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Zassenhaus conjecture for A_5

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Abstract. We develop a criterion for rational conjugacy of torsion units of the integral group ring $\mathbb{Z}G$ of a finite group G, as also a necessary condition for an element of $\mathbb{Z}G$ to be a torsion unit, and apply them to verify the Zassenhaus conjecture in case when $G = A_5$.

Keywords. Zassenhaus conjecture; rational conjugacy; torsion unit.

1. A criterion for rational conjugacy

Let U be a complex $n \times n$ matrix with $U^k = 1$, $k \geqslant 1$. Let Z be a primitive kth root of unity, and let μ_l be the multiplicity of Z^l as an eigenvalue of U. Then

$$\mu_l = \frac{1}{k}\sum_{r=0}^{k-1} \mathrm{Tr}(U^r)\mathrm{Z}^{-lr}; \quad l = 0, 1, \ldots, k-1. \tag{1}$$

In particular, the numbers on the right hand side of (1) are non-negative integers with sum n.

This follows at once on noting that

$$\mathrm{Tr}(U^r) = \mu_0 1 + \mu_1 \mathrm{Z}^r + \cdots + \mu_{k-1}\mathrm{Z}^{(k-1)r}.$$

Let G be a finite group. Two torsion units u and v of $\mathbb{Z}G$ are rationally conjugate if and only if in each irreducible representation their matrices have the same characteristic polynomials. This is a consequence of Lemma 5 of [4] coupled with the fact that these matrices, being of finite order, are diagonalizable.

Let C be a conjugacy class in G. For an element $\alpha = \sum \alpha(g)g$ in $\mathbb{C}G$, we define its partial augmentation $\varepsilon_C(\alpha)$ over C by setting

$$\varepsilon_C(\alpha) = \sum_{g \in C} \alpha(g).$$

One checks immediately that $\varepsilon_C(\alpha\beta) = \varepsilon_C(\beta\alpha)$, and hence conjugate units in $\mathbb{C}G$ have the same partial augmentations.

Let u be a unit in $\mathbb{Z}G$, $u^k = 1$, $k \geqslant 1$. Let χ be any character of G of degree n, and let R be the corresponding representation. The multiplicity $\mu_l(u;\chi)$ of Z^l as an eigenvalue of $R(u)$ is given by

$$\mu_l(u;\chi)=\frac{1}{k}\sum_{r=0}^{k-1}\chi(u^r)Z^{-lr};\quad l=0,1,\ldots,k-1.$$

Collecting together those r which have the same g.c.d. with k we get

$$\mu_l(u;\chi)=\frac{1}{k}\sum_{d|k}\sum_{\substack{r \bmod k/d\\(r,k/d)=1}}\chi(u^{dr})Z^{-dlr}.$$

Since $(u^d)^{k/d}=1$, $\chi(u^d)$ is a sum of n (k/d)th roots $\varepsilon_1,\ldots,\varepsilon_n$ of unity; therefore for $(r,k/d)=1$

$$\chi(u^{dr})=\varepsilon_1^r+\cdots+\varepsilon_n^r=(\chi(u^d))^{\sigma_r},$$

where σ_r is the automorphism $Z^d\to Z^{dr}$ of $\mathbb{Q}(Z^d)$. It follows that

$$\mu_l(u;\chi)=\frac{1}{k}\sum_{d|k}\mathrm{Tr}_{\mathbb{Q}(Z^d)/\mathbb{Q}}(\chi(u^d)Z^{-dl});\quad l=0,1,\ldots,k-1. \tag{2}$$

In particular, we have:

Theorem 1. *Suppose that u is an element of $\mathbb{Z}G$ satisfying $u^k=1$, $k\geqslant 1$. Let Z be a primitive kth root of unity. Then for every integer l and every character χ of G, the number*

$$\frac{1}{k}\sum_{d|k}\mathrm{Tr}_{\mathbb{Q}(Z^d)/\mathbb{Q}}(\chi(u^d)Z^{-dl})$$

is a non-negative integer.

