_P$ and $H(x)=H_P(x)$ $(x\in G)$. Every $x\in G$ can be written uniquely in the form $x=kman$ where $k\in K$, $m\in M\cap\exp\mathfrak{p}$, $a\in A$, $n\in N$. Put $k=\kappa(x)$ and $m=\mu(x)$. As usual let $\bar P=\theta(P)$ and $\bar N=\theta(N)$.

Theorem 1. *$c_{\bar P|P}(1:\nu)$ and $c_{P|P}(1:-\nu)$ extend to holomorphic functions of ν on $\mathfrak{F}_c(P)$ and they are given by the following integrals.*

$$(c_{\bar P|P}(1:\nu)\psi)(m)=\int_{\bar N}\tau(\kappa(\bar n))\,\psi(\mu(\bar n)\,m)\,e^{((-1)^{1/2}\nu-\rho)(H(\bar n))}\,d\bar n,$$

$$(c_{\bar P|P}(1:-\nu)\psi)(m)=\int_{\bar N}\psi(m\,\mu(\bar n)^{-1})\,\tau(\kappa(\bar n))^{-1}\,e^{((-1)^{1/2}\nu-\rho)(H(\bar n))}\,d\bar n.$$

Here $\psi\in L$, $\nu\in\mathfrak{F}_c(P)$, $m\in M$ and the Haar measure $d\bar n$ on $\bar N$ is so normalized that

$$\int_{\bar N}e^{-2\rho(H(\bar n))}\,d\bar n=1.$$

We need some preparation. Observe that $G=KP$ and $\bar NP$ is an open dense subset of G whose complement is of Haar measure zero. Let $d_l p$ and $d_r p$ denote the left- and right-invariant Haar measures respectively on P so that $d_r p = d_l p^{-1}$. Then $d_r p = \delta(p)\, d_l p$ where δ is a homomorphism of P into $\mathbf{R}_+^\times$. We can normalize the Haar measures dx and $d\bar{n}$ on G and $\bar{N}$ respectively in such a way that

$$\int_G f(x)\, dx = \int_{\bar{N}\times P} f(\bar{n} p)\, d\bar{n}\, d_r p = \int_{K\times P} f(k p)\, dk\, d_r p$$

for $f \in C_c(G)$. Put

$$\bar{f}(\bar{x}) = \int_P f(x p)\, d_l p \qquad (x \in G)$$

where $x \mapsto \bar{x}$ is the natural projection of G on $\bar{G} = G/P$. Note that

$$\bar{G} = \bar{K} = K/K \cap P = K/K_M$$

and put $\bar{f}(k) = \bar{f}(\bar{k})$ $(k \in K)$. Then

$$\int_K \bar{f}(k)\, dk = \int f(k p)\, dk\, d_l p = \int f(kp)\, \delta(p)^{-1}\, dk\, d_r p.$$

Since $K \cap P$ lies in the kernel of δ, we can extend δ on G by defining $\delta(k p) = \delta(p)$ $(k \in K,\ p \in P)$. Then $\delta(y p) = \delta(y)\, \delta(p)$ for $y \in G$, $p \in P$ and therefore

$$\begin{aligned}\int_K \bar{f}(k)\, dk &= \int f(x)\, \delta(x)^{-1}\, dx = \int f(\bar{n} p)\, \delta(\bar{n} p)^{-1}\, d\bar{n}\, d_r p \\ &= \int f(\bar{n} p)\, \delta(\bar{n})^{-1}\, d\bar{n}\, d_l p.\end{aligned}$$

On the other hand $\bar{N} \cap P = \{1\}$ and so we may identify $\bar{N}$ with its image under the projection of G on $\bar{G}$. Then the above relation becomes

$$\int_K \bar{f}(k)\, dk = \int_{\bar{N}} \bar{f}(\bar{n})\, \delta(\bar{n})^{-1}\, d\bar{n}.$$

But since $f \mapsto \bar{f}$ is a surjective mapping of $C_c(G)$ on $C(G/P)$, we have obtained the following result.

Lemma 1. *We can normalize the Haar measure $d\bar{n}$ in such a way that*

$$\int_K \phi(k)\, dk = \int_{\bar{N}} \phi(\bar{n})\, \delta(\bar{n})^{-1}\, d\bar{n}$$

for all $\phi \in C(G/P) = C(K/K_M)$.

It is easy to verify that

$$\delta(x) = e^{2\rho(H(x))} \qquad (x \in G).$$

Hence taking $\phi = 1$ in the above lemma we get the following result.

Corollary. *Under the above normalization of $d\bar{n}$ we have*

$$\int_{\bar{N}} e^{-2\rho(H(\bar{n}))}\, d\bar{n} = 1.$$

Now we come to the proof of Theorem 1. It follows from [1(e), Corollary of Lemma 32.2] that the two integrals converge uniformly when ν varies in a compact subset of $\mathfrak{F}_c(P)$. Therefore (see the proof of Theorem 18.1), it would be enough to verify the two equations for $\nu \in \mathfrak{F}_c(P) \cap \mathfrak{F}_c'(\delta)$. We prove only the first since the proof of the second is quite similar.

