Gravity and the Thermodynamics of Horizons

T. Padmanabhan

IUCAA, Post Bag 4, Ganeshkhind, Pune 411 007, India.

Abstract

Spacetimes with horizons show a resemblance to thermodynamic systems and it is possible to associate the notions of temperature and entropy with them. Several aspects of this connection are reviewed in a manner appropriate for broad readership. The approach uses two essential principles: (a) the physical theories must be formulated for each observer entirely in terms of variables any given observer can access and (b) consistent formulation of quantum field theory requires analytic continuation to the complex plane. These two principles, when used together in spacetimes with horizons, are powerful enough to provide several results in a unified manner. Since spacetimes with horizons have a generic behaviour under analytic continuation, standard results of quantum field theory in curved spacetimes with horizons can be obtained directly (Sections III to VII). The requirements (a) and (b) also put strong constraints on the action principle describing the gravity and, in fact, one can obtain the Einstein-Hilbert action from the thermodynamic considerations (Section VIII). The review emphasises the thermodynamic aspects of horizons, which could be obtained from general principles and is expected to remain valid, independent of the microscopic description ('statistical mechanics') of horizons.

We combine probabilities by multiplying, but we combine the actions ... by adding; ...... since the logarithm of the probability is necessarily negative, we may identify action provisionally with minus the logarithm of the statistical probability of the state...

<div align="right">Eddington (1920) [1]</div>

The mathematicians can go beyond this Schwarzschild radius and get inside, but I would maintain that this inside region is not physical space, ... and should not be taken into account in any physical theory.

<div align="right">Dirac (1962) [2]</div>

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Email address: nabhan@iucaa.ernet.in (T. Padmanabhan).
URL: http://www.iucaa.ernet.in/ paddy (T. Padmanabhan).

1   Introduction

The simplest solution to Einstein's equations in general relativity — the Schwarzschild solution — exhibits a singular behaviour when expressed in the most natural coordinate system which makes the symmetries of the solution obvious. One of the metric coefficients (g_{tt}) vanishes on a surface $\mathcal{H}$ of finite area while another (g_{rr}) diverges on the same surface. After some initial confusion, it was realized that these singularities are due to bad choice of coordinates. But the surface $\mathcal{H}$ brought in new physical features which have kept physicists active in the field for decades.

Detailed investigations in the 1970s showed that the Schwarzschild solution and its generalisations (with horizons) have an uncanny relationship with laws of thermodynamics. [A description of classical aspects of black hole thermodynamics can be found in [3,4] and [5]]. The work of Bekenstein moved these ideas forward [6,7,8] and one was initially led to a system with entropy but no temperature. This paradox was resolved when the black hole evaporation was discovered [9] and it was very soon realized that there is an intimate connection between horizons and temperature [10,11,12].

Later work over three decades has re-derived these results and extended them in many different directions but — unfortunately — without any further insight. It is probably fair to say that the "deep" relation between thermodynamics, quantum theory and general relativity, which was hoped for, is still elusive in the conventional approaches.

This review focuses on certain specific aspects of thermodynamics of horizons and attempts to unravel a deeper relationship between thermodynamics of horizons and gravity. Most of the material is aimed at a broader readership than the experts in the field. In order to keep the review self contained and of reasonable length, it is necessary to concentrate on some simple models (mentioning generalisations, when appropriate, only briefly) and deal directly with semi classical and quantum mechanical aspects. (Hence many of the beautiful results of classical black hole thermodynamics will not be discussed

here. Approaches based on string theory and loop gravity will be only briefly touched upon.) The broad aim of the review will be to analyse the following important conceptual issues:

- What is the key physics (*viz.* the minimal set of assumptions) which leads to the association of a temperature with a horizon ? Can one associate a temperature with *any* horizon ?
- Do all horizons, which hide information, possess an entropy ? If so, how can one understand the entropy and temperature of horizons in a broader context than that of, say, black holes ? What are the microscopic degrees of freedom associated with this entropy ?
- Do all observers attribute a temperature and entropy to the horizon in spite of the fact that the amount of information accessible to different observers is different ? If the answer is "no", how does one reconcile dynamical effects related to, say, black hole evaporation, with general covariance ?
- What is the connection between the above results and gravity, since horizons of certain kind can exist even in flat spacetime in the absence of gravity ?

All these issues are subtle and controversial to different degrees. Current thinking favours — correctly — the view that a temperature can be associated with any horizon and the initial sections of the review will concentrate on this question. The second set of issues raised above are not really settled in the literature and fair diversity of views prevails. We shall try to sort this out and clarify matters though there are still several open issues. The answer to the question raised in the third item above is indeed "no" and one requires serious rethinking about the concept of general covariance in quantum theory. We will describe, in the latter half of the review, a possible reinterpretation of the formalism so that each observer will have a consistent description. This analysis also leads to a deeper connection between gravity and spacetime thermodynamics, thereby shedding light on the last issue.

The logical structure of our approach (summarized in the last Section and Fig. 4 in page 82) will be as follows: Families of observers exist in any spacetime, who — classically — have access to only limited portions of the spacetime because of the existence of horizons. This leads to two effects when the horizon is (at least approximately) static:

- The Euclidean version of the quantum field theory needs to be formulated in an *effective* spacetime manifold obtained by removing the region blocked by the horizon. When the horizon is static, this *effective* manifold will have a nontrivial topology and leads to the association of a temperature with the horizon (Sections III to VI). This arises because the quantum theory contains information which classical theory does not have, due to non-zero correlation functions on a spacelike hypersurface across the horizon.
- The gravitational action functional, when formulated in terms of the vari-

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ables the family of observers can access, *will* have a boundary term proportional to the horizon area. This is equivalent to associating a constant entropy per unit area of any horizon. Further, it is possible to obtain the Einstein-Hilbert action using the structure of the boundary term. Among other things, this *clarifies a peculiar relation between the boundary and surface terms of the Einstein-Hilbert action* (Section VIII). This idea lends itself to further generalisations and leads to specific results in the semiclassical limit of quantum gravity.

Throughout the discussion, we emphasize the 'thermodynamical' aspects of horizons rather than the 'statistical mechanics' based on microscopic models, like string theory or loop gravity. While there has been considerable amount of work in recent years in the latter approaches (briefly discussed in Section 7.1), most of the results obtained by these approaches are necessarily model dependent. On the other hand, since any viable microscopic model for quantum gravity reduces to Einstein gravity in the long wavelength limit, it is possible to obtain several general results in the semi-classical limit of the theory which are independent of the microscopic details. This is analogous to the fact that the thermodynamical description of a gas, say, is broadly independent of the microscopic Hamiltonian which describes the behaviour of molecules in the gas. While such a microscopic description is definitely worth pursuing, one also needs to appreciate how much progress one can make in a reasonably model independent manner using essentially the structure of classical gravity. As we shall see, one can make significant progress in understanding the thermodynamics of horizon by this approach which should be thought of as complementing the more microscopic descriptions like the ones based on string theory.

We follow the sign conventions of [13] with the signature $(-+++)$ and use units with $G = \hbar = c = 1$. But, unlike [13], we let the Latin indices cover 0,1,2,3 while the Greek indices cover 1,2,3. The background material relevant to this review can be found in several text books [14,15,16] and review articles [17,18,19,20,21,22,23].

2 Horizon for a family of observers

Classical and quantum theories based on non-relativistic physics use the notion of absolute time and allow for information to be transmitted with arbitrarily large velocity. An event $\mathcal{P}(T_0, X_0^\alpha)$ can, in principle, influence *all* events at $T \geq T_0$ and be influenced by all events at $T \leq T_0$. There is no horizon limiting one's region of influence in non-relativistic theories.

The situation changes in special relativity, which introduces a maximal speed

c (equal to unity in our choice of units) for the propagation of signals. An event $\mathcal{P}(T_0, \mathbf{X}_0)$ can now acquire information only from the events $\mathcal{P}(T, \mathbf{X})$ in the "backward" light cone $|\mathbf{X}_0 - \mathbf{X}| \leq (T_0 - T)$ and can send information only to events in the "forward" light cone $|\mathbf{X} - \mathbf{X}_0| \leq (T - T_0)$. The light cones $\mathcal{C}(\mathcal{P})$ at $\mathcal{P}$, defined by the equation $\mathcal{C}(X^a) \equiv |\mathbf{X} - \mathbf{X}_0|^2 - (T - T_0)^2 = 0$, divide the spacetime into two regions which are either causally connected or causally disconnected to $\mathcal{P}$. This light cone structure is invariant under Lorentz transformations. The normal $n_a = \partial_a \mathcal{C}(T - T_0, \mathbf{X} - \mathbf{X}_0)$ to the light cone $\mathcal{C}(\mathcal{P})$ is a null vector ($n^a n_a = 0$) and the light cone is a null surface.

Consider now a timelike curve $X^a(t)$ in the spacetime, parametrised by the proper time t of the clock moving along that curve. We can construct past light cone $\mathcal{C}(t)$ for each event $\mathcal{P}[X^a(t)]$ on this trajectory. The union U of all these past light cones $\{\mathcal{C}(t), -\infty \leq t \leq \infty\}$ determines whether an observer on the trajectory $X^a(t)$ can receive information from all events in the spacetime or not. If U has a nontrivial boundary, there will be regions in the spacetime from which this observer cannot receive signals. (We shall always use the term "observer" as synonymous to a time-like curve in the spacetime, without any other additional, implied, connotations.) In fact, one can extend this notion to a family of timelike curves which fill a region of spacetime. We shall call such a family of curves with reasonable notions of smoothness a "congruence"; it is possible to define this concept with greater level of abstraction (see e.g. [24]) which is not required for our purpose. Given a congruence of time-like curves ("family of observers"), the boundary of the union of their causal pasts (which is essentially the boundary of the union of backward light cones) will define a *horizon* for this set of observers. We will assume that each of the timelike curves has been extended to the maximum possible value for the proper time parametrising the curve. If the curves do not hit any spacetime singularity, then this requires extending the proper time to infinite values. This horizon is dependent on the family of observers that is chosen, but is coordinate independent. We shall call the horizon defined by the above procedure as *causal* horizon in order to distinguish it from horizons defined through other criteria, some of which we will discuss in Section 2.5.

An important example (in flat spacetime) of a set of observers with horizon, which we shall repeatedly come across as a prototype, is a class of trajectories $X^i(t) = (T(t), X(t), 0, 0)$:

$$\kappa T = N \sinh(\kappa t), \quad \kappa X = N \cosh(\kappa t), \tag{1}$$

where N and κ are constants. The quantity (Nt) is the proper time of the clock carried by the observer with the trajectory $N = $ constant. Physically, for finite t, these trajectories (for different N) represent observers moving with (different) uniform acceleration (κ/N) along the X-axis. The velocity $(dX/dT) = \tanh(\kappa t)$ approaches the speed of light as $t \to \pm\infty$.

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For all $N > 0, \kappa > 0$, these trajectories are hyperbolas confined to the 'right wedge' of the spacetime ($\mathcal{R}$) defined by $X > 0, |T| < X$ and these observers cannot access any information in the region $T > X$. Hence, for this class of observers, the null light cone surface, $(T - X) = 0$, acts as a horizon. An inertial observer with the trajectory $(T = t, X = x, 0, 0)$ for all t will be able to access information from the region $T > X$ at sufficiently late times. The accelerated observer, on the other hand, will not be able to access information from half the spacetime even when $t \to \infty$.

Similarly, Eq. (1) with $N < 0$ represents a class of observers accelerating along negative x-axis and confined to the 'left wedge' ($\mathcal{L}$) defined by $X < 0, |T| < |X|$ who will not have access to the region $(T + X) > 0$. This example shows that the horizon structure is "observer dependent" and arises because of the nature of timelike congruence which is chosen to define it.

These ideas generalise in a straight forward manner to curved spacetime. As a simple example, consider a class of spacetimes with the metric

$$ds^2 = \Omega^2(X^a)(-dT^2 + dX^2) + dL_\perp^2 \tag{2}$$

where $\Omega(X^a)$ is a nonzero, finite, function everywhere (except possibly on events at which the spacetime has curvature singularities) and $dL_\perp^2$ vanishes on the $T - X$ plane. For light rays propagating in the $T - X$ plane, with $ds^2 = 0$, the trajectories are lines at 45°, just as in flat space time. The congruence in Eq. (1) will again have a horizon given by the surface $(T - X) = 0$ in this spacetime. Another class of observers with the trajectories $(T = t, X = x, 0, 0)$ for all t will be able to access information from the region $T > X$ at sufficiently late times (provided the trajectory can be extended without hitting a spacetime singularity). Once again, it is clear that the horizon is linked to the choice of a congruence of timelike curves.

Given any family of observers in a spacetime, it is most convenient to interpret the results of observations performed by these observer in a frame in which these observers are at rest. So the natural coordinate system $(t, \mathbf{x})$ attached to any timelike congruence is the one in which each trajectory of the congruence corresponds to $\mathbf{x} = $ constant. (This condition, of course, does not uniquely fix the coordinate system but is sufficient for our purposes.) For the accelerated observers introduced above, such a coordinate system is already provided by Eq. (1) itself with (t, N, Y, Z) now being interpreted as a new coordinate system, related to the inertial coordinate system (T, X, Y, Z) with all the coordinates taking the range $(-\infty, \infty)$. The transformations in Eq. (1) do not leave the form of the line interval $ds^2 = -dT^2 + |d\mathbf{X}|^2$ invariant; the line interval in the new coordinates is given by

$$ds^2 \equiv g_{ab}(x)dx^a dx^b = -N^2 dt^2 + dN^2/\kappa^2 + dL_\perp^2 \tag{3}$$

The light cones $T^2 = |X|^2$ in the $(Y, Z) = $ constant sector, now corresponds to the surface $N = 0$ in this new coordinate system (usually called the Rindler frame). Thus the Rindler frame is a static coordinate system with the $g_{00} = 0$ surface — which is just the light cone through the origin of the inertial frame — dividing the frame into two causally disconnected regions. Since the transformations in Eq. (1) covers only the right and left wedges, the metric in Eq. (3) is valid only in these two regions. Both the branches of the light cone $X = +T$ and $X = -T$ collapse to the line $N = 0$. The top wedge, $\mathcal{F}(|X| < T, T > 0)$ and the bottom wedge $\mathcal{P}(|X| < T, T < 0)$ of the Minkowski space *disappear* in this representation. (We shall see below how similar coordinates can be introduced in $\mathcal{F}, \mathcal{P}$ as well; see Eq. (13)).

The metric in Eq. (3) is static even though the transformations in Eq. (1) appear to depend on time in a nontrivial manner. This static nature can be understood as follows: The Minkowski spacetime possesses invariance under translations, rotations and Lorentz boosts which are characterised by the existence of a set of ten Killing vector fields. Consider any linear combination V^i of these Killing vector fields which is timelike in a sub-region $\mathcal{S}$ of Minkowski spacetime. The integral curves to this vector field V^i will define timelike curves in $\mathcal{S}$. If one treats these curves as the trajectories of a family of hypothetical observers, then one can set up an appropriate coordinate system for this observer. Since the four velocity of the observer is along the Killing vector field, it is obvious that the metric components in this coordinate system will not depend on the time coordinate. A sufficiently general Killing vector field which incorporates the effects of translations, rotations and boosts can be written as $V^i = (1 + \kappa X, \kappa T - \lambda Y, \lambda X - \rho Z, \rho Y)$ where κ, λ and ρ are constants. When $\lambda = \rho = 0$, the field V^i generates the effects of Lorentz boost along the $X-$axis and the trajectories in Eq. (1) are the integral curves of this Killing vector field. The static nature of Eq. (3) reflects the invariance under Lorentz boosts along the $X-$axis. One simple way of proving this is to note that Lorentz boosts along $X-$axis "corresponds to" a rotation in the $X - T$ plane by an imaginary angle; or, equivalently, Lorentz boost will "correspond to" rotation in terms of the imaginary time coordinates $T_E = iT, t_E = it$. In Eq. (1) $t \to t + \epsilon$ does represent a rotation in the $X - T_E$ plane on a circle of radius N. Clearly, Eq. (1) is just one among several possible trajectories for observers such that the resulting metric [like the one in Eq. (3)] will be static. (For example, the Killing vector field with $\rho = 0$ corresponds to a rotating observer while $\lambda = \kappa, \rho = 0$ leads to a cusped trajectory.) Many of these are analysed in literature (see, for example, [25,26,27,19]) but none of them lead to results as significant as Eq. (3). This is because Eq. (3) is a good approximation to a very wide class of metrics near the horizon. We shall now discuss this feature.

Motivated by Eq. (3), let us consider a more a general class of metrics which are: (i) static in the given coordinate system, $g_{0\alpha} = 0, g_{ab}(t, \mathbf{x}) = g_{ab}(\mathbf{x})$; (ii)

$g_{00}(\mathbf{x}) \equiv -N^2(\mathbf{x})$ vanishes on some 2-surface $\mathcal{H}$ defined by the equation $N^2 = 0$, (iii) $\partial_\alpha N$ is finite and non zero on $\mathcal{H}$ and (iv) all other metric components and curvature remain finite and regular on $\mathcal{H}$. The line element will now be:

$$ds^2 = -N^2(x^\alpha)dt^2 + \gamma_{\alpha\beta}(x^\alpha)dx^\alpha dx^\beta \tag{4}$$

The comoving observers in this frame have trajectories $\mathbf{x} = $ constant, four-velocity $u_a = -N\delta_a^0$ and four acceleration $a^i = u^j\nabla_j u^i = (0, \mathbf{a})$ which has the purely spatial components $a_\alpha = (\partial_\alpha N)/N$. The unit normal n_α to the $N = $ constant surface is given by $n_\alpha = \partial_\alpha N(g^{\mu\nu}\partial_\mu N\partial_\nu N)^{-1/2} = a_\alpha(a_\beta a^\beta)^{-1/2}$. A simple computation now shows that the normal component of the acceleration $a^i n_i = a^\alpha n_\alpha$, 'redshifted' by a factor N, has the value

$$N(n_\alpha a^\alpha) = (g^{\alpha\beta}\partial_\alpha N\partial_\beta N)^{1/2} \equiv Na(\mathbf{x}) \tag{5}$$

where the last equation defines the function a. From our assumptions, it follows that on the horizon $N = 0$, this quantity has a finite limit $Na \to \kappa$; the κ is called the surface gravity of the horizon.

These static spacetimes, however, have a more natural coordinate system defined in terms of the level surfaces of N. That is, we transform from the original space coordinates x^μ in Eq.(4) to the set (N, y^A), $A = 2, 3$ by treating N as one of the spatial coordinates. The y^A denotes the two transverse coordinates on the $N = $ constant surface. (Upper case Latin letters go over the coordinates 2,3 on the $t = $ constant, $N = $ constant surface). This can be always done locally, but possibly not globally, because N could be multiple valued etc. We, however, need this description only locally. The components of acceleration in the (N, y^A) coordinates are

$$a^N = a^\mu\partial_\mu N = Na^2, a^B = a^\mu\frac{\partial y^B}{\partial x^\mu}, a_B = 0, a_N = \frac{1}{N} \tag{6}$$

Using these we can express the metric in the new coordinates as

$$g^{NN} = \gamma^{\mu\nu}\partial_\mu N\partial_\nu N = N^2 a^2; \; g^{NA} = Na^A \tag{7}$$

etc. The line element now becomes:

$$ds^2 = -N^2 dt^2 + \frac{dN^2}{(Na)^2} + \sigma_{AB}(dy^A - \frac{a^A dN}{Na^2})(dy^B - \frac{a^B dN}{Na^2}) \tag{8}$$

The original 7 degrees of freedom in $(N, \gamma_{\mu\nu})$ are now reduced to 6 degrees of freedom in (a, a^A, σ_{AB}), because of our choice for g_{00}. This reduction is similar to what happens in the synchronous coordinate system which makes $N = 1$, but the synchronous frame loses the static nature [28]. In contrast, Eq.(8) describes the spacetime in terms of the magnitude of acceleration a, the transverse components a^A and the metric σ_{AB} on the two surface and

maintains the $t-$independence. The N is now merely a coordinate and the spacetime geometry is described in terms of (a, a^A, σ_{AB}) all of which are, in general, functions of (N, y^A). In well known, spherically symmetric spacetimes with horizon, we will have $a = a(N), a^A = 0$ if we choose $y^A = (\theta, \phi)$. Important features of dynamics are usually encoded in the function $a(N, y^A)$.

Near the $N \to 0$ surface, $Na \to \kappa$, the surface gravity, and the metric reduces to the Rindler form in Eq.(3):

$$ds^2 = -N^2 dt^2 + \frac{dN^2}{(Na)^2} + dL_\perp^2 \simeq -N^2 dt^2 + \frac{dN^2}{\kappa^2} + dL_\perp^2 \qquad (9)$$

where the second equality is applicable close to $\mathcal{H}$. Thus the metric in Eq. (3) is a good approximation to a large class of static metrics with g_{00} vanishing on a surface. (It is, of course, possible for N to vanish on more than one surface so that the spacetime has multiple horizons; this is a more complicated situation and requires a different treatment, which we will discuss in Section 6.3).

There is an interesting extension of the metric in Eq. (3) or Eq. (9) which is worth mentioning. Changing to the variable from N to l with

$$dl = \frac{dN}{a} = \frac{NdN}{Na}; \qquad l \approx \frac{1}{2\kappa}N^2 \qquad (10)$$

where the second relation is applicable near the horizon with $Na \approx \kappa$, we can cast the line element in the form

$$ds^2 = -f(l)dt^2 + \frac{dl^2}{f(l)} + dL_\perp^2 \approx -2\kappa l \ dt^2 + \frac{dl^2}{2\kappa l} + dL_\perp^2 \qquad (11)$$

where the second equation is applicable near the horizon with $l \approx (1/2\kappa)N^2$. More generally, the function $f(l)$ is obtained by expressing N in terms of l. Many examples of horizons in curved spacetime we come across have this structure with $g_{00} = -g^{11}$ and hence this is a convenient form to use.

There is a further advantage in using the variable l. The original transformations from (T, X) to (t, N) given by Eq. (1) maps the right and left wedges $(\mathcal{R}, \mathcal{L})$ into $(N > 0, N < 0)$ regions. Half of Minkowski spacetime contained in the future light cone $(\mathcal{F})$ through the origin $(|X| < T, T > 0)$ and past light cone $(\mathcal{P})$ through the origin $(|X| < T, T < 0)$ is *not* covered by the (t, N) coordinate system of Eq. (1) at all. But, if we now extend l to negative values then it is possible to use this (t, l) coordinate system to cover all the four quadrants of the Minkowski spacetime. The complete set of transformations we need are:

$$\kappa T = \sqrt{2\kappa l} \sinh(\kappa t); \quad \kappa X = \pm\sqrt{2\kappa l} \cosh(\kappa t) \qquad (12)$$

for $|X| > |T|$ with the positive sign in $\mathcal{R}$ and negative sign in $\mathcal{L}$ and

$$\kappa T = \pm\sqrt{-2\kappa l}\cosh(\kappa t); \quad \kappa X = \sqrt{-2\kappa l}\sinh(\kappa t) \qquad (13)$$

for $|X| < |T|$ with the positive sign in $\mathcal{F}$ and negative sign in $\mathcal{P}$. Clearly, $l < 0$ is used in $\mathcal{F}$ and $\mathcal{P}$. Note that t is timelike and l is spacelike in Eq. (11) only for $l > 0$ with their roles reversed for $l < 0$. A given value of (t, l) corresponds to a pair of points in $\mathcal{R}$ and $\mathcal{L}$ for $l > 0$ and to pair of points in $\mathcal{F}$ and $\mathcal{P}$ for $l < 0$. Figure 1 shows the geometrical features of the coordinate systems.

The following crucial difference between the (t, N) coordinates and (t, l) coordinates must be stressed: In the (t, N) coordinates, t is everywhere timelike (see the second equation of Eq. (9)) and the two regions $N > 0$ and $N < 0$ are completely disconnected. In the (t, l) coordinates, t is timelike where $l > 0$ and spacelike where $l < 0$ (see Eq. (11)) and the surface $l = 0$ acts as a "one-way membrane"; signals can go from $l > 0$ to $l < 0$ but not the other way around. When we talk of $l = 0$ surface as a horizon, we often have the interpretation based on this feature.
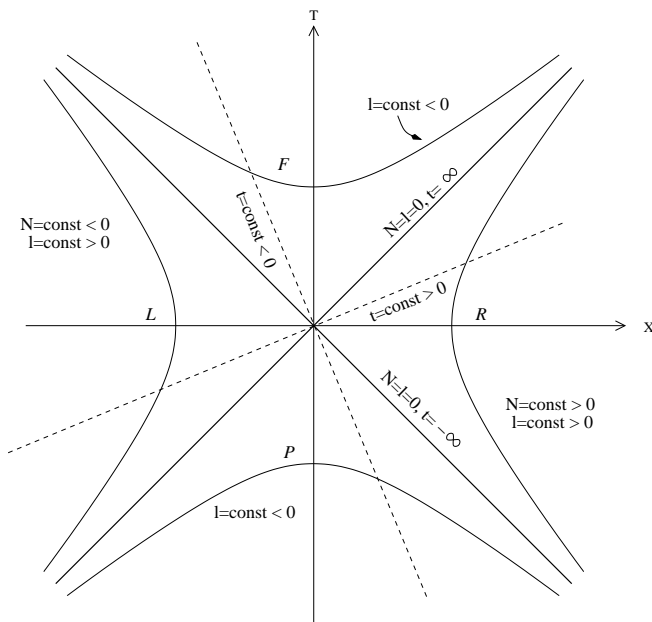
Fig. 1. The global manifold with different coordinate systems in the four quadrants. See text for discussion.

In Eq. (4), (11) etc., we have defined N and l such that the horizon is at $N = l = 0$. This, of course, is not needed and our results continue to hold when $f = 0$ at some finite $l = l_H$. In spherically symmetric spacetimes it is often convenient to take $0 \leq l < \infty$ and have the horizon at some finite value $l = l_H$.

Metrics of the kind in Eq. (4) could describe either genuinely curved spacetimes or flat spacetime in some non inertial coordinate system. The local physics of

the horizons really does not depend on whether the spacetime is curved or flat and we shall present several arguments in favour of the "democratic" treatment of horizons. In that spirit, we do not worry whether Eq. (4) represents flat or curved spacetime.

We have assumed that the spacetime in Eq. (8) is static. It is possible to generalise some of our results to *stationary* spacetimes, which have $g_{0\mu} \neq 0$ but with all metric coefficients remaining time independent. A uniformly rotating frame as well as curved spacetimes like Kerr metric belong to this class and pose some amount of mathematical difficulties. These difficulties can be overcome, but only by complicating the formalism and obscuring the simple physical insights. It is more difficult to extend the results to general, time dependent, horizons (for a discussion of issues involved in providing a general definition of horizon, see e.g., [29,30]). If one considers the static horizons as analogous to equilibrium thermodynamics then the analogue of time dependent horizons will be non equilibrium thermodynamics. The usual approach in thermodynamics is to begin with the study of equilibrium thermodynamics in order to define different thermodynamical variables etc. and then proceed to time dependent non equilibrium processes. These extreme limits are connected by quasi-static systems, which can again be handled by a straight forward generalisation of the static case. We shall adopt a similar philosophy in our study of horizons and develop the notion of thermodynamical variables like temperature, entropy etc. for the horizons using static spacetimes of the form in Eq. (8) thereby precluding from consideration, stationary metrics like that of rotating frame or Kerr spacetime. While stationary and time dependent metrics will be more complicated to analyse, we do not expect any new serious conceptual features to arise due to time dependence. What is more, the static horizons themselves have a rich amount of physics which needs to be understood.

The coordinate systems having metrics of the form Eq. (9) have several interesting, generic, features which we shall now briefly describe.

2.1 Horizon and infinite redshift

In the metrics of the form in Eq. (9), the $N = 0$ surface acts as a horizon and the coordinates (t, N) and (t, l) are badly behaved near this surface. This is most easily seen by considering the light rays traveling along the $N-$direction in Eq. (9) with $y^A = $ constant. These light rays are determined by the equation $(dt/dN) = \pm(1/N^2 a)$ and as $N \to 0$, we get $(dt/dN) \approx \pm(1/N\kappa)$. The slopes of the light cones diverge making the $N = 0$ surface act as a one way membrane in the (t, l) coordinates and as a barrier dividing the spacetime into two causally disconnected regions in the (t, N) coordinates. This difference arises

because the light cone $T = X$, for example, separates $\mathcal{R}$ from $\mathcal{F}$ and both regions are covered by the (t, l) coordinates; in contrast, the region $\mathcal{F}$ (and $\mathcal{P}$) are not covered in the (t, N) coordinates.

This result is confirmed by the nature of the trajectories of material particles with constant energy, near $N = 0$. The Hamilton-Jacobi (HJ) equation for the action A describing a particle of mass m is $\partial_a A \partial^a A = -m^2$. In a spacetime with the metric in Eq.(8) the standard substitution $A = -Et + f(x^\alpha)$, reduces it to:

$$N^4 a^2 \left(\frac{\partial f}{\partial N} \right)^2 = E^2 - N^2[m^2 + (\partial_\perp f)^2] \tag{14}$$

where $(\partial_\perp f)^2$ is the contribution from transverse derivatives. Near $N = 0$, the solution is universal, independent of m and the transverse degrees of freedom:

$$A \approx -Et \pm E \int \frac{dN}{N^2 a} \approx -E(t \pm \xi) \tag{15}$$

where

$$\xi \equiv \int \frac{dN}{N^2 a} = \int \frac{dl}{f(l)} \tag{16}$$

is called the *tortoise coordinate* and behaves as $\xi \simeq (1/\kappa) \ln N$ near the horizon. The trajectories are $N \cong (\text{constant}) \exp(\pm \kappa t)$ clearly showing that the horizon (at $N = 0$) cannot be reached in finite time t from either side.

Let us next consider the redshift of a photon emitted at (t_e, N_e, y^A), where N_e is close to the horizon surface $\mathcal{H}$, and is observed at (t, N, y^A). The frequencies at emission $\omega(t_e)$ and detection $\omega(t)$ are related by $[\omega(t)/\omega(t_e)] = [N_e/N]$. The trajectory of the out-going photon is given by

$$t - t_e = \int_{N_e}^N \frac{dN}{N^2 a} = -\frac{1}{\kappa} \ln N_e + \text{constant} \tag{17}$$

where we have approximated the integral by the dominant contribution near $N_e = 0$. This gives $N_e \propto \exp(-\kappa t)$, leading to the exponentially redshifted frequency $\omega(t) \propto N_e \propto \exp(-\kappa t)$.

2.2   *Inertial coordinate system near the horizon*

The bad behaviour of the metric near $N = 0$ is connected with the fact that the observers at constant-$\mathbf{x}$ perceive a horizon at $N = 0$. Given a congruence of timelike curves, with a non-trivial boundary for their union of past light cones, there will be trajectories in this congruence which are arbitrarily close to the boundary. Since each trajectory is labelled by a $\mathbf{x} = $ constant curve in the comoving coordinate system, it follows that the metric in this coordinate system will behave badly at the boundary.

