

# ON A PROBLEM RELATING TO A TETRAHEDRON.

BY K. S. K. IYENGAR

AND

K. V. IYENGAR.

(From the Department of Mathematics, University of Mysore, Bangalore.)

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1. IN two papers published in the previous number of these *Proceedings*, we solved the problem of determining the tetrahedron of maximum volume, the areas of whose faces are given. It was proved that the maximum is attained when the tetrahedron is orthogonal, and that—as one of us (K. S. K. Iyengar) succeeded in proving—in the particular case of three dimensions there is only one such tetrahedron. The analogous problem in any number of dimensions was also solved partially. By an interesting transformation of the same problem we now prove the above results of our earlier papers in a far more elegant way and we prove further that in the case of  $n$  dimensions also there is only one such orthogonal simplex, with assigned values for the  $(n - 1)$ -dimensional volumes of its face-simplexes. Finally, we solve by these methods an isoperimetrical problem of Steinitz.

2. Let  $OP_1P_2 \cdots P_n$  be a simplex in  $n$  dimensions and  $a_1 \cdots a_n, a_{n+1}$  the  $(n - 1)$ -dimensional volumes of the face-simplexes  $OP_2 \cdots P_n, OP_1P_3 \cdots P_n, OP_1P_2 \cdots P_{n-1}, P_1P_2 \cdots P_n$ . The problem is to determine the extrema for the volume of the simplex when the  $a_r$ 's are all given. We first prove that a real simplex exists if and only if the sum of any  $n$  of the  $a_r$ 's is greater than the remaining one. We also prove that the lower bound of the volume is zero and that the maximum is attained when the simplex is orthogonal, which is uniquely determined when the  $a_r$ 's are given. For proving these results the following transformation into another which is also an isoperimetric one is fundamental. The artifice employed is to obtain another simplex, the  $(n + 1)$  sides of which, forming a one-dimensional closed chain, are of lengths  $a_1, a_2 \cdots a_{n+1}$  and whose volume is a multiple of the  $(n - 1)$ th power of the volume of the original simplex. It will be seen that by this means the solution of our problem becomes almost immediate. Let  $\vec{OP}_1, \vec{OP}_2, \cdots \vec{OP}_n$  be represented by the vectors  $\vec{A}_1, \vec{A}_2, \cdots \vec{A}_n$  and let the  $(n - 1)$ -dimensional volumes of the faces

$OP_2P_3 \cdots P_n, OP_1P_3 \cdots P_n, \cdots, P_1P_2 \cdots P_n$  of the simplex  $OP_1P_2 \cdots P_n$  be  $a_1, a_2 \cdots a_n, a_{n+1}$  respectively. If we represent by  $\vec{A}_1'$  the vector product  $(\vec{A}_2\vec{A}_3 \cdots \vec{A}_n)$  and similarly the other vector products, it is easy to see that

$$(i) |\vec{A}_1'| = (n-1)! a_1, \cdots, |\vec{A}_n'| = (n-1)! a_n$$

(ii) if  $-\vec{A}'_{n+1} = \vec{A}_1' + \vec{A}_2' + \cdots + \vec{A}_n'$ , then the vector  $\vec{A}'_{n+1}$  will be perpendicular to the face  $P_1P_2 \cdots P_n$ , and  $|\vec{A}'_{n+1}| = (n-1) a_{n+1}$ . If therefore we draw  $OT_1, T_1T_2, \cdots, T_{n-1}T_n$  to represent the vectors  $\frac{1}{(n-1)!} \vec{A}_1', \frac{1}{(n-1)!} \vec{A}_2', \cdots, \frac{1}{(n-1)!} \vec{A}_n'$  respectively, then  $OT_n$  will represent the vector  $\frac{1}{(n-1)!} \vec{A}'_{n+1}$ , since  $\sum_{r=1}^{n+1} \vec{A}_r' = 0$ .