Fix $\psi \in L$, $\nu \in \mathfrak{F}_c(P) \cap \mathfrak{F}'_c(\delta)$ and put

$$\phi = E(P: \psi: \nu).$$

Then

$$\phi(x) = \int \psi(x\kappa(\bar{n}))\, \tau(\kappa(\bar{n}))^{-1} \exp\{((-1)^{1/2}\nu - \rho)(H(x\kappa(\bar{n}))) - 2\rho(H(\bar{n}))\}\, d\bar{n}$$

for $x \in G$, from Lemma 1. Now

$$\bar{n} = \kappa(\bar{n})\, \mu(\bar{n}) \exp H(\bar{n}) \cdot n$$

where $n \in N$. Hence if $m \in M_1 = MA$,

$$\psi(m^{-1}\kappa(\bar{n})) = \psi(m^{-1}\bar{n}\mu(\bar{n})^{-1}),$$
$$H(m^{-1}\kappa(\bar{n})) = H(m^{-1}\bar{n}) - H(\bar{n}).$$

Take $x = m^{-1}$, replace $\bar{n}$ by $\bar{n}^m$ inside the integral and observe that

$$d\bar{n}^m = e^{-2\rho(H(m))}\, d\bar{n}.$$

Then we obtain

$$e^{\nu_+(H(m))}\, \phi(m^{-1}) = \int_{\bar{N}} \psi(\bar{n}m^{-1}\, \mu(\bar{n}^m)^{-1})\, \tau(\kappa(\bar{n}^m))^{-1}\, e^{\nu_-(H(\bar{n})) - \nu_+(H(\bar{n}^m))}\, d\bar{n}$$

where $\nu_- = (-1)^{1/2}\nu - \rho$ and $\nu_+ = (-1)^{1/2}\nu + \rho$. On the other hand, we can choose $c > 0$ such that

$$|\psi(m)| \leqq c\, \Xi_M(m)$$

for all $m \in M_1$. Now let $m = m_0^{-1} a$ where $m_0 \in M$ and $a \in A$. Keep m_0 fixed and let $a \underset{P}{\to} \infty$. Then

$$H(\bar{n}^m) = H(m_0^{-1}\bar{n}^a) = H(m_0^{-1}\kappa(\bar{n}^a)) + H(\bar{n}^a).$$

Hence $H(\bar{n}^m) - H(\bar{n}^a)$ remains bounded. Moreover

$$|\psi(\bar{n}m^{-1}\, \mu(\bar{n}^m)^{-1})| = |\psi(\mu(\bar{n})\, m_0\, \mu(m_0^{-1}\bar{n}^a m_0)^{-1})|.$$

Now

$$\bar{n}^a \in \kappa(\bar{n}^a)\, \mu(\bar{n}^a)\, AN.$$

Hence

$$m_0^{-1}\bar{n}^a m_0 \in m_0^{-1}\kappa(\bar{n}^a)\, m_0 \cdot \mu(\bar{n}^a)^{m_0^{-1}} \cdot AN$$

and therefore

$$\mu(m_0^{-1}\bar{n}^a m_0) \in K_M \cdot \mu(m_0^{-1}\kappa(\bar{n}^a)\, m_0)\, \mu(\bar{n}^a)^{m_0^{-1}}.$$

This shows that

$$m_0\, \mu(m_0^{-1}\bar{n}^a m_0)\, m_0^{-1} \in C\mu(\bar{n}^a)$$

where C is a compact subset of M. Hence

$$\mu(\bar{n})\, m_0\, \mu(m_0^{-1}\bar{n}^a m_0)^{-1} \in \mu(\bar{n})\, \mu(\bar{n}^a)^{-1}\, C^{-1}\, m_0.$$

Therefore we can choose $c_1 > 0$ such that

$$|\psi(\bar{n} m^{-1} \mu(\bar{n}^m)^{-1})| \leqq c_1 \, \Xi_M(\mu(\bar{n}) \, \mu(\bar{n}^a)^{-1})$$

for all $\bar{n} \in \bar{N}$ and $a \in A$. By Lemma 20.1, we can take the limit inside the integral and conclude that

$$\begin{aligned} \lim_{a \underset{P}{\to} \infty} e^{\nu + (\log a)} \phi(m_0 a^{-1}) &= \int_{\bar{N}} \psi(\bar{n} m_0) \, e^{\nu - (H(\bar{n}))} \, d\bar{n} \\ &= \int_{\bar{N}} \tau(\kappa(\bar{n})) \, \psi(\mu(\bar{n}) \, m_0) \, e^{\nu - (H(\bar{n}))} \, d\bar{n}. \end{aligned}$$

The required result now follows from the corollary of Lemma 17.5.

We shall now derive some consequences of Theorem 1.

Lemma 2. *Fix $\omega \in \mathscr{E}_2(M)$. Then $L(\omega)$ is stable under $c_{\bar{P}|P}(1:\nu)$ and $c_{P|P}(1:\nu)$.*

Since $\mathscr{C}_\omega(M)$ is stable under both left and right translations of M, this is obvious from Theorem 1.

The following result was pointed out to me by Langlands.

