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Evolution, symmetry, and canonical structure in dynamics

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The behavior of symmetries of classical equations of motion under quantization is studied from a new point of view. GL(3,R), which is an invariance group of the linear equations of motion for the nonrelativistic free particle as well as the isotropic harmonic oscillator, is imposed as a group of automorphisms on acceptable Poisson brackets, and the consequences are examined in detail. The six independent variables of the classical system arrange themselves, in each acceptable bracket, into one canonical pair and four neutral elements. Consequences of this for the equations of motion, existence of a Hamiltonian, breakup of the states of motion into superselected sectors due to existence of neutral elements, and determination of the canonically realized subgroup of GL(3,R) are all discussed. The possible relevance of this manner of symmetry breakdown for solid state and particle physics is pointed out.

I. INTRODUCTION

It is by now well recognized that the equations of motion of a classical dynamical system may possess a larger group of symmetries than are captured in a Hamiltonian description with associated canonical structure.[1,2] The passage from a classical to the "corresponding" quantum-mechanical description of a system requires that the former be given in Lagrangian, or equivalently Hamiltonian, terms. If one adopts the viewpoint that the specification of the Lagrangian defines the classical system, then one is automatically led to the usual Poisson-bracket structure on the phase space; this is so whether the Lagrangian be nonsingular or singular. Then those (infinitesimal) transformations of the system that leave the Lagrangian invariant up to a total time derivative will be symmetries of the equations of motion, representable as canonical transformations on phase space with well-defined phase-space functions for generators. Symmetries of the equations of motion lying outside this set do not preserve the Poisson-bracket structure. They have neither constants of motion nor infinitesimal generators associated with them, and so are lost or broken when one passes to the quantum theory on the basis of the chosen Lagrangian.[1-3]

One can take an alternate point of view and say that a classical system is fully defined by its equations of motion (and, of course, the physical meaning attached to the variables). The attempt to find a Hamiltonian description is then motivated solely by the desire to set up a corresponding quantum system. This attitude exposes a new and deep ambiguity in the passage from classical to quantum theory. It is, in general, possible to set up several inequivalent Poisson-bracket structures on phase space, each with an associated Hamiltonian, all of which reproduce the same equations of motion. Depending on which of these structures one chooses, a corresponding subgroup of the group of symmetries of the equations of motion is singled out as the one realized via canonical transformations, and so after quantization via unitary transformations.[1,2]

We shall present in this paper yet another approach to this complex of questions, which is the following. Starting with a definite set of equations of motion, one can pick a specific subgroup of the symmetry group of the equations of motion and search for all possible definitions of (generalized) Poisson brackets on phase space with the property of being formally preserved under the chosen subgroup. Even for the simplest of systems one finds that this procedure of insisting that the Poisson brackets admit a given group as a group of automorphisms leads to a rich and varied set of results. An allowed bracket structure can be singular in the sense that one has nontrivial neutral elements (that is, functions whose Poisson brackets with *all* functions vanish), so that the true number of canonical degrees of freedom is less than the dimension of the phase space one started with. Correspondingly, while the Poisson brackets are preserved in form under the chosen group, it is a further subgroup that acts as a set of inner automorphisms or as canonical transformations with well-defined functions for generators. And this subgroup can vary from one allowed Poisson bracket to another. After all permissible bracket structures have been found and analyzed in this way, one can then go back to the equations of motion and search for those bracket definitions that are stable with respect to time evolution. This stability is a necessary, but not sufficient, prerequisite for the existence of a Hamiltonian. Contrasted with the approach described in the previous paragraph, here the automorphism group of the Poisson brackets is specified first, and the existence of a Hamiltonian is investigated later.