Hence the necessary and sufficient condition for the existence of simplexes having the given numbers  $a_1, a_2, \cdots, a_{n+1}$  for face-volumes is the existence of a closed linkage (polygon)  $OT_1T_2 \cdots T_nO$  in space such that  $|OT_1| = a_1, |T_1T_2| = a_2 \cdots |T_{n-1}T_n| = a_n$  and  $|OT_n| = a_{n+1}$ . But this means that the sum of any  $n$  of the numbers  $a_1, a_2, \cdots, a_{n+1}$  should be greater than the remaining one; or, if  $a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1}$  then the sole condition for the existence of simplexes with given  $(n-1)$ -dimensional volumes of the faces is  $a_1 + a_2 + \cdots + a_n > a_{n+1}$ .

3. Now the volume of the simplex  $OP_1P_2 \cdots P_n$  is

$$V = \frac{1}{n!} \|\vec{A}_1\vec{A}_2 \cdots \vec{A}_n\|$$

and that of the simplex  $OT_1T_2 \cdots T_n$  is

$$\begin{aligned} \bar{V} &= \frac{1}{n!} \left\| \frac{\vec{A}_1'}{(n-1)!}, \frac{\vec{A}_2'}{(n-1)!}, \cdots, \frac{\vec{A}_n'}{(n-1)!} \right\| \\ \therefore \bar{V} &= \frac{1}{n! \{(n-1)!\}^n} \left\| \vec{A}_1' \vec{A}_2' \cdots \vec{A}_n' \right\| \\ &= \frac{1}{n! \{(n-1)!\}^n} \left\| \vec{A}_1 \vec{A}_2 \cdots \vec{A}_n \right\|^{n-1} \\ &= \frac{1}{n! \{(n-1)!\}^n} \left\{ (n!) V \right\}^{n-1} \end{aligned}$$

whence

$$V = \frac{\{(n-1)!\}^2}{n^{n-2}} \bar{V},$$

so that the problem of finding the bounds of the volumes of all simplexes having given face-volumes  $a_1, a_2, \cdots, a_{n+1}$  such as  $OP_1P_2 \cdots P_n$  is

completely equivalent to the problem of finding the bounds of the volumes of all simplexes  $OT_1T_2 \cdots T_n$  such that  $|OT_1| = a_1$ ,  $(T_1T_2) = a_2 \cdots |T_{n-1}T_n| = a_n$ ,  $|OT_n| = a_{n+1}$ .

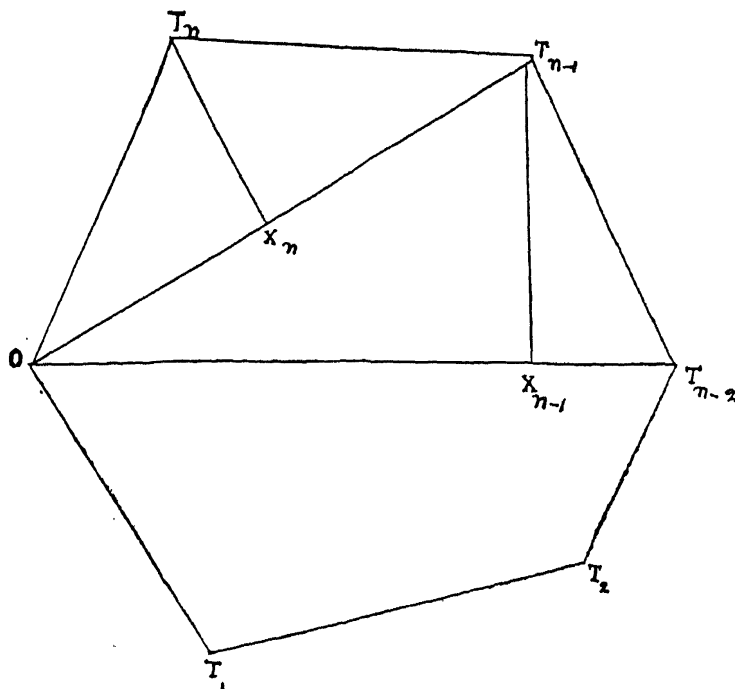


FIG. 1.

Let  $\phi_n$  be the angle between  $T_n\bar{X}_n$  and the perpendicular to the space  $OT_1T_2 \cdots T_{n-1}$  in the  $n$ -space, where  $T_n\bar{X}_n$  is the perpendicular to  $OT_{n-1}$  in the plane  $OT_{n-1}T_n$ .