Lemma 3. $\det c_{P|P}(1:\nu)$ *is not identically zero.*

Put

$$c(t) = \int_{\bar{N}} e^{-t\rho(H(\bar{n}))} \, d\bar{n} \qquad (t \geqq 2)$$

and

$$\alpha_t(\bar{n}) = c(t)^{-1} \, e^{-t\rho(H(\bar{n}))} \qquad (\bar{n} \in \bar{N}).$$

The proof is based on the following simple fact.

Lemma 4. *Let f be a continuous function on $\bar{N}$ which is integrable with respect to $d\bar{n}$. Then*

$$\lim_{t \to +\infty} \int_{\bar{N}} \alpha_t \, f \, d\bar{n} = f(1).$$

We shall prove this in §21.

Now fix $\nu \in \mathfrak{F}_c(P)$ and put

$$\nu_t = \nu + (-1)^{1/2} \, t\rho, \qquad C(t) = c(t)^{-1} \, c_{P|P}(1:-\nu_t) \qquad (t \geqq 2).$$

Then $\nu_t \in \mathfrak{F}_c(P)$ and $C(t) \in \operatorname{End} L$. Fix $\psi \in L$. Then it follows from Theorem 1 that

$$(C(t)\,\psi)(m) = \int \psi(m \, \mu(\bar{n})^{-1}) \, \tau(\kappa(\bar{n}))^{-1} \, e^{((-1)^{1/2}\nu - \rho)(H(\bar{n}))} \, \alpha_t(\bar{n}) \, d\bar{n}.$$

Hence

$$\lim_{t \to +\infty} C(t) \, \psi = \psi$$

from Lemma 3. This proves that $C(t) \to 1$ and therefore $\det C(t) \to 1$. Hence $\det C(t) \neq 0$ for t sufficiently large.

Combining Theorem 1 with Theorem 13.2, we can now obtain the following result.

Theorem 2. *Fix* $\psi \in L$, $\alpha \in C_c^\infty(\mathfrak{F}')$, $P_1, P_2 \in \mathscr{P}(\mathfrak{a})$ *and put*

$$\phi_\alpha(x) = \int_{\mathfrak{F}} \alpha(\nu)\, E(P_1 : \psi : \nu : x)\, d\nu \qquad (x \in G).$$

Then $\phi_\alpha \in \mathscr{C}(G, \tau)$ *and*

$$\phi_\alpha^{(P_2)}(ma) = \gamma(P_2) \int_{\mathfrak{F}} e^{(-1)^{1/2}\nu(\log a)} \sum_{s \in \mathfrak{w}} \alpha(s^{-1}\nu)(c_{\bar{P}_2|P_2}(1 : \nu)\, c_{P_2|P_1}(s : s^{-1}\nu)\, \psi)(m)\, d\nu$$

for $m \in M$, $a \in A$. *Here*

$$\gamma(P_2) = \int_{\bar{N}_2} e^{-2\rho(H(\bar{n}))}\, d\bar{n},$$

the integrand having the same meaning as in Theorem 1 *for* $P = P_2$.

There is no loss of generality in assuming that $\psi \in L(\omega)$ for some $\omega \in \mathscr{E}_2(M)$. Put

$$\phi(\nu : x) = E(P_1 : \psi : \nu : x) \qquad (\nu \in \mathfrak{F},\ x \in G).$$

Then it follows from Lemma 17.1 that ϕ is a function on $\mathfrak{F} \times G$ of type $II(\lambda)$ for a suitable $\lambda \in (-1)^{1/2}\, \mathfrak{h}_I^*$ (see the proof of Theorem 18.1). Therefore since $\operatorname{Supp} \alpha \subset \mathfrak{F}'$, it follows from Theorem 18.1 that the function

$$(\nu, x) \mapsto \alpha(\nu)\, \phi(\nu : x)$$

is of type $I'(\lambda)$ (§13). Hence we conclude from Theorem 13.1 that $\phi_\alpha \in \mathscr{C}(G, \tau)$.

Now put $P = P_2$ and let us use the notation of Theorem 13.2. Since $\rho(H(\bar{n})) \geqq 0$, it is clear from this that

$$\phi_\alpha^{(P)}(m) = \lim_{\varepsilon \to 0} \int_{\bar{N}} e^{-(1+\varepsilon)\rho(H(\bar{n}))}\, \phi_{P,\alpha}(\bar{n} m)\, d\bar{n} \qquad (m \in M_1).$$

(Here $\varepsilon > 0$.) But

$$\phi_{P,\alpha}(\bar{n} m) = \int_{\mathfrak{F}} \alpha(\nu)\, \tau(\kappa(\bar{n}))\, E_P(P_1 : \psi : \nu : \mu(\bar{n})\, m \exp H(\bar{n}))\, d\nu.$$

