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Haldane Exclusion Statistics and the Boltzmann Equation

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We generalize the collision term in the one-dimensional Boltzmann–Nordheim transport equation for quasiparticles that obey the Haldane exclusion statistics. For the equilibrium situation, this leads to the "golden rule" factor for quantum transitions. As an application of this, we calculate the density response function of a one-dimensional electron gas in a periodic potential, assuming that the particle–hole excitations are quasiparticles obeying the new statistics. We also calculate the relaxation time of a nuclear spin in a metal using the modified golden rule.

KEY WORDS: Haldane exclusion statistics; Boltzmann equation; Fermi golden rule; density response function.

1. INTRODUCTION

Recently, Haldane has proposed a new exclusion statistics for confined quasiparticles that interpolates between the Fermi and Bose quantum behaviors and gives rise to partial blocking of a singly occupied state.(1) There has been considerable theoretical activity concerning the study of statistical and thermodynamic properties of systems of particles obeying the generalized exclusion statistics. Examples of quasiparticles obeying this new statistics include spinons in an antiferromagnetic spin chain with inverse–square exchange interaction,(2) and two-dimensional anyons restricted to the lowest Landau level in a strong magnetic field.(3–5) Other

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examples are the Calogero–Sutherland model[6,7] in one dimension and particles interacting with a two-body short-range potential in two dimensions in the thermodynamic limit.[8] Haldane[9] has analyzed a Heisenberg spin chain in one dimension with inverse-square coupling, and shown that under certain conditions the quasiparticle excitations obey a particularly simple form of the generalized statistics. Despite all these theoretical studies, it is not entirely clear under what conditions a physical or chemical system will experimentally manifest this generalized statistics. The best evidence so far comes from recent neutron inelastic scattering experiment[10] on the compound $KCuF_3$, which is a one-dimensional Heisenberg antiferromagnet above 40 K. The observed inelastic scattering is best fitted by spinon excitations in a spin chain whose interactions fall off as the inverse square of the lattice distance.[11] The dynamic spin correlation function for such a system has been calculated by Haldane and Zirnbauer.[12]

The present paper addresses a question of current interest, namely how to calculate transport properties of particles obeying the generalized exclusion statistics. Specifically, how does the Boltzmann transport equation get modified in such a situation? It may be recalled that the collision term in the Boltzmann equation was modified by Nordheim[13] to incorporate the effect of quantum statistics. Account was taken of the Pauli blocking of the final state in binary collision events for particles obeying Fermi statistics. On the other hand, an enhancement factor was introduced (as in the Einstein coefficient of induced emission; see, e.g., ref. 14) in the final-state occupancy for Bose statistics. The suppression and enhancement factors $(1-f)$ and $(1+f)$ appropriate for fermions and bosons, respectively, are routinely used in many-body calculations. What are the corresponding factors for the systems of particles obeying the new statistics? We present the answer to this question. Future calculations of transport properties of such systems will find these results relevant. Besides generalizing the Boltzmann–Nordheim equation, we have also generalized the Fermi golden rule.

The plan of this paper is as follows. In Section 2, we examine changes in the structure of the collision term in the Boltzmann equation due to the incorporation of the new statistics. This yields the "golden rule" factor for quantum transitions. As an application of this, we examine in Section 3 the response of a one-dimensional electron gas to an external periodic potential and calculate its density response function. Whereas the ground state of the electron gas is taken to be the conventional Fermi system, we assume that the particle–hole excitations are quasiparticles obeying the new statistics. The magnitude of the density response function is then shown to be appreciably enhanced, thereby lowering the Peierls transition temperature. In Section 4, we generalize the Fermi golden rule for quasiparticles obeying the new statistics and apply it to the calculation of a relaxation time of a nuclear spin in a metal. Both applications are schematic, with a view to exploring qualitative changes from the standard results.