Similarly let  $\phi_{n-1}$  be the angle between  $T_{n-1}\bar{X}_{n-1}$  and the perpendicular to the space  $OT_1T_2 \cdots T_{n-2}$  in the  $(n-1)$ -space defined by  $OT_1T_2 \cdots T_{n-1}$ , where  $T_{n-1}\bar{X}_{n-1}$  is defined similarly to  $T_n\bar{X}_n$ ; and so on. Then the volume of the simplex is given by

$\frac{1}{n} \cdot T_n\bar{X}_n \cdot (n-1)$ -dimensional volume of  $OT_1T_2 \cdots T_{n-1} \cdot \cos \phi_n$  and by repeated application of this result we find for the volume the expression

$$\frac{1}{n!} \cdot T_n\bar{X}_n \cdot T_{n-1}\bar{X}_{n-1} \cdots T_2\bar{X}_2 \cdot OT_1 \cos \phi_n \cos \phi_{n-1} \cdots \cos \phi_3.$$

Let us denote  $OT_2$  by  $x_1$ ,  $OT_3$  by  $x_2 \cdots OT_{n-1}$  by  $x_{n-2}$  and the area of any triangle with sides  $a, b, c$  by the symbol  $(a, b, c)$ .

Then

$$T_n\bar{X}_n = \frac{2(a_{n+1}, a_n, x_{n-2})}{x_{n-2}}$$

$$T_{n-1}\bar{X}_{n-1} = \frac{2(x_{n-2}, x_{n-3}, a_{n-1})}{x_{n-3}}$$

.....

$$T_2\bar{X}_2 = \frac{2(a_1 a_2 x_1)}{OT_1}.$$

The volume of the simplex  $OT_1T_2 \cdots T_n$  is therefore given by

$$\frac{2^{n-1} (a_{n+1}, a_n, x_{n-2}) (x_{n-2}, x_{n-3}, a_{n-1}) \cdots (a_1, a_2, x_1)}{n! x_1 x_2 \cdots x_{n-2}} \times \cos \phi_n \cos \phi_{n-1} \cdots \cos \phi_3.$$

We see at once that the lower bound of the volumes of all such simplexes is zero, because any one of the angles  $\phi_3, \phi_4 \cdots \phi_n$  or all of them can be made as nearly equal to  $\frac{\pi}{2}$  as we like. We also see at once that in the maximum simplex, all the angles  $\phi_n, \phi_{n-1} \cdots \phi_3$  are zero. In other words, going back to the vector notation, since  $OT_1 = \vec{A}_1'$ , etc., we see that

$$[\vec{A}_1' \vec{A}_2'] \text{ and } [\vec{A}_3', \vec{A}_1' + \vec{A}_2']$$

are perpendicular to each other, the vector products being defined in the space given by  $\vec{A}_1', \vec{A}_2', \vec{A}_3'$ .

Instead of taking the simplex in the order  $OT_1T_2 \cdots T_n$  where  $OT_1 = \vec{A}_1', OT_2 = \vec{A}_2'$ , etc., we could form another simplex  $OT_1'T_2' \cdots T_n'$  such that  $OT_1' = \vec{A}_r', OT_2' = \vec{A}_s'$ , etc., and it is easy to see that

$$\text{Volume } (OT_1' \cdots T_n') = \text{Volume } (OT_1 \cdots T_n)$$

and we get for the maximum simplex corresponding conditions. Therefore for the maximum simplex, all the conditions are given by the following:—

The vectors  $[\vec{A}_r' \vec{A}_s']$ ,  $[\vec{A}_p', \vec{A}_r' + \vec{A}_s']$  should be perpendicular to each other, the vector-products being defined in the space given by  $\vec{A}_r', \vec{A}_s', \vec{A}_p'$ .

In the first papers of ours published in the previous number of these *Proceedings* we showed that for the maximum simplex with given faces, opposite edges are mutually perpendicular. This follows at once from the above if we go back from the vectors  $\vec{A}_r'$  to the vectors  $\vec{A}_r$ .