Fix $\varepsilon > 0$ and put $\nu_\varepsilon = \nu + (-1)^{1/2}\varepsilon\rho$ for $\nu \in \mathfrak{F}$. Then $\nu_\varepsilon \in \mathfrak{F}_c(P)$ and we conclude from [1(e), Corollary of Lemma 32.2] and Theorems 1 and 18.1 that

$$\int_{\bar{N}} e^{-(1+\varepsilon)\rho(H(\bar{n}))}\, \phi_{P,\alpha}(\bar{n} m a)\, d\bar{n}$$
$$= \gamma(P) \int_{\mathfrak{F}} \alpha(\nu) \sum_{s \in \mathfrak{w}} (c_{\bar{P}|P}(1 : (s\nu)_\varepsilon)\, c_{P|P_1}(s : \nu)\, \psi)(m)\, e^{(-1)^{1/2} s\nu(H(a))}\, d\nu$$

for $m \in M$ and $a \in A$. But $c_{\bar{P}|P}(1 : \nu)$ is holomorphic on $\mathfrak{F}_c'(\delta)$. Therefore since $\operatorname{Supp} \alpha \subset \mathfrak{F}'$, we obtain by making $\varepsilon \to 0$ that

$$\phi_\alpha^{(P)}(ma) = \sum_{s \in \mathfrak{w}} \gamma(P) \int_{\mathfrak{F}} \alpha(\nu)(c_{\bar{P}|P}(1 : s\nu)\, c_{P|P_1}(s : \nu)\, \psi)(m)\, e^{(-1)^{1/2} s\nu(H(a))}\, d\nu$$

and this is equivalent to the required result.

§ 20. A Result on Uniform Convergence

Let $P=MAN$ be a psgp of G. Define ρ, $H(x)$, $\mu(x)$ $(x\in G)$, A^+ and $\bar{N}$ as usual.

Lemma 1. *Fix $\nu\in\mathfrak{a}^*$ such that $\langle\nu,\alpha\rangle>0$ for every root α of (P,A) and put $\nu_+=\nu+\rho$, $\nu_-=\nu-\rho$. Then the integral*

$$\int_{\bar{N}} e^{-\nu_+(H(\bar{n}))+\nu_-(H(\bar{n}^a))}\,\Xi_M(\mu(\bar{n})\,\mu(\bar{n}^a)^{-1})\,d\bar{n}$$

converges uniformly for $a\in A^+$.

The present form of this lemma is due to Langlands [2, Lemma 3.12]. My original formulation was more complicated.

We first need an auxiliary result. Let $P_0=M_0A_0N_0$ be a minimal psgp of G contained in P and let us use the notation of [1(e), § 30].

Lemma 2. *Let $x, y\in G$. Then*

$$\Xi(xy^{-1})=\int_{\bar{N}_0} e^{-\rho_0(H_0(x\bar{n}_0)+H_0(y\bar{n}_0))}\,d\bar{n}_0,$$

where the Haar measure $d\bar{n}_0$ on $\bar{N}_0$ is so normalized that

$$\int_{\bar{N}_0} e^{-2\rho_0(H_0(\bar{n}_0))}\,d\bar{n}_0=1.$$

Let $\kappa_0(x)$ $(x\in G)$ denote the component of x in K corresponding to the Iwasawa decomposition $G=KA_0N_0$. Put $k_y=\kappa_0(yk)$ $(k\in K)$. Then $k\mapsto k_y$ is a diffeomorphism of K and [1(a), p. 281]

$$e^{2\rho_0(H_0(yk))}\,dk_y=dk.$$

Now

$$\Xi(xy^{-1})=\int_K e^{-\rho_0(H_0(xy^{-1}k))}\,dk.$$

Replacing k by k_y and observing that

$$H_0(xy^{-1}k_y)=H_0(xk)-H_0(yk),$$

we get

$$\Xi(xy^{-1})=\int_K e^{-\rho_0(H_0(xk)+H_0(yk))}\,dk$$

and the required result now follows from Lemma 19.1.

Let ${}^*P={}^*M\,{}^*A\,{}^*N$ be the minimal psgp of M corresponding to P_0 [1(e), Lemma 6.1] so that ${}^*P=M\cap P_0$. Put ${}^*\bar{N}=\theta({}^*N)$. Then $\bar{N}$ is a normal subgroup of $\bar{N}^0$ and the mapping

$$(\bar{n}, {}^*\bar{n})\mapsto\bar{n}_0=\bar{n}\cdot{}^*\bar{n}$$

defines a diffeomorphism of $\bar{N}\times{}^*\bar{N}$ onto $\bar{N}^0$. Let $d\bar{n}$ and $d^*\bar{n}$ denote the corresponding Haar measures. Then $d\bar{n}\cdot d^*\bar{n}=c\,d\bar{n}_0$ where c is a positive constant.

Let us now use the notations of [1(e), § 30].

Lemma 3. *We can normalize $d\bar{n}$ and $d^*\bar{n}$ in such a way that*

$$\int_{\bar{N}} e^{-2\rho(H(\bar{n}))}\, d\bar{n} = \int_{{}^*\bar{N}} e^{-2{}^*\rho({}^*H({}^*\bar{n}))}\, d^*\bar{n} = 1.$$

Then $d\bar{n}_0 = d\bar{n}\, d^*\bar{n}$ where $d\bar{n}_0$ is normalized as in Lemma 2.