The action functional in Eq. (15) corresponds to a particle with constant energy in the $(t, \mathbf{x})$ coordinate system, since we have separated the HJ equation with $(\partial A/\partial t) = -E = $ constant. Since this coordinate system is badly behaved at the horizon, the trajectory takes infinite coordinate time to reach the horizon from either direction. In a different coordinate system which is regular at the horizon, the trajectories can cross the horizon at finite time. This is clear from the fact that one can introduce a local inertial frame even near the horizon; the observers at rest in this frame (freely falling observers) will have regular trajectories which will cross the horizon. If we use a coordinate system in which freely falling observers are at rest and use their clocks to measure time, there will be no pathology at the horizon. In case of flat spacetime, the freely falling trajectories are obtained by choosing the action functional which behaves as $A = -E'T + F(\mathbf{X})$. The corresponding "good" coordinate system is, of course, the global inertial frame.

In the general case, the required transformation is

$$\kappa X = e^{\kappa \xi} \cosh \kappa t; \ \ \kappa T = e^{\kappa \xi} \sinh \kappa t \qquad (18)$$

where ξ is defined by Eq. (16). This result can be obtained as follows: We first transform the line element in Eq. (11) to the tortoise coordinate ξ:

$$ds^2 = N^2(\xi)(-dt^2 + d\xi^2) + dL_\perp^2; \qquad (19)$$

Introducing the null coordinates $u = (t - \xi), v = (t + \xi)$, we see that near the horizon, $N \approx \exp[\kappa \xi] = \exp[(\kappa/2)(v - u)]$ which is singular as $\xi \to -\infty$. This suggests the transformations to two new null coordinates (U, V) with $\kappa V = \exp[\kappa v], \kappa U = -\exp[-\kappa u]$ which are regular at horizon. The corresponding T and X given by $U = (T - X), V = (T + X)$. Putting it all together, we get the result in Eq. (18). The metric in terms of (T, X) coordinates has the form

$$ds^2 = \frac{N^2}{\kappa^2(X^2 - T^2)}(-dT^2 + dX^2) + dL_\perp^2 \qquad (20)$$

where N needs to be expressed in terms of (T, X) using the coordinate transformations. In general, this metric will be quite complicated and will *not* even be static. The horizon at $N = 0$ corresponds to the light cones $T^2 - X^2 = 0$ in these coordinates and $[N^2/\kappa^2(T^2 - X^2)]$ is finite on the horizon by construction. Thus the (T, X) coordinates are the locally inertial coordinates near $\mathcal{H}$.

The transformations in Eq. (18) show that $(X^2 - T^2)$ is purely a function of N (or l) while (X/T) is a function of t. Thus $t = $ constant curves are radial lines through the origin with the $X = 0$ plane coinciding with $N = 0$ plane. Curves of $N = $ constant are hyperbolas (see figure 1).

By very construction, the line element in the (T, X) coordinates is well behaved near the horizon, while the line element is pathological in the (t, N)

or (t, l) coordinates because the transformations in Eq. (18) are singular at $N = l = 0$. In the examples which we study the spacetime manifold will be well behaved near the horizon and this fact will be correctly captured in the (T, X) coordinates. The singular transformation from (T, X) coordinates to (t, l) coordinates is the cause for the bad behaviour of metric near $l = 0$ in these coordinates. But the family of observers, with respect to whom the horizon is defined to exist, will find it natural to use the (t, N) coordinate system and the "bad" behaviour of the metric tensor implies some *non-trivial physical phenomena* for these observers. Since any family of observers has a right to describe physics in the coordinate frame in which they are at rest, we need to take these coordinates seriously. (We will also see that (t, l) coordinates often have other interesting features which are not shared by the (T, X) coordinates. For example, the metric can be static in (t, l) coordinates but time dependent in (T, X) coordinates.)

The transformation in Eq. (18) requires the knowledge of the surface gravity κ on the horizon. If N vanishes at more than one surface — so that the spacetime has multiple horizons — then we need different transformations of the kind in Eq. (18) near each horizon with, in general, different values for κ. We shall comment on this feature in Section 6.3.

2.3   Classical wave with exponential redshift

The fact that the time coordinates used by the freely falling and accelerated observers are related by a nonlinear transformation Eq. (18) leads to an interesting consequence. Consider a monochromatic out-going wave along the X-axis, given by $\phi(T, X) = \exp[-i\Omega(T - X)]$ with $\Omega > 0$. Any other observer who is inertial with respect to the $X = $ constant observer will see this as a *monochromatic* wave, though with a different frequency. But an accelerated observer, at $N = N_0 = $ constant using the proper time co-ordinate $\tau \equiv N_0 t$ will see the same mode as varying in time as

$$\phi = \phi(T(t), X(t)) = \exp[i\Omega q e^{-\kappa t}] = \exp[i\Omega q \exp -(\kappa/N_0)\tau] \qquad (21)$$

where we have used Eq. (18) and $q \equiv \kappa^{-1} \exp(\kappa\xi)$. This is clearly not monochromatic and has a frequency which is being exponentially redshifted in time. The power spectrum of this wave is given by $P(\nu) = |f(\nu)|^2$ where $f(\nu)$ is the Fourier transform of $\phi(\tau)$ with respect to τ:

$$\phi(\tau) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} f(\nu) e^{-i\nu\tau} \qquad (22)$$

Because of the exponential redshift, this power spectrum will *not* vanish for $\nu < 0$. Evaluating this Fourier transform (by changing to the variable

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$\Omega q \exp[-(\kappa/N_0)\tau] = z$ and analytically continuing to Im z) one gets:

$$f(\nu) = (N_0/\kappa)(\Omega q)^{i\nu N_0/\kappa}\Gamma(-i\nu N_0/\kappa)e^{\pi\nu N_0/2\kappa} \qquad (23)$$

This leads to the the remarkable result that the power, per logarithmic band in frequency, at negative frequencies is a Planckian at temperature $T = (\kappa/2\pi N_0)$:

$$\nu|f(-\nu)|^2 = \frac{\beta}{e^{\beta\nu} - 1}; \quad \beta = \frac{2\pi N_0}{\kappa} \qquad (24)$$

and, more importantly,

$$|f(-\nu)|^2/|f(\nu)|^2 = \exp(-\beta\nu). \qquad (25)$$

Though $f(\nu)$ in Eq. (23) depends on Ω, the power spectrum $|f(\nu)|^2$ is independent of Ω; monochromatic plane waves of any frequency (as measured by the freely falling observers at $X = $ constant) will appear to have Planckian power spectrum in terms of the (negative) frequency ν, defined with respect to the proper time of the accelerated observer located at $N = N_0 = $ constant. The scaling of the temperature $\beta^{-1} \propto N_0^{-1} \propto |g_{00}|^{-1/2}$ is precisely what is expected in general relativity for temperature.

We saw earlier (see Eq. 17) that waves propagating from a region near the horizon will undergo exponential redshift. An observer detecting this exponentially redshifted radiation at late times ($t \to \infty$), originating from a region close to $\mathcal{H}$ will attribute to this radiation a Planckian power spectrum given by Eq. (24). This result lies at the foundation of associating temperature with horizons. [The importance of exponential redshift is emphasised by several people including [31,32,33,34,35,36].]

The Planck spectrum in Eq. (24) is in terms of the frequency and β has the (correct) dimension of time; no $\hbar$ appears in the result. If we now switch the variable to energy, invoking the basic tenets of quantum mechanics, and write $\beta\nu = (\beta/\hbar)(\hbar\nu) = (\beta/\hbar)E$, then one can identify a temperature $k_B T = (\kappa\hbar/2\pi c)$ which scales with $\hbar$. This "quantum mechanical" origin of temperature is superficial because it arises merely because of a change of units from ν to E. An astronomer measuring frequency rather than photon energy will see the spectrum in Eq. (24) as Planckian without any quantum mechanical input.

It is fairly straightforward to construct different time evolutions for a wave $\phi(t)$ such that the corresponding power spectrum $|f(\nu)|^2$ has the Planckian form. While the trajectory in Eq. (1) was never constructed for this purpose and leads to this result in a natural fashion, it is difficult to understand the physical origin of temperature or the Bose distribution for photons in this approach purely classically, especially since we started with a complex wave form. (The results for a real cosine wave is more intriguing; see [37,38]). The

true importance of the above result lies in the fact that, the mathematical operation involved in obtaining Eq. (24), acquires physical meaning in terms of positive and negative frequency modes in quantum field theory which we shall discuss later. This is suggested by Eq. (25) itself. In the quantum theory of radiation, the amplitudes of the wave, with frequencies differing in sign, cause absorption and emission of radiation by a system with two energy levels differing by $\delta E = h\nu$. Hence any system, which comes into steady state with this radiation in the accelerated frame, will have the ratio of populations in the two levels to be $\exp(-\beta E)$, giving an operational meaning to this temperature.

2.4   Field theory near the horizon: Dimensional reduction

The fact that $N \to 0$ on the horizon leads to interesting conclusions regarding the behaviour of any classical (or quantum) field near the horizon. Consider, for example, an interacting scalar field in a background spacetime described by the metric in Eq.(8), with the action:

$$A = -\int d^4x \sqrt{-g} \left(\frac{1}{2} \partial_a \phi \partial^a \phi + V \right) \tag{26}$$

$$= \int dt dN d^2 y \frac{\sqrt{\sigma}}{N^2 a} \times \left[\frac{\dot{\phi}^2}{2} - N^4 a^2 \left(\frac{\partial \phi}{\partial N} \right)^2 - N^2 \left[\frac{(\partial_\perp \phi)^2}{2} + V \right] \right]$$

where $(\partial_\perp \phi)^2$ denotes the contribution from the derivatives in the transverse directions including cross terms of the type $(\partial_N \phi \partial_\perp \phi)$. Near $N = 0$, with $Na \to \kappa$, the action reduces to the form

$$A \approx \int \sqrt{\sigma} d^2 x_\perp \int dt \int d\xi \left\{ \frac{1}{2} \left[\dot{\phi}^2 - \left(\frac{\partial \phi}{\partial \xi} \right)^2 \right] \right\} \tag{27}$$

where we have changed variable to ξ defined in Eq. (16) [which behaves as $\xi \approx (1/\kappa) \ln N$] and ignored terms which vanish as $N \to 0$. Remarkably enough this action represents a two dimensional free field theory in the (t, ξ) coordinates which has the enhanced symmetry of invariance under the conformal transformations $g_{ab} \to f^2(t, \xi) g_{ab}$ [see e.g., Section 3 of [39]]. The solutions to the field equations near $\mathcal{H}$ are plane waves in the (t, ξ) coordinates:

$$\phi_\pm = \exp[-i\omega(t \pm \xi)] = N^{\pm i\omega/\kappa} e^{-i\omega t} \tag{28}$$

These modes are the same as $\phi = \exp iA$ where A is the solution Eq. (15) to the Hamilton-Jacobi equation; this is because the divergence of $(1/N)$ factor near the horizon makes the WKB approximation almost exact near the horizon. The mathematics involved in this phenomenon is fundamentally the same as

the one which leads to the "no-hair-theorems" (see, eg., [40]) for the black hole.

There are several symmetry properties for these solutions which are worth mentioning:

(a) The Rindler metric and the solution near $\mathcal{H}$ is invariant under the rescaling $N \to \lambda N$, in the sense that this transformation merely adds a phase to ϕ. This scale invariance can also be demonstrated by studying the spatial part of the wave equation [41] near $\mathcal{H}$, where the equation reduces to a Schrodinger equation for the zero energy eigenstate in the potential $V(N) = -\omega^2/N^2$. This Schrodinger equation has the natural scale invariance with respect to $N \to \lambda N$ which is reflected in our problem.

(b) The relevant metric $ds^2 = -N^2 dt^2 + (dN/\kappa)^2$ in the $t - N$ plane is also invariant, up to a conformal factor, to the metric obtained by $N \to \rho = 1/N$:

$$ ds^2 = -N^2 dt^2 + \frac{dN^2}{\kappa^2} = \frac{1}{\rho^4}(-\rho^2 dt^2 + \frac{d\rho^2}{\kappa^2}) \tag{29} $$

Since the two dimensional field theory is conformally invariant, if $\phi(t, N)$ is a solution, then $\phi(t, 1/N)$ is also a solution. This is clearly true for the solution in Eq. (28). Since N is a coordinate in our description, this connects up the infrared behaviour of the field theory with the ultraviolet behaviour.

(c) More directly, we note that the symmetries of the theory enhance significantly near the $N = 0$ hypersurface. Conformal invariance, similar to the one found above, occurs in the gravitational sector as well. Defining $q = -\xi$ by $dq = -dN/N(Na)$, we see that $N \approx \exp(-\kappa q)$ near the horizon, where $Na \approx \kappa$. The space part of the metric in Eq.(8) becomes, near the horizon $dl^2 = N^2(dq^2 + e^{2\kappa q} dL_\perp^2)$ which is conformal to the metric of the anti-De Sitter (AdS) space. The horizon becomes the $q \to \infty$ surface of the AdS space. These results hold in any dimension.

(d) Finally, one can construct the metric in the bulk by a Taylor series expansion, from the form of the metric near the horizon, along the lines of exercise 1 (page 290) of [28]. These ideas work only because, algebraically, $N \to 0$ makes certain terms in the diffeomorphisms vanish and increases the symmetry. There is a strong indication that most of the results related to horizons will arise from the enhanced symmetry of the theory near the $N = 0$ surface (see e.g. [42,43,44] and references cited therein).

Metric	Rindler	Schwarzschild	De Sitter
$f(l)$	$2\kappa l$	$\left[1 - \frac{2M}{l}\right]$	$(1 - H^2 l^2)$
$\kappa = \frac{1}{2}f'(l_H)$	κ	$\dfrac{1}{4M}$	$-H$
ξ	$\frac{1}{2\kappa}\ln \kappa l$	$l + 2M\ln\left[\frac{l}{2M} - 1\right]$	$\frac{1}{2H}\ln\left(\frac{1-Hl}{1+Hl}\right)$
κX	$\sqrt{2\kappa l}\cosh \kappa t$	$e^{\frac{l}{4M}}\left[\frac{l}{2M} - 1\right]^{1/2}\cosh\left[\frac{t}{4M}\right]$	$\left(\frac{1-Hl}{1+Hl}\right)^{1/2}\cosh Ht$
κT	$\sqrt{2\kappa l}\sinh \kappa t$	$e^{\frac{l}{4M}}\left[\frac{l}{2M} - 1\right]^{1/2}\sinh\left[\frac{t}{4M}\right]$	$\left(\frac{1-Hl}{1+Hl}\right)^{1/2}\sinh Ht$

Table 1
Properties of Rindler, Schwarzschild and De Sitter metrics

2.5 Examples of spacetimes with horizons

While it is possible to have different kinds of solutions to Einstein's equations with horizons, some of the solutions have attracted significantly more attention than others. Table 1 summarises the features related to three of these solutions. In each of these cases, the metric can be expressed in the form Eq. (11) with different forms of $f(l)$ given in the table. All these cases have only one horizon at some surface $l = l_H$ and the surface gravity κ is well defined. (We have relaxed the condition that the horizon occurs at $l = 0$; hence κ is defined as $(1/2)f'$ evaluated at the location of the horizon, $l = l_H$.) The coordinates (T, X) are well behaved near the horizon while the original coordinate system (t, l) is singular at the horizon. Figure 1 describes all the three cases of horizons which we are interested in, with suitable definition for the coordinates.

In all the cases the horizon at $l = l_H$ corresponds to the light cones through the origin $(T^2 - X^2) = 0$ in the freely falling coordinate system. it is conventional to call the $T = X$ surface as the future horizon and the $T = -X$ surface as the past horizon. Also note that the explicit transformations to (T, X) given in Table 1 corresponds to $l > 0$ and the right wedge, $\mathcal{R}$. Changing l to $-l$ in these equations with $l < 0$ will take care of the left wedge, $\mathcal{L}$. The future and past regions will require interchange of sinh and cosh factors. These are direct generalisation of the transformations in Eq. (12) and Eq. (13).

The simplest case corresponds to flat spacetime in which (T, X) are the Minkowski coordinates and (t, l) are the Rindler coordinates. The range of coordinates extends to $(-\infty, \infty)$. The g_{00} does not go to (-1) at spatial infinity in (t, l) coordinates and the horizon is at $l = 0$.

The second case is that of a Schwarzschild black hole. The full manifold is described in the (T, X) coordinates, (called the Kruskal coordinates, which are analogous to the inertial coordinates in flat spacetime) but the metric is *not* static in terms of the Kruskal time T. The horizon at $X^2 = T^2$ divides the black hole manifold into the four regions $\mathcal{R}, \mathcal{L}, \mathcal{F}, \mathcal{P}$. In terms of the Schwarzschild coordinates, the metric is independent of t and the horizon is at $l = 2M$ where M is the mass of the black hole. The standard Schwarzschild coordinates (t, l) is a 2-to-1 map from the Kruskal coordinates (T, X). The region $l > 2M$ which describes the exterior of the black hole corresponds to $\mathcal{R}$ and $\mathcal{L}$ and the region $0 < l < 2M$, that describes the interior of the black hole, corresponds to $\mathcal{F}$ and $\mathcal{P}$. The transverse coordinates are now (θ, ϕ) and the surfaces $t =$ constant, $l =$ constant are 2-spheres.

In the case of a black hole formed due to gravitational collapse, the Schwarzschild solution is applicable to the region outside the collapsing matter, if the collapse is spherically symmetric. The surface of the collapsing matter will be a timelike curve cutting through $\mathcal{R}$ and $\mathcal{F}$, making the whole of $\mathcal{L}, \mathcal{P}$ (and part of $\mathcal{R}$ and $\mathcal{F}$) irrelevant since they will be inside the collapsing matter. In this case, the past horizon does not exist and we are only interested in the future horizon. Similar considerations apply whenever the actual solution corresponds only to part of the full manifold.

There are five crucial differences between the Rindler and Schwarzschild coordinates: (i) The Rindler coordinates represents flat spacetime which is a non singular manifold. The Schwarzschild coordinates describe a black hole manifold which has a physical singularity at $l = 0$ corresponding to $T^2 - X^2 = 16M^2$. Thus a world line $X =$ constant, crosses the horizon and hits the singularity in finite T. The region $T^2 - X^2 > 16M^2$ is treated as physically irrelevant in the manifold. (ii) In the Rindler metric, g_{ab} does not tend to η_{ab} when $|\mathbf{x}| \to \infty$ while in the Schwarzschild metric it does. (iii) The Rindler metric is independent of the t coordinate just as the Schwarzschild metric is independent of the t coordinate. Of course, the flat spacetime is static in T coordinate as well while the black hole spacetime is not static in the Kruskal coordinates. (iv) The surfaces with $t =$ constant, $l =$ constant are 2-spheres with finite area in the case of Schwarzschild coordinates; for example, the horizon at $l = 2M$ has the area $16\pi M^2$. In contrast, the transverse dimensions are non-compact in the case of Rindler coordinates and the horizon at $l = 0$ has infinite transverse area. (v) There is a non trivial, time dependent, dynamics in the black hole manifold which is not easy to see in the Schwarzschild coordinates but is obvious in the Kruskal coordinates. The geometrical structure of the full manifold contains two asymptotically flat regions connected by a worm-hole like structure [13].

Because of these features, the (t, l) Schwarzschild coordinate system has an intuitive appeal which Kruskal coordinate system lacks, in spite of the math-

ematical fact that Kruskal coordinate system is analogous to the inertial coordinate system while the Schwarzschild coordinate system is like the Rindler coordinate system.

The third spacetime listed in Table 1 is the De Sitter spacetime which, again, admits a Schwarzschild type coordinate system and a Kruskal type coordinate system. The horizon is now at $l = H^{-1}$ and the spacetime is *not* asymptotically flat. There is also a reversal of the roles of "inside" and "outside" of the horizon in the case of De Sitter spacetime. If the Schwarzschild coordinates are used on the black hole manifold, an observer at large distances ($l \rightarrow \infty$) from the horizon ($l = 2M$) will be stationed at nearly flat spacetime and will be confined to $\mathcal{R}$. The corresponding observer in the De Sitter spacetime is at $l = 0$ which is again in $\mathcal{R}$. Thus the nearly inertial observer in the De Sitter manifold is near the origin, "inside" the horizon, while the nearly inertial observer in the black hole manifold is at a large distance from the horizon and is "outside" the horizon; but both are located in the region $\mathcal{R}$ in figure 1 making this figure to be of universal applicability to all these three metrics. The transverse dimensions are compact in the case of De Sitter manifold as well.

The De Sitter manifold, however, has a high degree of symmetry and in particular, homogeneity [45,24]. It is therefore possible to obtain a metric of the kind given in Table 1 with any point on the manifold as the origin. (This is in contrast with the black hole manifold where the origin is fixed by the source singularity and the manifold is not homogeneous.) The horizon is different for different observers thereby introducing an observer dependence into the description. This is not of any deep significance in the approach we have adopted, since we have always defined the horizon with respect a family of observers.

It is certainly possible to provide *a* purely geometrical definition of horizon in some spacetimes like, for example, the Schwarzschild spacetime. The boundary of the causal past of the future time-like infinity in Schwarzschild spacetime will provide an intrinsic definition of horizon. But there exists time-like curves (like those of observers who fall into the black holes) for which this horizon does not block information. The comments made above should be viewed in the light of whether it is physically relevant and *necessary* to define horizons as geometric entities rather than whether it is *possible* to do so in certain spacetimes. In fact, a purely geometric definition of horizon actually hides certain physically interesting features. It is better to define horizons with respect to a family of observers (congruence of timelike curves) as we have done.

As an aside, it may be noted that our definition of horizon ("causal horizon") is more general than that used in the case of black hole spacetimes etc. in the following sense: (a) these causal horizons are always present in any space-time

for suitable choice of observers and (b) there is no notion of any "marginally trapped surfaces" involved in their definition. There is also no restriction on the topology of the two-dimensional surfaces (suitably defined sections of the boundary of causal past). Essentially, the usual black hole horizons are causal horizons but not conversely. For our purpose, the causal horizon defined in the manner described earlier turns out to be most appropriate. This is because it provides a notion of regions in spacetimes which are not accessible to a *particular class of observers* and changes with the class of observers under consideration. While more geometrical notions of horizons defined without using a class of observers definitely have their place in the theory, the causal horizon incorporates structures like Rindler horizon which, as we shall see, prove to be very useful. We stress that, though causal horizons depend on the family of time like curves which we have chosen — and thus is foliation dependent — it is generally covariant. Ultimately, definitions of horizons are dictated by their utility in discussing the issue we are interested in and for our discussion causal horizon serves this purpose best.

While the three metrics in Table 1 act as prototypes in our discussion, with sufficient amount of similarities *and differences* between them, most of our results are applicable to more general situations. The key features which could be extracted from the above examples are the following: There is a Killing vector field which is timelike in part of the manifold with the components $\xi^a = (1, 0, 0, 0)$ in the Schwarzschild-type static coordinates. The norm of this field $\xi^a \xi_a$ vanishes on the horizon which arises as a bifurcation surface $\mathcal{H}$. Hence, the points of $\mathcal{H}$ are fixed points of the killing field. There exists a spacelike hypersurface Σ which includes $\mathcal{H}$ and is divided by $\mathcal{H}$ into two pieces Σ_R and Σ_L, the intersection of which is in fact $\mathcal{H}$. (In the case of black hole manifold, Σ is the $T = 0$ surface, Σ_R and Σ_L are parts of it in the right and left wedges and $\mathcal{H}$ corresponds to the $l = 2M$ surface.) The topology of Σ_R and $\mathcal{H}$ depends on the details of the spacetime but $\mathcal{H}$ is assumed to have a non-zero surface gravity. Given this structure it is possible to generalise most of the results we discuss in the coming Sections.

The analysis in Section 2.3 shows that it is possible to associate a temperature with each of these horizons. In the case of a black hole manifold, an observer at $l = R \gg 2M$ will detect radiation at late times $(t \to \infty)$ which originated from near the horizon $l = 2M$ at early times. This radiation will have a temperature $T = (\kappa/2\pi) = (1/8\pi M)$ [9]. In the case of De Sitter spacetime, an observer near the *origin* will detect radiation at late times which originated from near the horizon at $l = H^{-1}$. The temperature in this case will be $T = (H/2\pi)$ [46]. In each of the cases, the temperature of this radiation, $T = \kappa/2\pi$, is determined by the surface gravity of the horizon.

3 Quantum field theory in singular gauges and thermal ambience

Horizons introduce new features in quantum theory as one proceeds from non-relativistic quantum mechanics (NRQM) to relativistic quantum theory. NRQM has a notion of absolute time t (with only $t \rightarrow c_1 t + c_2$, $c_1 > 0$ being the allowed symmetry transformation) and exhibits invariance under the Galilean group. In the path integral representation of non-relativistic quantum mechanics, one uses only the causal paths $X^\alpha(t)$ which "go forward" in this absolute time coordinate t.

This restriction has to be lifted in special relativity and the corresponding path integrals use paths $X^a(s) = (X^0(s), X^\alpha(s))$, which go forward in the proper-time s but either forward or backward in coordinate time X^0. In the path integral, this requires summing over paths which could intersect the $X^0 = $ constant plane on several points, going forwards and backwards. For such a path, the particle could be located at infinitely many points on the $X^0 = $ constant hypersurface, which is equivalent to having a many-particle state at any given time X^0. So if we demand a description in which causality is maintained and information on the $X^0 = $ constant hypersurface could be used to predict the future, such a description should be based on a system which is mathematically equivalent to infinite number of non relativistic point particles, located at different spatial locations, at any given time. Thus combining special relativity, quantum mechanics and causality *requires* the use of such constructs with infinite number of degrees of freedom and quantum fields are such constructs (see, for example, [47]). In the case of a free particle, this result is summarised by:

$$G_F(Y, X) \equiv \int_0^\infty ds e^{-ims} \int \mathcal{D}Z^a e^{iA[Y,s;X,0]} = \langle 0|T[\phi(Y)\phi(X)]|0\rangle \qquad (30)$$

Here $A[Y, s; X, 0]$ is the action for the relativistic particle to propagate from X^a to Y^a in the proper time s and the path integral is over all paths $Z^a(\tau)$ with these boundary conditions. The integral over all values of s (with the phase factor $\exp(-iEs) = \exp(-ims)$ corresponding to the energy $E = m$ conjugate to proper time s) gives the amplitude for the particle to propagate from X^a to Y^a. There is no notion of a quantum field in at this juncture; the second equality shows that the same quantity can be expressed in terms of a field.

It should be stressed that $G_F(Y, X) \neq 0$ when X^a and Y^a are separated by a spacelike interval; the propagation amplitude for a relativistic particle to cross a light cone (or horizon) is *non zero* in quantum field theory. Conventionally, this amplitude is reinterpreted in terms of particle-anti particle pairs. There is a well-defined way of ensuring covariance under Lorentz transformations for this interpretation and since all inertial observers see the same light cone

structure it is possible to construct a Lorentz invariant quantum field theory.

The description in Eq. (30) is (too) closely tied to the existence of a global time coordinate T (and those obtained by a Lorentz transformation from that). One can decompose the field operator $\phi(T, \mathbf{X})$ into positive frequency modes [which vary as $\exp(-i\Omega T)$] and negative frequency modes [which vary as $\exp(+i\Omega T)$] in a Lorentz invariant manner and use corresponding creation and annihilation operators to define the vacuum state. Two observers related by a Lorentz transformation will assign different (Doppler shifted) frequencies to the same mode but a positive frequency mode will always be seen as a positive frequency mode by any other inertial observer. The quantum state $|0\rangle$ in Eq. (30), interpreted as the vacuum state, is thus Lorentz invariant. There is also a well defined way of implementing covariance under Lorentz transformation in the Hilbert space so that the expectation values are invariant. The standard procedure for implementing a classical symmetry in quantum theory is to construct a unitary operator U corresponding to the symmetry and change the states of the Hilbert space by $|\psi\rangle \to U|\psi\rangle$ and change the operators by $O \to UOU^{-1}$ so that the expectation values are unaltered. This can be done in the case of Lorentz transformations.

The next logical step will be to extend these ideas to curvilinear coordinates in flat spacetime (thereby extending the invariance group from Lorentz group to general coordinate transformation group) and to curved spacetime. Several difficulties arise when we try to do this.

(i) If the background metric depends on time in a given coordinate system, then the quantum field theory reduces to that in an external time dependent potential. In general, this will lead to production of particles by the time dependent background gravitational field. On many occasions, like in the case of an expanding universe, this is considered a "genuine" physical effect [48,49]. If, on the other hand, the metric is static in a given coordinate system, one would have expected that a vacuum state could be well defined and no particle production can take place. This is true as long as the spacetime admits a global timelike Killing vector field throughout the manifold. If this is not the case, and the Killing vector field is timelike in one region and spacelike in another, then the situation becomes more complex. The usual examples are those with horizons where the norm of the Killing vector vanishes on the bifurcation surface which, in fact, acts as the horizon. In general, it is possible to provide different realizations of the algebra of commutators of field operators, each of which will lead to a different quantum field theory. These different theories will be (in general) unitarily inequivalent and the corresponding quantum states will be elements of different Hilbert spaces. If we want to introduce general covariance as a symmetry in quantum theory, we need unitary operators which could act on the states in the Hilbert space. *This procedure, however, is impossible to implement.* Mathematically, the elements

of general coordinate transformation group (which is an infinite dimensional Lie group) cannot be handled in the same way as the elements of Lorentz group (which can be obtained by exponentiating elements close to identity or as the products of such exponentials).

(ii) The standard QFT requires analytic continuation into complex plane of independent variables for its definition. It is conventional to provide a prescription such that the propagator G_F propagates positive frequency modes of the field forward in time and the negative frequency modes backward in time. This can be done either (a) through an $i\epsilon$ prescription or (b) by defining G_F in the Euclidean sector and analytically continuing to Minkowski spacetime. Both these procedures (implicitly) select a global time coordinate [more precisely an equivalence class of time coordinates related by Lorentz transformations]. *This procedure is not generally covariant.* The analytic continuation $t \to it$ and the general coordinate transformation $t \to f(t', \mathbf{x}')$ do not commute and one obtains different quantum field theories in different coordinate systems.

(iii) One can also define G_F as a solution to a differential equation, but $\langle \psi | T[\phi(Y)\phi(X)] | \psi \rangle$ for any state $|\psi>$ satisfies the same differential equation and the hyperbolic nature of this wave equation requires additional prescription to choose the appropriate G_F. This can be done by the methods (a) or (b) mentioned in (ii) above, in case of inertial frames in flat spacetime. But in curvilinear coordinate system or in curved spacetime, this wave operator defining G_F can be ill-defined at coordinate singularities (like horizons) and one requires extra prescriptions to handle this.

We shall now study several explicit manifestations of these difficulties, their resolutions and the physical consequences.