2. GENERALIZED BOLTZMANN–NORDHEIM EQUATION

We now derive the Boltzmann equation (specifically, the collision term) for particles obeying Haldane's statistics. For the derivation of the collision term, we make the usual assumptions of (i) the dominance of binary collisions, (ii) the hypothesis of molecular chaos ("stosszahlansatz"), and (iii) the slow variation of the distribution function $f(x, p, t)$ over distances and times of the order of characteristic interaction lengths and durations, respectively. Recall that $f(x, p, t)\, dx\, dp/(2\pi)$ is the number of particles in the phase-space volume element $dx\, dp$ at time t. (We set $\hbar = 1$ throughout this paper.) We wish to derive an expression for the collision term C in the Boltzmann equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \mathscr{F}\frac{\partial f}{\partial p} = C \tag{1}$$

C is the rate of change of f caused by collisions. Collisions of the type $p'p'_1 \to pp_1$ tend to increase the population of particles having momentum p, while those of the type $pp_1 \to p'p'_1$ tend to decrease it. Let $w(p'p'_1 \to pp_1)\, dp\, dp_1$ be the transition probability per unit volume and per unit time that two particles having incoming momenta p' and p'_1 are scattered with outgoing momenta in the ranges $(p, p+dp)$ and (p_1, p_1+dp_1). Then according to the hypothesis of molecular chaos, for *classical* (Maxwell–Boltzmann, MB) particles the net increase in the number of particles in $dx\, dp$ due to collisions occurring during dt is[15]

$$\int [f'\, dp' \cdot f'_1\, dp'_1 \cdot w(p'p'_1 \to pp_1)\, dp\, dp_1\, dx\, dt$$

$$-f\, dp \cdot f_1\, dp_1 \cdot w(pp_1 \to p'p'_1)\, dp'\, dp'_1\, dx\, dt]$$

where $f' \equiv f(x, p', t)$, etc., and the integration is only over $dp'\, dp'_1\, dp_1$. Using the detailed-balance property $w(p'p'_1 \to pp_1) = w(pp_1 \to p'p'_1)$ and inserting a factor 1/2 to take into account the fact that a final state with momenta (p', p'_1) is indistinguishable from that with momenta (p'_1, p'), we get

$$C = \frac{1}{2}\int dp_1\, dp'\, dp'_1\, w(pp_1 \to p'p'_1)(f'f'_1 - ff_1) \tag{2}$$

For particles obeying *quantum* statistics, however, the above derivation needs to be modified because now the transition probability depends also on the occupancy of the final state. Thus, $(f'f'_1 - ff_1)$ in (2) should be replaced by $f'f'_1 F(f)\, F(f_1) - ff_1 F(f')\, F(f'_1)$, where F takes account of the effect of the quantum statistics on the accessibility of the final state. For example, in the case of Fermi–Dirac (FD) statistics, $F(f) = (1 - f)$ to incorporate Pauli blocking, and for Bose–Einstein (BE) statistics, $F(f) = (1 + f)$ to take care of Einstein enhancement. According to the H-theorem in statistical mechanics, the entropy production vanishes if and only if

$$f'f'_1 F(f)\, F(f_1) = ff_1 F(f')\, F(f'_1) \tag{3}$$

Solution of this equation provides the shape of the equilibrium distribution function f_{eq}. Conversely, as we proceed to show, if f_{eq} is known, (3) can be used to deduce the functional form $F(f)$. Rewriting (3) in the form

$$\frac{f}{F(f)} \cdot \frac{f_1}{F(f_1)} = \frac{f'}{F(f')} \cdot \frac{f'_1}{F(f'_1)}$$

taking logarithms of both sides, and using the fact that energy E is the appropriate summational invariant in the collision, we get

$$\ln[f/F(f)] = -a(E - b)$$

where a and b are constants to be identified with inverse temperature T and chemical potential μ, respectively. Thus, we get finally

$$F(f) = \exp[a(E - b)]\, f = \zeta(E)\, f, \qquad \zeta(E) \equiv \exp[(E - \mu)/T]$$

where we have set the Boltzmann constant to unity. For the MB, FD, and BE statistics, the equilibrium distribution functions f are ζ^{-1}, $1/(\zeta + 1)$, and $1/(\zeta - 1)$, respectively. Hence, the corresponding F's are $1, (1 - f)$, and $(1 + f)$ as expected.