The proof of the first part is the same as that of the corollary of Lemma 19.1. Since $d\bar{n}\, d^*\bar{n} = c\, d\bar{n}_0$, we have

$$c = \int_{{}^*\bar{N}} d^*\bar{n} \int_{\bar{N}} e^{-2\rho_0(H_0(\bar{n}\,{}^*\bar{n}))}\, d\bar{n}.$$

Fix ${}^*\bar{n} \in {}^*\bar{N}$. Then ${}^*\bar{n} = kan$ ($k \in K_M$, $a \in {}^*A$, $n \in {}^*N$) and

$$H_0(\bar{n}\,{}^*\bar{n}) = H_0(\bar{n}k) + \log a = H_0(k^{-1}\bar{n}k) + {}^*H({}^*\bar{n}).$$

But since K_M normalizes $\bar{N}$, we conclude that

$$\int_{\bar{N}} e^{-2\rho_0(H_0(\bar{n}\,{}^*\bar{n}))}\, d\bar{n} = e^{-2{}^*\rho({}^*H({}^*\bar{n}))} \int_{\bar{N}} e^{-2\rho_0(H_0(\bar{n}))}\, d\bar{n}.$$

On the other hand

$$H_0(\bar{n}\,{}^*k) = H(\bar{n}) + {}^*H(\mu(\bar{n})\,{}^*k)$$

for ${}^*k \in K_M$. Hence if d^*k is the normalized Haar measure on K_M, we conclude from [1(a), Corollary p. 261] that

$$\int_{K_M} e^{-2\rho_0(H_0(\bar{n}\,{}^*k))}\, d^*k = e^{-2\rho(H(\bar{n}))} \int_{K_M} e^{-2{}^*\rho({}^*H(\mu(\bar{n})\,{}^*k))}\, d^*k$$
$$= e^{-2\rho(H(\bar{n}))}.$$

Therefore

$$\int_{\bar{N}} e^{-2\rho_0(H_0(\bar{n}))}\, d\bar{n} = \int_{\bar{N}} d\bar{n} \int_{K_M} e^{-2\rho_0(H_0({}^*k^{-1}\bar{n}\,{}^*k))}\, d^*k$$
$$= \int_{\bar{N}} e^{-2\rho(H(\bar{n}))}\, d\bar{n} = 1$$

and this proves that

$$c = \int_{{}^*\bar{N}} e^{-2{}^*\rho({}^*H({}^*\bar{n}))}\, d^*\bar{n} = 1.$$

Corollary. $\Xi_M(m_1 m_2^{-1}) = \int_{{}^*\bar{N}} e^{-{}^*\rho({}^*H(m_1\,{}^*\bar{n}) + {}^*H(m_2\,{}^*\bar{n}))}\, d^*\bar{n}$ *for $m_1, m_2 \in M$.*

This follows by applying Lemma 2 to $(M, {}^*P)$ in place of (G, P_0).

Now we come to the proof of Lemma 1. Fix $0 < \varepsilon \leqq 1$ such that

$$\langle \rho_0 - \varepsilon \nu, \alpha_0 \rangle \geqq 0$$

for every root α_0 of (P_0, A_0). Note that

$$-\nu_+(H(\bar{n})) + \nu_-(H(\bar{n}^a)) = \nu(H(\bar{n}^a) - H(\bar{n})) - \rho(H(\bar{n}^a) + H(\bar{n}))$$

and it follows from [1(e), Lemma 30.4] that we can choose $c \geqq 0$ such that

$$\nu(H(\bar{n}^a) - H(\bar{n})) \leqq c$$

for all $\bar{n}\in\bar{N}$ and $a\in A^+$. Put $\nu'=\varepsilon\nu$. Then

$$-\nu_+(H(\bar{n}))+\nu_-(H(\bar{n}^a))\leqq(1-\varepsilon)\,c-\nu'_+(H(\bar{n}))+\nu'_-(H(\bar{n}^a)).$$

Hence it would be enough to prove Lemma 1 for $\varepsilon\nu$ instead of ν.

So we may now assume that

$$\langle\rho_0-\nu,\alpha_0\rangle\geqq 0$$

for every root α_0 of (P_0, A_0). Let $\bar{n}_0=\bar{n}\cdot{}^*\bar{n}$ where $\bar{n}\in\bar{N}$ and ${}^*\bar{n}\in{}^*\bar{N}$. Then

$$H_0(\bar{n}_0)=H(\bar{n})+{}^*H(\mu(\bar{n})\cdot{}^*\bar{n}),$$
$$H_0(\bar{n}_0^a)=H(\bar{n}^a)+{}^*H(\mu(\bar{n}^a)\cdot{}^*\bar{n})\qquad(a\in A).$$

Therefore

$$\begin{aligned}(\nu-\rho_0)(H_0(\bar{n}_0^a))-(\nu+\rho_0)(H_0(\bar{n}_0))\\ =\nu_-(H(\bar{n}^a))-\nu_+(H(\bar{n}))\\ -{}^*\rho({}^*H(\mu(\bar{n})\,{}^*\bar{n}))-{}^*\rho({}^*H(\mu(\bar{n}^a)\,{}^*\bar{n})).\end{aligned}$$