3.1 Singular gauge transformations and horizon

In many manifolds with horizon, like those discussed in Section 2.5, one can usually introduce a global coordinate system covering the full manifold in which the metric is non singular though (possibly) not static. A clear example is the Kruskal coordinate system in the black hole manifold in which the metric depends on the Kruskal time coordinate. Quantum field theory in such a coordinate system will require working with a time dependent Hamiltonian; no natural vacuum state exists on such a global manifold because of this time dependence.

Many of these manifolds also allow transformation to another coordinate system (like the Schwarzschild coordinate system) in which the metric is independent of the new time coordinate. There exists a well defined family of observers who will be using this coordinate system and the question arises as to how

they will describe the quantum field theory. The metric in the new coordinates is singular on the horizon and we need to ask how that singularity needs to regularised and interpreted. These singular coordinate transformations require careful, special handling since they cannot be obtained by "exponentiating" infinitesimal, non-singular coordinate transformations.

To see this issue clearly, it is better to use the concept of gauge transformations rather than coordinate transformations. In the standard language of general relativity, one has a manifold with a metric and different choices can be made for the coordinate charts on the manifold. When one changes from a coordinate chart x^i to $\bar{x}^i$, the metric coefficients (and other tensors) change in a specified manner. In the language of particle physics, the same effect will be phrased differently. The coordinate chart and a back ground metric can be fixed at some fiducial value at first; the theory is then seen to be invariant under some infinitesimal transformations $g_{ij} \to g_{ij} + \delta g_{ij}$ where δg_{ij} can be expressed in terms of four gauge functions $\xi^a(x)$ by $\delta g_{ij} = -\nabla_i \xi_j - \nabla_j \xi_i$. The translation between the two languages is effected by noticing that the infinitesimal coordinate transformation $x^i \to x^i + \xi^i(x)$ will lead to the same δg_{ab} in the general relativistic language.

It is now clear that there are two separate types of gauge (or coordinate) transformations which we need to consider: the *infinitesimal* ones and the *large* ones. The *infinitesimal* gauge transformations of the theory, induced by the four gauge functions ξ^i have the form $\delta g_{ij} = -\nabla_i \xi_j - \nabla_j \xi_i$. For example, the transformation induced by $\xi^a_{(R)} = (-\kappa X T, -(1/2)\kappa T^2, 0, 0)$ changes the flat space-time metric $g_{ab} = (-1, 1, 1, 1)$ to the form $g_{ab} = (-(1 + 2\kappa X), 1, 1, 1)$, up to first order in ξ. This could be naively thought of as the infinitesimal version of the transformation to the accelerated frame. (It is naive because the "small" parameters here are $(\kappa X, \kappa T)$ and we run into trouble at large (X, T).) Obviously, one cannot describe a situation in which $N \to 0$ within the class of infinitesimal transformations.

The classical theory is also invariant under finite transformations, which are more "dangerous". Of particular importance are the *large gauge transformations*, which are capable of changing $N > 0$ in a non-singular coordinate system to a nontrivial function $N(x^a)$ that vanishes on a hypersurface. The transformation from the $(T, \mathbf{X})$ to the Schwarzschild type coordinates belongs to precisely this class. In particular, the coordinate transformation which changes the metric from $g_{ab} = (-1, 1, 1, 1)$ to $g_{ab} = (-(1 + \kappa X)^2, 1, 1, 1)$ is the "large" version of the infinitesimal version generated by $\xi^a_{(R)}$. Given such large gauge transformations, we can discuss regions arbitrarily close to the $N = 0$ surface.

A new issue, which is conceptually important, comes up while doing quantum field theory in a spacetime with a $N = 0$ surface. All physically relevant results in the spacetime will depend on the combination Ndt rather than on

the coordinate time dt. The Euclidean rotation $t \to te^{i\pi/2}$ can equivalently be thought of as the rotation $N \to Ne^{i\pi/2}$. This procedure becomes ambiguous on the horizon at which $N = 0$. But the family of observers with a horizon, *will* indeed be using a comoving co-ordinate system in which $N \to 0$ on the horizon. Clearly we need a new physical principle to handle quantum field theory as seen by this family of observers.

To resolve this ambiguity, it is necessary to work in complex plane in which the metric singularity can be avoided. This, in turn, can be done either by analytically continuing in the time coordinate t or in the space coordinate x. The first procedure of analytically continuing in t is well known in quantum field theory but not the second one since one rarely works with space dependent Hamiltonian in standard quantum field theory. We shall briefly describe these two procedures and use them in the coming Sections.

Let us consider what happens to the coordinate transformations in Eq. (18) and the metric near the horizon, when the analytic continuation $T \to T_E = Te^{i\pi/2}$ is performed. The hyperbolic trajectory in Eq. (1) for $N = 1$ (for which t measures the proper time), is given in parametric form as $\kappa T = \sinh \kappa t, \kappa X = \cosh \kappa t$. This becomes a circle, $\kappa T_E = \sin \kappa t_E, \kappa X = \cos \kappa t_E$ with, $-\infty < t_E < +\infty$ on analytically continuing in both T and t. The mapping $\kappa T_E = \sin \kappa t_E$ is many-to-one and limits the range of κT_E to $\kappa|T_E| \leq 1$ for $(-\infty < t_E < \infty)$.

Further, the *complex plane probes the region which is classically inaccessible* to the family of observers on $N = $ constant trajectory. The transformations in (1) with $N > 0, -\infty < t < \infty$ cover *only* the right hand wedge $[|X| > |T|, X > 0]$ of the Lorentzian sector; one needs to take $N < 0, -\infty < t < \infty$ to cover the left hand wedge $[|X| > |T|, X < 0]$. Nevertheless, *both $X > 0$ and $X < 0$* are covered by different ranges of the "angular" coordinate t_E. The range $(-\pi/2) < at_E < (\pi/2)$ covers $X > 0$ while the range $(\pi/2) < at_E < (3\pi/2)$ covers $X < 0$. The light cones of the inertial frame $X^2 = T^2$ are mapped into the origin of the $T_E - X$ plane. The region "inside" the horizon $|T| > |X|$ simply *disappears* in the Euclidean sector. Mathematically, Eq. (18) shows that $\kappa t \to \kappa t - i\pi$ changes X to $-X$, ie., the complex plane contains information about the physics beyond the horizons through imaginary values of t.

This fact is used in one way or another in several derivations of the temperature associated with the horizon [50,51,46,52,53,54,55,56]. Performing this operation twice shows that $\kappa t \to \kappa t - 2i\pi$ is an identity transformation implying periodicity in the imaginary time $i\kappa t = \kappa t_E$. More generally, all the events $\mathcal{P}_n \equiv (t = (2\pi n/\kappa), \mathbf{x})$ [where $n = \pm 1, \pm 2, ...$] which correspond to *different* values of T and X will be mapped to the *same* point in the Euclidean space.

This feature arises naturally when we analytically continue in the time coordinate t to the Euclidean sector. If we take $t_E = it$, then the metric near the

horizon becomes:

$$ds^2 \approx N^2 dt_E^2 + (dN/\kappa)^2 + dL^2 \tag{31}$$

Near the origin of the $t_E - N$ plane, this is the metric on the surface of a cone. The conical singularity at the origin can be regularised by assuming that t_E is an angular coordinate with $0 < \kappa t_E \leq 2\pi$. When we analytically continue in t and map the $N = 0$ surface to the origin of the Euclidean plane, the ambiguity of defining Ndt on the horizon becomes similar to the ambiguity in defining the θ direction of the polar coordinates at the origin of the plane. This can be resolved by imposing the periodicity in the angular coordinate (which, in the present case, is the imaginary time coordinate).

This procedure of mapping $N = 0$ surface to the origin of Euclidean plane will play an important role in later discussion (see Section 8). To see its role in a broader context, let us consider a class of observers who have a horizon. A natural interpretation of general covariance will require that these observers will be able to formulate quantum field theory entirely in terms of an "effective" spacetime manifold made of regions which are accessible to them. Further, since the quantum field theory is well defined only in the Euclidean sector [or with an $i\epsilon$ prescription] it is necessary to construct an effective spacetime manifold *in the Euclidean sector* by removing the part of the manifold which is hidden by the horizon. For a wide class of metrics with horizon, the metric close to the horizon can be approximated by Eq. (31) in which (the region inside) the horizon is reduced to a point which we take to be the origin. The region close to the origin can be described in Cartesian coordinates (which correspond to the freely falling observers) or in polar coordinates (which would correspond to observers at rest in a Schwarzschild-type coordinates) in the Euclidean space. The effective manifold for the observers with horizon can now be thought to be the Euclidean manifold with the origin removed. This principle is of very broad validity since it only uses the form of the metric very close to the horizon where it is universal. The structure of the metric far away from the origin can be quite complicated (there could even be another horizon elsewhere) but the key topological features are independent of this structure. It seems reasonable, therefore, to postulate that the physics of the horizons need to be tackled by using an effective manifold, the topology of which is non trivial because a point (corresponding to the region blocked by the horizon) is removed. We will pursue this idea further in Section 8 and show how it leads to a deeper understanding of the link between gravity and thermodynamics.

There is a second, equivalent, alternative for defining the theories in singular static manifolds. This is to note that the Euclidean rotation is equivalent to the $i\epsilon$ prescription in which one uses the transformation $t \to t(1 + i\epsilon)$ which, in turn, translates to $N \to N(1 + i\epsilon)$. Expanding this out, we get

$$N \to N + i\epsilon \, \text{sign}(N) \tag{32}$$

Near the origin, the above transformation is equivalent to $l \to l(1 + i\epsilon)$ [since $l \propto N^2$]. Hence,

$$l \to l + i\epsilon \, \mathrm{sign}(l) \qquad (33)$$

This procedure involves analytic continuation in the *space* coordinate N while the first procedure uses analytic continuation in the time coordinate. Both the procedures will lead to identical conclusions but in different manners. We shall now explore how this arises.

3.2 Propagators in singular gauges

Let us begin by computing the amplitude for a particle to propagate from an event P to another event P' with an energy E [51]. From the general principles of quantum mechanics, this is given by the Fourier transform of the Green's function $G_F[P \to P']$ with respect to the time coordinate. The vital question, of course, is which time coordinate is used as a conjugate variable to energy E. Consider, for example, the flat spacetime situation with P' being some point on $T = t = 0$ axis in $\mathcal{R}$ and P being some event in $\mathcal{F}$ with the *Rindler* coordinates $(t, l, 0, 0)$. The amplitude $G_F[P \to P']$ will now correspond to a particle propagating from the inside of the horizon to the outside. (See Fig. 2; the fact that this amplitude is non zero in quantum field theory is a necessary condition for the rest of the argument.) The amplitude for this propagation to take place with the particle having an energy E — when measured with respect to the *Rindler time coordinate* — is given by

$$\mathcal{Q}(E; P \to P') = \int_{-\infty}^{\infty} dt \, e^{-iEt} G_F[P(t, \mathbf{y}) \to P'(0, \mathbf{x})] \qquad (34)$$

(The notation in the left hand side should be interpreted as being *defined* by the right hand side; obviously, the *events* P and P' can be specified only when the time coordinate is fixed but we are integrating over the time coordinate to obtain the corresponding amplitude in the energy space. The minus sign in $\exp(-iEt)$ is due to the fact that t is the time coordinate of the *initial* event P.) Shifting the integration by $t \to t - i(\pi/\kappa)$ in the integral we will pick up a prefactor $\exp(-\pi E/\kappa)$; further, the event P will become the event P_R obtained by reflection at the origin of the inertial coordinates [see Eqs. (12),(13)]. We thus get

$$\mathcal{Q}(E; P \to P') = e^{-\pi E/\kappa} \int_{-\infty}^{\infty} dt \, e^{-iEt} G_F[P_R \to P'] = e^{-\pi E/\kappa} \mathcal{Q}(E; P_R \to P') \qquad (35)$$

The reflected event P_R is in the region $\mathcal{P}$; the amplitude $\mathcal{Q}(E; P_R \to P')$ corresponds to the emission of a particle by the past horizon ("white hole" in the case of Schwarzschild spacetime) into the region $\mathcal{R}$. By time reversal invariance, the corresponding probability is also the same as the probability

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P_{abs} for the black hole to absorb a particle. It follows that the probability for emission and absorption of a particle with energy E across the horizon are related by

$$P_{\mathrm{em}} = P_{\mathrm{abs}} \exp\left(-\frac{2\pi E}{\kappa}\right) \tag{36}$$

This result can be directly generalised to any other horizon since the ingredients which we have used are common to all of them. The translation in time coordinates $t \to t - i(\pi/\kappa)$ requires analyticity in a strip of width (π/κ) in the complex plane but this can be proved in quite general terms.
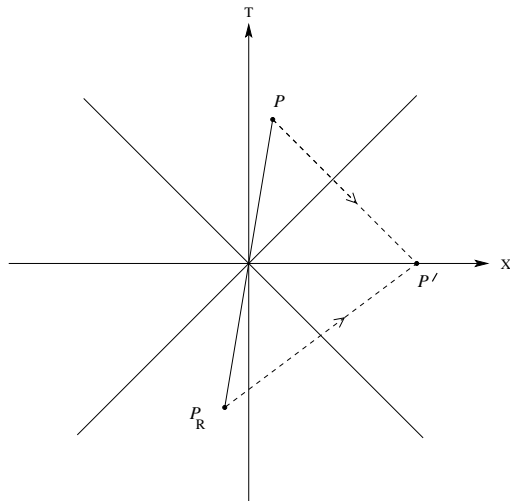

Fig. 2. The relation between absorption and emission probabilities across the horizon. See text for details.

The fact that the propagation amplitudes between two events in flat spacetime can bear an exponential relationship is quite unusual. The crucial feature is that the relevant amplitude is defined at constant energy E, which in turn involves Fourier transform of the Green's function with respect to the Rindler time coordinate t. It is this fact which leads to the Boltzmann factor in virtually every derivation we will discuss.

To see this result more explicitly, let us ask how the amplitude in Eq. (30) in flat spacetime will be viewed by observers following the trajectories in Eq. (1) for $N = 1$. For mathematical simplicity, let us consider a massless particle, for which $G_F(Y,X) = -(4\pi^2)^{-1}[s^2(Y,X) - i\epsilon]^{-1}$ where $s(Y,X)$ is the spacetime interval between the two events. Consider now $G_F(Y,X)$ between two events along the trajectory in Eq. (1) with $N = 1$. Treating $G_F(Y,X)$ as a scalar, we find that

$$G_{\mathrm{F}}(Y(t), X(t')) = -\frac{1}{4\pi^2} \frac{(\kappa/2)^2}{\sinh^2\left[\kappa(t-t')/2\right] - i\epsilon} \tag{37}$$

The first striking feature of this amplitude is that it is periodic in the imaginary time under the change $it \to it + 2\pi/\kappa$ which arises from the fact that Eq.(1) has this property. In the limit of $\kappa \to 0$, the G_F is proportional to $[(t - t')^2 - i\epsilon]^{-1}$ which is the usual result in inertial coordinates. Next, using the

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series expansion for $\mathrm{cosech}^2 z$, we see that the propagator in Eq. (37) can be expressed as a series:

$$G_F(\tau) = -\frac{1}{4\pi^2} \sum_{n=-\infty}^{n=\infty} \left[\left(\tau + 2\pi i n \kappa^{-1}\right)^2 - i\epsilon \right]^{-1} \tag{38}$$

where $\tau = (t - t')$. The $n = 0$ term corresponds to the inertial propagator (for $\kappa = 0$) and the other terms describe the new effects. If we interpret the Fourier transform of $G(t-t')$ as the amplitude for propagation in energy space, Eq.(38) will give an amplitude

$$\Delta G(|E|) \equiv \int_{-\infty}^{+\infty} d\tau e^{iE\tau} \Delta G(\tau) = \frac{1}{2\pi} \frac{|E|}{\exp(\beta|E|) - 1} \tag{39}$$

in which the Δ indicates that the $n = 0$ term has been dropped.

The new feature which has come about is the following: In computing $G_F(P, P')$ using Eq. (30) we sum over paths which traverses all over the $X - T$ plane even though the two events are in the right wedge. The paths which have contributed in Eq. (30) do criss-cross the horizon several times even though the region beyond the horizon is inaccessible to the observers following the trajectories in Eq. (1). The net effect of paths crossing the horizon leads to the extra term in Eq. (39). In fact, the $n > 0$ terms in Eq. (38) contribute for $E < 0$, while $n < 0$ terms contribute for $E > 0$. The result in Eq. (39) also shows that

$$\frac{\Delta G(|E|)}{\Delta G(-|E|)} = \exp(-\beta|E|) \tag{40}$$

which can be interpreted as the probability for a particle to cross the horizon in two different directions.

These features emerges more dramatically in the Euclidean sector [57,58,59]. The Euclidean Green's function is $G_E \propto R^{-2}$ where R^2 is the Euclidean distance between the two points. To express the same Euclidean Green's function in terms of t and t', we need to analytically continue in t as well by $t \to t_E = te^{i\pi/2}$. The Green's function now becomes, in terms of t_E, t'_E,

$$G_{\mathrm{E}}(Y_E(t_E), X_E(t'_E)) = \frac{1}{4\pi^2} \frac{(\kappa/2)^2}{\sin^2\left[\kappa(t_E - t'_E)/2\right]} \tag{41}$$

and can be expressed as a series:

$$G_E(t_E - t'_E) = \left(\frac{\kappa^2}{4\pi^2}\right) \sum_{n=-\infty}^{n=\infty} \left[\theta - \theta' + 2\pi n\right]^{-2} \tag{42}$$

with $\theta \equiv at_E$. Clearly, each term in the sum can be interpreted as due to a loop which winds n times around the circle of radius $x = 1/\kappa$ in the θ direction. But note that these winding paths go over the $X < 0$ region of Minkowski

space. Paths which wind around the origin in the Euclidean sector contains information about the region beyond the horizon (the left wedge) even though $x > 0$.

As we said before, these results emerge naturally once we realise that the physical theory (in this case the quantum field theory) should be formulated in an effective Euclidean manifold from which the region inaccessible to the chosen family of observers are removed. Here, this family is made of $N = $ constant observers and the inaccessible region corresponds to the origin of the Euclidean plane. The winding numbers for different paths as well as the fact that these paths probe the region beyond the horizon make the quantum field theory nontrivial.

3.3   Going around the horizon: complex plane

The above analysis involved analytic continuation in the time coordinate t which allowed one to probe the region beyond the horizon, that was classically inaccessible in Re-t. As we discussed in Section 3.1, the same results must also be obtainable from analytic continuation in N since only the combination $N dt$ is physically relevant. However, because $N \to 0$ on the horizon, we know (see Section 2.4) that the modes which vary as $\exp[-i\omega t]$ diverge on the horizon. The analytic continuation in N should regularise and interpret this behaviour meaningfully. In particular, Eq.(39) suggests that the probability for a particle with energy E to go from $l = -\delta$ to $l = \delta$ should have an exponential dependence in βE. It is interesting to see how this result can be interpreted in the "bad" coordinates (t, x).

This amplitude, for the outgoing mode ϕ_- in Eq. (28), is given by the ratio $\mathcal{Q} = [\phi_-(\delta)/\phi_-(-\delta)] \approx (-1)^{-i\omega/\kappa}$ which depends on the nature of the regulator used for defining this quantity. For $l < 0$, our prescription in Eq. (33) requires us to interpret l as having a small, negative imaginary part: $(l - i\epsilon)$. (The outgoing mode with positive frequency $\phi_- = \exp{-i\omega(t-\xi)} \propto \exp{i\omega\xi}$ is analytic in the upper half of complex-ξ plane and will pick up contributions only from poles in the upper half; to obtain nonzero contribution we need to shift the pole from $l = 0$ to $l = i\epsilon$ which is precisely the interpretation used above). This is same as moving along the l-axis in the lower half of the complex plane so that (-1) becomes $\exp(-i\pi)$. Then $\mathcal{Q} = \exp(-\omega(\pi/\kappa))$ and the probability is $|\mathcal{Q}|^2 = \exp(-\omega(2\pi/\kappa)) = \exp(-\beta\omega)$ which is the Boltzmann factor that we would have expected.

More formally, the above result can be connected up with the concept of anti-particles in field theory being particles traveling "backward in time" [50]. If we take ϕ_- as the outgoing particle state with positive frequency, then analytic

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continuation can be used to provide the corresponding anti-particle state. The standard field theory rule is that, if $\phi_-(l)$ describes a particle state $\phi_-(l - i\epsilon)$ will yield an anti particle state. Using the result

$$(l - i\epsilon)^{-i\omega/\kappa} = l^{-i\omega/\kappa}\theta(l) + |l|^{-i\omega/\kappa}e^{-\pi\omega/\kappa}\theta(-l) \tag{43}$$

this procedure splits the wave into two components which could be thought of as a particle-anti particle pair. The square of the relative weights of the two terms in the above equation, $e^{-2\pi\omega/\kappa}$ gives the Boltzmann factor. In fact, this relation can be used to interpret the amplitude for a particle to go from inside the horizon to outside in terms of a pair of particles being produced just outside the horizon with one falling into the horizon and the other escaping to infinity.

The analyticity arguments used above contain the gist of thermal behaviour of horizons. Since the positive frequency mode $\exp(-i\Omega U)$ (with $\Omega > 0$) is analytic in the lower half of complex $U \equiv (T - X)$ plane, any arbitrary superposition of such modes with different (positive) values of Ω will also be analytic in the lower half of complex U plane. Conversely, if we construct a mode which is analytic in the lower half of complex U plane, it can be expressed as a superposition of purely positive frequency modes [60]. From the transformations in Eq. (18), we find that the positive frequency wave mode near the horizon, $\phi = \exp(-i\omega u)$ can be expressed as $\phi \propto U^{i\omega/\kappa}$ for $U < 0$. If we interpret this mode as $\phi \propto (U - i\epsilon)^{i\omega/\kappa}$ then, this mode is analytic throughout the lower half of complex U plane. Using Eq. (43) with l replaced by U, we can interpret the mode as

$$(U - i\epsilon)^{i\omega/\kappa} = \begin{cases} e^{[i(\omega/\kappa)\ln U]} & \text{(for } U > 0) \\ e^{\pi\omega/\kappa}e^{[(i\omega/\kappa)\ln |U|)} & \text{(for } U < 0) \end{cases} \tag{44}$$

This interpretation of $\ln(-U)$ as $\ln|U| - i\pi = \kappa u - i\pi = \kappa t - \xi - i\pi$ is consistent with the procedure adopted in Section 3.2, viz., using $\kappa t \to \kappa t - i\pi$ to go from $X > 0$ to $X < 0$.

Similar results arise in a more general context for *any* system described by a wave function $\Psi(t, l; E) = \exp[iA(t, l; E)]$ in the WKB approximation [61]. The dependence of the quantum mechanical probability $P(E) = |\Psi|^2$ on the energy E can be quantified in terms of the derivative

$$\frac{\partial \ln P}{\partial E} \approx -\frac{\partial}{\partial E}2(\mathrm{Im}A) = -2\mathrm{Im}\left(\frac{\partial A}{\partial E}\right) \tag{45}$$

in which the dependence on (t, l) is suppressed. Under normal circumstances, action will be real in the leading order approximation and the imaginary part will vanish. (One well known example is in the case of tunneling in which the action acquires an imaginary part; Eq. (45) correctly describes the depen-

dence of tunneling probability on the energy.) For any Hamiltonian system, the quantity $(\partial A/\partial E)$ can be set to a constant t_0 thereby determining the trajectory of the system: $(\partial A/\partial E) = -t_0$. Once the trajectory is known, this equation determines t_0 as a function of E [as well as (t, l)]. Hence we can write

$$\frac{\partial \ln P}{\partial E} \approx 2\mathrm{Im}\,[t_0(E)] \tag{46}$$

From the trajectory Eq. (17) we note that $t_0(E)$ can pick up an imaginary part if the trajectory of the system crosses the horizon. In fact, since $\kappa t \to \kappa t - i\pi$ changes X to $-X$ [see Eqs. (12,13,18)], the imaginary part is given by $(-\pi/\kappa)$ leading to $(\partial \ln P/\partial E) = -2\pi/\kappa$. Integrating, we find that the probability for the trajectory of any system to cross the horizon, with the energy E will be given by the Boltzmann factor

$$P(E) \propto \exp\left[-\frac{2\pi}{\kappa}E\right] = P_0 \exp\left[-\beta E\right] \tag{47}$$

with temperature $T = \kappa/2\pi$. (For special cases of this general result see [62] and references cited therein.)

In obtaining the above result, we have treated κ as a constant (which is determined by the background geometry) independent of E. A more interesting situation develops if the surface gravity of the horizon changes when some amount of energy crosses it. In that case, we should treat $\kappa = \kappa(E)$ and the above result generalises to

$$P(E) \propto \exp -\int \frac{2\pi dE}{\kappa(E)} \equiv P(E_0) \exp[-(S(E) - S(E_0))] \tag{48}$$

where $dS \equiv (2\pi/\kappa(E))dE = dE/T(E)$ is very suggestive of an entropy function. An explicit example in which this situation arises is in the case of a spherical shell of energy E escaping from a a black hole of mass M. This changes the mass of the black hole to $(M - E)$ with the corresponding change in the surface gravity. The probability for this emission will be governed by the difference in the entropies $S(M) - S(M - E)$. When $E \ll M$ we recover the old result with $S(M) - S(M - E) \approx (\partial S/\partial M)E = \beta E$. (We shall say more about this in Section 7.)

Finally, it is interesting to examine how these results relate to the more formal approach to quantum field theory. The relation between quantum field theories in two sets of coordinates $(t, \mathbf{x})$ and $(T, \mathbf{X})$, related by Eq. (18), with the metric being static in the $(t, \mathbf{x})$ coordinates can be described as follows: Static nature suggests a natural decomposition of wave modes as

$$\phi(t, \mathbf{x}) = \int d\omega[a_\omega f_\omega(\mathbf{x})e^{-i\omega t} + a_\omega^\dagger f_\omega^*(\mathbf{x})e^{i\omega t}] \tag{49}$$

in $(t, \mathbf{x})$ coordinates. But, as we saw in Section 2.4, these modes are going to

behave badly (as $N^{\pm i\omega/\kappa}$) near the horizon since the metric is singular near the horizon in these coordinates. We could, however, expand $\phi(t, \mathbf{x})$ in terms of some other set of modes $F_\nu(t, \mathbf{x})$ which are well behaved at the horizon. This could, for example, be done by solving the wave equation in $(T, \mathbf{X})$ coordinates and rewriting the solution in terms of $(t, \mathbf{x})$. This gives an alternative expansion for the field:

$$\phi(t, \mathbf{x}) = \int d\nu [A_\nu F_\nu(t, \mathbf{x}) + A_\nu^\dagger F_\nu^*(t, \mathbf{x})] \tag{50}$$

Both these sets of creation and annihilation operators define two different vacuum states $a_\omega |0\rangle_a = 0, A_\nu |0\rangle_A = 0$. The modes $F_\nu(t, \mathbf{x})$ will contain both positive and negative frequency components with respect to t while the modes $f_\omega(\mathbf{x})e^{-i\omega t}$ are pure positive frequency components. The positive and negative frequency components of $F_\nu(t, \mathbf{x})$ can be extracted through the Fourier transforms:

$$\alpha_{\omega\nu} = \int_{-\infty}^{\infty} dt\ e^{i\omega t} F_\nu(t, \mathbf{x}_f); \quad \beta_{\omega\nu} = \int_{-\infty}^{\infty} dt\ e^{-i\omega t} F_\nu(t, \mathbf{x}_f) \tag{51}$$

where $\mathbf{x}_f$ is some convenient fiducial location far away from the horizon. One can think of $|\alpha_{\omega\nu}|^2$ and $|\beta_{\omega\nu}|^2$ as similar to unnormalised transmission and reflection coefficients. (They are very closely related to the Bogoliubov coefficients usually used to relate two sets of creation and annihilation operators.) The $a-$particles in the $|0\rangle_A$ state is determined by the quantity $|\beta_{\omega\nu}/\alpha_{\omega\nu}|^2$. If the particles are uncorrelated, then the normalised flux of out going particles will be

$$\mathcal{N} = \frac{|\beta_{\omega\nu}/\alpha_{\omega\nu}|^2}{1 - |\beta_{\omega\nu}/\alpha_{\omega\nu}|^2} \tag{52}$$

If the F modes are chosen to be regular near the horizon, varying as $\exp(-i\Omega U)$ etc., then Eq. (18) shows that $F_\nu(t, \mathbf{x}_f) \propto \exp(-i\Omega q e^{-\kappa t})$ etc. The integrals in Eq. (51) again reduces to the Fourier transform of an exponentially redshifted wave and we get $|\beta_{\omega\nu}/\alpha_{\omega\nu}|^2 = e^{-\beta\omega}$ and Eq. (52) leads to the Planck spectrum. This is the quantum mechanical version of Eq. (21) and Eq. (24).

When we can use WKB approximation we can also set $F_\nu(t, x) = \exp[iA_\nu(t, x)]$ in the integrals in Eq. (51) and use the saddle point approximation. The saddle point is to be determined by the condition

$$\pm\omega + \frac{\partial A_\nu}{\partial t} = 0 \tag{53}$$

where the upper sign is for $\alpha_{\omega\nu}$ and the lower sign is for $\beta_{\omega\nu}$. The upper sign corresponds to a saddle point trajectory with energy $E = \omega$ but, for $\beta_{\omega\nu}$ we get the condition $E = -\omega$ so that the trajectory has negative energy. Writing

the saddle point trajectory as $x_\pm(t)$ it is easy to show that

$$|\alpha_{\omega\nu}|^2 = \exp\left[-2\mathrm{Im}\int_{x_+0}^{x_f} p_+(x)dx\right]; \quad |\beta_{\omega\nu}|^2 = \exp\left[-2\mathrm{Im}\int_{x_-0}^{x_f} p_-(x)dx\right]$$
(54)

This result contains essentially the same mathematics as Eq. (46) since one can relate the imaginary part of t_0 to the imaginary part of $p = (\partial A/\partial x)$ through the HJ equation. Since positive energies are allowed while negative energies are classically forbidden, this will often lead to $|\alpha|^2 \approx 1$ and $|\beta|^2$ to be an exponentially small number.

The same result arises when one studies the problem of over-the-barrier reflection in the $(1/x^2)$ potential — to which the field theory near the horizon can be mapped because of scale invariance — using the method of complex paths [see, e.g., Eq. (A36) of [41]]. While the literature in this subject often uses the term "tunneling" [see eg., [53,63]] to describe the emergence of an imaginary part to p, A etc., in the context of horizons it is more appropriate to think of this process as "over-the-barrier reflection". Both the processes are governed by an exponential involving an integral of $p(x)$ over dx. In tunneling, $p(x)$ becomes imaginary when $p^2(x) \propto E - V(x)$ becomes negative. In the over the barrier reflection, $E > V$ and the transmission coefficient remains close to unity because the process is classically allowed. The imaginary part, leading to an exponentially small reflection coefficient, arises because one needs to analytically continue x into the complex plane just as we have done [64]. In Eq. (46) as well as in Eq. (54) the imaginary part arises because the path $x(t)$ needs to be deformed into the complex plane [41] rather than because the momentum p becomes complex.