Now, for particles obeying the Haldane statistics, the equilibrium distribution function, as shown by Wu,[5] is

$$f(E) = 1/[\omega(E) + \alpha] \tag{4}$$

where $\alpha (0 \leqslant \alpha \leqslant 1)$ is the statistical interpolation parameter ($\alpha = 0$ corresponds to bosons and $\alpha = 1$ to fermions) and $\omega(E)$ satisfies

$$\omega(E)^{\alpha}\, [1 + \omega(E)]^{1 - \alpha} = \exp[(E - \mu)/T] \tag{5}$$

It is straightforward to show that, in this case,

$$F(f) = (1 - \alpha f)^{\alpha} \left[1 + (1 - \alpha) f \right]^{1 - \alpha} \tag{6}$$

In particular, for semions ($\alpha = 1/2$) we get

$$\begin{aligned} f &= 1/(\tfrac{1}{4} + \zeta^2)^{1/2} \\ F(f) &= (1 - f/2)^{1/2} (1 + f/2)^{1/2} \end{aligned} \tag{7}$$

Thus, the Boltzmann equation (1) for particles obeying Haldane's statistics is given by

$$\begin{aligned} &\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \mathscr{F} \frac{\partial f}{\partial p} \\ &\quad = \frac{1}{2} \int dp_1\, dp'\, dp'_1\, w(pp_1 \to p'p'_1)[f'f'_1 F(f)\, F(f_1) - ff_1 F(f')\, F(f'_1)] \end{aligned}$$

where F is given by (6).

3. DENSITY RESPONSE FUNCTION

In calculating the density response function $\chi(Q)$ of the one-dimensional electron gas, we follow the treatment given by Kagoshima *et al.*[16] For a periodic potential $V(r) = \sum_Q V_Q e^{iQr}$, the charge distribution of the electrons in a length L is distorted by $\delta\rho(r) = (1/L) \sum_Q \rho_Q e^{iQr}$, and $\chi(Q)$ is just the linear response function, obtained from

$$\rho_Q = -V_Q \chi(Q)$$

by calculating $\delta\rho(r)$ perturbatively. Consider an electron in the initial state k, absorbing a phonon of wave number Q ($-Q$), reaching the final state $k + Q$ ($k - Q$). Then, using FD statistics, we obtain the usual expression for ρ_Q[16]

$$\rho_Q = \frac{V_Q}{N} \sum_k \left[\frac{f_k(1 - f_{k+Q})}{E_k - E_{k+Q}} + \frac{f_k(1 - f_{k-Q})}{E_k - E_{k-Q}} \right] \tag{8}$$

where we have neglected the phonon energy $\hbar\omega_Q$ in comparison to the electron energy $E_k = \hbar^2 k^2/2m$, and N is the number of electrons in length L. The dummy variable k in the second term may be replaced by $(k + Q)$, resulting in the cancellation of the bilinear terms in fs, and yielding

$$\chi(Q) = \frac{1}{N} \sum_k \frac{f_{k+Q} - f_k}{E_k - E_{k+Q}} \tag{9}$$

At temperature $T=0$, we have $f_k=1$ and $f_{k+Q}=0$. In one dimension, this results in a logarithmic singularity for transitions from $k=\pm k_F$ to $k=\mp k_F$ with $Q=\mp 2k_F$. This is a manifestation of the well-known Peierls instability. At nonzero temperature $T\equiv\beta^{-1}$, the singularity is replaced by a finite result, as may be verified by a straightforward calculation:

$$\chi(Q=2k_F)=\frac{D(E_F)}{2N}\int_0^{\varepsilon_B\beta/2}\frac{\tanh x}{x}\,dx \tag{10}$$

where $D(E_F)$ is the density of states at the Fermi level and the energy integration is carried out only in the range $|E-E_F|\leqslant\varepsilon_B\ll E_F$. We repeat this calculation for χ by replacing the factors $f(1-f)$ in (8) by the new factors $fF(f)$ derived in Section 2. There is of course, no *a priori* justification for assuming that these excitations obey the new statistics. Nevertheless, if they did, what would be the experimental signals in a one-dimensional system? The enhancement of $\chi(Q)$ near $Q=2k_F$ has a direct bearing on the estimation of the Peierls metal–insulator transition temperature, as well