The system we study is a nonrelativistic particle in three dimensions obeying equations of motion linear in (Cartesian) position and momentum. The cases of physical interest are the free particle for which

$$\dot{\vec{q}}=\vec{p}/m\,,\quad \dot{\vec{p}}=0 \tag{1}$$

and the isotropic harmonic oscillator

$$\dot{\vec{q}}=\vec{p}/m\,,\quad \dot{\vec{p}}=-m\omega^2\vec{q}\,. \tag{2}$$

In the former case one normally uses the expression for the conserved kinetic energy as the Lagrangian and this then leads to the conventional Poisson brackets (PB's)

$$\{q_j,q_k\}=\{p_j,p_k\}=0\,,\quad \{q_j,p_k\}=\delta_{jk}\,. \tag{3}$$

The Euclidean group of rotations and translations in configuration space, under which Eqs. (1) are preserved, preserves the PB's (3) as well and, in fact, acts as a group of canonical transformations generated by the linear and angular momenta. [For the isotropic oscillator (2) one normally uses the same PB's (3), and the canonically realized symmetry group is SU(3): this includes transformations with a quadrupole generator mixing $\vec{q}$ and $\vec{p}$ linearly.] However, both sets of equations of motion (1) and (2) are also invariant under the general real linear group GL(3,R), that is, when $\vec{q}$ and $\vec{p}$ are subjected to one and the same real nonsingular linear transformation

$$q_j\to q_j'=M_{jk}q_k\,,\quad p_j\to p_j'=M_{jk}p_k\,,\quad \det M\neq 0\,. \tag{4}$$

[Note that unlike the SU(3) symmetry group of the oscillator, there is *no mixing* of $\vec{q}$ and $\vec{p}$ here.] Elements of GL(3,R) outside the rotation subgroup violate the PB's (3) and are not realized as canonical transformations. We take GL(3,R) as the basis of our study. In Sec. II we pose and solve the problem of finding all possible definitions of generalized PB's on three q's and three p's such that they are form invariant when GL(3,R) acts as in Eq. (4). It turns out that the solutions can be profitably divided into five distinct types; the basis for this is explained in Sec. II. All solutions are singular. In Sec. III we examine each type of solution to discover how many independent neutral elements it possesses, and then expose the balance of true canonical variables. In all cases we find four independent neutral elements and one canonical pair, adding up to six variables. The subgroup of GL(3,R) that acts via inner automorphisms in each case, and the way in which it so acts, is also analyzed in Sec. III. In Sec. IV we search for those GL(3,R)-invariant PB's that are stable with respect to the free-particle equation of motion (1). For this purpose it is necessary to divide each of the five families of solutions into smaller subfamilies, and one then finds that only three cases are stable. In each of these cases we are able to exhibit a Hamiltonian, and we also briefly examine the transformations generated by the linear and angular momenta. Section V contains a corresponding stability analysis for the oscillator case. In the concluding section we comment on the lessons to be drawn from this study and especially its significance for quantization and symmetry breaking.

II. GL(3,R)-INVARIANT POISSON BRACKETS

With respect to the action (4) of GL(3,R) on $\vec{q}$ and $\vec{p}$ there are no bilinear invariants or invariant numerical tensors. So the only way we can possibly have GL(3,R)-invariant brackets is to choose the brackets between the basic variables to be themselves bilinear expressions. The most general possibility is then given by

$$\begin{aligned}&\{q_j,q_k\}=\alpha(q_jp_k-q_kp_j)\,,\\&\{p_j,p_k\}=\beta(q_jp_k-q_kp_j)\,,\\&\{q_j,p_k\}=\lambda q_jq_k+\mu p_jp_k+\nu q_jp_k+\rho q_kp_j\,.\end{aligned} \tag{5}$$

This involves six real parameters α, β, λ, μ, ν, ρ; and hereafter specific values for these parameters will always be listed in this sequence. Brackets among functions of $\vec{q}$ and $\vec{p}$ are evaluated using the derivation property

$$\{AB,C\}=A\{B,C\}+\{A,C\}B\,. \tag{6}$$

The Jacobi identities now impose constraints on the six parameters. We find that the identities for three q's or three p's, i.e.,

$$\sum_{\text{cyclic}}\{q_j,\{q_k,q_l\}\}=\sum_{\text{cyclic}}\{p_j,\{p_k,p_l\}\}=0 \tag{7}$$

are automatically satisfied: this is obviously because with just two GL(3,R) vectors $\vec{q}$ and $\vec{p}$ it is impossible to form a totally antisymmetric third-rank tensor. The remaining two Jacobi identities