4 Thermal Density Matrix from tracing over modes hidden by Horizon

In the previous few Sections, we have derived the thermality of horizons from the geometry of the line element in the Euclidean spacetime. The key idea has been the elimination of the region inaccessible in Re-t to a family of observers (the origin in the Euclidean plane) and using Im-t to probe these regions. If these ideas are consistent, the same effect should arise, when we construct the quantum field theory in the accessible region (in $N > 0$, say) by integrating out the information contained in $N < 0$. That is, one family of observers may describe the quantum state in terms of a wave function $\Psi(f_L, f_R)$ which depends on the field modes both on the "left" $(N < 0)$ and "right" $(N > 0)$ sides of the horizon while another family of observers will describe the same system by a density matrix obtained by integrating out the modes f_L in the inaccessible region. We shall now show that this is indeed the case using an

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adaptation of the analysis by [65](also see, [66]).

On the $T = t = 0$ hypersurface one can define a vacuum state $|\text{vac}\rangle$ of the theory by giving the field configuration for the whole of $-\infty < X < +\infty$. This field configuration, however, separates into two disjoint sectors when one uses the (t, N) coordinate system. Concentrating on the (T, X) plane and suppressing Y, Z coordinates in the notation for simplicity, we now need to specify the field configuration $\phi_R(X)$ for $X > 0$ and $\phi_L(X)$ for $X < 0$ to match the initial data in the global coordinates; given this data, the vacuum state is specified by the functional $\langle \text{vac} | \phi_L, \phi_R \rangle$.
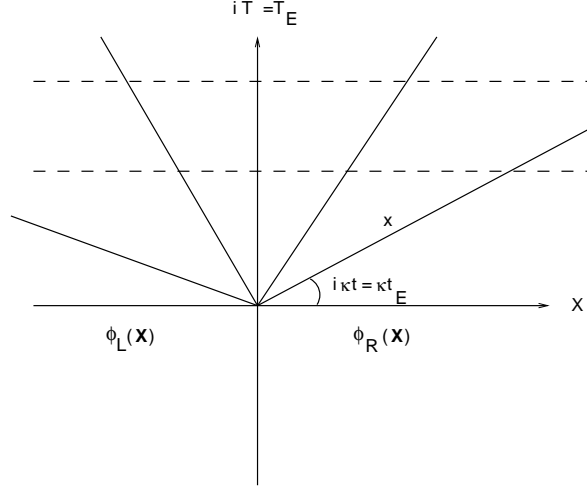

Fig. 3. Thermal effects due to a horizon; see text for a discussion.

Let us next consider the *Euclidean* sector corresponding to the (T_E, X) plane where $T_E = iT$. The QFT in this plane can be defined along standard lines. The analytic continuation in t, however, is a different matter; we see from Eq. (31) that the coordinates $(\kappa t_E = i\kappa t, x)$ are like polar coordinates in (T, X) plane with t_E having a periodicity of $(2\pi/\kappa)$. Figure 3 now shows that evolution in κt_E from 0 to π will take the system configuration from $X > 0$ to $X < 0$. This allows one to prove that $\langle \text{vac} | \phi_L, \phi_R \rangle \propto \langle \phi_L | e^{-\pi H/\kappa} | \phi_R \rangle$; normalisation now fixes the proportionality constant, giving

$$\langle \text{vac} | \phi_L, \phi_R \rangle = \frac{\langle \phi_L | e^{-\pi H/\kappa} | \phi_R \rangle}{\left[\text{Tr}(e^{-2\pi H/\kappa}) \right]^{1/2}} \tag{55}$$

To provide a simple proof of this relation, let us consider the ground state wave functional $\langle \text{vac} | \phi_L, \phi_R \rangle$ in the extended spacetime expressed as a path integral. The ground state wave functional can be represented as a Euclidean path integral of the form

$$\langle \text{vac} | \phi_L, \phi_R \rangle \propto \int_{T_E=0; \phi=(\phi_L, \phi_R)}^{T_E=\infty; \phi=(0,0)} \mathcal{D}\phi \, e^{-A} \tag{56}$$

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where $T_E = iT$ is the Euclidean time coordinate. From Fig. 3 it is obvious that this path integral could also be evaluated in the polar coordinates by varying the angle $\theta = \kappa t_E$ from 0 to π. When $\theta = 0$ the field configuration corresponds to $\phi = \phi_R$ and when $\theta = \pi$ the field configuration corresponds to $\phi = \phi_L$. Therefore

$$\langle \text{vac} | \phi_L, \phi_R \rangle \propto \int_{\kappa t_E = 0; \phi = \phi_R}^{\kappa t_E = \pi; \phi = \phi_L} \mathcal{D}\phi e^{-A} \tag{57}$$

But in the Heisenberg picture, this path integral can be expressed as a matrix element of the Hamiltonian H_R (in the (t, N) coordinates) giving us the result:

$$\langle \text{vac} | \phi_L, \phi_R \rangle \propto \int_{\kappa t_E = 0; \phi = \phi_R}^{\kappa t_E = \pi; \phi = \phi_L} \mathcal{D}\phi e^{-A} = \langle \phi_L | e^{-(\pi/\kappa)H_R} | \phi_R \rangle \tag{58}$$

Normalising the result properly gives Eq. (55).

This result, in turn, implies that for operators $\mathcal{O}$ made out of variables having support on $\mathcal{R}$, the vacuum expectation values $\langle \text{vac} | \mathcal{O}(\phi_R) | \text{vac} \rangle$ become thermal expectation values. This arises from straightforward algebra of inserting a complete set of states appropriately:

$$
\begin{aligned}
\langle \text{vac} | \mathcal{O}(\phi_R) | \text{vac} \rangle &= \sum_{\phi_L} \sum_{\phi_R^1, \phi_R^2} \langle \text{vac} | \phi_L, \phi_R^1 \rangle \langle \phi_R^1 | \mathcal{O}(\phi_R) | \phi_R^2 \rangle \langle \phi_R^2, \phi_L | \text{vac} \rangle \\
&= \sum_{\phi_L} \sum_{\phi_R^1, \phi_R^2} \frac{\langle \phi_L | e^{\frac{-\pi H_R}{\kappa}} | \phi_R^1 \rangle \langle \phi_R^1 | \mathcal{O} | \phi_R^2 \rangle \langle \phi_R^2 | e^{\frac{-\pi H_R}{\kappa}} | \phi_L \rangle}{Tr(e^{-2\pi H_R/\kappa})} \\
&= \frac{Tr(e^{-2\pi H_R/\kappa} \mathcal{O})}{Tr(e^{-2\pi H_R/\kappa})} \tag{59}
\end{aligned}
$$

Thus, tracing over the field configuration ϕ_L behind the horizon leads to a thermal density matrix $\rho \propto \exp[-(2\pi/\kappa)H]$ for observables in $\mathcal{R}$.

The main ingredients which have gone into this result are the following. (i) The singular behaviour of the (t, x) coordinate system near $x = 0$ separates out the $T = 0$ hypersurface into two separate regions. (ii) In terms of *real* (t, x) coordinates, it is not possible to distinguish between the points (T, X) and $(-T, -X)$ but the *complex* transformation $t \to t \pm i\pi$ maps the point (T, X) to the point $(-T, -X)$. As usual, a rotation in the complex plane (Re t, Im t) encodes the information contained in the full $T = 0$ plane.

The formalism developed above can be used to express $|\text{vac}\rangle$ formally in terms of quantum states defined in $\mathcal{R}$ and $\mathcal{L}$. It can be easily shown that

$$|\text{vac}\rangle = \prod_{k_\perp, \omega} \sqrt{1 - e^{-2\pi\omega/\kappa}} \sum_{n=0}^{\infty} |n\rangle_{\mathcal{R}} |n\rangle_{\mathcal{L}} \, e^{-\pi n\omega/\kappa} \tag{60}$$

The result in Eq. (60) shows that when the vacuum state $|\text{vac}\rangle$ is "partitioned" by the horizon at $x = 0$, it can be expressed as a highly correlated combination of states defined in $\mathcal{R}$ and $\mathcal{L}$. While this result is suggestive, it is — unfortunately — somewhat formal. One can rigorously prove [67] that the states $|n\rangle$ on either $\mathcal{R}$ or $\mathcal{L}$ are orthogonal to all the states of the standard Fock space of Minkowski quantum field theory.

The results in Eq. (55) and Eq. (59) are completely general and we have not assumed any specific Lagrangian for the field. For *free* field theories in static spacetimes, it is possible to give a more explicit demonstration of the fact that the vacuum state appears as a thermal density matrix. To do this, we begin by noting that in any spacetime, with a metric which is independent of the time coordinate and $g_{0\alpha} = 0$, the wave equation for a massive scalar field $(\Box - m^2)\phi = 0$ can be separated in the form $\phi(t, \mathbf{x}) = \psi_\omega(\mathbf{x})e^{-i\omega t}$ with the modes $\psi_\omega(\mathbf{x})$ satisfying the equation

$$\frac{|g_{00}|}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\, g^{\alpha\beta}\partial_\beta\psi_\omega) = -\omega^2\psi_\omega \tag{61}$$

The normalisation may be chosen using the conserved scalar product:

$$(\psi_\omega, \psi_\nu) \equiv \int d^3x\, \sqrt{-g}|g^{00}|\psi_\omega\psi_\nu^* = \delta_{\omega\nu} \tag{62}$$

Using this relation in the field equation, it can be easily deduced that

$$\int d^3x\, \sqrt{-g}\partial_\alpha\psi_\omega^*\partial^\alpha\psi_\nu = \omega^2\delta_{\omega\nu} \tag{63}$$

Expanding the field as $\phi(t, \mathbf{x}) = \sum_\omega q_\omega(t)\psi_\omega(\mathbf{x})$ and substituting into the free field action, we find that the action reduces to that of a sum of harmonic oscillators:

$$A = -\frac{1}{2}\int \sqrt{-g}\, dt\, d^3x\, (\partial_a\phi\partial^a\phi + m^2\phi^2) = \frac{1}{2}\sum_\omega \int dt\left[|\dot{q}_\omega|^2 - (\omega^2 + m^2)|q_\omega|^2\right] \tag{64}$$

Let us now apply this result to the quantum field theory decomposed into oscillators in: (i) the (T, X) space as well as in (ii) the (t, x) coordinate system on the right and (iii) the left hand side.

On the $T = 0$ surface, we expand the field in terms of a set of mode functions $F_\Omega(X, \mathbf{X}_\perp)$ with coefficients Q_Ω; that is, $\phi = \sum_\Omega Q_\Omega F_\Omega(X, \mathbf{X}_\perp)$. Similarly, the field can be expanded in terms of a set of modes in $\mathcal{R}$ and $\mathcal{L}$:

$$\phi(X > 0, \mathbf{X}_\perp) = \sum_\omega a_\omega f_\omega(X, \mathbf{X}_\perp); \quad \phi(X < 0, \mathbf{X}_\perp) = \sum_\omega b_\omega g_\omega(X, \mathbf{X}_\perp). \tag{65}$$

The functional integral in Eq. (56) now reduces to product over a set of independent harmonic oscillators and thus the ground state wave functional can

be expressed in the form

$$\Psi[Q] = \langle \mathrm{vac}|\phi(\mathbf{X})\rangle = \prod_\Omega \langle \mathrm{vac}|Q_\Omega\rangle \propto \exp\left[-\sum_\omega A_E(T_E = \infty, 0; T_E = 0, Q_\Omega)\right]$$
(66)

where A_E is the Euclidean action with the boundary conditions as indicated. On the other hand, we have shown that this ground state functional is the same as $\langle \phi_R, \kappa t_E = \pi | \phi_L, \kappa t_E = 0\rangle$. Hence

$$\Psi[a, b] = \langle \mathrm{vac}|\phi(\mathbf{X})\rangle \propto \exp\left[-\sum_\Omega A_E(\kappa t_E = \pi, a_\omega; \kappa t_E = 0, b_\omega)\right]$$
(67)

The Euclidean action for a harmonic oscillator q with boundary conditions $q = q_1$ at $t_E = 0$ and $q = q_2$ at $t_E = \beta$ is given by

$$A_E(q_1, 0; q_2, \beta) = \frac{\omega}{2}\left[\frac{\cosh\omega\beta}{\sinh\omega\beta}(q_1^2 + q_2^2) - \frac{2q_1 q_2}{\sinh\omega\beta}\right]$$
(68)

Equation (66) corresponds to $\beta = \infty, q_2 = 0, q_1 = Q_\Omega$ giving $A_E(T_E = \infty, 0; T_E = 0, Q_\Omega)] = (\Omega/2)Q_\Omega^2$ leading to the standard ground state wave functional. The more interesting one is, of course, the one in Eq. (67) corresponding to $\beta = (\pi/\kappa), q_1 = a_\omega, q_2 = b_\omega$. This gives

$$A_E(a_\omega, 0; b_\omega, (\pi/\kappa)) = \frac{\omega}{2}\left[\frac{\cosh(\pi\omega/\kappa)}{\sinh(\pi\omega/\kappa)}(a_\omega^2 + b_\omega^2) - \frac{2a_\omega b_\omega}{\sinh(\pi\omega/\kappa)}\right]$$
(69)

An observer confined to $\mathcal{R}$ will have observables made out of a_ωs. Let $\mathcal{O}(a_\omega)$ be any such observable. The expectation value of $\mathcal{O}$ in the state Ψ is given by

$$\langle \mathcal{O}\rangle = \int \prod_\omega da_\omega \int \prod_\omega db_\omega \Psi^*(a_\omega, b_\omega)\mathcal{O}\Psi(a_\omega, b_\omega) \equiv \int \prod_\omega da_\omega \rho(a_\omega, a_\omega)\mathcal{O}(a_\omega)$$
$$= \mathrm{Tr}(\rho\mathcal{O})$$
(70)

where

$$\rho(a'_\omega, a_\omega) \equiv \int \prod_\omega db_\omega \Psi^*(a'_\omega, b_\omega)\Psi(a_\omega, b_\omega)$$
(71)

$$= C\exp-\sum_\omega\left\{\frac{\omega}{2}\left[\frac{\cosh(2\pi\omega/\kappa)}{\sinh(2\pi\omega/\kappa)}(a_\omega^2 + a_\omega'^2) - \frac{2a_\omega a'_\omega}{\sinh(2\pi\omega/\kappa)}\right]\right\}$$

is a thermal density matrix corresponding to the temperature $T = (\kappa/2\pi)$.

The fact that the exponential in the density matrix in Eq. (72) is similar to that in Eq. (69), with π replaced by 2π, is noteworthy and this result can be

obtained more directly from an alternative argument. The matrix element of ρ can be expressed as the integral

$$\langle \phi'_R | \rho | \phi''_R \rangle = \int \mathcal{D}\phi_L \langle \phi_L \phi'_R | 0 \rangle \langle 0 | \phi_L \phi''_R \rangle \qquad (72)$$

Each of the two terms in the integrand can be expressed in terms of A_E using Eq. (57). In one of them, we shall take $\kappa t_E = \epsilon$ (with ϵ being infinitesimal and positive) at the lower limit of the integral and in the other, we will take $\kappa t_E = -\epsilon$ at the lower limit of the integral. Hence the product which occurs in the integrand of Eq. (72) can be thought of as evolving the field from a configuration ϕ''_R at $\kappa t_E = +\epsilon$ to a configuration ϕ'_R at $\kappa t_E = -\epsilon$ rotating in κt_E in the anti clockwise direction from ϵ to $(2\pi - \epsilon)$. In the limit of $\epsilon \to 0$, this is same as evolving the system by the angle $\kappa t_E = 2\pi$. So we can set $\beta = (2\pi/\kappa), q_1 = a_\omega, q_2 = a'_\omega$ in Eq. (68) leading to Eq. (72). In arriving at equation Eq. (69) we have evolved the same system from $\kappa t_E = 0$ to $\kappa t_E = \pi$ in order to go from $x > 0$ to $x < 0$. This explains the correspondence between Eq. (72) and Eq. (69).

To avoid misunderstanding, we stress that the temperature associated to a horizon is not directly related to the question of what a given non-inertial detector will measure. In the case of a uniformly accelerated detector in flat spacetime, it turns out that the detector results will match with the temperature of the horizon [11,60,68]. There are, however, several other situations in which these two results do not match [25,26,27,19]. The physics of a non inertial detector is well understood and there are no unresolved issues [69,70].

5   Asymptotically static horizons and Hawking radiation

The association of a temperature with a horizon, by itself, does not mean that the horizon radiates energy in an irreversible manner or that a black hole "evaporates". In fact, the metrics mentioned in Section 2.5 (leading to horizons and temperature) are all trivially invariant under $t \to -t$. The horizons in these spacetimes exist "forever"; the most natural vacuum states of the theory share this invariance and describe a situation in thermal equilibrium. There is no net radiation flowing to regions far away from the horizon.

A completely different class of physical phenomena arises if the spacetime metric is time dependent, like, for example, in the case of an expanding universe. Then the natural choice of mode functions and the corresponding vacuum states at $t \to -\infty$ and $t \to \infty$, usually called $|\text{in}\rangle$ and $|\text{out}\rangle$, will be different and the $|\text{in}\rangle$ vacuum will contain "out-particles". In general, the spectrum of particles produced will depend on the detailed nature of the time evolution. The result will not have the same kind of universality as the results we have

discussed so far and each case needs to be addressed separately.

One important exception to this general rule is when the metric (in some coordinate system) evolves from a geometry which has no horizon in the asymptotic past ($t \to -\infty$) to a geometry with a horizon in the asymptotic future ($t \to +\infty$). Then the late time behaviour of modes, in a coordinate system appropriate for the family of observers who has a horizon, is exponentially redshifted and will lead to a thermal spectrum of particles. It must be stressed that we are now dealing with an explicitly time dependent situation, the physics of which is different from the static horizons discussed in the previous Sections. Time reversal invariance need not hold and there could be a genuine flow of created particles from one region to another. This can arise in different contexts, three of which are of primary interest to us because of their connection with the corresponding static metrics:

(a) One can introduce coordinate systems in flat spacetimes which smoothly interpolates between inertial coordinates at $t \to -\infty$ to the Rindler coordinates at $t \to +\infty$. Such a coordinate system will appropriately describe a family of observers with time dependent acceleration. The clock time t of this observer with variable acceleration will match with inertial time coordinate in the asymptotic past and with the Rindler time coordinate in the asymptotic future and the metric will be static in both the limits. It is straightforward to show that the vacuum state in the asymptotic past, $|in\rangle$ will contain a thermal distribution of out-particles.

(b) A spherically symmetric distribution of matter, collapsing and forming a black hole, represents another case in which the horizon develops asymptotically. A family of observers at constant (large) radii outside will notice a horizon forming as $t \to \infty$. The vacuum state of the asymptotic past will be populated by a thermal distribution of out-particles in the future.

(c) The De Sitter spacetime also allows a time dependent generalisation which is most easily obtained by using the cosmological (Friedmann) coordinates to describe the De Sitter metric. In these coordinates, the dynamics of the spacetime is described in terms of an expansion factor $a(t)$. If $a(t)$ has a power law behaviour at small and moderate t and evolves into $a(t) \to \exp(Ht)$ as $t \to \infty$, the geometry will describe a universe which is asymptotically De Sitter. [There is some observational evidence to suggest that our universe is indeed evolving in this manner; for a review, see e.g., [71].]

Most of the techniques used in the previous Sections are not applicable when the spacetime is explicitly time dependent but the results based on infinite redshift will survive. We have seen in Section 2.3 that a wave mode undergoing exponential redshift can lead to a thermal distribution of particles. At late times and far away from the horizon, only modes which emanate from near

the horizon at early times will contribute significantly. These modes would have undergone exponential redshift in all the three cases described above and will lead to a thermal spectrum.

5.1 Asymptotically Rindler observers in flat spacetime

Let us begin with the case of a time dependent Rindler metric in flat spacetime, which corresponds to an observer who is moving with a variable acceleration [27,39]. The transformation from the flat inertial coordinates $(T, \mathbf{X})$ to the proper coordinates $(t, \mathbf{x})$ of an observer with variable acceleration is effected by $Y = y, Z = z$ and

$$X = \int^{'} \sinh \mu(t) dt + x \cosh \mu(t); \quad T = \int^{'} \cosh \mu(t) dt + x \sinh \mu(t) \quad (73)$$

where the function $\mu(t)$ is related to the time dependent acceleration $g(t)$ by $g(t) = (d\mu/dt)$. The form of the metric in the accelerated frame is remarkably simple:

$$ds^2 = -(1 + g(t)x)^2 dt^2 + dx^2 + dy^2 + dz^2 \quad (74)$$

We will treat $g(t)$ to be an arbitrary function except for the limiting behaviour $g(t) \to 0$ for $t \to -\infty$ and $g(t) \to g_0$=constant for $t \to +\infty$. Hence, at early times, the line element in Eq. (74) represent the standard inertial coordinates and the positive frequency modes $\exp(-i\omega t)$ define the standard Minkowski vacuum, $|in\rangle$. At late times, the metric goes over to the Rindler coordinates and we are interested in knowing how the initial vacuum state will be interpreted at late times. The wave equation $(\Box - m^2)\phi = 0$ for a massive scalar field can be separated in the transverse coordinates as $\phi(t, x, y, z) = f(t, x)e^{ik_y y}e^{ik_z z}$ where f satisfies the equation

$$-\frac{1}{(1 + g(t)x)} \frac{\partial}{\partial t} \left(\frac{1}{(1 + g(t)x)} \frac{\partial f}{\partial x} \right) = \chi^2 f \quad (75)$$

with $\chi^2 \equiv m^2 + k_y^2 + k_z^2$. It is possible to solve this partial differential equation with the ansatz

$$f(x, t) = \exp i \left(\int \alpha(t) dt + \beta(t)x \right) \quad (76)$$

where α and β satisfy the equations $\alpha^2(t) - \beta^2(t) = \chi^2; \dot{\beta} = g(t)\alpha; \dot{\alpha} = g(t)\beta;$ these are solved uniquely in terms of $\mu(t)$ to give $\alpha(t) = \chi \cosh[\mu(t) - \eta]; \beta(t) = \chi \sinh[\mu(t) - \eta]$ where η is another constant. The final solution for the mode labelled by $(\mathbf{k}_\perp, \eta)$ is now given by

$$f_{k_y k_z \eta}(x, t) = \exp -i\chi \left[\int \cosh(\mu - \eta) dt + x \sinh(\mu - \eta) \right] \quad (77)$$

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For the limiting behaviour we have assumed for $g(t)$, we see that $\mu(t)$ vanishes at early times and varies as $\mu(t) \approx (g_0 t + \text{constant})$ at late times. Correspondingly, the mode f will behave as

$$f(x,t) \to \exp -i\chi \left[t \cosh \eta - x \sinh \eta \right] \tag{78}$$

at early times $(t \to -\infty)$ which is just the standard Minkowski positive frequency mode with $\omega = \chi \cosh \eta$, $k_x = \chi \sinh \eta$. At late times the mode evolves to

$$f(x,t) \to \exp -i \left[(\chi/2g_0)(1 + g_0 x) e^{g_0 t} \right] \tag{79}$$

We are once again led to a wave mode with exponential blueshift at any given x. The metric is static in t at late times and the out-vacuum will be defined in terms of modes which are positive frequency with respect to t. The Bogoliubov transformations between the mode in Eq. (79) and modes which vary as $\exp(-i\nu t)$ will involve exactly the same mathematics as in Eq. (24). We will get a thermal spectrum at late times.

5.2 Hawking radiation from black holes

The simplest model for the formation of the black hole is based on a spherical distribution of mass M which collapses under its own weight to form a black hole. Since only the exponential redshift of the modes at late times is relevant as far as the thermal spectrum is concerned, the result should be independent of the detailed nature of the collapsing matter [9,12,60]. Further, the angular coordinates do not play a significant role in this analysis, allowing us to work in the two dimensional (t, r) subspace. The line element exterior to the spherically symmetric distribution of matter can be taken to be $ds^2 = -C(r) du dv$ where

$$\xi = \int dr \, C^{-1}; \quad u = t - \xi + R_0^*; \quad v = t + \xi - R_0^* \tag{80}$$

and R_0^* is a constant. In the interior, the line element is taken to be $ds^2 = -B(U,V) dU dV$ with $U = \tau - r + R_0$, $V = \tau + r - R_0$ and R_0 and R_0^* are related in the same manner as r and ξ. Let us assume that, for $\tau < 0$, matter was at rest with its surface at $r = R_0$ and for $\tau > 0$, it collapses inward along the trajectory $r = R(\tau)$. The coordinates have been chosen so that at the onset of collapse $(\tau = t = 0)$ we have $u = U = v = V = 0$ at the surface. Let the coordinate transformations between the interior and exterior be given by the functional forms $U = f(u)$ and $v = h(V)$. Matching the geometry along the trajectory $r = R(\tau)$, it is easy to show that

$$\frac{dU}{du} = (1 - \dot{R}) C \left(\left[BC(1 - \dot{R}^2) + \dot{R}^2 \right]^{1/2} - \dot{R} \right)^{-1} \tag{81}$$

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$$\frac{dv}{dV} = \frac{1}{C(1+\dot{R})} \left(\left[BC(1-\dot{R}^2) + \dot{R}^2 \right]^{1/2} + \dot{R} \right) \tag{82}$$

As the modes propagate inwards they will reach $r = 0$ and re-emerge as out-going modes. In the (t, r) plane, this requires reflection of the modes on the $r = 0$ line, which corresponds to $V = U - 2R_0$. The solutions to the two dimensional wave equations $\Box \phi = 0$ which (i) vanish on the line $V = U - 2R_0$ and (ii) reduce to standard exponential form in the remote past, can be determined by noting that, along $r = 0$ we have

$$v = h(V) = h[U - 2R_0] = h[f(u) - 2R_0] \tag{83}$$

Hence the solution is

$$\Phi = \frac{i}{\sqrt{4\pi\omega}} \left(e^{-i\omega v} - e^{-i\omega h[f(u)-2R_0]} \right) \tag{84}$$

(The second term, which is the "reflected wave" at $r = 0$ can, in fact, be entirely interpreted in terms of Doppler shift arising from a fictitious moving surface having the trajectory $r = 0$.) Given the trajectory $R(\tau)$, one can integrate Eq. (81) to obtain $f(u)$ and use Eq. (84) to completely solve the problem. This will describe time-dependent particle production from some collapsing matter distribution and — in general — the results will depend on the details of the collapse.

The analysis, however, simplifies considerably and a universal character emerges if the collapse proceeds to form a horizon on which $C \to 0$. Near $C = 0$, equations (81) and (82) simplifies to

$$\frac{dU}{du} \approx \frac{\dot{R}-1}{2\dot{R}}C(R); \quad \frac{dv}{dV} \approx \frac{B(1-\dot{R})}{2\dot{R}} \tag{85}$$

where we have used the fact that $(\dot{R}^2)^{1/2} = -\dot{R}$ for the collapsing solution. Further, near $C = 0$, we can expand $R(\tau)$ as $R(\tau) = R_h + \nu(\tau_h - \tau) + \mathcal{O}[(\tau_h - \tau)^2]$ where $R = R_h$ at the horizon and $\nu = -\dot{R}(\tau_h)$. Integrating Eq. (85) treating B approximately constant, we get

$$au \approx -\ln|U + R_h - R_0 - \tau_h| + \text{const} \tag{86}$$

where $\kappa = (1/2)(\partial C/\partial r)_{R_h}$ is the surface gravity. and

$$v \approx \text{constant} - BV(1+\nu)/2\nu \tag{87}$$

It is clear that: (i) The relation between v and V is linear and hence holds no surprises; it also depends on B. (ii) The relation between U and u, which can be written as $U \propto \exp(-\kappa u)$ is universal (independent of B) and signifies the exponential redshift we have alluded to several times. The late time behaviour

of out-going modes can now be determined using Eq. (86) and Eq. (87) in Eq. (84). We get:

$$\Phi \cong \frac{i}{\sqrt{4\pi\omega}} \left(e^{-i\omega v} - \exp\left(i\omega \left[ce^{-\kappa u} + d \right] \right) \right) \tag{88}$$

where c, d are constants. This mode with exponential redshift, when expressed in terms of $\exp(-i\nu u)$ will lead to a thermal distribution of particles with temperature $T = \kappa/2\pi$. For the case of a black hole, if we take $\kappa = 1/4M$, then the Bogoliubov coefficients are given by

$$\alpha_{\omega\nu} = \frac{-2iM\nu e^{-i\omega d} e^{i\nu t_0}}{2\pi\sqrt{\nu\omega}} \left(\frac{-e^{-(t_0+d)/4M}}{\omega c} \right)^{-4iM\nu} e^{2M\pi\nu} \Gamma(-4iM\nu) \tag{89}$$

and $\beta_{\omega\nu} = e^{-4M\pi\nu}\alpha_{\omega\nu}^*$. Note that these quantities *do* depend on c, d, t_0 etc; but the modulus

$$|\beta_{\omega\nu}|^2 = \frac{1}{2} \frac{4M}{[\exp(8\pi M\nu) - 1)]} \tag{90}$$

is independent of these factors. [The mathematics is essentially the same as in Eqs.(23),(24).] This shows that the vacuum state at early times will be interpreted as containing a thermal spectrum of particles at late times.

5.3 Asymptotically De Sitter spacetimes

The De Sitter universe is a solution to Einstein's equations $G_b^a = 8\pi T_b^a$ with a source given by $T_b^a = \Lambda\delta_b^a$. The spacetime metric given in Table 1 is given in terms of the parameter $H^2 = \Lambda/3$ and is useful for providing easy comparison with Schwarzschild and Rindler metrics. But this coordinate system hides the symmetries of the De Sitter manifold. Since the source is homogeneous, isotropic and constant in space and time, the metric can be cast as a section of a maximally symmetric manifold. Using the Friedmann-Robertson-Walker coordinates, which is appropriate for describing maximally symmetric 3-space, one can express the De Sitter spacetime in the form

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2) \right] \tag{91}$$

with $k = 0, a(t) = \exp(Ht)$ or with $k = 1, a(t) = H^{-1}\cosh(Ht)$. (There is also a solution with $k = -1$ which we do not need).

To proceed from such an "eternal" De Sitter universe, to an asymptotically De Sitter universe, we only have to add normal matter or radiation to the source of the Einstein's equations. At sufficiently late times the energy densities of matter or radiation will be diluted exponentially leading to the De Sitter solution at late times. (This occurs in a wide class of dark energy models

[72,73].) Mathematically, this will correspond to $a(t)$ which is a power law at small t tending to $a(t) \propto \exp(Ht)$ for $Ht \gtrsim 1$.