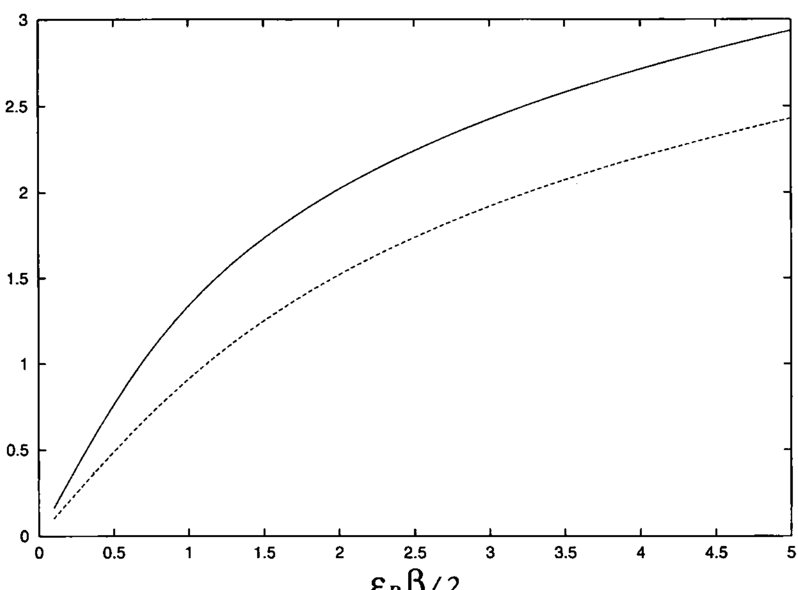

Fig. 1. Dashed and solid lines represent integrals occurring in (10) and (12), respectively, as a function of $\varepsilon_B\beta/2$, showing an enhancement of the density response function $\chi(Q=2k_F)$ at various temperatures as one goes from fermionic to semionic statistics.

as on the Kohn anomaly. It may therefore be a worthwhile exercise to reexamine $\chi(Q)$ with the modified golden rule.

It is simplest to illustrate the calculation with semions ($\alpha = 1/2$), where the equilibrium distribution is explicitly known [see (7)]. The expression in the square brackets in (8) is then replaced by

$$\left[\frac{f_k(1-\frac{1}{4}f_{k+Q}^2)^{1/2}}{E_k - E_{k+Q}} + \frac{f_k(1-\frac{1}{4}f_{k-Q}^2)^{1/2}}{E_k - E_{k-Q}}\right] \tag{11}$$

Note that on changing the dummy variable $k \to k+Q$ in the second term in (11), there is no cancellation of the bilinear terms in f as earlier. However, at $T=0$, the quasiparticle excitations near the Fermi surface have $f_k = 2, f_{k+Q} = 0$, and hence there is an overall enhancement of $\chi(Q)$ by a factor of 2, again with a logarithmic singularity at $Q = 2k_F$. At nonzero temperature, the semionic distribution function f given by (7) may be used to evaluate $\chi(Q)$, and one obtains after straightforward algebra

$$\chi(Q=2k_F) = \frac{D(E_F)}{2N} 2 \int_0^{\epsilon_B \beta/2} \frac{dx}{x} \frac{2(e^{2x} - e^{-2x})}{(1+4e^{-4x})^{1/2}(1+4e^{4x})^{1/2}} \tag{12}$$

which should be compared with (10). The integrals in (10) and (12) are plotted as a function of $\epsilon_B \beta/2$ in Fig. 1, and again show an enhancement of $\chi(Q)$, but by a factor somewhat greater than 2. Although there is an enhancement in the response function in our simple model, because its temperature dependence does not change, it may not be possible to detect this effect experimentally.