$$\begin{aligned}&\{q_j,\{q_k,p_l\}\}+\{q_k,\{p_l,q_j\}\}+\{p_l,\{q_j,q_k\}\}=0\,,\\&\{p_j,\{p_k,q_l\}\}+\{p_k,\{q_l,p_j\}\}+\{q_l,\{p_j,p_k\}\}=0\end{aligned} \tag{8}$$

lead to the first pair and the second pair, respectively, of the following four equations:

$$\begin{aligned}&\mu\lambda+2\lambda\alpha+\alpha\beta=\rho^2\,,\\&(\rho+\nu)\alpha=(\rho-\nu)\mu\,,\\&\lambda\mu+2\mu\beta+\beta\alpha=\rho^2\,,\\&(\rho+\nu)\beta=(\rho-\nu)\lambda\,.\end{aligned} \tag{9}$$

Any solution of these four equations, other than the identically vanishing one, leads to an acceptable GL(3,R)-invariant nontrivial PB among three q's and three p's.

Let us denote by S the set of all solutions to Eqs. (9). In order to be able to survey and classify the members of S in a compact way, we proceed as follows. With $\vec{q}$ and $\vec{p}$ obeying Eq. (5) corresponding to some element of S, let us subject $\vec{q}$ and $\vec{p}$ to a general real linear transformation in two dimensions:

$$q_j' = aq_j + bp_j\,, \quad p_j' = cq_j + dp_j\,, \qquad \Delta \equiv ad - bc \neq 0\,. \tag{10}$$

This commutes with the action (4) of GL(3,R). If we therefore compute the PB's among $\vec{q}'$ and $\vec{p}'$ and express the results in terms of $\vec{q}'$ and $\vec{p}'$, they will have the same form as Eqs. (5) but with altered parameter values. These new values will, of course, be a solution to Eqs. (9). Thus transformations (10) on $\vec{q}$ and $\vec{p}$ give rise in a natural way to linear transformations preserving Eqs. (9) and thus acting on S. A common scale change of $\vec{q}$ and $\vec{p}$ by a positive factor leaves the parameters $\alpha, \beta, \ldots$ unchanged, so we need only consider $\Delta = \pm 1$ in Eq. (10). If $\Delta = +1$, we have in (10) the transformations of the group SL(2,R). But here again, a simultaneous change of sign of $\vec{q}$ and $\vec{p}$ [which *is* an SL(2,R) transformation] leads to no change at all in the parameters. Thus the faithful action of SL(2,R) on $\vec{q}$ and $\vec{p}$ as in Eq. (10) leads to an action of this group on $\alpha, \beta, \ldots$ which is not faithful, since two elements of SL(2,R) differing but in sign act identically on S. The transformations on S [corresponding to $\Delta = +1$ in Eq. (10)] must therefore form a faithful action of the proper orthochronous homogeneous Lorentz group in three dimensions, denoted by $SO(2,1)_+$.[4] The parameters transform as follows:

$$\begin{aligned}
\alpha' &= a^2\alpha + b^2\beta + ab(\nu - \rho)\,,\\
\beta' &= c^2\alpha + d^2\beta + cd(\nu - \rho)\,,\\
\lambda' &= d^2\lambda + c^2\mu - cd(\nu + \rho)\,,\\
\mu' &= b^2\lambda + a^2\mu - ab(\nu + \rho)\,,\\
\nu' &= ac(\alpha - \mu) + bd(\beta - \lambda) + (ad + bc)\nu\,,\\
\rho' &= -ac(\alpha + \mu) - bd(\beta + \lambda) + (ad + bc)\rho\,.
\end{aligned} \tag{11}$$

We shall rewrite this in a more transparent form in a moment. As for the possibility $\Delta = -1$ in (10), it suffices to consider the single transformation of physical time reversal

$$T\colon\ \vec{q} \to \vec{q}\,,\ \vec{p} \to -\vec{p}\,, \tag{12}$$

which has the following effect on the parameters:

$$T\colon\ \alpha' = -\alpha\,,\ \beta' = -\beta\,,\ \lambda' = -\lambda\,, \qquad \mu' = -\mu\,,\ \nu' = \nu\,,\ \rho' = \rho\,. \tag{13}$$