In the asymptotic future, one can introduce the static Schwarzschild coordinates in the manifold and define a vacuum state. However, it is not possible to assign a natural (or unique) vacuum state in the asymptotic past if $a(t)$ is time dependent and one needs to invoke some extra prescription to define a vacuum state. This issue has been extensively discussed in the literature and several possible prescriptions based on different criteria have been explored (see e.g., [14]). One of the simplest choices will be to choose modes which vary as $\exp(-i\omega t)$ near $t \approx t_0$ will define an instantaneous vacuum state around $t \cong t_0$. At late times, the frequency of the wave mode will vary as $\omega(t) \propto a(t)^{-1} \propto e^{-Ht}$ in the WKB approximation. Fourier transforming these modes with respect to another instantaneous vacuum state defined through the modes which vary as $\exp(-i\omega t)$ near $t \to +\infty$, one can recover a thermal spectrum of particles at late times in the initial vacuum state [46]. It is clear from this discussion that the asymptotically De Sitter spacetime requires a somewhat different approach compared to the other two cases because of explicit time dependence.

6 Expectation values of Energy-Momentum tensor

The flow of radiation at late times, away from the horizon, is the new feature which arises when horizon forms in the asymptotic future. A formal way of describing this result is to use the expectation value $\langle \psi | T_{ab} | \psi \rangle$ of the energy momentum tensor of the matter field T_{ab}. If the quantum state is time reversal invariant, then expectation values of flux, $\langle T_0^\alpha \rangle$, will vanish, though the expectation value of energy density, $\langle T_0^0 \rangle$, can be nonzero and correspond to thermal radiation at some equilibrium temperature, related to the surface gravity of horizon.

It is clear that a new element, the quantum state $|\psi\rangle$, has entered the discussion. In a given spacetime with a horizon, one can, of course, make different choices for this state, even if we nominally decide it should be a "vacuum state". The expectation value of various operators, including T_{ab} will be quite different in each of these states and there is no assurance that they will even be finite near the horizon (or at infinity) in an arbitrary state. Similarly, if the mode functions are not invariant under time reversal, then the expectation value of energy-momentum tensor in the corresponding vacuum state may show a flux of radiation.

This new feature allows us to mimic the effects of formation of asymptotic horizons by choosing a quantum state which is not time reversal invariant.

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That is, we can identify quantum states which will contain flux of radiation emanating from the horizon at late times even though we are working in a static spacetime with a metric which is invariant under time reversal. This is possible only because the late time behaviour in the case of spacetimes with asymptotic horizons (discussed in the previous Section) is independent of the details of the metric during the transient phase. We shall now see how such quantum states and the expectation values of T_{ab} in those states can be constructed.

6.1   The $\langle T_{ab} \rangle$ in two-dimensional field theory

A purely technical difficulty in such an approach arises from the fact that the mode functions in four dimensional spacetime are fairly complicated in form and the expectation value $\langle T_{ab} \rangle$ is usually not tractable analytically. However, the situation simplifies enormously in two dimensions and since the results in two dimension capture the essence of physics, we shall use this approach to explain the choice of vacuum states and the corresponding results.

In the (1+1) dimension, the metric has three independent components while the freedom of two coordinate transformations allows us to impose two conditions on them. Hence we can reduce any two dimensional metric to a conformally flat form locally. Consider such a spacetime with signature $(-, +)$ and line element expressed as

$$ds^2 = -C(x^+, x^-)dx^+dx^-; \qquad x^\pm = t \pm x \tag{92}$$

A massless scalar field in this spacetime is described by the action

$$A = -\frac{1}{2} \int d^2x \sqrt{-g}\, g^{ab} \partial_a \phi\, \partial_b \phi = \frac{1}{2} \int dt dx [\dot{\phi}^2 - \phi'^2] \tag{93}$$

since $\sqrt{-g}\, g^{ab} = \eta^{ab}$ for the metric in Eq. (92). The field equation $(\partial^2 \phi / \partial x^+ \partial x^-) = 0$ has the general solution:

$$\phi(x^+, x^-) = \phi_1(x^+) + \phi_2(x^-) \tag{94}$$

with $\phi_1(x^+) = \phi_1(t + x)$ being the 'in-going' (or 'left moving') mode and $\phi_2(x^-) = \phi(t-x)$ being the 'outgoing' (or 'right moving') mode. The expansion of the scalar field, in terms of the normalised plane wave mode functions, is given by

$$\phi = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi|k|}} \left[a(k)e^{i(kx-|k|t)} + h.c. \right] \tag{95}$$

It is more convenient to rewrite this expansion in terms of the in-going and outgoing modes (as in Eq. (94)) and label them by the frequency $\omega = |k|$. This

is easily done by separating the integration range in Eq. (95) into $(-\infty, 0)$ and $(0, \infty)$ and changing the variable of integration from k to $-k$ in the first range. This gives

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left[a(\omega)e^{-i\omega x^-} + b(\omega)e^{-i\omega x^+} + h.c. \right] \tag{96}$$

which is of the form in Eq. (94). There is a direct correspondence between the set of modes $[\exp(-i\omega x^\pm)]$ and the vacuum state $|x^\pm\rangle$ annihilated by the operators $a(\omega), b(\omega)$.

The stress-tensor for the scalar field is given by $T_{ab} = \partial_a\phi\,\partial_b\phi - (1/2)g_{ab}(\partial^c\phi\,\partial_c\phi)$. To evaluate its expectation value, it is convenient to relate it to the Feynman Green function in this vacuum state $G_F(x^\pm, y^\pm) = \langle x^\pm|T(\phi(x^\pm)\phi(y^\pm))|x^\pm\rangle$, by:

$$\langle T_{ab}(x)\rangle = \lim_{x\to x'} \left[\partial_a\partial'_b - \frac{1}{2}g_{ab}\partial^c\partial'_c \right] G_F(x, x') \tag{97}$$

Using this procedure, it can be shown that the (regularised) expectation values are given by (see e.g.,[14])

$$\langle x^\pm|T_{\pm\pm}|x^\pm\rangle = -\frac{1}{12\pi}C^{1/2}\partial_\pm^2 C^{-1/2} \tag{98}$$

$$\langle T_{+-}\rangle = \frac{C}{96\pi}R; \quad R = 4C^{-1}\partial_+\partial_- \ln C \tag{99}$$

We shall now use the results in Eq. (98), Eq. (99) to evaluate $\langle T_{ab}\rangle$ in space-times with horizons.

6.2 Vacuum states and $\langle T_{ab}\rangle$ in the presence of Horizons

Since the mode functions are plane waves in conformally flat (1+1) space-time, we can immediately identify two natural sets of modes and corresponding vacuum states. The out-going and in-going modes of the form given by $(4\pi\omega)^{-1/2}[\exp(-i\omega u), \exp(-i\omega v)]$ define a static vacuum state (called Boulware vacuum in the case of Schwarzschild black hole [74] but can be defined in any other spacetime) natural to the (t, x) or (t, l) coordinates. The modes of the kind $(4\pi\omega)^{-1/2}[\exp(-i\omega U), \exp(-i\omega V)]$ define another vacuum state [called Hartle-Hawking vacuum in the case of Schwarzschild black hole [51]] natural to the (T, X) coordinates. (Note that these two coordinate frames (T, X) and (t, x) are related by Eq. (18).) Finally, the modes of the kind $(4\pi\omega)^{-1/2}[\exp(-i\omega U), \exp(-i\omega v)]$ define a third vacuum state [called the Unruh vacuum [60]] which is natural to the situation in which a horizon forms asymptotically, as in the case of gravitational collapse. This is obvious from the discussion in Section 5.2 [see Eq. (88)] which shows how these modes originate in the collapse scenario.

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Using the result that, in any conformally flat coordinate system of the form $ds^2 = -C(x^+, x^-)dx^+dx^-$, the expectation values of the stress-tensor component are given by Eq. (98), Eq. (99), we can explicitly evaluate the various expectation values. In the cases of interest to us the conformal factor only depends on the tortoise coordinate $\xi = (1/2)(x^+ - x^-)$. For example, in the Boulware vacuum we get

$$\langle B|T_{--}|B\rangle = \langle B|T_{++}|B\rangle = \frac{1}{96\pi}\left[CC'' - \frac{1}{2}(C')^2\right] \tag{100}$$

(where the prime denotes derivative with respect ξ) while in the Hartle-Hawking vacuum we get

$$\langle HH|T_{--}|HH\rangle = \langle HH|T_{++}|HH\rangle = \langle B|T_{--}|B\rangle + \frac{\kappa^2}{48\pi}. \tag{101}$$

In both these cases, there is no flux since $\langle T_{xt}\rangle = 0$. Near the horizon, we have

$$\langle B|T_{\pm\pm}|B\rangle \approx -\frac{\kappa^2}{48\pi}; \quad \langle HH|T_{\pm\pm}|HH\rangle \approx 0. \tag{102}$$

The coordinate system used by an inertial observer near the horizon will have U instead of u and hence the actual values measured by an inertial observer near the horizon will vary as $\langle B|T_{uu}|B\rangle(du/dU)^2$ and will *diverge* on the horizon if we choose the vacuum state $|B\rangle$.

A more interesting situation arises in the case of Unruh vacuum which differs from the Boulware vacuum only in the outgoing modes. If the coordinate x^- is replaced by $X^- \equiv F(x^-)$, the conformally flat nature of the line element is maintained and the only stress tensor component which changes is $\langle T_{--}\rangle$. Using this fact, we find that

$$\langle U|T_{--}|U\rangle = \langle HH|T_{--}|HH\rangle; \quad \langle U|T_{++}|U\rangle = \langle B|T_{++}|B\rangle \tag{103}$$

thereby making $\langle U|T_{--}|U\rangle \neq \langle U|T_{++}|U\rangle$. This leads to a flux of radiation with

$$\langle U|T_{\xi t}|U\rangle = -(\kappa^2/48\pi) \tag{104}$$

It is also clear that the energy density, as measured by inertial observers, is finite near the future horizon in $|U\rangle$.

In the case of eternal black hole (or eternal De Sitter), there are *two* horizons in the full manifold corresponding to $T = \pm X$. So far we have discussed the behaviour near the future horizon, $T = X$ (in global coordinates). One can perform a similar analysis at the past horizon $T = -X$ for each of these quantum states. The stress-tensor expectation value in $|HH\rangle$ is finite at both horizons. In contrast, the expectation value in $|B\rangle$ diverges at both horizons while the expectation value in $|U\rangle$ [which is finite at the future horizon $(T = X)$] diverges in the past horizon $(T = -X)$. Since we require the expectation values

to be finite at both horizons, $|HH\rangle$ is a suitable choice in the case of eternal black hole etc. However, when a black hole forms due to gravitational collapse, the past horizon does not exist since it is covered by the internal metric of the collapsing matter. Therefore, both $|HH\rangle$ and $|U\rangle$ are acceptable choices for a black hole formed due to gravitational collapse. The (time symmetric) Hartle-Hawking state describes thermal equilibrium and zero flux and the (time-asymmetric) Unruh vacuum describes a state with a flux of radiation.

In the case of a Schwarzschild black hole, the explicit formulas for the stress-tensor expectation value are given by

$$\langle T_{++}\rangle_U = \langle T_{++}\rangle_B = \langle T_{--}\rangle_B = \frac{\pi}{12}T_H^2\left[\frac{48M^4}{r^4} - \frac{32M^3}{r^3}\right] \qquad (105)$$

$$\langle T_{--}\rangle_U = \frac{\pi}{12}T_H^2\left[1 - \frac{2M}{r}\right]^2\left[1 + \frac{4M}{r} + \frac{12M^2}{r^2}\right] \qquad (106)$$

where $T_H = (\kappa/2\pi) = (1/8\pi M)$. At $r \to \infty$, there is a constant flux of magnitude $(\pi/12)T_H^2$ which is the flux at the temperature T_H.

Though these results are valid only in $(1+1)$ spacetime, the results for the four dimensional spacetime in the $r - t$ sector can be approximated by $\langle T_{ab}^{4D}\rangle \approx (1/4\pi r^2)\langle T_{ab}^{2D}\rangle$. Since the net flux across a spherical surface of constant r in $4D$ is given by $4\pi r^2\langle T_{ab}^{4D}\rangle$, we can directly interpret $\langle T_{ab}^{2D}\rangle$ as the net flux in the $4D$ case. Our results then imply that the energy flowing to infinity per second is given by $(\pi/12)T_H^2$.

While the above results are generally accepted and is taken to imply the radiation of energy from a collapsing black hole to infinity at late times, there are some serious unresolved issues related to situations with asymptotic horizons. These issues are particularly important for the general case rather than for black hole since in the latter the asymptotic flatness of the spacetime helps to alleviate the problems somewhat. We shall now briefly discuss these issues.

We saw in Section 5 that one can construct a coordinate system even in flat spacetime such that certain quantum states exhibit a flux of radiation away from the horizon. But in De Sitter or Rindler spacetimes there is *no* natural notion of "energy source", analogous to the mass of the black-hole, which could decrease as the radiation flows away from the horizon. The conventional view is to assume that: (1) In the case of black-holes, one considers the collapse scenario as "physical" and the natural quantum state is the Unruh vacuum. The notion of evaporation, etc. then follow in a concrete manner. The eternal black-hole (and the Hartle-Hawking vacuum state) is taken to be just a mathematical construct not realized in nature. (2) In the case of Rindler, one may like to think of a time-symmetric vacuum state as natural and treat the situation as one of thermal equilibrium. This forbids using quantum states

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with outgoing radiation in the Minkowski spacetime.

The real trouble arises for spacetimes which are asymptotically De Sitter. Does it "evaporate" ? The analysis in the earlier Sections show that it is imperative to associate a temperature with the De Sitter horizon but the idea of the cosmological constant changing due to evaporation of the De Sitter spacetime seems too radical. Unfortunately, there is no clear mathematical reason for a dichotomous approach as regards a collapsing black-hole and an asymptotically De Sitter spacetime, since the mathematics is identical. Just as collapsing black hole leads to an asymptotic event horizon, a universe which is dominated by cosmological constant at late times will also lead to a horizon. Just as we can mimic the time dependent effects in a collapsing black hole by a time asymmetric quantum state (say, Unruh vacuum), we can mimic the late time behaviour of an asymptotically De Sitter universe by a corresponding time asymmetric quantum state. Both these states will lead to stress tensor expectation values in which there will be a flux of radiation. The energy source for expansion at early times (say, matter or radiation) is irrelevant just as the collapse details are irrelevant in the case of a black-hole.

If one treats the De Sitter horizon as a 'photosphere' with temperature $T = (H/2\pi)$ and area $\mathcal{A}_H = 4\pi H^{-2}$, then the radiative luminosity will be $(dE/dt) \propto T^4 \mathcal{A}_H \propto H^2$. If we take $E = (1/2)H^{-1}$ (which will br justified in Section 8.2), this will lead to a decay law [75] for the cosmological constant of the form:

$$\Lambda(t) = \Lambda_i \left[1 + k(L_P^2 \Lambda_i)(\sqrt{\Lambda_i}(t - t_i))\right]^{-2/3} \propto (L_P^2 t)^{-2/3} \qquad (107)$$

where k is a numerical constant and the second proportionality is for $t \to \infty$. It is interesting that this naive model leads to a late time cosmological constant which is independent of the initial value (Λ_i). Unfortunately, its value is still far too large. These issues are not analysed in adequate detail in the literature and might have important implications for the cosmological constant problem. (For some recent work and references to earlier literature, see [76,77]; for an interesting connection between thermality in Rindler and DeSitter spacetime, see [78,79].)

6.3 Spacetimes with multiple horizons

A new class of mathematical and conceptual difficulties emerge when the spacetime has more than one horizon. For example, metrics in the form in Eq. (11) with $f(r)$ having simple zeros at $r = r_i, i = 1, 2, 3, ...$, exhibit coordinate singularities at $r = r_i$. The coordinate t alternates between being timelike and spacelike when each of these horizons are crossed. Since all curvature invariants are well behaved at the horizons, it will be possible to introduce

coordinate patches such that the metric is also well behaved at the horizon. This is done exactly as in Eq. (18) near each horizon $r = r_i$ with κ replaced by $a_i = N'(r_i) = f'(r_i)/2$.

When there is more than one horizon, we need to introduce one Kruskal like coordinate patch for *each* of the horizons; the (u, v) coordinate system is unique in the manifold but the (U_i, V_i) coordinate systems are different for each of the horizons since the transformation in Eq. (18) depends explicitly on a_i's which are (in general) different for each of the horizons. In such a case, there will be regions of the manifold in which more than one Kruskal like patch can be introduced. The compatibility between these coordinates leads to new constraints.

Consider, for example, the region between two consecutive horizons $r_n < r < r_{n+1}$ in which t is timelike. The coordinates (U_i, V_i) with $i = n, n+1$ overlaps in this region. Euclideanisation of the metric can be easily effected in the region $r_n < r < r_{n+1}$ by taking $\tau = it$. This will lead to the transformations

$$
\begin{aligned}
U_{n+1} &= -U_n \exp[(a_{n+1} + a_n)(-i\tau - \xi)]; \\
V_{n+1} &= -V_n \exp[-(a_{n+1} + a_n)(-i\tau + \xi)]
\end{aligned}
\tag{108}
$$

Obviously, single valuedness can be maintained only if the period of τ is an integer multiple of $2\pi/(a_{n+1} + a_n)$. More importantly, we get from Eq. (18) the relation

$$
U_i + V_i = \frac{2}{a_i} \exp\left(a_i \xi\right) \sinh\left(-i a_i \tau\right)
\tag{109}
$$

which shows that (U_i, V_i) can be used to define values of τ only up to integer multiples of $2\pi/a_i$ in each patch. But since (U_n, V_n) and (U_{n+1}, V_{n+1}) are to be well defined coordinates in the overlap, the periodicity $\tau \to \tau + \beta$ which leaves both the sets (U_n, V_n) and (U_{n+1}, V_{n+1}) invariant must be such that β is an integer multiple of both $2\pi/a_n$ and $2\pi/a_{n+1}$. This will require $\beta = 2\pi n_i/a_i$ for all i with n_i being a set of integers. This, in turn, implies that $a_i/a_j = n_i/n_j$ making the ratio between any two surface gravities a rational number, which is the condition for a non singular Euclidean extension to exist.

These issues also crop up when one attempts to develop a quantum field theory based on different mode functions and vacuum states (see, for example, [80]). It is easy to develop the quantum field theory in the $t - r$ plane if we treat it as a $(1 + 1)$ dimensional spacetime. In a region between two consecutive horizons $r_n < r < r_{n+1}$, we can use (at least) three sets of coordinates: $(u, v), (U_n, V_n), (U_{n+1}, V_{n+1})$ all of which maintain the conformally flat nature of the $(1 + 1)$ dimensional metric, allowing us to define suitable mode functions and vacuum state in a straightforward manner. The outgoing and in-going modes of the kind $(4\pi\omega)^{-1/2} [\exp(-i\omega u), \exp(-i\omega v)]$ define a static (global) Boulware vacuum state. The modes of the kind $(4\pi\omega)^{-1/2}[\exp(-i\omega U_i),$

$\exp(-i\omega V_i)$] with $i = (n, n+1)$ define *two different* Hartle-Hawking vacua. As regards the Unruh type vacua, we now have three different choices. The mode functions $\mathcal{U}_n = (4\pi\omega)^{-1/2}[\exp(-i\omega U_n), \exp(-i\omega v)]$ define the analogue of Unruh vacuum for the horizon at $r = r_n$. Similarly, $\mathcal{U}_{n+1} = (4\pi\omega)^{-1/2}[\exp(-i\omega u),$ $\exp(-i\omega V_{n+1})]$ define another vacuum state corresponding to the horizon at $r = r_{n+1}$. What is more, we can now also define another set of modes and a vacuum state based on $\mathcal{U}_{n,n+1} = (4\pi\omega)^{-1/2}[\exp(-i\omega U_n), \exp(-i\omega V_{n+1})]$. The physical meaning of these three vacua can be understood from the radiative flux $\langle\psi|T_{\xi t}|\psi\rangle$ in each of these states. We find that $\langle\mathcal{U}_n|T_{\xi t}|\mathcal{U}_n\rangle = -(a_n^2/48\pi)$; $\langle\mathcal{U}_{n+1}|T_{\xi t}|\mathcal{U}_{n+1}\rangle = (a_{n+1}^2/48\pi)$ and

$$\langle\mathcal{U}_{n,n+1}|T_{\xi t}|\mathcal{U}_{n,n+1}\rangle = \frac{a_{n+1}^2 - a_n^2}{48\pi} \tag{110}$$

It is clear that the quantum state $|\mathcal{U}_{n,n+1}\rangle$ corresponds to one with radiative flux at two different temperatures arising from the two different horizons; in the case of Schwarzschild-De Sitter spacetime, one flux will correspond to radiation flowing outward from the black hole horizon and the other to radiation flowing inward from the De Sitter horizon. A detector kept between the horizons will respond as though it is immersed in a radiation bath containing two distinct Planck distributions with different temperatures [81].

In addition to the coordinate systems we have defined, it is also possible to introduce a global non singular coordinate system for the SdS metric. (The method works for several other metrics with similar structure, but we shall concentrate on SdS for definiteness.) Let the horizons be at r_1 and r_2 which are the roots of $(1 - 2M/r - H^2r^2) = 0$ with surface gravities κ_1, κ_2. We introduce the two sets of Kruskal-like coordinates $(U_1, V_1), (U_2, V_2)$ by the usual procedure. The global coordinate system in which the metric is well behaved at both the horizons is given by

$$\bar{U} = \frac{1}{\kappa_1}\tanh\kappa_1 U_1 + \frac{1}{\kappa_2}\tanh\kappa_2 U_2; \quad \bar{V} = \frac{1}{\kappa_1}\tanh\kappa_1 V_1 + \frac{1}{\kappa_2}\tanh\kappa_2 V_2 \tag{111}$$

in the region I ($U_1 < 0, V_1 > 0, U_2 > 0, V_2 < 0$). Similar definitions can be introduced in all other regions of the manifold [80,82,83] maintaining continuity and smoothness of the metric. The resulting metric in the $\bar{U}, \bar{V}$ coordinates has a fairly complicated form and depends explicitly on the time coordinate $\bar{T} = (1/2)(\bar{U} + \bar{V})$. In general, the metric coefficients are *not* periodic in the imaginary time; however, if the ratio of the surface gravities is rational with $\kappa_2/\kappa_1 = n_2/n_1$, then the metric is periodic in the imaginary time with the period $\beta = 2\pi n_2/\kappa_2 = 2\pi n_1/\kappa_1$. Since the physical basis for such a condition is unclear, it is difficult to attribute a single temperature to spacetimes with multiple horizons. This demand of $\kappa_2/\kappa_1 = n_2/n_1$ is related to an expectation of thermal equilibrium which is violated in spacetimes with multiple horizons having different temperatures. Hence, such spacetimes will not — in general

— have a global notion of temperature.

7 Entropy of Horizons

The analytic properties of spacetime manifold in the complex plane directly lead to the association of a temperature with a generic class of horizons. In Section 5 we also saw that there exist quantum states in which a flux of thermal radiation will flow away from the horizon if the horizon forms asymptotically. Given these results, it is natural to enquire whether one can attribute other thermodynamic variables, in particular entropy, to the horizons. We shall now discuss several aspects of this important — and not yet completely resolved — issue.

The simplest and best understood situation arises in the case of a Schwarzschild black hole formed due to gravitational collapse of matter. In this case, one can rigorously demonstrate the flow of thermal flux of radiation to asymptotic infinity at late times, which can be collected by observers located in (near) flat spacetimes at $r \to \infty$. Given a temperature and a change in energy, one can invoke classical thermodynamics to define the *change in* the entropy via $dS = dE/T(E)$. Integrating this equation will lead to the function $S(E)$ except for an additive constant which needs to be determined from additional considerations. In the Schwarzschild spacetime, which is asymptotically flat, it is possible to associate an energy $E = M$ with the black-hole. Though the calculation was done in a metric with a fixed value of energy $E = M$, it seems reasonable to assume that — as the energy flows to infinity at late times — the mass of the black hole will decrease. *If* we make this assumption, *then* one can integrate the equation $dS = dM/T(M)$ to obtain the entropy of the black-hole to be

$$S = 4\pi M^2 = \frac{1}{4}\left(\frac{\mathcal{A}_H}{L_P^2}\right) \tag{112}$$

where $\mathcal{A}_H = 4\pi(2M)^2$ is the area of the event horizon and $L_P = (G\hbar/c^3)^{1/2}$ is the Planck length. This integration constant is fixed by the additional assumption that S should vanish when $M = 0$.[1]

The fact that entropy of the Schwarzschild black hole is *proportional* to the horizon area was conjectured [6,7,8] even before it was known that black holes have a temperature. The above analysis fixes the proportionality constant

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[1] One may think that this assumption is eminently reasonable since the Schwarzschild metric reduces to the Lorentzian metric when $M \to 0$. But note that in the same limit of $M \to 0$, the temperature of the black-hole diverges. Treated as a limit of Schwarzschild spacetime, normal flat spacetime has infinite — rather than zero — temperature.

between area and entropy to be $(1/4)$ in Planck units. It is also obvious that the entropy is purely a quantum mechanical effect and diverges in the limit of $\hbar \to 0$. Nevertheless, even in the *classical* processes involving black holes, the horizon area does act in a manner similar to entropy. For example, when two black holes coalesce and settles down to a final steady state (if they do), the sum of the areas of horizons does not decrease. Similarly, in some simple processes in which energy is dumped into the black hole, one can prove an analogue for first law of thermodynamics involving the combination TdS. While both T and S depend on $\hbar$ the combination TdS is independent of $\hbar$ and can be described in terms of classical physics.

The next natural question is whether the entropy defined by Eq. (112) is the same as "usual entropy". If so, one should be able to show that for any processes involving matter and black holes, we must have $d(S_{BH}+S_{matter})/dt \geq 0$ which goes under the name generalised second law (GSL). One simple example in which the area (and thus the entropy) of the black hole *decreases* is the Hawking evaporation; but the GSL holds since the thermal radiation produced in the process has entropy. It is generally believed that GSL always holds though a completely general proof is difficult to obtain. Several thought experiments, when analysed properly, uphold this law [84] and a proof is possible under certain restricted assumptions regarding the initial state [85]. All these suggest that the area of the black hole corresponds to an entropy which is same as the "usual entropy".

In the case of normal matter, entropy can be provided a statistical interpretation as the logarithm of the number of available microstates that are consistent with the macroscopic parameters which are held fixed. That is, $S(E)$ is related to the degrees of freedom (or phase volume) $g(E)$ by $S(E) = \ln g(E)$. Maximisation of the phase volume for systems which can exchange energy will then lead to equality of the quantity $T(E) \equiv (\partial S/\partial E)^{-1}$ for the systems. It is usual to identify this variable as the thermodynamic temperature. (This definition works even for self-gravitating systems in microcanonical ensemble; see eg., [86].)

Assuming that the entropy of the black hole should have a similar interpretation, one is led to the conclusion that the density of states for a black hole of energy E should vary as

$$g(E) \propto \exp\left(\frac{1}{4}\frac{\mathcal{A}_H}{L_P^2}\right) = \exp\left[4\pi\left(\frac{E}{E_P}\right)^2\right] \qquad (113)$$

Such a growth implies, among other things, that the Laplace transform of $g(E)$ does not exist so that no canonical partition function can be defined (without some regularization).

This brings us to the next question: What are the microscopic states by count-

ing which one can obtain the result in Eq. (113) ? That is, what are the degrees of freedom (or the missing information content) which lead to this entropy ?

There are two features that need to be stressed regarding these questions. First, classically, the black hole is determined by its charge, mass and angular momentum and hence has "no hair" (for a review, see e.g., [40]). Therefore, the degrees of freedom which could presumably account for all the information contained in the initial (pre-collapse) configuration cannot be classical. Second, the question is intimately related to what happens to the matter that collapses to form the black hole. If the matter is "disappearing" in a singularity then the information content of the matter can also "disappear". But since singularities are unacceptable in physically correct theories, we expect the classical singularity to be replaced by some more sophisticated description in the correct theory. Until we know what this description is, it is impossible to answer in a convincing manner what happens to the information and entropy which is thrown into the black hole or was contained in the initial pre collapse state.

In spite of this fact, several attempts have been made in the literature to understand features related to entropy of black holes. A statistical mechanics derivation of entropy was originally attempted in [87]; the entropy has been interpreted as the logarithm of: (a) the number of ways in which black hole might have been formed [7,88]; (b) the number of internal black hole states consistent with a single black hole exterior [7,89,88] and (c) the number of horizon quantum states [90,91,92]. There are also other approaches which are more mathematical — like the ones based on Noether charge [93,94,95], deficit angle related to conical singularity [96,97], entanglement entropy [98,99,100] and thermo field theory and related approaches [101,102,103]. Analog models for black holes which might have some relevance to this question are discussed in [104,105,106]. There are also attempts to compute the entropy using the Euclidean gravitational action and canonical partition function [52]. However, since we know that canonical partition function does not exist for this system these calculations require a non trivial procedure for their interpretation. In fact, once the answer is known, it seems fairly easy to come up with very imaginative derivations of the result. We shall comment on a few of them.

To begin with, the thermal radiation surrounding the black hole has an entropy which one can attempt to compute. It is fairly easy to see that this entropy will proportional to the horizon area but will diverge quadratically. We saw in Section 2.4 that, near the horizon, the field becomes free and solutions are simple plane waves. It is the existence of such a continuum of wave modes which leads to infinite phase volume for the system. More formally, the number of modes $n(E)$ for a scalar field ϕ with vanishing boundary conditions at two

radii $r = R$ and $r = L$ is given by

$$n(E) = \frac{2}{3\pi} \int_R^L \frac{r^2 dr}{(1 - 2M/r)^2} \left[E^2 - \left(1 - \frac{2M}{r} \right) m^2 \right]^{3/2} \tag{114}$$

in the WKB limit. [This result is essentially the same as the one contained in Eq.(14); see [107,108].] This expression diverges as $R \to 2M$ showing that a scalar field propagating in a black hole spacetime has infinite phase volume. The corresponding entropy computed using the standard relations:

$$S = \beta \left[\frac{\partial}{\partial \beta} - 1 \right] F; \qquad F = - \int_0^\infty dE \frac{n(E)}{e^{\beta E} - 1}, \tag{115}$$

is quadratically divergent: $S = (\mathcal{A}_H/l^2)$ with $l \to 0$. The divergences described above occur around any infinite redshift surface and is a geometric (covariant) phenomenon.

The same result can also be obtained from what is known as "entanglement entropy" arising from the quantum correlations which exist across the horizon. We saw in Section 4 that if the field configuration inside the horizon is traced over in the vacuum functional of the theory, then one obtains a density matrix ρ for the field configuration outside [and vice versa]. The entropy $S = -Tr(\rho \ln \rho)$ is usually called the entanglement entropy. This is essentially the same as the previous calculation and, of course, S diverges quadratically on the horizon [109,110,98,99,100]. Much of this can be done without actually bringing in gravity anywhere; all that is required is a spherical region inside which the field configurations are traced out [111,112]. Physically, however, it does not seem reasonable to integrate over all modes without any cut off in these calculations. By cutting off the mode at $l \approx L_P$ one can obtain the "correct" result but in the absence of a more fundamental argument for regularising the modes, this result is not of much significance. The cut off can be introduced in a more sophisticated manner by changing the dispersion relation near Planck energy scales but again there are different prescriptions that are available [113,114,115,116] and none of them are really convincing.