4. We shall now prove that the maximum simplex  $OT_1T_2 \cdots T_n$  with given sides  $OT_1 = a_1$ , etc., is unique.

Sufficient amount of complexity is introduced if we restrict ourselves to 5 dimensions, the proof for  $n$  dimensions being the same as for 5.

Let the maximum simplex be  $OT_1T_2 \cdots T_5$  where  $OT_1 = a_1, OT_2 = a_5, \cdots OT_5 = a_6$ .

Let  $OT_2 = x, OT_3 = y, OT_4 = z$ . Then the volume of the simplex is

$$\bar{V} = K \cdot \frac{(a_1 a_2 x) (x a_3 y) (y a_4 z) (z a_5 a_6)}{xyz},$$

where  $K$  is a constant.

$$\bar{V}^2 = K^2 \cdot \frac{\{(a_1 + a_2)^2 - x^2\} \{x^2 - (a_1 - a_2)^2\} \cdots \{ \quad \}}{x^2 y^2 z^2}.$$

Put  $x^2 = u, y^2 = v, z^2 = w$  and designate  $(a_1 a_2 x)^2$  by  $[a_1 a_2 u]$ , etc. We shall prove that  $\log \bar{V}$  is a convex function of  $(u, v, w)$ .

Now  $2 \log V = \text{constant} + \log [a_1, a_2, u] + \log \{[ua_3 v]/u\} + \log \{[va_4 w]/v\} + \log \{[wa_5 a_6]/w\}$ .

It is easy to prove that if  $f(u, v, w)$  is convex so is also  $\log f(u, v, w)$ . We now prove that  $[a_1 a_2 u]$  is convex; for

$$[a_1 a_2 u] = K \{(a_1 + a_2)^2 - u\} \{u - (a_1 - a_2)^2\}, [K = \frac{1}{18}] \text{ and so } \frac{d^2}{du^2} [a_1, a_2, u] < 0.$$

$$\begin{aligned} \text{Similarly } [wa_5 a_6]/w &= Kw^{-1} \{(a_5 + a_6)^2 - w\} \{w - (a_5 - a_6)^2\} \\ &= A + Bw - \frac{K(a_5 + a_6)^2(a_5 - a_6)^2}{w} \end{aligned}$$

$$\therefore \frac{d^2}{dw^2} \{[wa_5 a_6] w^{-1}\} < 0.$$

Consider now  $f = u^{-1} [ua_3 v]$

$$\begin{aligned} &= \frac{K \{u - (a_3 - \sqrt{v})^2\} \{(a_3 + \sqrt{v})^2 - u\}}{u} \\ &= \frac{[-u^2 + u \{(a_3 - \sqrt{v})^2 + (a_3 + \sqrt{v})^2\} - (a_3^2 - v)^2] K}{u} \\ &= \left[ -u + 2(a_3^2 + v) - \frac{(a_3^2 - v)^2}{u} \right] K \end{aligned}$$

$$\therefore \frac{1}{K} \frac{\partial^2 f}{\partial u^2} = -\frac{2(a_3^2 - v)^2}{u^3}; \quad \frac{1}{K} \frac{\partial^2 f}{\partial v^2} = -\frac{2}{u}; \quad \frac{1}{K} \frac{\partial^2 f}{\partial u \partial v} = -\frac{2(a_3^2 - v)}{u^2}$$

so that

$$\frac{\partial^2 f}{\partial u^2} \frac{\partial^2 f}{\partial v^2} = \left( \frac{\partial^2 f}{\partial u \partial v} \right)^2.$$

The quadratic form formed by the second partial differential coefficients of the logarithm of each of the functions is therefore negative definite and hence  $\log V$  is a convex function of  $u, v, w$ . Now in the maximum simplex  $\frac{\partial \bar{V}}{\partial u} = \frac{\partial \bar{V}}{\partial v} = \frac{\partial \bar{V}}{\partial w} = 0$  (the angles  $\phi_5, \phi_4, \phi_3$  being zero as before). The range of admissible values of  $u, v, w$  forms a continuum  $K$ . If there are two maxima in the continuum at  $P_1$  and  $P_2$  we can join  $P_1 P_2$  by a curve  $L$ , lying in  $K$  and in that curve there must be a minimum of  $\log V$  for variations along  $L$ . But that is impossible since the quadratic form of the second order differential coefficients of  $\log V$  is negative definite.