The fact that we have in Eq. (11) a six-dimensional real linear representation of $SO(2,1)_+$, with transformation coefficients which are bilinear in the elements of the SL(2,R) matrix, suggests strongly that $\alpha, \beta, \ldots$ are combinations of components of two vectors under $SO(2,1)_+$. This is indeed so. We define two vectors x and y in a fictitious three-dimensional space-time, with signature $+--$, as

$$x_0 = \alpha + \beta\,,\ x_1 = \alpha - \beta\,,\ x_2 = \rho - \nu\,, \qquad y_0 = \mu + \lambda\,,\ y_1 = \mu - \lambda\,,\ y_2 = \rho + \nu\,. \tag{14}$$

Then the transformation (11) on the parameters, induced by the SL(2,R) action (10) on $\vec{q}$ and $\vec{p}$, can be depicted as

$$x \to x' = \Lambda x\,,\quad y \to y' = \Lambda y\,,$$
$$\Lambda = \begin{pmatrix} \frac{1}{2}(a^2+b^2+c^2+d^2) & \frac{1}{2}(a^2-b^2+c^2-d^2) & -(ab+cd)\\ \frac{1}{2}(a^2+b^2-c^2-d^2) & \frac{1}{2}(a^2-b^2-c^2+d^2) & cd-ab\\ -(ac+db) & bd-ac & ad+bc \end{pmatrix}. \tag{15}$$

Note that $SO(2,1)_+$ does *not* include "fictitious" time reversal $x_0 \to -x_0$, $x_1 \to x_1$, $x_2 \to x_2$ which is not unimodular. The time-reversal transformation T of Eq. (13) translates into

$$x_0 \to -x_0\,,\ x_1 \to -x_1\,,\ x_2 \to x_2 \tag{16}$$

and similarly for y. This too is not included in $SO(2,1)_+$ since it is not continuously connected to the identity. Combining the action of $SO(2,1)_+$ as in (15) and of T as in (16) we get the group SO(2,1) of all unimodular Lorentz transformations in three dimensions acting faithfully on S. We shall write G for this group and use its action to classify the members of S. Equations (9) defining S become, in terms of x and y,

$$x \wedge y = 0\,,\quad (x+y)^2 = 0\,. \tag{17}$$

At this point one recognizes the existence of further transformations on x and y preserving Eqs. (17), and so on $\alpha, \beta, \ldots$ preserving Eqs. (9),

and thus defined on S. These are: (i) a common scale transformation applied to both x and y, and so to all the parameters $\alpha,\beta,\ldots$, (ii) the interchange $x\leftrightarrow y$, corresponding to the interchanges $\alpha\leftrightarrow\mu$, $\beta\leftrightarrow\lambda$, $\nu\to-\nu$, $\rho\to\rho$, (iii) fictitious time reversal $x_0\to -x_0$, $x_1\to x_1$, $x_2\to x_2$ (and similarly for y) corresponding to $\alpha\to-\beta$, $\beta\to-\alpha$, $\lambda\to-\mu$, $\mu\to-\lambda$, $\nu\to\nu$, $\rho\to\rho$. We might consider including these transformations and so enlarging the group used to classify the elements of S.
However, none of the transformations just listed appears as the result of a transformation defined on $\vec{q}$ and $\vec{p}$ and then elevated to act on S, so we do not consider them further and confine ourselves to the action of $G=\mathrm{SO}(2,1)$ on S.

Leaving aside the totally trivial case $x=y=0$, one sees that the following types of solutions to Eqs. (17) exist: (i) one of the vectors x,y vanishes and the other is lightlike, (ii) both are nonvanishing, they are parallel, and their sum (and so each one) is lightlike, (iii) the sum $x+y$ is null, allowing x to be timelike, lightlike, or spacelike. With the help of the group G, a solution of any type can be transformed into a standard form. In other words, S splits into disjoint orbits under the action of G, and we can display one representative element from each distinct orbit. We label the orbits and choose representative elements in this way:

Orbit A: $x=0,\ y^2=0$; representative $x=0,\ y=(1,0,1)$, (18a)

Orbit B: $x^2=0,\ y=0$; representative $x=(1,0,1),\ y=0$, (18b)