The entropy computed using any non gravitational degrees of freedom will scale in proportion with the number, g_s, of the species of fields which exist in nature. This does not cause a (separate) problem since one can re-absorb it in the renormalisation of gravitational constant G. In any calculation of effective action for a quantum field in curved spacetime, one will obtain a term proportional to R with a quadratically divergent coefficient. This coefficient is absorbed by renormalising the gravitational constant and this procedure will also take care of g_s.

In conventional description, entropy is also associated with the amount of missing information and one is tempted to claim that information is missing

inside the horizon of black hole thereby leading to the existence of non zero entropy. It is important to distinguish carefully the separate roles played by the horizon and singularity in this case; let us, for a moment, ignore the black hole singularity inside the horizon. Then the fact that a horizon hides information is no different from the fact that the information contained in a room is missing to those who refuse to enter the room. The observers at $(r, \theta, \phi) = $ constant in the Schwarzschild metric do not venture into the horizon and hence cannot access the information at $r < 2M$. Observers who are comoving with the collapsing matter, or even those who plunge into the horizon later on, can access (at least part of) the information which is not available to the standard Schwarzschild observers at $r > 2M$. In this respect, there is no difference between a Rindler observer in flat spacetime and a $(r, \theta, \phi) = $ constant observer in the Schwarzschild spacetime (see figure 1) and it is irrelevant what happens to the information content of matter which has collapsed inside the event horizon. The information missing due to a horizon is observer dependent since — as we have stressed before — the horizon is defined with respect to a congruence of timelike curves ("family of observers"). If one links the black hole entropy with the missing information then the entropy too will become observer dependent.

In the examples which we have discussed in the previous Sections, the thermal density matrix and temperature of the horizon indeed arose from the integration of modes which are hidden by the horizon. In the case of a black hole formed by collapse, there is a well defined, non singular, description of physics in the asymptotic past. As the system evolves, the asymptotic future is made of two parts. One part is outside the horizon and the other part (classically) hits a singularity inside the horizon. The initial quantum state has now evolved to a correlated state with one component inside the horizon and one outside. If we trace over the states inside the horizon, the outside will be described by a density matrix. None of this is more mystifying than the usual phenomenon in quantum theory of starting with a correlated quantum state of a system with two parts (say, two electrons each having two spin states), spatially separating the two components and tracing over one of them in describing the (spatially) localised measurements made on the other. There is no real information loss paradox in such systems.

In the case of the black hole there is an additional complication that the matter collapses to a singularity classically taking the information along with it. In this description, some of the information will be missing even to those observers who dare to plunge inside the event horizon. But, as we said before, this issue cannot be addressed until the problem of final singularity is solved. We have no idea what happens to the matter (or the wave modes of the quantum field) near the singularity and as such it is not possible to do a book keeping on the entropy content of matter inside the black hole. As the black hole evaporates, its mass will decrease but such a semi-classical calculation

cannot be trusted at late stages.

There is considerable discussion in the literature on the "information loss problem" related to this issue. Broadly speaking, this problem arises because the evolution seems to take a pure quantum state to a state with significant amount of thermal radiation. It is, however, difficult even to attempt to tackle this problem properly since physics loses its predictive power at a singularity. One cannot meaningfully ask what happens to the information encoded in the matter variables which collapses to a singularity. So to tackle this question, one needs to know the correct theory which replaces the singularity. If for example, a Planck size remnant is formed inside the event horizon then one needs to ask whether a freely falling observer can retrieve most of the information at late stages from this remnant. [Some of the discussions in the literature also mixes up results obtained in different domains with qualitative arguments for the concurrent validity. For example, one key assumption in the information loss paradox is that the initial state is pure. It is far from obvious that in a fully quantum gravitational context a pure state will collapse to form a black hole [117]].

One immediate consequence, of linking entropy of horizons to the information hidden by them, is that all horizons must be attributed an entropy proportional to its area, with respect to the observers who perceive this horizon. More precisely, given a congruence of timelike curves in a spacetime we define the horizon to be the boundary of the union of the causal pasts of the congruence. Assuming this is non trivial surface, observers moving on this congruence will attribute a constant entropy per unit area $(1/4L_P^2)$ to this horizon. (We shall say more about this in Section 8.) The analysis given in Section 3.3 [see Eq. (48)] shows that whenever a system crosses the horizon with energy E, the probability picks up a Boltzmann factor related to the entropy. In the case of a spherically symmetric horizon, one can imagine thin shells of matter carrying some amount of energy being emitted by the horizon. This will lead to the correct identification of entropy for the horizon. It is conceivable that similar effect occurs whenever a packet of energy crosses the horizon even though it will be difficult to estimate its effect on the surface gravity of the horizon. Naive attempts to compute the corresponding results for other geometries will not work and a careful formalism using the entropy density of horizons — which is currently not available — will be required.

7.1 Black hole entropy in quantum gravity models

The above discussion highlights the fact that any model for quantum gravity, which has something to say about the black hole singularity, will also make definite predictions about the entropy of the black hole. There has been

considerable amount of work in this direction based on different candidate models for quantum gravity. We will summarise some aspects of this briefly. [More extensive discussions as well as references to original literature can be found in the reviews [118,119]]. The central idea in any of these approaches is to introduce microscopic degrees of freedom so that one can attribute large number of microscopic states to solutions that could be taken to represent a classical black hole configuration. By counting these microscopic states, if one can show that $g(E) \propto \exp(\alpha E^2)$, then it is usually accepted as an explanation of black hole entropy.

In standard string theory this is done as follows: There are certain special states in string theory, called BPS states [120], that contain electric and magnetic charges which are equal to their mass. Classical supergravity has these states as classical solutions, among which are the extremal black holes with electric charge equal to the mass (in geometric units). These solutions can be expressed as a Reissner-Nordstrom metric with both the roots of $g_{00} = 0$ coinciding: obviously, the surface gravity at the horizon, proportional to $g'_{00}(r_H)$ vanishes though the horizon has finite area. Thus these black holes, classically, have zero temperature but finite entropy. Now, for certain compactification schemes in string theory (with $d = 3, 4, 5$ flat directions), in the limit of $G \to 0$, there exist BPS states which have the same mass, charge and angular momentum of an extremal black hole in d dimensions. One can explicitly count the number of such states in the appropriate limit and one finds that the result gives the exponential of black hole entropy with correct numerical factors [121,119,122]. This is done in the weak coupling limit and a duality between strong coupling and weak coupling limits [123,124,125] is used to argue that the same result will arise in the strong gravity regime. Further, if one perturbs the state slightly away from the BPS limit, to get a *near* extremal black hole and construct a thermal ensemble, one obtains the standard Hawking radiation from the corresponding near extremal black hole [122].

While these results are intriguing, there are several issues which are still open: First, the extremality or near extremality was used crucially in obtaining these results. We do not know how to address the entropy of a normal Schwarzschild black hole which is far away from the extremality condition. Second, in spite of significant effort, we do not still have a clear idea of how to handle the classical singularity or issues related to the information loss paradox. This is disappointing since one might have hoped that these problems are closely related. Finally, the result is very specific to black holes. One does not get any insight into the structure of other horizons, especially De Sitter horizon, which does not fit the string theory structure in a natural manner.

The second approach in which some success related to black hole entropy is claimed, is in the loop quantum gravity (LQG). While string theory tries to incorporate all interactions in a unified manner, loop quantum gravity [126,127]

has the limited goal of providing a canonically quantised version of Einstein gravity. One key result which emerges from this programme is a quantisation law for the areas. The variables used in this approach are like a gauge field A_a^i and the Wilson lines associated with them. The open Wilson lines carry a quantum number J_i with them and the area quantisation law can be expressed in the form: $\mathcal{A}_H = 8\pi G\gamma \sum \sqrt{J_i(J_i+1)}$ where J_i are spins defined on the links i of a spin network and γ is free parameter called Barbero-Immirizi parameter. The J_i take half-integral values if the gauge group used in the theory is SU(2) and take integral values if the gauge group is SO(3). These quantum numbers, J_i, which live on the links that intersect a given area, become undetermined if the area refers to a horizon. Using this, one can count the number of microscopic configurations contributing to a given horizon area and estimate the entropy. One gets the correct numerical factor (only) if $\gamma = \ln m/2\pi\sqrt{2}$ where $m = 2$ or $m = 3$ depending on whether the gauge group SU(2) or SO(3) is used in the theory [128,129,130,131].

Again there are several unresolved issues. To begin with, it is not clear how exactly the black hole solution arises in this approach since it has been never easy to arrive at the low energy limit of gravity in LQG. Second, the answer depends on the Immirizi parameter γ which needs to be adjusted to get the correct answer, if we know the correct answer from elsewhere. Even then, there is an ambiguity as to whether one should have SU(2) with $\gamma = \ln 2/2\pi\sqrt{2}$ or SO(3) with $\gamma = \ln 3/2\pi\sqrt{2}$. The SU(2) was the preferred choice for a long time, based on its close association with fermions which one would like to incorporate in the theory. However, recently there has been some rethinking on this issue due to the following consideration: For a classical black hole, one can define a class of solutions to wave equations called quasi normal modes [see e.g.,[132,133,134,135]]. These modes have discrete frequencies which are complex, given by

$$\omega_n = i\frac{n+(1/2)}{4M} + \frac{\ln(3)}{8\pi M} + \mathcal{O}(n^{-1/2}) \tag{116}$$

The $\ln(3)$ in the above equation is not negotiable [136,137,138,139]. If one chooses SO(3) as the gauge group, then one can connect up the frequency of quanta emitted by a black hole when the area changes by one quantum in LQG with the quasi normal mode frequency [140,141]. It is not clear whether this is a coincidence or of some significance. Third, most of the details of the LQG are probably not relevant to the computation of the entropy. Suppose we have *any* formalism of quantum gravity in which there is a minimum quantum for length or area, of the order of L_P^2. Then, the horizon area $\mathcal{A}_H$ can be divided into $n = (\mathcal{A}_H/c_1 L_P^2)$ patches where c_1 is a numerical factor. If each patch has k degrees of freedom (due to the existence of a surface field), then the total number of microscopic states are k^n and the resulting entropy is $S = n\ln k = (4\ln k/c_1)(\mathcal{A}_H/4L_P^2)$ which will give the standard result if we choose $(4\ln k/c_1) = 1$. The essential ingredients are only discreteness of the

area and existence of certain degrees of freedom in each one of the patches.

Another key issue in counting the degrees of freedom is related to the effective dimensionality. If we repeat the above argument with the volume $V \propto M^3$ of the black hole then one will get an entropy proportional to the volume rather than area. It is clear that, near a horizon, only a region of length L_P across the horizon contributes the microstates so that in the expression (V/L_P^3), the relevant V is $M^2 L_P$ rather than M^3. It is possible to interpret this as due to the entanglements of modes across the horizon over a length scale of L_P, which — in turn — induces a nonlocal coupling between the modes on the surface of the horizon. Such a field will have one particle excitations, which have the same density of states as black hole [113,114]. While this is suggestive of why we get the area scaling rather than volume scaling, a complete understanding is lacking. The area scaling of entropy has also led to different proposals of holographic bounds [see, e.g. [20]] which is beyond the scope of this review.

8   The thermodynamic route to gravity

Given the fact that entropy of a system is closely related to accessibility of information, it is inevitable that there will be some connection between gravity and thermodynamics. To bring this out, it is useful to recollect the way Einstein handled the principle of equivalence and apply it in the present context. Einstein did not attempt to "derive" principle of equivalence in the conventional sense of the word. Rather, he accepted it as a key feature which must find expression in the way gravity is described — thereby obtaining a geometrical description of gravity. Once the geometrical interpretation of gravity is accepted, it follows that there *will* arise surfaces which act as one-way-membranes for information and will thus lead to some connection with thermodynamics. It is, therefore, more in tune with the spirit of Einstein's analysis to *accept* an inevitable connection between gravity and thermodynamics and ask what such a connection would imply. We shall now describe this procedure in detail.

The existence of a class of observers with limited access to spacetime regions, because of the existence of horizons, is a generic feature. This, a priori, has nothing to do with the dynamics of general relativity or gravity; such examples exist even in flat spacetime. But when the spacetime is flat, one can introduce an additional "rule" that only the inertial coordinates must be used to describe physics. While this appears to be artificial and ad hoc, it is logically tenable. It is the existence of gravitational interaction, which makes spacetime curved, that removes this option and forces us to consider different curvilinear coordinate systems. Further, gravity makes these phenomena related to horizons appear more natural in certain contexts, as in the case of black holes. A region

of spacetime, described in some coordinate system with a non-trivial metric tensor $g_{ab}(x^k)$, can then have a light cone structure such that information about one sub-region is not accessible to observers in another region.

Such a limitation is *always* dependent on the family of observers with respect to which the horizon is defined. To appreciate this fact, let us note that the freedom of choice of the coordinates allows 4 out of 10 components of the metric tensor to be pre-specified, which we shall take to be $g_{00} = -N^2$, $g_{0\alpha} = N_\alpha$. These four variables allow us to characterise the observer-dependent information. For example, with the choice $N = 1, N_\alpha = 0, g_{\alpha\beta} = \delta_{\alpha\beta}$, the $\mathbf{x} = $ constant trajectories correspond to a class of inertial observers in flat spacetime while with $N = (ax)^2, N_\alpha = 0, g_{\alpha\beta} = \delta_{\alpha\beta}$ the $\mathbf{x} = $ constant trajectories represent a class of accelerated observers with a horizon at $x = 0$. We only need to change the form of N to make this transition in which a class of time-like trajectories, $\mathbf{x} = $ constant, acquire a horizon. Similarly observers plunging into a black hole will find it natural to describe the Schwarzschild metric in the synchronous gauge with $N = 1, N_\alpha = 0$ (see e.g., [28]) in which they can indeed access the information contained inside the horizon. The less masochistic observers will use a more standard foliation which has $N^2 = (1 - 2M/r)$ and the surface $N = 0$ will act as the horizon which restricts the flow of information from $r < 2M$ to the observers at $r > 2M$.

This aspect, viz. that different observers (defined as different families of time-like curves) may have access to different regions of space-time and hence differing amount of information, introduces a very *new* feature into physics. It is now necessary to ensure that *physical theories in a given coordinate system are formulated entirely in terms of the variables that an observer using that coordinate system can access* [142]. This "principle of effective theory" is analogous to the renormalisation group arguments used in high energy physics which "protects" the low energy theories from the unknown complications of the high energy sector. For example, one can use QED to predict results at, say, 10 GeV without worrying about the structure of the theory at 10^{19} GeV, as long as one uses coupling constants and variables defined around 10 GeV and determined observationally. In this case, one invokes the effective field theory approach in the momentum space. We can introduce the same reasoning in coordinate space and demand—for example—that the observed physics outside a black hole horizon must not depend on the unobservable processes beyond the horizon.

In fact, this is a natural extension of a more conventional procedure used in flat spacetime physics. Let us recall that, in standard description of flat spacetime physics, one often divides the spacetime by a space-like surface $t = t_1 =$ constant. Given the necessary information on this surface, one can predict the evolution for $t > t_1$ *without knowing the details at $t < t_1$*. In the case of curved spacetime with horizon, similar considerations apply. For ex-

ample, if the spacetime contains a Schwarzschild black hole, say, then the light cone structure guarantees that the processes inside the black hole horizon cannot affect the outside events *classically*. What makes our demand non trivial is the fact that the situation in *quantum theory* is quite different. Quantum fluctuations of fields will have nontrivial correlations across the horizon which is indicated by the fact that the propagators do not vanish for spacelike separations. (Alternatively, QFT in the Euclidean sector probes the region beyond the horizon.) Our principle of effective theory states that it must be possible to "protect" the physical processes outside the horizon from such effects influencing it across the horizon.

For a wide class of horizons which we have discussed, the region inside the horizon (essentially the $\mathcal{F}$ and $\mathcal{P}$ of the maximally extended Kruskal-type coordinate systems) disappears "into" the origin of the Euclidean coordinate system. The principle of effective theory requires that one should deal with the corresponding effective manifold in which the region that is inaccessible to a family of observers is removed. In the examples studied in the earlier Sections, this required removing a point (say, the origin) from the $X - T$ plane in the Euclidean manifold. The standard results of quantum field theory in coordinate systems with static horizons can be obtained from this approach. We shall now proceed to study *gravity* from this approach.

In the case of gravity, the information regarding the region inside the horizon will now manifest in two different forms. First, as a periodicity in the imaginary time coordinate and non trivial winding number for paths which circle the point which is removed. Second, as a boundary term in the Euclidean action for gravity, since the Euclidean action needs to be defined carefully taking into account any contribution which arises from an infinitesimal region around the point which is removed.

The origin in the Euclidean spacetime translates to the horizon surface in the Lorentzian spacetime. If we choose to work entirely in the Lorentzian spacetime, we need to take care of the above two effects by: (i) restricting the time integration to a suitable (finite) range in defining the action and (ii) having a suitable surface term to the action describing gravitational dynamics which will get a contribution from the horizon. Since the horizon surface is the only common element to the inside and outside regions, the effect of the quantum entanglements across a horizon can only appear as a surface term in the action. So it is an inevitable consequence of principle of equivalence that the action functional describing gravity *must* contain certain boundary terms which are capable of encoding the information equivalent to that present beyond the horizon. We shall now see that this surface term can be determined from general principles and, in fact, one can deduce the form of the full action for gravity using this approach [143].

Before we begin the detailed discussion, we mention related approaches exploring the connection between thermodynamics and gravity at different levels. Many people have attempted to relate the thermodynamics of gravity and matter systems to the Euclidean action [52,144,145,146,147,148,149]. Some of these papers also discuss the derivation of laws of thermodynamics as applicable to matter coupled to gravity. An attempt to derive Einstein's equations from thermodynamics, which is closer in spirit to the discussion presented here, was made by [150] but this work did not unravel the structure of gravitational action functional. Several intriguing connections between not only gravitational systems but even other field theoretic phenomena and condensed matter systems have been brought out by [151,152].

Let us now proceed with our programme. In order to provide a local, Lagrangian, description of gravitational physics, this boundary term must be expressible as an integral of a four-divergence, allowing us to write the action functional for gravity formally as

$$A_{\text{grav}} = \int d^4x \sqrt{-g}\, L_{\text{grav}} = \int d^4x \sqrt{-g}\, \left(L_{\text{bulk}} + \nabla_i U^i \right) = A_{\text{bulk}} + A_{\text{sur}} \quad (117)$$

where L_{bulk} is quadratic in the first derivatives of the metric and we are using the convenient notation $\nabla_i U^i \equiv (-g)^{-1/2} \partial_i [(-g)^{1/2} U^i]$ irrespective of whether U^i is a genuine four vector or not. Since different families of observers will have different levels of accessibility to information, we do expect A_{sur} to depend on the foliation of spacetime. On the other hand, since the overall dynamics should be the same for all observers, A_{grav} should be a scalar. It follows that neither A_{bulk} nor A_{sur} are covariant but their sum should be a covariant scalar.

Let us first determine the form of A_{sur}. The horizon for a class of observers arises in a specific gauge and resultant A_{sur} will in general depend on the gauge variables N, N_α. Of the gauge variables N, N_α, the lapse function N plays a more important role in our discussion than N_α, and we can set $N_\alpha = 0$ without loss of generality. The residual gauge (co-ordinate) transformation that keeps $N_\alpha = 0$ but changes the other components of the metric is given by the *infinitesimal* space-time transformation $x^i \to x^i + \xi^i(x^j)$, with the condition $g_{\alpha\beta}\dot{\xi}^\beta = N^2(\partial \xi^0 / \partial x^\alpha)$, which is equivalent to

$$\xi^\alpha = \int dt\, N^2 g^{\alpha\beta} \frac{\partial \xi^0}{\partial x^\beta} + f^\alpha(x^\beta) \,. \quad (118)$$

Such transformations keep $N_\alpha = 0$, but change N and $g_{\alpha\beta}$ according to $\delta g_{ij} = -\nabla_i \xi_j - \nabla_j \xi_i$ (see e.g., [28], §97).

We next introduce a $(1+3)$ foliation with the standard notation for the metric components $(g_{00} = -N^2,\ g_{0\alpha} = N_\alpha)$. Let $u^i = (N^{-1}, 0, 0, 0)$ be the four-velocity of observers corresponding to this foliation, i.e. the normal to the foliation; $a^i = u^j \nabla_j u^i$ be the related acceleration; and $K_{ab} = -\nabla_a u_b - u_a a_b$

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be the extrinsic curvature of the foliation, with $K \equiv K_i^i = -\nabla_i u^i$. (With this standard definition, K_{ab} is purely spatial, $K_{ab}u^a = K_{ab}u^b = 0$; so one can work with the spatial components $K_{\alpha\beta}$ whenever convenient.)

Given this structure, we can list all possible vector fields U^i which can be used in Eq. (117). This vector has to be built out of u^i, g_{ab} and the covariant derivative operator ∇_j acting only once. The last restriction arises because the equations of motion should be of no order higher than two. Given these conditions, (i) there is only one vector field — viz., the u^i itself — which has no derivatives and (ii) only three vectors $(u^j\nabla_j u^i, u^j\nabla^i u_j, u^i\nabla^j u_j)$ which are linear in covariant derivative operator. The first one is the acceleration $a^i = u^j\nabla_j u^i$; the second identically vanishes since u^j has unit norm; the third can be written as $-u^i K$. Thus U^i in the surface term must be a linear combination of $u^i, u^i K$ and a^i at the lowest order. The corresponding term in the action *must* have the form

$$A_{\text{sur}} = \int d^4x \sqrt{-g}\nabla_i U^i = \int d^4x \sqrt{-g}\nabla_i \left[\lambda_0 u^i + \lambda_1 K u^i + \lambda_2 a^i\right] \qquad (119)$$

where λ's are numerical constants to be determined.

Let the region of integration be a four volume $\mathcal{V}$ bounded by two space-like surfaces Σ_1 and Σ_2 and two time-like surfaces $\mathcal{S}$ and $\mathcal{S}_1$. The space-like surfaces are constant time slices with normals u^i, and the time-like surfaces have normals n^i and we shall choose $n_i u^i = 0$. The induced metric on the space-like surface Σ is $h_{ab} = g_{ab} + u_a u_b$, while the induced metric on the time-like surface $\mathcal{S}$ is $\gamma_{ab} = g_{ab} - n_a n_b$. These two surfaces intersect on a two-dimensional surface $\mathcal{Q}$, with the induced metric $\sigma_{ab} = h_{ab} - n_a n_b = g_{ab} + u_a u_b - n_a n_b$. In this foliation, the first two terms of Eq. (119) contribute only on the $t = $ constant hypersurfaces (Σ_1 and Σ_2) while the third term contributes on $\mathcal{S}$ and hence on a horizon (which we shall treat as the null limit of a time-like surface $\mathcal{S}$, like the limit $r \to 2M+$ in the black hole spacetime). Hence we get, on the horizon,

$$A_{\text{sur}} = \lambda_2 \int d^4x\sqrt{-g}\ \nabla_i a^i = \lambda_2 \int_{\mathcal{S}} dt d^2x\ N\sqrt{|\sigma|}(n_\alpha a^\alpha) \qquad (120)$$

Further, in any static spacetime with a horizon: (i) The integration over t becomes multiplication by $\beta \equiv 2\pi/\kappa$ where κ is the surface gravity of the horizon, since there is a natural periodicity in the Euclidean sector. (ii) As the surface $\mathcal{S}$ approaches the horizon, the quantity $N(a_i n^i)$ tends to $-\kappa$ which is constant over the horizon. (see e.g., [153] as well as the discussion at the end of Section 2.5). [2] Using $\beta\kappa = 2\pi$, the surface term gives, on the horizon, the

---

[2] The minus sign in $(-\kappa)$ depends on the convention adopted for n_α. It arises naturally under two circumstances. First is when the region outside the horizon is treated as bounded on one side by the horizon and n_α is the outward normal as per-

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contribution

$$A_{\rm sur} = -\lambda_2 \kappa \int_0^\beta dt \int d^2x \sqrt{\sigma} = -2\pi\lambda_2 \mathcal{A}_H \tag{121}$$

where $\mathcal{A}_H$ is the area of the horizon.

It is interesting to ask how the above result arises if we choose to work entirely in Euclidean spacetime. Such an exercise is important for two reasons. First, the range of integration for time coordinate has a natural limit only in Euclidean sector and while obtaining Eq.(121) we have "borrowed" it and used it in the Lorentzian sector; it will be nice to see it in the proper context. Second, in the Euclidean sector, there is no light cone and horizon gets mapped to the origin of the $t_E - x$ plane. In the effective manifold, we would have removed this point and the surface term has to arise from a limiting procedure. It is important to see that it works correctly. We shall now briefly discuss the steps involved in this analysis.

Consider a simply connected, compact region of the Euclidean manifold $\mathcal{M}$ with two bounding surfaces $\mathcal{S}_0$ and $\mathcal{S}_\infty$, where $\mathcal{S}_0$ encloses a small region around the origin (which corresponds to the horizon in our coordinate system) and $\mathcal{S}_\infty$ is an outer boundary at large distance which we really do not care about. We assume that the region $\mathcal{M}$ is foliated by such surfaces and the normal to the surface defines a vector field u^i. The earlier arguments now show that the only non-trivial terms we can use in the Lagrangian are again of the form in Eq.(119) but the nature of boundary surfaces have now changed. We are interested in the contribution from the inner boundary near the origin, where we can take the metric to be approximately Rindler:

$$ds_E^2 \approx (\kappa x)^2 dt_E^2 + dx^2 + dL_\perp^2 \tag{122}$$

and the inner surface to be $S^1 \times R^2$ where S^1 is small circle around the origin in the $t_E - x$ plane and R^2 is the transverse plane. While evaluating Eq. (119), the integral of $\nabla_i a^i$ will now give $a_i u^i = 0$ on the boundary while the integral of $\nabla_i u^i$ will now give $u_i u^i = 1$, leading to the area of the boundary. In the limit of the radius of S^1 going to zero, this contribution from $\nabla_i u^i$ vanishes. The interesting contribution comes from the integral of $\nabla_i(Ku^i)$ term, which will give the integral of $K = -\nabla_i u^i$ on the boundary. Taking $u^i = \delta_x^i$ we get the contribution

$$-\lambda_2 \int d^2x_\perp \int_0^{2\pi/\kappa} dt_E \partial_x(\kappa x) = -2\pi\lambda_2 \mathcal{A}_H \tag{123}$$

---

ceived from the outside observers. Second, when we take the normal to the horizon to be pointing to the outside (like in the direction of unit vector $\hat{\mathbf{r}}$ in Schwarzschild geometry) but we take the contribution to the surface integral from two surfaces (at $r \to \infty$ and $r \to 2M$ in the Schwarzschild spacetime) and subtract one from the another. The horizon contributes at the lower limit of the integration and picks up a minus sign.

exactly as in Eq.(121). This analysis, once again, demonstrates the consistency of working in an effective manifold with the origin removed.

Treating the action as analogous to entropy, we see that the information blocked by a horizon, and encoded in the surface term, *must be* proportional to the area of the horizon. Taking into consideration the non compact horizons, like the Rindler horizon, we may state that the entropy (or the information content) per unit area of the horizon is a constant related to λ_2. Writing $\lambda_2 \equiv -(1/8\pi\mathcal{A}_P)$, where $\mathcal{A}_P$ is a fundamental constant with the dimensions of area, the entropy associated with the horizon will be $S_H = (1/4)(\mathcal{A}_H/\mathcal{A}_P)$. The numerical factor in λ_2 is chosen for later convenience; the sign is chosen so that $S \geq 0$.

Having determined the form of A_{sur} we now turn to the nature of A_{grav} and A_{bulk}. We need to express the Lagrangian $\nabla_i U^i$ as a difference between two Lagrangians L_{grav} and L_{bulk} such that: (a) L_{grav} is a generally covariant scalar. (b) L_{bulk} is utmost quadratic in the time derivatives of the metric tensor. (c) Neither L_{grav} nor L_{bulk} should contain four divergences since such terms are already taken into account in L_{sur}. This is just an exercise in differential geometry and leads to Einstein-Hilbert action. Thus it is possible to obtain the full dynamics of gravity purely from thermodynamic considerations [143]. We shall, however, obtain this result in a slightly different manner which throws light on certain peculiar features of Einstein-Hilbert action, as well as the role played by local Lorentz invariance.

8.1   Einstein-Hilbert action from spacetime thermodynamics

Since the field equations of gravity are generally covariant and of second order in the metric tensor, one would naively expect these equations to be derived from an action principle involving g_{ab} and its first derivatives $\partial_k g_{ab}$, analogous to the situation for many other field theories of physics. The arguments given in the last Section show that the existence of horizons (and the principle of effective theory) suggest that the gravitational Lagrangian will have a term $\nabla_i U^i$ [see Eq. (119)] which contains *second* derivative of g_{ab}.

While any such Lagrangian can describe the classical physics correctly, there are some restrictions which quantum theory imposes on Lagrangians with second derivatives. Classically, one can postulate that the equations of motion are obtained by varying an action with some arbitrary function $f(q, \dot{q})$ of q and $\dot{q}$ held fixed at the end points. Quantum mechanically, however, it is natural to demand that either q or $p \equiv (\partial L/\partial \dot{q})$ is held fixed rather than a mixture of the two. This criterion finds a natural description in the path integral approach to quantum theory. If one uses the coordinate representation in non-relativistic

quantum mechanics, the probability amplitude for the dynamical variables to change from q_1 (at t_1) to q_2 (at t_2) is given by

$$\psi(q_2, t_2) = \int dq_1 K\left(q_2, t_2; q_1, t_1\right) \psi(q_1, t_1) \ , \tag{124}$$

$$K\left(q_2, t_2; q_1, t_1\right) = \sum_{\text{paths}} \exp\left[\frac{i}{\hbar} \int dt \ L_q(q, \dot{q})\right] \ , \tag{125}$$

where the sum is over all paths connecting (q_1, t_1) and (q_2, t_2), and the Lagrangian $L_q(q, \dot{q})$ depends on $(q, \dot{q})$. It is, however, quite possible to study the same system in momentum space, and enquire about the amplitude for the system to have a momentum p_1 at t_1 and p_2 at t_2. From the standard rules of quantum theory, the amplitude for the particle to go from (p_1, t_1) to (p_2, t_2) is given by the Fourier transform

$$G\left(p_2, t_2; p_1, t_1\right) \equiv \int dq_2 dq_1 \ K\left(q_2, t_2; q_1, t_1\right) \ \exp\left[-\frac{i}{\hbar}\left(p_2 q_2 - p_1 q_1\right)\right] \tag{126}$$

Using Eq. (125) in Eq. (126), we get

$$
\begin{aligned}
G\left(p_2, t_2; p_1, t_1\right) &= \sum_{\text{paths}} \int dq_1 dq_2 \exp\left[\frac{i}{\hbar}\left\{\int dt \ L_q - \left(p_2 q_2 - p_1 q_1\right)\right\}\right] \\
&= \sum_{\text{paths}} \int dq_1 dq_2 \exp\left[\frac{i}{\hbar} \int dt \left\{L_q - \frac{d}{dt}\left(pq\right)\right\}\right] \\
&\equiv \sum_{\text{paths}} \exp\left[\frac{i}{\hbar} \int L_p(q, \dot{q}, \ddot{q}) \ dt\right] \ .
\end{aligned}
\tag{127}
$$

where

$$L_p \equiv L_q - \frac{d}{dt}\left(q\frac{\partial L_q}{\partial \dot{q}}\right) \ . \tag{128}$$

In arriving at the last line of Eq. (127), we have (i) redefined the sum over paths to include integration over q_1 and q_2; and (ii) upgraded the status of p from the role of a parameter in the Fourier transform to the physical momentum $p(t) = \partial L/\partial \dot{q}$. This result shows that, given any Lagrangian $L_q(q, \partial q)$ involving only up to the first derivatives of the dynamical variables, it is *always* possible to construct another Lagrangian $L_p(q, \partial q, \partial^2 q)$ involving up to second derivatives, such that it describes the same dynamics but with different boundary conditions [154,155]. The prescription is given by Eq. (128). While using L_p, one keeps the *momenta* fixed at the endpoints rather than the *coordinates*. This boundary condition is specified by the subscripts on the Lagrangians. The result generalises directly to multi-component fields and provides a natural interpretation of Lagrangians with second derivatives.