ω being a measurable subset of $\bar{N}$, put $\omega_0=\omega\cdot{}^*\bar{N}$. Then integrating both sides, we get

$$\begin{aligned}\int_{\omega_0} e^{(\nu-\rho_0)(H_0(\bar{n}_0^a))-(\nu+\rho_0)(H_0(\bar{n}_0))}\,d\bar{n}_0\\ =\int_{\omega} e^{\nu_-(H(\bar{n}^a))-\nu_+(H(\bar{n}))}\,\Xi_M(\mu(\bar{n})\,\mu(\bar{n}^a)^{-1})\,d\bar{n}=I_\omega(a)\qquad\text{(say)}\end{aligned}$$

from the corollary of Lemma 3. On the other hand $M=K_M\cdot{}^*A\cdot{}^*N$ is an Iwasawa decomposition of M. Hence

$$\bar{n}_0^a=\bar{n}^a\cdot{}^*\bar{n}=\bar{n}^a\cdot{}^*k\cdot{}^*a\cdot{}^*n$$

where ${}^*k\in K_M$, ${}^*a\in{}^*A$, ${}^*n\in{}^*N$. Since M normalizes $\bar{N}$, it is clear that

$$H_0(\bar{n}_0^a)=H_0(\bar{n}')+H_0({}^*a)$$

where $\bar{n}'={}^*k^{-1}\cdot\bar{n}^a\cdot{}^*k\in\bar{N}$. Hence we conclude from [1(a), Lemma 43] that

$$(\rho_0-\nu)(H_0(\bar{n}^a))\geqq(\rho_0-\nu)(H_0({}^*a))={}^*\rho({}^*H({}^*\bar{n})).$$

Therefore

$$\begin{aligned}(\nu-\rho_0)(H_0(\bar{n}_0^a))-(\nu+\rho_0)(H_0(\bar{n}_0))\\ \leqq-{}^*\rho({}^*H({}^*\bar{n}))-\nu_+(H(\bar{n}))-{}^*\rho({}^*H(\mu(\bar{n})\,{}^*\bar{n})).\end{aligned}$$

Integrating both sides on ω_0 and applying Lemma 3 and its corollary, we find that

$$I_\omega(a)\leqq\int_\omega e^{-\nu_+(H(\bar{n}))}\,\Xi_M(\mu(\bar{n}))\,d\bar{n}\qquad(a\in A).$$

Now choose $\varepsilon>0$ so small that $\langle\nu,\alpha\rangle\geqq\varepsilon\langle\rho,\alpha\rangle$ for every root α of (P, A). Then

$$\nu^+(H(\bar{n}))\geqq(1+\varepsilon)\,\rho(H(\bar{n}))\qquad(\bar{n}\in\bar{N})$$

from [1(e), Lemma 30.4]. On the other hand

$$\int_{\bar{N}} e^{-(1+\varepsilon)\rho(H(\bar{n}))}\,\Xi_M(\mu(\bar{n}))\,d\bar{n}<\infty$$

from [1(e), Corollary of Lemma 32.2]. Therefore the assertion of Lemma 1 is now obvious.

§ 21. Proof of Lemma 19.4

For $T \geqq 0$, let $\bar{N}(T)$ denote the set of all points $\bar{n} \in \bar{N}$ such that $\rho(H(\bar{n})) \leqq T$. Then $\bar{N}(T)$ is a compact set and $\bar{N}(0) = \{1\}$. Let $(\alpha_1, \ldots, \alpha_l)$ be the system of simple roots of (P, A). Then

$$2\rho = m_1 \alpha_1 + \cdots + m_l \alpha_l$$

where m_i are positive integers. Put $m = m_1 + \cdots + m_l$.

Lemma 1. *There exists a number $c > 0$ such that*

$$\int_{\bar{N}(\varepsilon)} d\bar{n} \geqq c\,\varepsilon^{2m}$$

for $0 < \varepsilon \leqq 1$.

Put

$$\beta(a) = \inf_{1 \leqq i \leqq l} \alpha_i(\log a)/2 \qquad (a \in A^+).$$

Then

$$\rho(H(\bar{n}^a)) \leqq \log(1 + e^{1 - \beta(a)})$$

for $\bar{n} \in \bar{N}(1)$ and $a \in A^+$ from [1(e), Lemma 30.2]. Fix ε $(0 < \varepsilon \leqq 1)$ and choose $a \in A$ such that

$$\alpha_i(\log a) = 2(1 - \log \varepsilon) \qquad (1 \leqq i \leqq l).$$

Then $a \in A^+$ and

$$1 - \beta(a) = \log \varepsilon.$$

Hence

$$\rho(H(\bar{n}^a)) \leqq \log(1 + \varepsilon) \leqq \varepsilon$$

for $\bar{n} \in \bar{N}(1)$. Therefore

$$\int_{\bar{N}(\varepsilon)} d\bar{n} \geqq \int_{(\bar{N}(1))^a} d\bar{n} = e^{-2\rho(\log a)}\, c_0$$

where

$$c_0 = \int_{\bar{N}(1)} d\bar{n} > 0.$$

But

$$2\rho(\log a) = m\,\alpha_i(\log a) = 2m(1 - \log \varepsilon).$$

Hence

$$\int_{\bar{N}(\varepsilon)} d\bar{n} \geqq c\,\varepsilon^{2m}$$

where $c = c_0\, e^{-2m} > 0$.