Orbit C_s, $-\infty<s<\infty,\ s\neq 0$: $y=sx,\ x^2=0$; representative $x=(1,0,1),\ y=(s,0,s)$, (18c)

Orbit D_k, $0<k<\infty$: $x^2=k^2,\ y=-x$; representative $x=(k,0,0),\ y=(-k,0,0)$, (18d)

Orbit E_k, $0<k<\infty$: $x^2=-k^2,\ y=-x$; representative $x=(0,0,k),\ y=(0,0,-k)$. (18e)

Thus S is the union of the single orbit A, the single orbit B, the orbits C_s for all s, D_k for all k, and E_k for all k. The most general solution to Eqs. (9) or (17), and thus the most general GL(3,R)-invariant PB among q's and p's, arises by starting with any one of the representative solutions listed, applying any element of G to it, and then reading off the values of $\alpha,\beta,\ldots$. One could partially unify the description of the orbits A, B, C_s by defining an orbit F_ϕ for each value of an angle ϕ in the range $0\leq\phi<\pi$:

$$F_\phi:\ x\cos\phi+y\sin\phi=0,\quad x^2=y^2=0. \tag{19}$$

One obtains A when $\phi=0$; C_s for $s<0$ when $0<\phi<\pi/2$; B when $\phi=\pi/2$; and C_s for $s>0$ when $\pi/2<\phi<\pi$. However, we shall leave the classification in the form (18).

Analyzing the orbits a little further, one finds that the subgroup of G leaving the representative of A invariant (stability group) consists of elements corresponding to the SL(2,R) transformations

$$\pm\begin{pmatrix}1+u & u\\ -u & 1-u\end{pmatrix},\quad -\infty<u<\infty. \tag{20}$$

This means that A is the union of two disjoint pieces, each of which is connected and of dimension 2; and the transformation T takes one from one connected portion to the other. This structure is also geometrically evident, as A corresponds to x vanishing and y varying over the positive or negative light cone. Orbit B is similar in structure to orbit A, the roles of x and y being interchanged. In particular, the representative of B has the same stability group as that of A. Next, for each fixed s, the orbit C_s is again like A or B, the union of two disjoint pieces, each connected and of dimension 2. Its representative element has the same stability group as do those of A and B. The two pieces of C_s may be taken to correspond to x varying over the positive and negative light cones, respectively, with y always standing in a fixed relation to x. The set of all orbits C_s makes up a generic family which must be pictured as the union of four disjoint pieces, each connected and of dimension 3. These four pieces arise as x may lie on the positive or negative light cone and, independently, s may be positive or negative. The representative of the orbit D_k has a stability group whose elements correspond to the SL(2,R) transformations

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},\quad 0\leq\theta<2\pi. \tag{21}$$

Thus again each D_k is the union of two disjoint pieces connected by T, each piece of dimension 2. The two pieces correspond to x varying over the positive and negative timelike hyperboloids of "mass" k, respectively, while y is always the negative of x. The set of orbits D_k for all k constitutes a generic family made up of two disjoint pieces of dimension 3 each, corresponding to x lying within the positive or negative light cone (and $y=-x$). Finally, each orbit E_k consists of one connected two-dimensional piece, as x now varies over the spacelike hyperboloid of mass k, and $y=-x$. Its representative element has a stability group consisting of T, of $\mathrm{SO}(2,1)_+$ elements corresponding to the $\mathrm{SL}(2,R)$ transformations

$$\pm\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}, \quad a>0 \tag{22}$$

and products of the two. The set of all E_k is a generic, connected, three-dimensional family, with x varying outside the light cones and $y=-x$.

We conclude this section by noting that the three-dimensional nature of physical space played no role in the analysis. Thus what we have achieved is a complete classification of $\mathrm{GL}(n,R)$-invariant PB's among n q's and n p's.

III. CANONICAL CONTENT OF GL(3,R)-INVARIANT BRACKETS

Any two definitions of PB's corresponding to two points on the same G orbit in S will be both nonsingular or both singular to the same extent, i.e., they will possess the same numbers of independent neutral elements and residual canonical pairs. This is because one can pass from one PB to the other by an invertible transformation defined on the q's and p's directly. (*A priori* one cannot claim that PB's corresponding to points on different orbits will also share these properties but it happens in fact that they do.) It is therefore sufficient to examine the canonical content of the PB corresponding to each of the representative elements on orbits listed in Eq. (18).