To obtain another consequence of (2), we notice that $\chi(u^d)$ depends only on the numbers $\varepsilon_C(u^d)$ and not on u as such. Therefore we obtain the following criterion for rational conjugacy of torsion units of $\mathbb{Z}G$.

Theorem 2. *Let u and v be units in $\mathbb{Z}G$ with $u^k=1=v^k$, $k\geqslant 1$. Then u and v are rationally conjugate if and only if $\varepsilon_C(u^d)=\varepsilon_C(v^d)$ for every divisor d of k and every conjugacy class C of G.*

In particular, for units of prime order we have

COROLLARY 1

Two units u and v in $\mathbb{Z}G$ satisfying $u^p=1=v^p$, p a prime, are rationally conjugate if and only if they have the same partial augmentations.

These results enable us to check the Zassenhaus conjecture in A_5.

2. Torsion units in A_5

Theorem 3. *Every normalized torsion unit in $\mathbb{Z}A_5$ is rationally conjugate to a group element.*

Proof. We denote by C_1, C_2, C_3, C_4 and C_5 the conjugacy classes in A_5 of 1, $S=(1\ 2)(3\ 4)$, $T=(1\ 2\ 3)$, $V=(1\ 2\ \ 3\ 4\ 5)$ and $V^2=(1\ 3\ 5\ 2\ 4)$ respectively. The character table for A_5 is reproduced below [2, p. 319]:

	χ_1	χ_2	χ_3'	χ_4	χ_5
C_1	1	3	3	4	5
C_2	1	-1	-1	0	1
C_3	1	0	0	1	-1
C_4	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	0
C_5	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1	0

Let $u = \sum u(g)g$ be a normalized torsion unit in $\mathbb{Z}A_5$ of order $k > 1$, and let

$$\nu_i = \varepsilon_{C_i}(u), \quad i = 1, 2, \ldots, 5$$

be the partial augmentations of u. By Corollary 1.3 on page 45 of [5], we have

$$\nu_1 = 0; \tag{3}$$

since u is of augmentation 1,

$$\nu_2 + \nu_3 + \nu_4 + \nu_5 = 1. \tag{3'}$$

For any character χ of A_5 of degree n,

$$\chi(u) = \nu_2\chi(S) + \nu_3\chi(T) + \nu_4\chi(V) + \nu_5\chi(V^2). \tag{4}$$

The possible values of k are divisors of 60 [1]. To prove the above theorem we need to show that a normalized unit of order 2, 3 or 5 is rationally conjugate to a group element, and that there is no normalized unit of order 4, 6, 10 or 15.

By Theorem 2.7 of [3] we have

$$\left\{\begin{array}{ll} \nu_3 = \nu_4 = \nu_5 = 0,\ \nu_2 = 1, & \text{when } k = 2 \text{ or } 4 \\ \nu_2 = \nu_4 = \nu_5 = 0,\ \nu_3 = 1, & \text{when } k = 3 \\ \nu_2 = \nu_3 = 0 & \text{when } k = 5 \\ \nu_4 = \nu_5 = 0 & \text{when } k = 6 \\ \nu_3 = 0 & \text{when } k = 10 \\ \nu_2 = 0 & \text{when } k = 15. \end{array}\right\} \tag{5}$$

When $k = 2$ or 3, the partial augmentations of u are the same as those of S or T respectively; hence by Corollary 1, u is rationally conjugate to S or T. When $k = 5$, we have by (2), with $Z = \exp(2\pi i/5)$, $K = \mathbb{Q}(Z)$

$$\mu_l(u; \chi_3) = \tfrac{1}{5}[3 + \mathrm{Tr}_{K/\mathbb{Q}}(Z^{-l}(\nu_4\chi_3(V) + \nu_5\chi_3(V^2)))]$$
$$= \begin{cases} \nu_4 & \text{if } l = 1 \\ \nu_5 & \text{if } l = 2. \end{cases}$$

As the integers $\mu_l(u,\chi_3)$ are non-negative, the same is true of ν_4 and ν_5; since $\nu_4+\nu_5=1$, we have

$$\nu_4=1,\quad \nu_5=0 \quad \text{or} \quad \nu_4=0,\quad \nu_5=1.$$

Thus the partial augmentations of u are the same as those of V or V^2; hence u is rationally conjugate either to V or V^2.

