A NOTE ON A RESULT OF MAHLER'S

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In a recent paper, Mahler [2] proved that for any algebraic number field K of degree n and discriminant d there exists a constant C depending only on n and d such that for any ceiling $\lambda(\mathfrak{p})$ of K there exists a basis $\alpha_1, \cdots, \alpha_n$ of the corresponding ideal $\mathfrak{a}_\lambda$ such that

$$C^{-(n-1)}\lambda(\mathfrak{q}) \leqq |\alpha_k|_\mathfrak{q} \leqq C\lambda(\mathfrak{q}) \quad \text{for all } \mathfrak{q},$$
$$C^{-n}\lambda(\mathfrak{r}) \leqq |\alpha_k|_\mathfrak{r} \leqq \lambda(\mathfrak{r}) \quad \text{for all } \mathfrak{r};$$
$$(k = 1, 2, \cdots, n).$$

Moreover, if $p_\mathfrak{r}$ denotes the rational prime below $\mathfrak{r}$, then

$$|\alpha_k|_\mathfrak{r} = \lambda(\mathfrak{r})$$

for all $\mathfrak{r}$ for which $p_\mathfrak{r} > C^{n^2}$.

For notations and definitions we refer the reader to the above-mentioned paper of Mahler.

The object of this note is to prove the same result with a constant C, which differs from Mahler's constant by a factor which approaches zero exponentially as n approaches infinity. We remark that for small n, Mahler's constant is better than ours.

Let $\beta_1, \cdots, \beta_n$ be a basis of $\mathfrak{a}_\lambda$ and let

$$F(x) = F(x_1, \cdots, x_n) = \sum_\mathfrak{q} \lambda(\mathfrak{q})^{-1}|\beta_1 x_1 + \cdots + \beta_n x_n|_\mathfrak{q}.$$

Then $F(x)$ is a symmetric convex distance function and the volume V of the convex body

$$F(x) \leqq 1$$

is given by the formula:

$$V = \frac{2^n \pi^{r_2}}{n!|\sqrt{d}|}.$$

We now use a theorem of Mahler [1] and Hermann Weyl [3] to obtain a unimodular matrix (g_{ij}) such that

$$\prod_{h=1}^{n} F(g_{h1}, \dots, g_{hn}) \leqq \frac{\gamma_n}{V}$$

where γ_n is a certain constant depending only on n. If we write

$$(1) \qquad \begin{cases} \alpha_1 = g_{11}\beta_1 + \cdots + g_{1n}\beta_n \\ \alpha_n = g_{n1}\beta_1 + \cdots + g_{nn}\beta_n \end{cases}$$

and

$$(2) \qquad m_h = F(g_{h1}, \cdots, g_{hn}) = \sum_{\mathfrak{q}} \lambda(\mathfrak{q})^{-1} |\alpha_h|_{\mathfrak{q}}$$

then $\alpha_1, \cdots, \alpha_n$ is a basis of $\mathfrak{a}_\lambda$ and

$$(3) \qquad \prod_{h=1}^{n} m_h \leqq \gamma_n V^{-1} = n!\,\gamma_n 2^{-n} \pi^{-r_2} |\sqrt{d}|.$$

By the inequality on arithmetic and geometric means we get

$$\frac{1}{n} m_h = \frac{1}{n} \sum_{\mathfrak{q}} \lambda(\mathfrak{q})^{-1} |\alpha_h|_{\mathfrak{q}} \geqq [4^{-r_2} \prod_{\mathfrak{q}} \lambda(\mathfrak{q})^{-n(\mathfrak{q})} \cdot |N(\alpha_h)|]^{1/n}$$

i.e.,

$$N(\mathfrak{a}_\lambda)^{-1} |N(\alpha_h)| \leqq 4^{r_2} \left(\frac{m_h}{n}\right)^n.$$

Since $\alpha_h \in \mathfrak{a}_\lambda$, therefore $N(\mathfrak{a}_\lambda) \leqq |N(\alpha_h)|$ and the above inequality gives

$$(4) \qquad m_h \geqq n \cdot 4^{-r_2/n}.$$

Using (2), (3), and (4) we obtain

$$|\alpha_h|_{\mathfrak{q}} \leqq \lambda(\mathfrak{q}) m_h = \lambda(\mathfrak{q}) \frac{\prod_{i=1}^{n} m_i}{\prod_{i \neq h} m_i} \leqq \lambda(\mathfrak{q}) C$$

with

$$C = n!\,\gamma_n n^{1-n} \pi^{-r_2} 2^{-(r_1+(2r_2/n))} |\sqrt{d}|.$$

The remaining assertions of Mahler's theorem with this C now follow from his lemmas 1 and 2.

We remark finally that the ratio of Mahler's constant to the constant obtained here is

$$\frac{2^{-r_2/n} n^{\frac{1}{2}(1-n)} \gamma_n \pi^{-\frac{1}{2}n} \Gamma\left(\frac{n}{2}+1\right) |\sqrt{d}|}{n!\,\gamma_n n^{1-n} \pi^{-r_2} 2^{-(r_1+2r_2/n)} |\sqrt{d}|} \sim \frac{\sqrt{n}}{2\sqrt{2}} 2^{r_2/n} \left(\frac{4}{\pi}\right)^{\frac{1}{2}r_1} \left(\frac{e}{2}\right)^{\frac{1}{2}n}.$$

References

[1] K. Mahler, Acta Mathematica 68 (1937), 109–144.
[2] K. Mahler, *J. Austr. Math. Soc.* 4 (1964), 425–448.
[3] H. Weyl, *Proc. London Math. Soc.* 47 (1942), 268–289.

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