Thus, in the case of gravity, the *same* equations of motion can be obtained from A_{bulk} or from another (as yet unknown) action:

$$A' = \int d^4x \sqrt{-g} L_{\text{bulk}} - \int d^4x \partial_c \left[g_{ab} \frac{\partial \sqrt{-g} L_{\text{bulk}}}{\partial(\partial_c g_{ab})} \right]$$

$$\equiv A_{\text{bulk}} - \int d^4x \partial_c(\sqrt{-g} V^c) \tag{129}$$

where V^c is made of g_{ab} and Γ^i_{jk}. Further, V^c must be linear in the Γ's since the original Lagrangian L_{bulk} was quadratic in the first derivatives of the metric. (This argument assumes that we have fixed the relevant dynamical variables q of the system; in the case of gravity, we take these to be g_{ab}). Since Γs vanish in the local inertial frame and the metric reduces to the Lorentzian form, the action A_{bulk} cannot be generally covariant. However, the action A' involves the second derivatives of the metric and we shall see later that that the action A' is indeed generally covariant.

To obtain a quantity V^c, which is linear in Γs and having a single index c, from g_{ab} and Γ^i_{jk}, we must contract on two of the indices on Γ using the metric tensor. (Note that we require A_{bulk}, A' etc. to be Lorentz scalars and P^c, V^c etc. to be vectors under Lorentz transformation.) Hence the most general choice for V^c is the linear combination

$$V^c = \left(a_1 g^{ck} \Gamma^m_{km} + a_2 g^{ik} \Gamma^c_{ik} \right) \tag{130}$$

where $a_1(g)$ and $a_2(g)$ are unknown functions of the determinant g of the metric which is the only (pseudo) scalar entity which can be constructed from g_{ab}s and Γ^i_{jk}s. Using the identities $\Gamma^m_{km} = \partial_k(\ln \sqrt{-g})$, $\sqrt{-g} g^{ik} \Gamma^c_{ik} = -\partial_b(\sqrt{-g} g^{bc})$, we can rewrite the expression for $P^c \equiv \sqrt{-g} V^c$ as

$$P^c = \sqrt{-g} V^c = c_1(g) g^{cb} \partial_b \sqrt{-g} + c_2(g) \sqrt{-g} \partial_b g^{bc} \tag{131}$$

where $c_1 \equiv a_1 - a_2$, $c_2 \equiv -a_2$ are two other unknown functions of the determinant g. If we can fix these coefficients by using a physically well motivated prescription, then we can determine the surface term and — by integrating — the Lagrangian L_{bulk}.

To do this, let us consider a static spacetime in which all g_{ab}s are independent of x^0 and $g_{0\alpha} = 0$. Around any given event $\mathcal{P}$ one can construct a local Rindler frame with an acceleration of the observers with $\mathbf{x} = $ constant, given by $a^i = (0, \mathbf{a})$ and $\mathbf{a} = \nabla(\ln \sqrt{g_{00}})$. This Rindler frame will have a horizon which is a plane surface normal to the direction of acceleration and a temperature $T = |\mathbf{a}|/2\pi$ associated with this horizon. The result obtained in Eq.(121) shows that the entropy S associated with this horizon is proportional to its area or, more precisely,

$$\frac{dA_{\text{sur}}}{d\mathcal{A}_\perp} = \frac{1}{4\mathcal{A}_P} \tag{132}$$

where $\mathcal{A}_P$ is a fundamental constant with the dimensions of area. In particular, this result must hold in flat spacetime in Rindler coordinates. In the static

Rindler frame, the surface term is

$$A_{\text{sur}} = -\int d^4x\, \partial_c P^c = -\int_0^\beta dt \int_{\mathcal{V}} d^3x\, \nabla \cdot \mathbf{P} = \beta \int_{\partial\mathcal{V}} d^2x_\perp \hat{\mathbf{n}} \cdot \mathbf{P} \qquad (133)$$

The overall sign in the last equation depends on the choice of direction for $\hat{\mathbf{n}}$; we have chosen it to be consistent with the convention employed earlier in Eq. (121). We have restricted the time integration to an interval $(0, \beta)$ where $\beta = (2\pi/|\mathbf{a}|)$ is the inverse temperature in the Rindler frame, since the Euclidean action will be periodic in the imaginary time with the period β. We shall choose the Rindler frame such that the acceleration is along the $x^1 = x$ axis. The most general form of the metric representing the Rindler frame can be expressed in the form

$$ds^2 = -(1 + 2al)dt^2 + \frac{dl^2}{(1 + 2al)} + (dy^2 + dz^2) \qquad (134)$$

$$= -\left[1 + 2al(x)\right]dt^2 + \frac{l'^2 dx^2}{[1 + 2al(x)]} + (dy^2 + dz^2)$$

where $l(x)$ is an arbitrary function and $l' \equiv (dl/dx)$. Since the acceleration is along the x-axis, the metric in the transverse direction is unaffected. The first form of the metric is the standard Rindler frame in the (t, l, y, z) coordinates. We can, however, make any coordinate transformation from l to some other variable x without affecting the planar symmetry or the static nature of the metric. This leads to the general form of the metric given in the second line, in terms of the (t, x, y, z) coordinates. Evaluating the surface term P^c in (131) for this metric, we get the only non zero component to be

$$P^x = 2ac_2(g) + [1 + 2al(x)]\frac{l''}{l'^2}[c_1(g) - 2c_2(g)] \qquad (135)$$

so that the action in Eq. (133) becomes

$$A_{\text{sur}} = \beta P^x \int d^2x_\perp = \beta P^x \mathcal{A}_\perp \qquad (136)$$

where $\mathcal{A}_\perp$ is the transverse area of the $(y - z)$ plane. From Eq. (132) it follows that

$$2a\beta c_2(g) + \beta[c_1 - 2c_2](1 + 2al)\frac{l''}{l'^2} = \frac{1}{4\mathcal{A}_P} \qquad (137)$$

For the expression in the left hand side to be a constant independent of x for any choice of $l(x)$, the second term must vanish requiring $c_1(g) = 2c_2(g)$. An explicit way of obtaining this result is to consider a class of functions $l(x)$ which satisfy the relation $l' = (1 + 2al)^n$ with $0 \le n \le 1$. Then

$$\beta[c_1(l') - 2c_2(l')](1 + 2al)\frac{l''}{l'^2} = 2a\beta[c_1(l') - 2c_2(l')]n \qquad (138)$$

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which can be independent of n and x only if $c_1(g) = 2c_2(g)$. Further, using $a\beta = 2\pi$, we find that $c_2(g) = (16\pi\mathcal{A}_P)^{-1}$ which is a constant independent of g. Hence P^c has the form

$$
P^c = \frac{1}{16\pi\mathcal{A}_P} \left(2g^{cb}\partial_b\sqrt{-g} + \sqrt{-g}\partial_b g^{bc} \right) = \frac{\sqrt{-g}}{16\pi\mathcal{A}_P} \left(g^{ck}\Gamma^m_{km} - g^{ik}\Gamma^c_{ik} \right)
$$
$$
= -\frac{1}{16\pi\mathcal{A}_P} \frac{1}{\sqrt{-g}} \partial_b(gg^{bc}) \tag{139}
$$

The second equality is obtained by using the standard identities mentioned after Eq. (130) while the third equality follows directly by combining the two terms in the first expression.

The general form of P^c which we obtained in Eq. (131) is not of any use unless we can fix (c_1, c_2). In general, this will not have any simple form and will involve an undetermined range of integration over time coordinate. But in the case of gravity, two natural features conspire together to give an elegant form to this surface term. First is the fact that Rindler frame has a periodicity in Euclidean time and the range of integration over the time coordinate is naturally restricted to the interval $(0, \beta) = (0, 2\pi/a)$. The second is the fact that the surviving term in the integrand P^c is linear in the acceleration a thereby neatly canceling with the $(1/a)$ factor arising from time integration.

Given the form of P^c we need to solve the equation

$$
\left(\frac{\partial\sqrt{-g}L_{\text{bulk}}}{\partial g_{ab,c}} g_{ab} \right) = P^c = -\frac{1}{16\pi\mathcal{A}_P} \frac{1}{\sqrt{-g}} \partial_b(gg^{bc}) \tag{140}
$$

to obtain the first order Lagrangian density. It is straightforward to show that this equation is satisfied by the Lagrangian

$$
\sqrt{-g}L_{\text{bulk}} = \frac{1}{16\pi\mathcal{A}_P} \left(\sqrt{-g}\, g^{ik} \left(\Gamma^m_{i\ell}\Gamma^\ell_{km} - \Gamma^\ell_{ik}\Gamma^m_{\ell m} \right) \right). \tag{141}
$$

This is the second surprise. The Lagrangian which we have obtained is precisely the first order Dirac-Schrodinger Lagrangian for gravity (usually called the Γ^2 Lagrangian). Note that we have obtained it without introducing the curvature tensor anywhere in the picture.

Given the two pieces, the final second order Lagrangian follows from our Eq. (129) and is, of course, the standard Einstein-Hilbert Lagrangian:

$$
\sqrt{-g}L_{grav} = \sqrt{-g}L_{\text{bulk}} - \frac{\partial P^c}{\partial x^c} = \left(\frac{1}{16\pi\mathcal{A}_P} \right) R\sqrt{-g}. \tag{142}
$$

Thus our full second order Lagrangian *turns out* to be the standard Einstein-Hilbert Lagrangian. This result has been obtained, by relating the surface term

in the action to the entropy per unit area. This relation uniquely determines the gravitational action principle and gives rise to a generally covariant action; i.e., the surface terms dictate the form of the Einstein Lagrangian in the bulk. The idea that surface areas encode bits of information per quantum of area allows one to determine the nature of gravitational interaction on the bulk, which is an interesting realization of the holographic principle.

The solution to Eq. (140) obtained in Eq. (141) is not unique. However, self consistency requires that the final equations of motion for gravity must admit the line element in Eq. (134) as a solution. It can be shown, by fairly detailed algebra, that this condition makes the Lagrangian in Eq. (141) to be the only solution.

We stress the fact that there is a very peculiar identity connecting the Γ^2 Lagrangian L_{bulk} and the Einstein-Hilbert Lagrangian L_{grav}, encoded in Eq. (142). This relation, which is purely a differential geometric identity, can be stated through the equations:

$$
L_{grav} = L_{\text{bulk}} - \nabla_c \left[g_{ab} \frac{\partial L_{\text{bulk}}}{\partial(\partial_c g_{ab})} \right]; \quad L_{\text{bulk}} = L_{grav} - \nabla_c \left[\Gamma^j_{ab} \frac{\partial L_{grav}}{\partial(\partial_c \Gamma^j_{ab})} \right] \quad (143)
$$

This relationship defies any simple explanation in conventional approaches to gravity but arises very naturally in the approach presented here. The first line in Eq. (143) also shows that the really important degrees of freedom in gravity are indeed the surface degrees of freedom. To see this we merely have to note that at any given event, one can choose the local inertial frame in which $L_{\text{bulk}} \sim \Gamma^2$ vanishes; but the left hand side of the first line in Eq. (143) cannot vanish, being proportional to R. That is, in the local inertial frame all the geometrical information is preserved by the surface term in the right hand side, which cannot be made to vanish since it depends on the second derivatives of the metric tensor. In this sense, gravity is intrinsically holographic.

The approach also throws light on another key feature of the surface term in the Einstein-Hilbert action. To see this, consider the expansion of the action in terms of a graviton field by $g_{ab} = \eta_{ab} + \lambda h_{ab}$ where $\lambda = \sqrt{16\pi G}$ has the dimension of length and h_{ab} has the correct dimension of $(\text{length})^{-1}$ in natural units with $\hbar = c = 1$. Since the scalar curvature has the structure $R \simeq (\partial g)^2 + \partial^2 g$, substitution of $g_{ab} = \eta_{ab} + \lambda h_{ab}$ gives to the lowest order:

$$
L_{EH} \propto \frac{1}{\lambda^2} R \simeq (\partial h)^2 + \frac{1}{\lambda} \partial^2 h \quad (144)
$$

Thus the full Einstein-Hilbert lagrangian is non-analytic in λ because the surface term is non-analytic in λ! It is sometimes claimed in literature that one can obtain a correct theory for gravity by starting with a massless spin-2 field h_{ab} coupled to the energy momentum tensor T_{ab} of other matter sources to the lowest order, introducing self-coupling of h_{ab} to its own energy momentum

tensor at the next order and iterating the process. *It will be quite surprising if, starting from $(\partial h)^2$ and doing a honest iteration on λ, one can obtain a piece which is non-analytic in λ.* At best, one can hope to get the quadratic part of L_{EH} which gives rise to the Γ^2 action but not the four-divergence term involving $\partial^2 g$. The non-analytic nature of the surface term is vital for it to give a finite contribution on the horizon and the horizon entropy cannot be interpreted in terms of gravitons propagating around Minkowski spacetime. Clearly, there is lot more to gravity than gravitons (for a detailed discussion, see [156]).

The analysis leading to Eq. (142) can also be carried out in the Euclidean sector, starting from Eq,(123). It is shown in Appendix A that the integral of $\partial_c P^c$ with P^c given by Eq.(139), can be alternatively thought of as the integral of K over the boundaries [see Eq (A.11)]. The rest of the analysis is straight forward so we will not discuss it.

In the above discussion we split the Einstein-Hilbert action as a quadratic part and a surface term. There is a different way of expressing the Einstein-Hilbert action which will turn out to be useful for our later purposes. This is done by introducing the $(1+3)$ foliation and writing the the bulk Lagrangian as (see Appendix A):

$$R \equiv L_{\mathrm{EH}} = L_{\mathrm{ADM}} - 2\nabla_i(Ku^i + a^i) \equiv L_{\mathrm{ADM}} + L_{\mathrm{div}} \qquad (145)$$

where

$$L_{\mathrm{ADM}} = {}^{(3)}\mathcal{R} + (K_{ab}K^{ab} - K^2) \qquad (146)$$

is the ADM Lagrangian [157] quadratic in $\dot{g}_{\alpha\beta}$, and $L_{\mathrm{div}} = -2\nabla_i(Ku^i + a^i)$ is a total divergence. Neither L_{ADM} nor L_{div} is generally covariant. For example, u^i explicitly depends on N, which changes when one makes a coordinate transformation from the synchronous frame to a frame with $N \neq 1$.

There is a conceptual difference between the $\nabla_i(Ku^i)$ term and the $\nabla_i a^i$ term that occur in L_{div} in Eq.(145). This is obvious in the standard foliation, where Ku^i contributes on the constant time hypersurfaces, while a^i contributes on the time-like or null surface which separates the space into two regions (as in the case of a horizon). To take care of the Ku^i term more formally, we recall that the form of the Lagrangian used in functional integrals depends on the nature of the transition amplitude one is interested in computing, and one is free to choose a different representation. We shall now switch to the momentum representation of the action functional, as described earlier in the discussion leading to Eq. (128).

Since L_{ADM} is quadratic in $\dot{g}_{\alpha\beta}$, we can treat $g_{\alpha\beta}$ as the coordinates and obtain another Lagrangian L_π in the momentum representation along the lines of

Eq.(128). The canonical momentum corresponding to $q_A = g_{\alpha\beta}$ is

$$p^A = \pi^{\alpha\beta} = \frac{\partial(\sqrt{-g}\ L_{ADM})}{\partial \dot{g}_{\alpha\beta}} = -\frac{\sqrt{-g}}{N}(K^{\alpha\beta} - g^{\alpha\beta}K)\ , \qquad (147)$$

so that the term $d(q_A p^A)/dt$ is just the time derivative of

$$g_{\alpha\beta}\pi^{\alpha\beta} = -\frac{\sqrt{-g}}{N}(K - 3K) = \sqrt{-g}(2Ku^0)\ . \qquad (148)$$

Since

$$\frac{\partial}{\partial t}(\sqrt{-g}\ Ku^0) = \partial_i(\sqrt{-g}\ Ku^i) = \sqrt{-g}\ \nabla_i(Ku^i)\ , \qquad (149)$$

the combination $\sqrt{-g}\ L_\pi \equiv \sqrt{-g}[L_{\mathrm{ADM}} - 2\nabla_i(Ku^i)]$ describes the same system in the momentum representation with $\pi^{\alpha\beta}$ held fixed at the end points. (This result is known in literature [158] and can be derived from the action principle, as done in Appendix A. The procedure adopted here, which is based on Eq. (143) relating the bulk and surface terms, provides a clearer interpretation.) Switching over to this momentum representation, the relation between the action functionals corresponding to Eq. (145) can now be expressed as

$$A_{\mathrm{EH}} = A_\pi + A_{\mathrm{boun}}\ , \qquad (150)$$

$$A_\pi \equiv A_{\mathrm{ADM}} - \frac{1}{8\pi}\int \sqrt{-g}\ d^4x\ \nabla_i(Ku^i)\ . \qquad (151)$$

Here A_π describes the ADM action in the momentum representation, and

$$A_{\mathrm{boun}} = -\frac{1}{8\pi}\int d^4x\ \sqrt{-g}\ \nabla_i a^i = -\frac{1}{8\pi}\int dt \int_{\mathcal{S}} d^2x\ N\sqrt{\sigma}(n_\alpha a^\alpha) \qquad (152)$$

is the boundary term arising from the integral over the surface. In the last equality, $\sigma_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta$ is the induced metric on the boundary 2-surface with outward normal n_α, and the gauge $N_\alpha = 0$ has been chosen.

8.2 Einstein's equations as a thermodynamic identity

The fact that the information content, entangled across a horizon, is proportional to the area of the horizon arises very naturally in the above derivation. This, in turn, shows that the fundamental constant characterising gravity is the quantum of area $4\mathcal{A}_P$ which can hold approximately one bit of information. The conventional gravitational constant, given by $G = \mathcal{A}_P c^3/\hbar$ will, in fact, *diverge* if we take the limit $\hbar \to 0$ with $\mathcal{A}_p = $ constant. This is reminiscent of the structure of bulk matter made of atoms. Though one can describe bulk matter using various elastic constants etc., such a description cannot be considered as the strict $\hbar \to 0$ limit of quantum mechanics — since no atomic

system can exist in this limit. Similarly, spacetime and gravity are inherently quantum mechanical just as bulk solids are [159,160].

This suggests that spacetime dynamics is like the thermodynamic limit in solid state physics. In fact, this paradigm arises very naturally for any *static* spacetime with a horizon [161]. Such a spacetime has a metric in Eq. (4) with the horizon occurring at the surface $N = 0$ and its temperature β^{-1} determined by the surface gravity on the horizon. Consider a four-dimensional region of spacetime defined as follows: 3-dimensional spatial region is taken to be some compact volume $\mathcal{V}$ with boundary $\partial\mathcal{V}$. The time integration is restricted to the range $[0, \beta]$ since there is a periodicity in Euclidean time. We now define the entropy associated with the same spacetime region by:

$$S = \frac{1}{8\pi G} \int \sqrt{-g} d^4 x \nabla_i a^i = \frac{\beta}{8\pi G} \int_{\partial\mathcal{V}} \sqrt{\sigma} d^2 x (Nn_\mu a^\mu) \qquad (153)$$

The second equality is obtained because, for static spacetimes: (i) time integration reduces to multiplication by β and (ii) since only the spatial components of a^i are non zero, the divergence becomes a three dimensional one over $\mathcal{V}$ which is converted to an integration over its boundary $\partial\mathcal{V}$. If the boundary $\partial\mathcal{V}$ is a horizon, $(Nn_\mu a^\mu)$ will tend to a constant surface gravity κ and the using $\beta\kappa = 2\pi$ we get $S = \mathcal{A}/4G$ where $\mathcal{A}$ is the area of the horizon. (For convenience, we have chosen the sign of n^α such that $Na_\mu n^\mu \to \kappa$, rather than $-\kappa$.) Thus, in the familiar cases, this does reduce to the standard expression for entropy. Similar considerations apply to each piece of *any* area element when it acts as a horizon for some Rindler observer. Results obtained earlier show that the bulk action for gravity can be obtained from a surface term in the action, if we take the entropy of any horizon to be proportional to its area with an elemental area $\sqrt{\sigma} d^2 x$ contributing an entropy $dS = (Nn_\mu a^\mu)\sqrt{\sigma} d^2 x$. The definition given above in Eq. (153) is the integral expression of the same.

The total energy E in this region, acting as a source for gravitational acceleration, is given by the Tolman energy [162] defined by

$$E = 2 \int_{\mathcal{V}} d^3 x \sqrt{\gamma} N (T_{ab} - \frac{1}{2} T g_{ab}) u^a u^b \qquad (154)$$

The covariant combination $2(T_{ab} - (1/2)T g_{ab})u^a u^b$ [which reduces to $(\rho + 3p)$ for an ideal fluid] *is* the correct source for gravitational *acceleration*. For example, this will make geodesics accelerate away from each other in a universe dominated by cosmological constant, since $(\rho + 3p) < 0$. The factor N correctly accounts for the relative redshift of energy in curved spacetime. It is now possible to obtain some interesting relations between these quantities.

In any space time, there is differential geometric identity (see Eq. (A.13))

$$R_{bd} u^b u^d = \nabla_i (Ku^i + a^i) - K_{ab} K^{ab} + K_a^a K_b^b \qquad (155)$$

where K_{ab} is the extrinsic curvature of spatial hypersurfaces and K is its trace. This reduces to $\nabla_i a^i = R_{ab} u^a u^b$ in static spacetimes with $K_{ab} = 0$. Combined with Einstein's equations, this gives

$$\frac{1}{8\pi G} \nabla_i a^i = (T_{ab} - \frac{1}{2} T g_{ab}) u^a u^b \tag{156}$$

This equation deals directly with a^i which occur as the components of the metric tensor in Eq. (8). We now integrate this relation with the measure $\sqrt{-g} d^4 x$ over a four dimensional region chosen as before. Using Eq.(154),(153), the integrated form of Eq.(156) will read quite simply as

$$S = (1/2)\beta E, \tag{157}$$

Note that both S and E depend on the congruence of timelike curves chosen to define them through u^a. If these ideas are consistent, then the free energy of the spacetime must have direct geometrical meaning *independent of the congruence of observers used to define the entropy S and E*. It should be stressed that the energy E which appears in Eq.(154) is *not* the integral

$$U \equiv \int_{\mathcal{V}} d^3 x \sqrt{\gamma} N (T_{ab} u^a u^b) \tag{158}$$

based on $\rho = T_{ab} u^a u^b$ but the integral of $(\rho + 3p)$, since the latter is the source of gravitational acceleration in a region. The free energy, of course, needs to be defined as $F \equiv U - TS$, since pressure — which is an independent thermodynamic variable — should not appear in the free energy. This gives:

$$\beta F \equiv \beta U - S = -S + \beta \int_{\mathcal{V}} d^3 x \sqrt{\gamma} N (T_{ab} u^a u^b) \tag{159}$$

and using Eqs.(153),(156) and $R = -8\pi G T$, we find that

$$\beta F = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} R \tag{160}$$

which is just the Einstein-Hilbert action. The equations of motion obtained by minimising the action can be equivalently thought of as minimising the macroscopic free energy. For this purpose, it *is* important that F is generally covariant and is independent of the u^i used in defining other quantities.

The sign of E in Eq. (154) can be negative if matter with $\rho + 3p < 0$ dominates in the region $\mathcal{V}$. The sign of S in Eq. (153) depends on the convention chosen for the direction of the normal to $\partial\mathcal{V}$ but it is preferable to choose this such that $S > 0$. Then the sign of β will arrange itself so that Eq. (157) holds. (Of course, the temperature is $T = |\beta|^{-1} > 0$). As an illustration, consider the Schwarzschild spacetime and the De Sitter universe. For spherically symmetric metrics with a horizon, having $g_{00} = -g^{11}, g_{00}(r_H) = 0$, we can write

$g_{00} \approx g'_{00}(r_H)(r - r_H)$ near the horizon and $\beta = -4\pi/g'_{00}(r_H)$ in our signature convention. Hence $\beta = 8\pi M > 0$ for Schwarzschild while $\beta = -2\pi/H < 0$ for de Sitter. In the first case, $\beta = 8\pi M$ and we can take $E = M$ for any compact two surface $\partial\mathcal{V}$ that encloses the horizon. Since $Na = (M/r^2)$, Eq.(153) gives $S = 4\pi(M^2/G)$ for any $\partial\mathcal{V}$. This result agrees with Eq.(157). The de Sitter case is more interesting since it is nonempty. In the static coordinates with $-g_{00} = g^{rr} = (1 - H^2r^2)$, let us choose a spherical surface of radius $L < H$. We then have $E = -H^2L^3$ and $S = \pi HL^3$ from (154) and (153). Once again, equation (157) holds since $\beta = -2\pi/H$.

We should, therefore, be able to rewrite Einstein's equations in a form analogous to the $TdS - dU = PdV$ equation [75,163]. It is fairly straight forward to achieve this in the case of spacetimes of the form:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (161)$$

with $f(r) = 1 - 2m(r)/r$. This metric solves the Einstein's equations if the energy density $\rho(r)/8\pi$ and the transverse pressure $\mu(r)/8\pi$ are arranged to give $\rho(r) = (m'/2r^2); \mu(r) = \rho + (1/2)r\rho'(r)$ and the radial pressure is set equal to the energy density. If there is a horizon at $r = a$, with $f(a) = 0, f'(a) \equiv B$, then the temperature T is determined by $T^{-1} = \beta = 4\pi/B$. Further, we find that, for a spherical region of radius $r = a$,

$$S = \pi a^2; \quad E = \frac{1}{2}a^2 B, \quad |U| = \frac{a}{2} \qquad (162)$$

These relations hold on the horizon for a class of solutions parametrised by the function $m(r)$ with a determined as the root of the equation $2m(a) = a$. What is more, these relations, along with the fact that radial pressure is equal to the energy density, allow us to write Einstein's' equations as

$$dU = TdS - PdV \qquad (163)$$

where the differentials are interpreted as $dU = (dU/da)da$ etc. In these spacetimes, $S \propto U^2$ giving the density of states $g(U) = \exp(cU^2)$ where c is a constant.

The above results are of particular importance to a horizon which is *not* associated with a black hole, viz. De Sitter horizon. In this case, $f(r) = (1 - H^2r^2)$, $a = H^{-1}, B = -2H < 0$ so that the temperature — which should be positive — is $T = |f'(a)|/(4\pi) = (-B)/4\pi$. For horizons with $B = f'(a) < 0$ (like the De Sitter horizon) we have $f(a) = 0, f'(a) < 0$, and it follows that $f > 0$ for $r < a$ and $f < 0$ for $r > a$; that is, the "normal region" in which t is timelike is inside the horizon as in the case of, for example, the De Sitter metric. The Einstein's equations for the metric in Eq.(161) evaluated at the horizon $r = a$

reads as:

$$\frac{-B}{4\pi}d\left(\frac{1}{4}4\pi a^2\right) + \frac{1}{2}da = -T_r^r(a)d\left(\frac{4\pi}{3}a^3\right) = P(-dV) \tag{164}$$

The first term on the left hand side is again of the form TdS (with positive temperature and entropy). The term on the right hand side has the correct sign since the inaccessible region (where $f < 0$) is now outside the horizon and the volume of this region changes by $(-dV)$. Once again, we can use Eq. (164) to identify the entropy and the energy:

$$S = \frac{1}{4}(4\pi a^2) = \frac{1}{4}\mathcal{A}_{\text{horizon}}; \quad U = -\frac{1}{2}H^{-1} \tag{165}$$

As a byproduct, our approach provides an interpretation of energy for the De Sitter spacetime and a consistent thermodynamic interpretation of De Sitter horizon.

Our identification, $U = -(1/2)H^{-1}$ is also supported by the following argument: If we use the "reasonable" assumptions $S = (1/4)(4\pi H^{-2}), V \propto H^{-3}$ and $U = -PV$ in the equation $TdS - PdV = dU$ and treat U as an unknown function of H, we get the equation

$$H^2\frac{dU}{dH} = -(3UH + 1) \tag{166}$$

which integrates to give precisely $U = -(1/2)H^{-1}$. Note that we only needed the proportionality, $V \propto H^{-3}$ in this argument since $PdV \propto (dV/V)$. The ambiguity between the coordinate and proper volume is irrelevant.

These results can be stated more formally as follows: In standard thermodynamics, we can consider two equilibrium states of a system differing infinitesimally in the extensive variables volume, energy and entropy by dV, dU and dS while having *same* values for the intensive variables temperature (T) and pressure (P). Then, the first law of thermodynamics asserts that $TdS = PdV + dU$ for these states. In a similar vein, we can consider two spherically symmetric solutions to Einstein's equations with the radius of the horizon differing by da while having the same source T_{ik} and the same value for B. Then the entropy and energy will be infinitesimally different for these two spacetimes; but the fact that both spacetimes satisfy Einstein's equations shows that TdS and dU will be related to the external source T_{ik} and da by Einstein's equations. Just as in standard thermodynamics, this relation could be interpreted as connecting a sequence of quasi-static equilibrium states.

The analysis is classical except for the crucial periodicity argument which is used to identify the temperature uniquely. This is again done locally by approximating the metric by a Rindler metric close to the horizon and identifying the Rindler temperature. This idea bypasses the difficulties in defining

and normalising Killing vectors in spacetimes which are not asymptotically flat.