Since A_5 has no element of order 4, we see by Theorem 2.1 on page 177 of [5] that there are no normalized units of order 4 in $\mathbb{Z}A_5$. Thus it only remains to prove that k cannot be 6, 10 or 15.

Let $k=6$, ω a primitive cube root of unity, and $\mathrm{Z}=-\omega$. We have, by the foregoing results, for any character χ of degree n,

$$\mu_l(u;\chi)=\tfrac{1}{6}[n+\chi(S)\mathrm{Z}^{-3l}+\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\chi(T)\mathrm{Z}^{-2l}+\chi(u)\mathrm{Z}^{-l})].$$

It is clear from the character table of A_5 that $\chi(S)$, $\chi(T)$ and hence $\chi(u)=\nu_2\chi(S)+\nu_3\chi(T)$ are integers. Thus

$$\mu_l(u,\chi)=\tfrac{1}{6}[n+(-1)^l\chi(S)+\chi(T)\,\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}\omega^l+(-1)^l\chi(u)\,\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}\omega^{-l}]. \tag{6}$$

Taking $\chi=\chi_3$, we obtain

$$\mu_l(u;\chi_3)=\tfrac{1}{6}[3+(-1)^{l+1}+(-1)^{l+1}\nu_2\,\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}\omega^{-l}].$$

These being non-negative integers, we obtain on taking $l=0$, 1 and 2 that $\nu_2=-2$ and hence $\nu_3=3$. Now take $\chi=\chi_5$ in (6) to obtain

$$\mu_l(u;\chi_5)=\tfrac{1}{6}[5+(-1)^l-\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\omega^l+5(-1)^l\omega^{-l})]$$

giving $\mu_0(u,\chi_5)=-1$ which is impossible. Thus there are no units of order 6.

Now let $k=10$; then u^2 is rationally conjugate to either V or V^2. Replacing u by u^3, if necessary, we may assume that u^2 is rationally conjugate to V. Let $\zeta=\exp(2\pi i/5)$ and $\mathrm{Z}=-\zeta$. We have for a character χ of degree n

$$\mu_l(u,\chi)=\tfrac{1}{10}[n+(-1)^l\chi(S)+\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{3l}\chi(V))+\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}((-1)^l\zeta^{4l}\chi(u))].$$

Taking $\chi=\chi_5$, we find that for every l

$$\tfrac{1}{10}[5+(-1)^l+(-1)^l\nu_2\,\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{4l})]$$

is a non-negative integer. One easily checks on taking $l=0,1$ and 5 that this is impossible.

Finally let $k=15$; then u^3 is rationally conjugate to V or V^2. Replacing u by u^2, if necessary, we may assume that u^3 is conjugate to V. Let $\zeta=\exp(2\pi i/5)$, $\omega=\exp(2\pi i/3)$ and $\mathrm{Z}=\omega\zeta$.

We have, for a character χ of degree n

$$\mu_l(u;\chi)=\tfrac{1}{15}[n+\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\chi(T)\omega^l)+\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(V)\zeta^{2l})+\mathrm{Tr}_{\mathbb{Q}(\mathrm{Z})/\mathbb{Q}}(\chi(u)\mathrm{Z}^{-l})].$$

Taking $\chi=\chi_5$ and $l=0$, 3 we see that

$$\frac{3-8\nu_3}{15} \quad \text{and} \quad \frac{3+2\nu_3}{15}$$

are non-negative integers. This being obviously impossible, we conclude that there is no normalized unit of order 15. This completes the proof of the theorem.

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