This therefore proves that the maximum simplex is unique. Going back from the simplex  $OT_1T_2 \cdots T_n$  to the simplex  $OP_1P_2 \cdots P_n$  we can state the theorem as follows:—

(1) All the stationary simplexes  $OP_1P_2 \cdots P_n$  having given volumes [( $n - 1$ )-dimensional] for its faces are orthogonal (*vide* also K. S. K. Iyengar's paper on this in the preceding number of the *Proceedings*).

(2) There is only one such orthogonal simplex which is the maximum.

5. We will now shew that, by this method of ours, we can prove a result of Steinitz that if the sum of the four areas of a tetrahedron be given then the tetrahedron has maximum volume when all the faces are equal and it is regular (*incidentally orthogonal*). This result could be easily generalised to  $n$  dimensions. As the method is the same for all dimensions we will give here a proof for 3 dimensions.

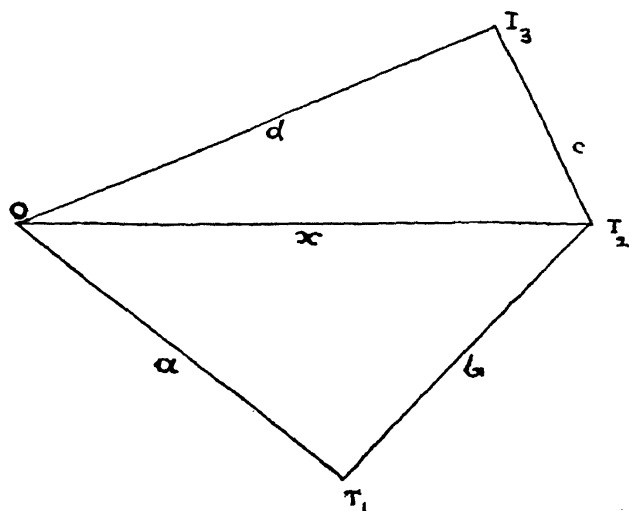


FIG. 2.

Adopting our old notation, if the volume of the tetrahedron  $OP_1P_2P_3$  with given areas  $a, b, c, d$  for its faces be  $V$ , then  $V^2 = \frac{4}{3} \bar{V}$ ,

where  $\bar{V} =$  volume of  $(OT_1T_2T_3)$  in our usual notation.

Let  $\bar{V}$  be the volume of the maximum simplex with sides  $a, b, c, d$ .

Then

$$\begin{aligned} \bar{V} &= K. \frac{(\text{area } OT_1T_2) \cdot (\text{area } OT_2T_3)}{OT_2} \\ &= K. \frac{(a, b, x) (c, d, x)}{x}. \end{aligned}$$

[Note that  $x$  is a function of  $a, b, c, d$  and that the angle between the planes  $OT_1T_2, OT_2T_3$  is  $\frac{\pi}{2}$ .]

Let us now deform the triangle  $OT_1T_2$  to another  $OT_1'T_2$  such that

$$OT_1' = T_1'T_2 \text{ and } OT_1' + T_1'T_2 = a + b,$$

( $T_1'$  being in the same plane  $OTT_2$ ).

Then the volume of the simplex  $OT_1'T_2T_3$

$$= K. \frac{\left(\frac{a+b}{2}, \frac{a+b}{2}, x\right)(c, d, x)}{x}$$

$$\geq \text{Volume } OT_1T_2T_3$$

since

$$\left(\frac{a+b}{2}, \frac{a+b}{2}, x\right) \geq (a, b, x).$$

It therefore follows at once that of all the tetrahedra  $OT_1T_2T_3$  such that  $OT_1 + T_1T_2 + T_2T_3 + T_3O = 4a$ , the one for which

(i)  $OT_1 = T_1T_2 = \dots = a$ ,

(ii) the angles between the planes  $(OT_1T_2, OT_3T_2)$  and also the angles between the planes  $(OT_1T_3, T_1T_2T_3)$  are right angles

is the maximum. Going back to the original tetrahedron  $OP_1P_2P_3$  the maximum tetrahedron is readily seen to be a regular one.