Finally we mention that this framework also imposes a strong constraints on the form of action functional A_{grav} in *semi-classical* gravity. It can be shown that, the area of the horizon, as measured by *any* observer blocked by that horizon, will be quantised [142]. In normal units, $\mathcal{A}_{\text{horizon}} = 8\pi m(G\hbar/c^3) = 8\pi m L_{\text{Planck}}^2$ where m is an integer. (Incidentally, this will match with the result from loop quantum gravity, for the high-j modes, if the Immirizi parameter is unity.) In particular, any flat spatial surface can be made a horizon for a suitable Rindler observer, and hence all area elements (in even flat space-time) must be intrinsically quantised. In the quantum theory, the area operator for one observer need not commute with the area operator of another observer, and there is no inconsistency in all observers measuring quantised areas. The changes in area, as measured by any observer, are also quantised, and the minimum detectable change is of the order of L_{Planck}^2. It can be shown, from very general considerations, that there is an operational limitation in measuring areas smaller than L_{Planck}^2, when the principles of quantum theory and gravity are combined [164]; our result is consistent with this general analysis. (The Planck length plays a significant role in different approaches which combine the principles of quantum theory and gravity; see, for example, [165,166].) While there is considerable amount of literature suggesting that the area of a *black hole* horizon is quantised [for a small sample of references, see [167,7,141,168,169,170,171,172,173,174,175,40,176,177,178,179] as well as papers cited in Section 7.1] the result mentioned above is more general and is applicable to any static horizon.

9 Conclusions and Outlook

We shall now take stock of the results discussed in this review from a broader perspective and will attempt to provide an overall picture.

Combining the principles of quantum theory with special relativity (and Lorentz invariance) required a fairly drastic change in the description of physical systems. Similarly, it is natural for new issues to arise when we take the next step of combining quantum theory with the concept of general covariance or when we attempt to do quantum field theory in a curved background spacetime. However, one would have naively expected these issues to be *kinematical* in the sense that they are independent of the field equations or the action for gravity. Our discussion shows that there is a strong link between the kinematical aspects and the dynamics of gravity because of the structure of classical general relativity. While it may be convenient to distinguish between the kinematical aspects (discussed in Sections 2 to 6) and the dynamical aspects
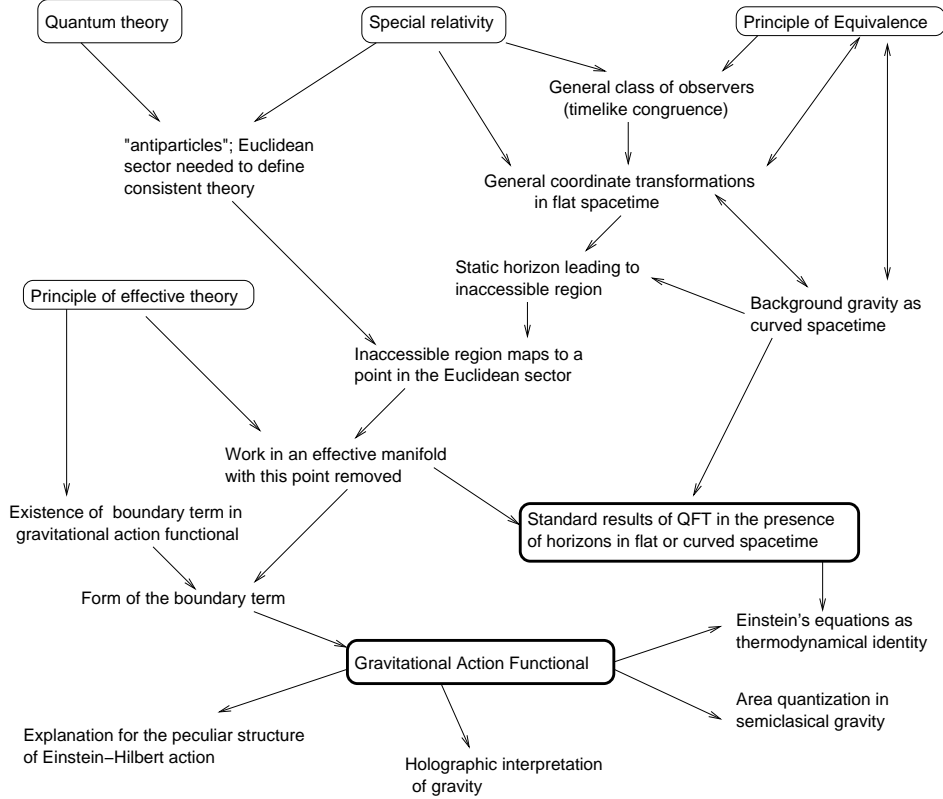
Fig. 4. Summary of the logical structure of the approach adopted in this review

(discussed in Sections 7 – 8), each by itself can only give a partial picture. The overall structure of the theory should allow a seamless transition across these two aspects.

In this review this was attempted by (i) noting that one needs to use the Euclidean sector to incorporate the new ingredients which arise when special relativity is combined with quantum mechanics and (ii) using the fact that when quantum theory is formulated in the Euclidean sector, a unique structure emerges in the presence of horizons. Using a congruence of timelike curves to define a horizon, one finds that it is possible to incorporate the kinematical effects of (at least static) horizons in a general manner and associate the notion of temperature with the horizons. This is achieved by using a coordinate system in which the spacetime region hidden by a horizon is mapped to a single point in the Euclidean sector and constructing an effective manifold for a family of observers by removing this point. The resulting non trivial topology leads to the standard results of quantum field theory in curved spacetimes with horizons.

The importance of the above point of view lies in its ability to provide a deeper relationship between gravity and thermodynamics, as shown in Section 8. If one accepts the idea — that the physical theory for a class of observers should be formulated in an effective manifold in which the region inaccessible to

those observers is removed — then one is led to enquire what it implies for the dynamics of gravity. Using the fact that the horizon is the common element between the inaccessible and accessible regions, it is possible to argue that the action functional for gravity must contain (i) a well defined surface term and (ii) a bulk term which is related to the surface term in a specific manner. Hence, this point of view allows one to determine the action functional for gravity from thermodynamic considerations. What is more, it links the kinematical and dynamical aspects of the theory in an interesting manner.

This approach is very similar in spirit to that of renormalisation group theory (RGT) in particle physics. When an experimenter does not have information about the model at scales $k > \Lambda$, say, in *momentum space*, the RGT allows one to use an effective low energy theory with the coupling constants readjusted to incorporate the missing information. This, in turn, puts restrictions on the nature of the theory as well as the "running" of the coupling constants. Similarly, when a given family of observers has limited information because they are blocked by a horizon (in *real* space rather than momentum space) it is necessary to add certain boundary terms in the action functional in order to provide a consistent description. Just as the RGT contains nontrivial information about the low energy sector of the theory, our approach allows us to determine the form of the action in the long wavelength limit of gravity. As far as the loss of information due to a horizon is concerned, there is no need to distinguish between the uniformly accelerated observers in flat spacetime and, say, the observers located permanently at $r > 2M$ in the Schwarzschild spacetime.

There are some new insights that arise in this approach which are worth exploring further.

- Einstein's equations for gravity can be obtained from a variety of action functionals, any two of which differ by a surface term. In the case of Einstein-Hilbert action, the surface term is related in a very specific manner to the bulk term. (See e.g., Eq. (143); it is rather intriguing that this relation has not been explored in the literature before.) *This relation is so striking that it demands an explanation* which is indeed provided by the thermodynamic paradigm described in Section 8.
- The approach makes gravity "holographic" in a specific sense of the word. The Einstein-Hilbert Lagrangian has the structure $L_{EH} = L_1 + L_2$ where $L_1 \sim (\partial g)^2, L_2 \sim \partial^2 g$. Along any world line, one can choose a coordinate system such that $(\partial g)^2 \to 0$ suggesting that the dynamics of the theory is actually contained in the $L_2 \sim \partial^2 g$ term which leads to the surface term in the action. We saw in Section 8.1 that one could determine the bulk term from the surface term under certain assumptions. This fact, that the structure of the surface term in an action determines the theory, provides a possible interpretation of holographic principle (which is somewhat different

from the conventional interpretation of the term).

- The approach supports the paradigm that the spacetime is similar to the continuum limit of a solid that is obtained when one averages over the underlying microscopic degrees of freedom [159]. As described in Section 8.2, this strongly indicates the possibility that gravity is intrinsically quantum mechanical at all scales just as solids cannot exist in the strict $\hbar \to 0$ limit. Just as the bulk properties of solids can be described without reference to the underlying atomic structure, much of classical and semi classical gravity (including the entropy of black holes) will be independent of the underlying description of the microscopic degrees of freedom. Clear signs of this independence emerges from the study of Einstein-Hilbert action which contains sufficient structure to lead to many of the results involving the horizon thermodynamics. Hence any microscopic description of gravity which leads to Einstein-Hilbert action as the long wavelength limit will also incorporate much of horizon physics.

Acknowledgements

A   Gravitational Action Functional

This appendix summarises several aspects of action functionals used in gravity and derives some of the results not readily available in the literature.

The conventional action principle for general relativity is the Einstein-Hilbert action given by

$$A_{\text{EH}} \equiv \frac{1}{16\pi} \int R\sqrt{-g}d^4x \qquad (A.1)$$

Straightforward algebra shows that the scalar curvature can be expressed in the form

$$R\sqrt{-g} = \frac{1}{4}\sqrt{-g}M^{abcijk}g_{ab,c}g_{ij,k} - \partial_j P^j \equiv \sqrt{-g}L_{\text{quad}} - \partial_j P^j \qquad (A.2)$$

where

$$M^{abcijk} = g^{ck}\left[g^{ab}g^{ij} - g^{ai}g^{bj}\right] + 2g^{cj}\left[g^{ai}g^{bk} - g^{ki}g^{ba}\right] \qquad (A.3)$$

and

$$P^j = \sqrt{-g}g_{ac,i}(g^{ac}g^{ji} - g^{ia}g^{cj}) \equiv \sqrt{-g}V^j \qquad (A.4)$$

This result is equivalent to a more conventional expression for the gravitational action written in terms of Christoffel symbols with:

$$L_{\text{quad}} = g^{ab}\left(\Gamma^i_{ja}\Gamma^j_{ib} - \Gamma^i_{ab}\Gamma^j_{ij}\right) \tag{A.5}$$

and

$$P^c = \sqrt{-g}\left(g^{ck}\Gamma^m_{km} - g^{ik}\Gamma^c_{ik}\right) = -\frac{1}{\sqrt{-g}}\partial_b(gg^{bc}) \tag{A.6}$$

The manner in which P^c is expressed hides its geometrical interpretation. To bring this out, note that the integral of $\partial_c P^c$ can be evaluated in a given coordinate system, most simply by:

$$\int d^4x\,\partial_c P^c = \int dx^0 dx^1 dx^2 dx^3(\partial_0 P^0 + \partial_1 P^1 + \cdots)$$
$$= \int_{x^0} dx^1 dx^2 dx^3 P^0 + \int_{x^1} dx^0 dx^2 dx^3 P^1 + \cdots \tag{A.7}$$

where the subscript on the integral indicates the coordinate that is held constant. To study the integral of P^n on the $x^n = $ constant surface, let us choose a coordinate system in which the metric has the form

$$ds^2 = g_{nn}(dx^n)^2 + g^{\perp}_{ab}dx^a dx^b \tag{A.8}$$

where $n = 0, 1, 2, 3$ and for each choice of n the a, b run over the other three coordinates. (We have assumed that the cross terms vanish to simplify the computation.) The P^c in this coordinate system can be computed using the last expression in Eq. (A.6). We get:

$$P^n = -\frac{1}{\sqrt{g_{nn}}}\frac{1}{\sqrt{g^{\perp}}}\partial_n\left(g_{nn}g^{\perp}\frac{1}{g_{nn}}\right) = -\frac{2}{\sqrt{g_{nn}}}\partial_n\sqrt{g^{\perp}} \tag{A.9}$$

The normal to the surface $x^n = $ constant is given by $n^a = g_{nn}^{-1/2}\delta^a_n$ and the trace of the extrinsic curvature of the $x^n = $ constant is

$$K = -\nabla_a n^a = -\frac{1}{\sqrt{g_{nn}}}\frac{1}{\sqrt{g^{\perp}}}\partial_n\left(\sqrt{g_{nn}}\sqrt{g^{\perp}}\frac{1}{\sqrt{g_{nn}}}\right) = -\frac{1}{\sqrt{g^{\perp}}}\frac{1}{\sqrt{g_{nn}}}\partial_n\sqrt{g^{\perp}} \tag{A.10}$$

Hence we get the result

$$\int_{\mathcal{V}} d^4x\,\partial_c P^c = \sum_{\partial\mathcal{V}} 2\int K\sqrt{g^{\perp}}d^3x \tag{A.11}$$

where the sum is over all the bounding surfaces. Thus the total divergence term can be expressed as the sum over the integrals of the extrinsic curvatures on

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each boundary and the Einstein-Hilbert action in Eq.(A.1) becomes

$$A_{\text{EH}} = \int_{\mathcal{V}} \frac{d^4x}{16\pi} \sqrt{-g} L_{\text{quad}} - \int_{\partial\mathcal{V}} \frac{d^3x}{8\pi} \sqrt{g^\perp} K \qquad (A.12)$$

This result can be obtained in a more geometrical fashion, which is instructive. We foliate the space-time by a series of space-like hypersurfaces Σ with normals u^i. Next, from the relation $R_{abcd}u^d = (\nabla_a\nabla_b - \nabla_b\nabla_a)u_c$, we obtain the identity

$$\begin{aligned}
R_{bd}u^b u^d &= g^{ac} R_{abcd}u^b u^d = u^b\nabla_a\nabla_b u^a - u^b\nabla_b\nabla_a u^a \\
&= \nabla_a(u^b\nabla_b u^a) - (\nabla_a u^b)(\nabla_b u^a) - \nabla_b(u^b\nabla_a u^a) + (\nabla_b u^b)^2 \\
&= \nabla_i(Ku^i + a^i) - K_{ab}K^{ab} + K_a^a K_b^b \qquad (A.13)
\end{aligned}$$

where $K_{ij} = K_{ji} = -\nabla_i u_j - u_i a_j$, is the extrinsic curvature with $K \equiv K_i^i = -\nabla_i u^i$ and $K_{ij}K^{ij} = (\nabla_i u^j)(\nabla_j u^i)$. Further using

$$R = -R \; g_{ab}u^a u^b = 2(G_{ab} - R_{ab})u^a u^b \;, \qquad (A.14)$$

and the identity

$$2 \, G_{ab}u^a u^b = K_a^a K_b^b - K_{ab}K^{ab} + {}^{(3)}\mathcal{R} \;, \qquad (A.15)$$

where ${}^{(3)}\mathcal{R}$ is the scalar curvature of the 3-dimensional space, we can write the scalar curvature as

$$R = {}^{(3)}\mathcal{R} + K_{ab}K^{ab} - K_a^a K_b^b - 2\nabla_i(Ku^i + a^i) \equiv L_{\text{ADM}} - 2\nabla_i(Ku^i + a^i) \;, \quad (A.16)$$

where L_{ADM} is the ADM Lagrangian.

Let us now integrate Eq. (A.16) over a four volume $\mathcal{V}$ bounded by two space-like hypersurfaces Σ_1 and Σ_2 and a time-like hypersurface $\mathcal{S}$. The space-like hypersurfaces are constant time slices with normals u^i, and the time-like hypersurface has normal n^i orthogonal to u^i. The induced metric on the space-like hypersurface Σ is $h_{ab} = g_{ab} + u_a u_b$, while the induced metric on the time-like hypersurface $\mathcal{S}$ is $\gamma_{ab} = g_{ab} - n_a n_b$. The Σ and $\mathcal{S}$ intersect along a 2-dimensional surface $\mathcal{Q}$, with the induced metric $\sigma_{ab} = h_{ab} - n_a n_b = g_{ab} + u_a u_b - n_a n_b$. With $g_{00} = -N^2$, we get

$$\begin{aligned}
A_{\text{EH}} = \frac{1}{16\pi} \int_{\mathcal{V}} d^4x \; \sqrt{-g} \; R &= \frac{1}{16\pi} \int_{\mathcal{V}} d^4x \; \sqrt{-g} \; L_{\text{ADM}} - \frac{1}{8\pi} \int_{\Sigma_1}^{\Sigma_2} d^3x \; \sqrt{h} \; K \\
&\quad - \frac{1}{8\pi} \int_{\mathcal{S}} dt \; d^2x \; N \; \sqrt{\sigma}(n_i a^i) \;. \qquad (A.17)
\end{aligned}$$

Let the hypersurfaces $\Sigma, \mathcal{S}$ as well as their intersection 2-surface $\mathcal{Q}$ have the corresponding extrinsic curvatures K_{ab}, Θ_{ab} and q_{ab}. To express the Einstein-

Hilbert action in the form in Eq.(A.12), as a term having only the first derivatives, plus an integral of the trace of the extrinsic curvature over the bounding surfaces, we use the foliation condition $n_i u^i = 0$ between the surfaces, and note that

$$n_i a^i = n_i u^j \nabla_j u^i = -u^j u^i \nabla_j n_i = (g^{ij} - h^{ij}) \nabla_j n_i = q - \Theta \qquad (A.18)$$

where $\Theta \equiv \Theta_a^a$ and $q \equiv q_a^a$ are the traces of the extrinsic curvature of the surfaces, when treated as embedded in the 4-dimensional or 3-dimensional enveloping manifolds. Using Eq. (A.18) to replace $(n_i a^i)$ in the last term of Eq. (A.17), we get the result

$$A_{\text{EH}} + \frac{1}{8\pi} \int_{\Sigma_1}^{\Sigma_2} d^3x \sqrt{h} K - \frac{1}{8\pi} \int_{\mathcal{S}} dt d^2x N \sqrt{\sigma} \Theta$$
$$= \frac{1}{16\pi} \int_{\mathcal{V}} d^4x \sqrt{-g} L_{\text{ADM}} - \frac{1}{8\pi} \int_{\mathcal{S}} dt d^2x N \sqrt{\sigma} q \qquad (A.19)$$

The left hand side is in the form we want as the sum of A_{EH} and the traces of extrinsic curvatures on the bounding surfaces. In the right hand side, the first term, L_{ADM} is *not* purely quadratic in the first derivatives of the metric tensor, since it contains $^{(3)}\mathcal{R}$, which in turn contains second derivatives of the metric tensor. We can now use a formula, analogous to Eq. (A.2), to separate the second derivatives from $^{(3)}\mathcal{R}$. The relation is

$$^{(3)}\mathcal{R}\sqrt{h} = {}^{(3)}L_{\text{quad}}\sqrt{h} + \partial_\mu Q^\mu, \qquad (A.20)$$

where h is the determinant of the spatial metric, $^{(3)}L_{\text{quad}}$ is made from the spatial metric and its spatial derivatives and Q^μ is same as P^i but built from spatial metric. The sign reflects the fact that g is negative definite while h is positive definite. What we need in Eq.(A.19) is $\sqrt{-g}^{(3)}\mathcal{R} = N\sqrt{h}^{(3)}\mathcal{R}$ which becomes:

$$\sqrt{-g}^{(3)}\mathcal{R} = {}^{(3)}L_{\text{quad}}\sqrt{-g} + N\partial_\mu Q^\mu \qquad (A.21)$$
$$= {}^{(3)}L_{\text{quad}}\sqrt{-g} - \sqrt{-g}\left(\frac{\partial_\mu N}{N}\right)\frac{\partial_\nu(hh^{\mu\nu})}{h} + \partial_\mu(NQ^\mu)$$

On integration, the last term becomes a surface integral and using the result analogous to Eq. (A.11), we find that

$$\int dt d^3x \partial_\mu(NQ^\mu) = \int dt d^2x N Q^\mu n_\mu = \int dt d^2x N \sqrt{\sigma} q \qquad (A.22)$$

When we substitute Eq.(A.21) into the L_{ADM} in Eq.(A.19), the terms with q cancel and we get the final result:

$$A_{\text{EH}} + \sum \frac{1}{8\pi} \int_{\Sigma_1}^{\Sigma_2} d^3x \ \sqrt{h} \ K = \frac{1}{16\pi} \int_{\mathcal{V}} d^4x \ \sqrt{-g}[(K_{ab}K^{ab} - K_a^a K_b^b)$$

$$+ \ ^{(3)}L_{\text{quad}} + \frac{\partial_\mu N}{Nh} \partial_\nu(hh^{\mu\nu})] \tag{A.23}$$

which is precisely A_{quad}. The terms with K_{ab} are quadratic in time derivatives of spatial metric, the $^{(3)}L_{\text{quad}}$ has quadratic terms of spatial derivatives of spatial metric and the last term gives a (quadratic) cross term between spatial derivatives of spatial metric and $g_{00,\mu}$. This is the standard result often used, which—unfortunately—misses the importance of the $(n_i a^i)$ term in the action by splitting it as in Eq. (A.18).

Let us now get back to some features of Eq. (A.2) which are not adequately emphasised in the literature. The first interesting result that can be obtained from Eq. (A.2) is a direct relation between P^j and L_{quad}. Differentiation of L_{quad} followed by contraction with g_{ab} gives

$$g_{ab}\frac{\partial L_{\text{quad}}}{\partial(g_{ab,c})} = g_{ij,k}\left[g^{ij}g^{ck} - g^{ik}g^{cj}\right] = V^c = \frac{1}{\sqrt{-g}}P^c \tag{A.24}$$

This remarkable result shows that the scalar curvature can be written in the form

$$R = L_{\text{quad}} - \frac{1}{\sqrt{-g}}\partial_c\left[\sqrt{-g}g_{ab}\frac{\partial L_{\text{quad}}}{\partial(g_{ab,c})}\right] \tag{A.25}$$

Comparing this result with Eq. (A.11), we get a more dynamical interpretation of K. We have

$$2K = n_c g_{ab}\frac{\partial L_{\text{quad}}}{\partial(g_{ab,c})} \equiv n_c g_{ab}\pi^{abc} \tag{A.26}$$

The quantity $\Pi^{ab} = n_c\pi^{abc}$ is the energy-momentum conjugate to g_{ab} with respect to the surface defined by the normal n_c.

If we take the Lagrangian to be $L(q_A, \partial_i q_A)$ which depends on a set of dynamical variables q_A where A could denote a collection of indices (in the case of gravity $q_A \rightarrow g_{ab}$ with A denoting a pair of indices), then one can obtain a second Lagrangian by

$$L_\pi = L - \partial_i\left[q_A\frac{\partial L}{\partial(\partial_i q_A)}\right] = L - \partial_i(q_A p^{Ai}) \tag{A.27}$$

Both will lead to the same equations of motion provided q_A is fixed while varying L and p^{Ai} is fixed while varying L_π. [See discussion leading to Eq. (128).] In the case of gravity, L corresponds to the quadratic Lagrangian while L_π corresponds to the Einstein-Hilbert Lagrangian and Eq.(A.27) corresponds to Eq.(A.25).

It is possible to understand Eq.(A.25) from the fact that L_{quad} has certain degrees of homogeneity in terms of g_{ab} and $g_{ab,c}$. The argument proceeds as

follows: Consider any Lagrangian $L(q_A, \partial_i q_A)$ which depends on a set of dynamical variables q_A where A could denote a collection of indices as before. Let the Euler-Lagrange function resulting from L be:

$$F^A \equiv \frac{\partial L}{\partial q_A} - \partial_i \left[\frac{\partial L}{\partial(\partial_i q_A)} \right] \tag{A.28}$$

Taking the contraction $q_A F^A$ and manipulating the terms we get

$$q_A F^A = q_A \frac{\partial L}{\partial q_A} - \partial_i \left[q_A \frac{\partial L}{\partial(\partial_i q_A)} \right] + (\partial_i q_A) \frac{\partial L}{\partial(\partial_i q_A)} \tag{A.29}$$

If L is a homogeneous function of degree μ in q_A and a homogeneous function of degree λ in $\partial_i q_A$, then the first term on the right hand side is μL and the third term is λL because of Euler's theorem. Hence

$$q_A F^A = (\lambda + \mu)L - \partial_i \left[q_A \frac{\partial L}{\partial(\partial_i q_A)} \right] \tag{A.30}$$

In the case of gravity, $F^A = -(R^{ab} - (1/2)g^{ab}R)\sqrt{-g}$ with the minus sign arising from the fact that F^A corresponds to contravariant indices. So

$$q_A F^A = g_{ab}[-(R^{ab} - \frac{1}{2}g^{ab}R)\sqrt{-g} = R\sqrt{-g} \tag{A.31}$$

Further, if we change $g_{ab} \to f g_{ab}$ then $g^{ab} \to f^{-1}g^{ab}$, $\sqrt{-g} \to f^2\sqrt{-g}$. If the first derivatives $g_{ab,c}$ are held fixed, the above changes will change $\sqrt{-g}L_{\text{quad}}$ in Eq. (A.2) by the factor $f^2 f^{-3} = f^{-1}$ showing that $\sqrt{-g}L_{\text{quad}}$ is of degree $\mu = -1$ in g_{ab}. When g_{ab} is held fixed and $g_{ab,c}$ is changed by a factor f, $\sqrt{-g}L_{\text{quad}}$ changes by factor f^2; so $\sqrt{-g}L_{\text{quad}}$ is of degree $\lambda = +2$ in the derivatives. Using $q_A F^A = R\sqrt{-g}$ and $\mu + \lambda = 1$ in Eq. (A.30) we get the result which is identical to Eq. (A.25).

From the relation Eq. (A.27), it is possible to derive the variations of A_{EH} and A_{quad} for arbitrary variations of δg_{ab}. We get:

$$\delta(16\pi A_{\text{EH}}) = \int_{\mathcal{V}} d^4 x \sqrt{-g}\, G_{ab}\delta g^{ab} + \int_{\partial\mathcal{V}} d^3 x\, h_{ab}\delta[\sqrt{h}(K^{ab} - h^{ab}K)]$$
$$= \int_{\partial\mathcal{V}} d^3 x\, h_{ab}\delta\Pi^{ab} \tag{A.32}$$

where $\Pi^{ab} = \sqrt{h}(K^{ab} - h^{ab}K)$ and the last equality holds when equation of motion ($G_{ab} = 0$) are satisfied ("on-shell"). Similarly,

$$\delta(16\pi A_{\text{quad}}) = \int_{\mathcal{V}} d^4 x \sqrt{-g}\, G_{ab}\delta g^{ab} - \int_{\partial\mathcal{V}} d^3 x[\sqrt{h}(K^{ab} - h^{ab}K)]\delta h_{ab}$$
$$= -\int_{\partial\mathcal{V}} d^3 x\, \Pi^{ab}\delta h_{ab} \tag{A.33}$$

with the last equality holding on shell. Subtracting one from the other, we have

$$16\pi\delta(A_{\text{quad}} - A_{\text{EH}}) = -\int_{\partial\mathcal{V}} d^3x(\Pi^{ab}\delta h_{ab} + h_{ab}\delta\Pi^{ab})$$
$$= -\int_{\partial\mathcal{V}} d^3x\delta(h_{ab}\Pi^{ab}) = 2\delta\int_{\partial\mathcal{V}} d^3x\sqrt{h}K \qquad (A.34)$$

irrespective of the equations of motion ("off-shell") which is precisely what is needed for consistency. Thus Einstein-Hilbert Lagrangian describes gravity in the momentum space and leads to the field equations when the momenta Π^{ab} are fixed at the boundaries while the quadratic Lagrangian describes gravity in the coordinate space with the metric h_{ab} fixed on the boundary.

Finally, we shall provide a direct derivation of the ADM form of the action starting from Eq.(A.2) and separating out the space and time components. To do this, we shall assume a metric of the form $g_{00} = -N^2, g_{0\alpha} = 0$ and $g_{\alpha\beta}$ arbitrary. In evaluating the kinetic energy term of the form $(1/4)M\partial g\partial g$ in Eq.(A.2), one can separate out the terms made of (i) the time derivatives of $g_{\alpha\beta}$, (ii) time derivatives of g_{00}, (iii) spatial derivatives of $g_{\alpha\beta}$, (iv) spatial derivatives of g_{00}, (v) mixed terms involving one spatial derivative of g_{00} and one spatial derivative of $g_{\alpha\beta}$. Of these, it is easy to verify that (ii) and (iv) vanishes identically since the corresponding component of M is zero. The remaining three terms give in L_{quad}:

$$L_{\text{quad}} = \frac{1}{4N^2}\dot{g}_{\alpha\beta}\dot{g}_{\mu\nu}\left[g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}\right] + \left(\frac{\partial_\mu N}{N}\right)\partial_\nu g_{\alpha\beta}\left[g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\beta\mu}\right] + (\cdots)$$
$$(A.35)$$

where $(\cdots)$ denote purely spatial terms. The first three terms in L_{quad} correspond to (i), (v) and (iii) respectively. The last term made entirely out of spatial derivatives of spatial metric is not explicitly written down. Next consider the terms that arise from $(-g)^{-1/2}\partial_c P^c$ which can be classified as follows: (a) The time derivative term arises from $c = 0$. (b) Spatial derivatives involving $\partial_\alpha g_{00}$. (c) In calculating the spatial derivative terms, one should note that $\sqrt{-g} = N\sqrt{h}$. This will give terms involving product of spatial derivatives of N and $g_{\alpha\beta}$. (d) Spatial derivatives of purely spatial metric. Working out the terms, we get

$$\frac{1}{\sqrt{-g}}\partial_c\left(\sqrt{-g}V^c\right) = \frac{1}{\sqrt{-g}}\partial_0\left(\sqrt{-g}g^{00}g^{\alpha\beta}\dot{g}_{\alpha\beta}\right) + \frac{2}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial_\beta N}{N}\right)$$
$$+ \frac{\partial_\mu N}{N}\partial_\nu g_{\alpha\beta}\left[g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\beta\mu}\right] + (\cdots) \qquad (A.36)$$

When Eq. (A.35) and Eq. (A.36) are added, the cross term involving $\partial_\mu N\partial_\nu g_{\alpha\beta}$ cancels out precisely. All the spatial terms combine together to give $^{(3)}\mathcal{R}$. This

leads to the result

$$R = \frac{1}{4N^2} \dot{g}_{\alpha\beta} \dot{g}_{\mu\nu} \left[g^{\alpha\beta} g^{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} \right] + {}^3R$$

$$- \frac{1}{\sqrt{-g}} \partial_0 \left[\sqrt{-g} g^{00} g^{\alpha\beta} \dot{g}_{\alpha\beta} \right] - \frac{2}{\sqrt{-g}} \partial_\alpha \left[\sqrt{-g} g^{\alpha\beta} \frac{\partial_\beta N}{N} \right] \qquad \text{(A.37)}$$

The terms in the first line give what is conventionally called the ADM Lagrangian L_{ADM}. The time derivative term (in the second line) leads to the integral of twice the trace of the extrinsic curvature K on the t =constant surfaces. The spatial derivative term leads to the integral of twice the normal component of the acceleration on the timelike boundaries. Incidentally, note that the last two terms can be expressed more symmetrically in the form

$$- \frac{1}{\sqrt{-g}} \left[\partial_0 \left(\sqrt{-g} g^{00} g^{\alpha\beta} \partial_0 g_{\alpha\beta} \right) - \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} g^{00} \partial_\beta g_{00} \right) \right] \qquad \text{(A.38)}$$

It is clear that the structure of Einstein-Hilbert Lagrangian is very special.

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