4. GENERALIZED FERMI GOLDEN RULE AND RELAXATION TIME

We now consider generalization of the well-known Fermi golden rule, for particles obeying the fractional exclusion statistics. Consider a two-particle scattering process $k + p \to k' + p'$ in one dimension. Let the two particles be distinct. We have in mind processes like the electron–nuclear interactions in metals. According to the Fermi golden rule, the transition probability per unit time for such a process is given by

$$w_{i \to f} = 2\pi \, |\langle i| \, H_{\text{int}} \, |f\rangle|^2 \, \delta(E_i - E_f)$$

where H_{int} is the interaction Hamiltonian and the delta function ensures energy conservation between the initial and the final states. Typically in electron–nuclear interactions (as in magnetic relaxation in solids) one can neglect the nuclear recoil to a very good approximation and therefore one can replace $\delta(E_i - E_f)$ by $\delta(E_k - E_{k'})$, k and k' being the initial and final momenta of the scattered electron. We work in this limit to keep the calculation simple and also because this is the relevant limit for many physical applications. The total transition probability per unit time at non-zero temperature is then given by[17]

$$W_{i \to f} = 2\pi \int dk\, dk'\, |\langle i|\, H_{\text{int}}\, |f\rangle|^2\, \delta(E_k - E_{k'})\, \rho(k)\, \rho(k')\, f(E_k)\, F(E_{k'})$$

where $\rho(k)$ denotes the density of states in the momentum space, $f(E)$ is as in (4), and $F(E)$ is what was denoted by $F(f)$ earlier; see (6). Performing one of the two integrations by using the delta function, we obtain

$$W_{i \to f} = 2\pi \int dE\, |\langle i|\, H_{\text{int}}\, |f\rangle|^2\, [\rho(E)]^2 f(E)\, F(E) \tag{13}$$

In order to evaluate the integral in (13), we note that

$$f(E)\, F(E) = [f(E)]^2\, e^{(E - E_{\text{F}})/T} \tag{14}$$

where E_{F} is the Fermi energy. Notice that in the low-temperature limit we have

$$f(E) = 1/\alpha, \qquad E \leqslant E_{\text{F}}$$

and zero otherwise. Thus at low enough temperatures, the system for an arbitrary α (except very close to the bosonic end) does exhibit a Fermi surface. We will be using this fact later in the calculation. Now for reasons that will become clear shortly, we wish to obtain an expression for the derivative df/dE. Using (4) and (5), we get

$$\frac{df}{dE} = -[f(E)]^2 \frac{d\omega}{dE} = -[f(E)]^2 \frac{\omega(1+\omega)}{T(\alpha+\omega)} \tag{15}$$

Substituting $[f(E)]^2$ from (15) into (14), we get

$$f(E)\, F(E) = -\frac{df}{dE} \frac{T(\alpha+\omega)}{\omega(1+\omega)}\, e^{(E - E_{\text{F}})/T} \tag{16}$$

We now consider the low-temperature ($T \ll E_{\text{F}}$) limit of the process under consideration. In this limit, $f(E) \approx \theta(E_{\text{F}} - E)/\alpha$, and hence $df/dE \approx -\delta(E_{\text{F}} - E)/\alpha$. Substituting this in (16), we find

$$f(E)\, F(E) \approx \frac{T[\alpha + \omega(E)]}{\alpha\omega(E)[1 + \omega(E)]}\, \delta(E_{\text{F}} - E) \tag{17}$$

Substituting (17) in (13) and performing the energy integration, we get the generalized Fermi golden rule:

$$W_{i\to f} = 2\pi\, |\langle i|\, H_{\text{int}}\, |f\rangle|^2\, [\rho(E_{\text{F}})]^2\, \frac{T[\alpha+\omega(E_{\text{F}})]}{\alpha\omega(E_{\text{F}})[1+\omega(E_{\text{F}})]}$$