Now we come to the proof of Lemma 19.4. Fix ε $(0<\varepsilon\leqq 1)$ and let $\bar{N}_r(\varepsilon)$ denote the complement of $\bar{N}((r-1)\varepsilon)$ in $\bar{N}(r\varepsilon)$ $(r\geqq 1)$. Then if $t\geqq 2$,

$$\int_{\bar{N}_r(\varepsilon)} e^{-t\rho(H(\bar{n}))}\,d\bar{n}\geqq e^{-r\varepsilon t}\int_{\bar{N}_r(\varepsilon)} d\bar{n}=e^{-r\varepsilon t}(\mu(r\varepsilon)-\mu((r-1)\varepsilon))$$

where

$$\mu(T)=\int_{\bar{N}(T)} d\bar{n} \qquad (T\geqq 0).$$

Therefore

$$c(t)=\int_{\bar{N}} e^{-t\rho(H(\bar{n}))}\,d\bar{n}\geqq \sum_{r\geqq 1} e^{-r\varepsilon t}(\mu(r\varepsilon)-\mu((r-1)\varepsilon)).$$

On the other hand

$$\mu(T)=\int_{\bar{N}(T)} d\bar{n}\leqq e^{2T}\int_{\bar{N}} e^{-2\rho(H(\bar{n}))}\,d\bar{n}=c(2)\,e^{2T}.$$

Hence if $t>2$,

$$e^{-r\varepsilon t}\mu(r\varepsilon)\to 0$$

as $r\to+\infty$. Therefore

$$\begin{aligned} c(t)&\geqq \sum_{r\geqq 1}\mu(r\varepsilon)\,e^{-r\varepsilon t}(1-e^{-\varepsilon t})\\ &\geqq \mu(\varepsilon)\,e^{-\varepsilon t}(1-e^{-\varepsilon t}). \end{aligned}$$

Now take $\varepsilon=t^{-1}$. Then it follows from Lemma 1 that

$$c(t)\geqq \mu(t^{-1})\,e^{-1}(1-e^{-1})\geqq c_0\,t^{-2m} \qquad (t>2),$$

where c_0 is a positive constant independent of t.

Now let U be any open neighborhood of 1 in $\bar{N}$. We have to show that

$$\int_{{}^cU} \alpha_t(\bar{n})\,d\bar{n}\to 0$$

as $t\to+\infty$. (As usual ${}^cU$ denotes the complement of U.) Fix ε $(0<\varepsilon\leqq 1)$ such that $\bar{N}(\varepsilon)\subset U$. Then if $t>2$,

$$\int_{{}^cU} \alpha_t(\bar{n})\,d\bar{n}\leqq \int_{{}^c\bar{N}(\varepsilon)} \alpha_t(\bar{n})\,d\bar{n}=c(t)^{-1}\int_{{}^c\bar{N}(\varepsilon)} e^{-t\rho(H(\bar{n}))}\,d\bar{n}.$$

But $c(t)^{-1}\leqq c_0^{-1}\,t^{2m}$ and

$$\begin{aligned} \int_{{}^c\bar{N}(\varepsilon)} e^{-t\rho(H(\bar{n}))}\,dt&\leqq e^{-(t-2)\varepsilon}\int_{{}^c\bar{N}(\varepsilon)} e^{-2\rho(H(\bar{n}))}\,d\bar{n}\\ &\leqq c(2)\,e^{-(t-2)\varepsilon}. \end{aligned}$$

Therefore

$$\int_{{}^cU} \alpha_t(\bar{n})\,d\bar{n}\leqq c_1\,t^{2m}\,e^{-t\varepsilon}\to 0$$

as $t\to+\infty$, where $c_1=c_0^{-1}\,c(2)\,e^{2\varepsilon}$. This proves Lemma 19.4.

§ 22. Appendix

Let $x=(x_1, \dots, x_n)$ denote a variable point in $E=\mathbf{R}^n$. Put $D_i=\partial/\partial x_i$ and $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$ for a multi-index $\alpha=(\alpha_1, \dots, \alpha_n)$. We write $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_n$, $|x|=\max_i |x_i|$ and denote by M the set of all multi-indices.

Let V and $\mathscr{S}(V)$ be as before (§ 6).

Lemma 1. *Let f be an element in $C^\infty(E, V)$ such that $f=0$ on the hyperplane $x_1=0$. Then $f=x_1 g$ where*

$$g(x)=\int_0^1 f_1(x_1 t, x_2, \dots, x_n)\, dt$$

and $f_1=D_1 f$. Hence $g\in C^\infty(E, V)$ and

$$|D^\alpha g(x)|_{\mathbf{s}} \leqq \sup_{|y|\leqq|x|} |D^\alpha f_1(y)|_{\mathbf{s}}$$

for all $x\in E$, $\alpha\in M$ and $\mathbf{s}\in\mathscr{S}(V)$.

This is obvious.