In the special case when $\alpha = 1$ (fermions), we have, from (5), $\omega(E_{\text{F}}) = 1$, and we get

$$W^{\text{F}}_{i\to f} = 2\pi\, |\langle i|\, H_{\text{int}}\, |f\rangle|^2\, [\rho(E_{\text{F}})]^2\, T$$

We can therefore, in general, write

$$W^{\alpha}_{i\to f} = W^{\text{F}}_{i\to f}\, \frac{\alpha+\omega(E_{\text{F}})}{\alpha\omega(E_{\text{F}})[1+\omega(E_{\text{F}})]}$$

We may now apply this result to specific cases. A straightforward application is to the calculation of the relaxation time of a nuclear spin in a metal. The relaxation time for arbitrary α is then given by[17]

$$\frac{1}{\tau}(\alpha) = \frac{1}{\tau}(\text{fermions})\, \frac{\alpha+\omega(E_{\text{F}})}{\alpha\omega(E_{\text{F}})[1+\omega(E_{\text{F}})]}$$

Thus the change due to fractional statistics is simply given by a multiplicative factor which depends on α. In particular, at $\alpha = 1/2$, this multiplicative factor can be explicitly calculated, and we get

$$\frac{1}{\tau}(\alpha) = \sqrt{5}\, \frac{1}{\tau}(\text{fermions})$$

Notice that in deriving the generalized Fermi golden rule we have made a number of simplifying assumptions. Real systems are likely to be more complicated. Nevertheless, the above derivation probably indicates the correct direction in which the transition rates move when fractional-exclusion-statistics particles are involved.

In summary, we have generalized the Boltzmann–Nordheim equation and the Fermi golden rule for quasiparticles obeying the Haldane exclusion statistics. As two simplified applications of these results, we have calculated the density response function and the relaxation time of a one-dimensional gas obeying the new statistics. Although the applications presented here are simple and exploratory, we believe that they are the first attempts in this area to make a connection of theory to the consideration of observable effects. Moreover, this could be attempted only because the transport equation was nontrivially modified.

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REFERENCES

1. F. D. M. Haldane, *Phys. Rev. Lett.* **67**:937 (1991).
2. F. D. M. Haldane, *Phys. Rev. Lett.* **60**:635 (1988); B. S. Shastry, *Phys. Rev. Lett.* **60**:639 (1988).
3. M. V. N. Murthy and R. Shankar, *Phys. Rev. Lett.* **72**:3629 (1994).
4. A. Dasnières de Veigy and S. Ouvry, *Phys. Rev. Lett.* **72**:600 (1994).
5. Y.-S. Wu, *Phys. Rev. Lett.* **73**:922 (1994).
6. F. Calogero, *J. Math. Phys.* **10**:2197 (1969).
7. B. Sutherland, *J. Math. Phys.* **12**:246, 251 (1971); *Phys. Rev. A* **4**:2019 (1971).
8. R. K. Bhaduri, M. V. N. Murthy, and M. K. Srivastava, McMaster University preprint (July 1995) cond-mat/9510085.
9. F. D. M. Haldane, *Phys. Rev. Lett.* **66**:1529 (1991).
10. R. A. Cowley, D. A. Tennant, S. E. Nagler, and T. Perring, *J. Magnetism Magnetic Materials* **140–144**:1651 (1995).
11. G. Mueller, H. Thomas, H. Beck, and J. C. Bonner, *Phys. Rev. B* **24**:1429 (1981).
12. F. D. M. Haldane and M. R. Zirnbauer, *Phys. Rev. Lett.* **71**:4055 (1993).
13. L. W. Nordheim, *Proc. R. Soc. Lond. A* **199**:689 (1928).
14. J. J. Sakurai, *Advanced Quantum Mechanics* (Benjamin, Menlo Park, California, 1984), p. 38.
15. S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).
16. S. Kagoshima, H. Nagasawa, and T. Sambongi, *One-Dimensional Conductors* (Springer-Verlag, Berlin, 1988), Chapter 2.
17. A. Abragam, *The Principles of Nuclear Magnetism* (Oxford, London, 1962), p. 358.

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