Let $p\neq 0$ be the product of N real linear forms on E and E' the set of all points $x\in E$ where $p(x)\neq 0$. A function f from E' to V is said to be locally bounded (on E), if for every compact set ω in E and $\mathbf{s}\in\mathscr{S}(V)$, $|f(x)|_{\mathbf{s}}$ remains bounded for $x\in\omega\cap E'$.

For $\alpha\in M$, $r\geqq 0$ and $\mathbf{s}\in\mathscr{S}(V)$, put

$$\mathbf{s}_{\alpha,r}(f)=\sup_E (1+|x|)^r |D^\alpha f|_{\mathbf{s}} \qquad (f\in C^\infty(E, V)).$$

If F is a finite subset of M, put

$$\mathbf{s}_{F,r}(f)=\sum_{\alpha\in F} \mathbf{s}_{\alpha,r}(f).$$

Let $\mathscr{C}(E, V)$ denote the set of all functions $f\in C^\infty(E, V)$ such that $\mathbf{s}_{\alpha,r}(f)<\infty$ for all $\alpha\in M$ and $r\geqq 0$.

Lemma 2. *Fix $\alpha\in M$ and let F denote the set of all $\beta\in M$ such that $|\beta|\leqq|\alpha|+N$. Then for every $r\geqq 0$, we can choose a number $c_r\geqq 1$ with the following property. Suppose $f\in\mathscr{C}(E, V)$ and $p^{-1}f$ is locally bounded. Then $f=pg$ where $g\in\mathscr{C}(E, V)$ and*

$$\mathbf{s}_{\alpha,r}(g)\leqq c_r\, \mathbf{s}_{F,r}(f)$$

for all $\mathbf{s}\in\mathscr{S}(V)$.

By an easy induction we are reduced to the case $N=1$. Hence we may assume that $p=x_1$. Then $f=x_1 g$ in the notation of Lemma 1. Let E_1 and E_2 be the sets of points $x\in E$ where $|x_1|\leqq 1$ and $|x_1|\geqq 1$ respectively. Then if $x\in E_1$, we have

$$1+|x|\leqq 2(1+\max_{i\geqq 2}|x_i|)$$

and therefore

$$(1+|x|)^r |D^\alpha g(x)|_{\mathbf{s}}\leqq 2^r \sup_{|y|\leqq|x|} |D^\alpha f_1(y)|(1+|y|)^r.$$

This means that

$$\sup_{E_1}(1+|x|)^r\,|D^\alpha g(x)|_s \leqq 2^r\,s_{\beta,r}(f)$$

where $\beta=(\alpha_1+1, \alpha_2, \ldots, \alpha_n)$.

On the other hand $g=x_1^{-1}f$ on E_2. Since $|x_1|\geqq 1$, it follows directly by differentiation that

$$|D^\alpha g(x)|_s \leqq \alpha_1! \sum_{0\leqq m\leqq \alpha_1} |D_1^m D^\beta f(x)|_s$$

on E_2 where $\beta=(0, \alpha_2, \ldots, \alpha_n)$. Therefore since $E=E_1\cup E_2$ the required result is obvious.

Let us now use the notation of Theorem 18.1.

Lemma 3. *Let H be an element in $\mathfrak{a}$ such that $\alpha(H)\neq 0$ for every root α of* $(\mathfrak{g}, \mathfrak{a})$. *Then $sH\neq H$ for every $s\neq 1$ in $\mathfrak{w}$.*

Extend $\mathfrak{a}$ to a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}$ and put $\mathfrak{w}_0=\mathfrak{w}(\mathfrak{a}_0)$. Let Q be the set of all roots of $(\mathfrak{g}, \mathfrak{a}_0)$ which vanish at H. Then if $\beta\in Q$, it is clear that $\beta=0$ on $\mathfrak{a}$.

Let $\mathfrak{w}_1$ be the stabilizer of H in $\mathfrak{w}_0$. Then $\mathfrak{w}_1$ is the subgroup of $\mathfrak{w}_0$ generated by the Weyl reflexions s_β for $\beta\in Q$. Hence every element of $\mathfrak{w}_1$ leaves $\mathfrak{a}$ fixed pointwise.

Now suppose $sH=H$ for some $s\in\mathfrak{w}$. We can choose $s_0\in\mathfrak{w}_0$ such that $s_0=s$ on $\mathfrak{a}$. But then $s_0\in\mathfrak{w}_1$ and hence $s_0=1$ on $\mathfrak{a}$. This proves that $s=1$.

References

1. Harish-Chandra: (a) Spherical functions on a semisimple Lie group I. Amer. J. Math. **80**, 241–310 (1958)

   (b) Spherical functions on a semisimple Lie group II. Amer. J. Math. **80**, 553–613 (1958)

   (c) Invariant differential operators and distributions on a semisimple Lie algebra. Amer. J. Math. **86**, 534–564 (1964)

   (d) Discrete series for semisimple Lie groups II. Acta Math. **116**, 1–111 (1966)

   (e) Harmonic analysis on real reductive groups I. J. Functional Analysis **19**, 104–204 (1975)

2. Langlands, R. P.: On the classification of irreducible representations of real algebraic groups, 1973, preprint

Received July 28, 1975

Harish-Chandra
The Institute for Advanced Study
School of Mathematics
Princeton, N. J. 08540
USA