In what follows, a caret on a vector denotes a unit vector, and the letters n, n' will denote neutral quantities.

We begin with the representative of orbit A, with parameter values $(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2})$. This leads to the $\mathrm{GL}(3,R)$-invariant PB's

$$\begin{aligned}&\{q_j,q_k\}=\{p_j,p_k\}=0\,,\\ &\{q_j,p_k\}=\tfrac{1}{2}(q_j+p_j)(q_k+p_k)\end{aligned} \tag{23}$$

or in terms of the combinations $q\pm p$,

$$\begin{aligned}&\{q_j+p_j,q_k+p_k\}=\{q_j-p_j,q_k-p_k\}=0\,,\\ &\{q_j-p_j,q_k+p_k\}=(q_j+p_j)(q_k+p_k)\,.\end{aligned} \tag{24}$$

The direction of $\vec{q}+\vec{p}$, and the components of $\vec{q}-\vec{p}$ perpendicular to $\vec{q}+\vec{p}$, are four independent quantities that cannot be altered by any canonical transformation generated with the help of the above PB's. We have in this case then four independent neutral quantities:

$$A:\ (\vec{q}+\vec{p})/|\vec{q}+\vec{p}|\,,\ (\vec{q}-\vec{p})\times(\vec{q}+\vec{p})/|\vec{q}+\vec{p}| \quad \text{neutral}. \tag{25}$$

There cannot be any more, so one canonical pair of variables, Q and P say, must lie hidden within the above PB structure. With a little algebra one finds the representation

$$A:\ \vec{q}+\vec{p}=\hat{n}/Q\,,\ \vec{q}-\vec{p}=\hat{n}P+\underline{\vec{n}}'\,,\ \ \hat{n}\cdot\underline{\vec{n}}'=0\,, \tag{26}$$

where $\hat{n}$ is a neutral unit vector and $\underline{\vec{n}}'$ another neutral vector orthogonal to $\hat{n}$. If one computes PB's among q_j and p_j regarding them as functions of Q, P for which one has, of course, $\{Q,P\}=1$, one reproduces Eqs. (23).

In exactly analogous fashion, one can find representations for $\vec{q}$ and $\vec{p}$ in each of the remaining four cases listed in Eq. (18). For the representative of orbit B, the parameter values are $(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0,\ -\frac{1}{2},\ \frac{1}{2})$ and the PB's are

$$\{q_j,q_k\}=\{p_j,p_k\}=-\{q_j,p_k\}=\tfrac{1}{2}(q_jp_k-q_kp_j)\,. \tag{27}$$

The corresponding neutral elements are

$$B:\ \vec{q}+\vec{p},\ \vec{q}\times\vec{p}/|\vec{q}\times\vec{p}| \quad \text{neutral} \tag{28}$$

and we have the representation

$$B:\ \vec{q}+\vec{p}=\underline{\vec{n}},\ \vec{q}=\underline{\vec{n}}Q+\hat{n}'e^{-P/2},\ \ \underline{\vec{n}}\cdot\hat{n}'=0\,. \tag{29}$$

The representative point in C_s has parameter values

$$\left(\frac{1}{2},\ \frac{1}{2},\ \frac{s}{2},\ \frac{s}{2},\ \frac{s-1}{2},\ \frac{s+1}{2}\right)$$

and PB's

$$\begin{aligned}&\{q_j,q_k\}=\{p_j,p_k\}=\tfrac{1}{2}(q_jp_k-q_kp_j)\,,\\ &\{q_j,p_k\}=\frac{s}{2}(q_j+p_j)(q_k+p_k)\\ &\qquad\qquad -\tfrac{1}{2}(q_jp_k-q_kp_j)\,.\end{aligned} \tag{30}$$

This leads, after some algebra, to the neutral combinations

$$\begin{aligned}C_s:\ &(\vec{q}+\vec{p})/|\vec{q}+\vec{p}|\,,\ \vec{q}\times\vec{p}/|\vec{q}\times\vec{p}|\,,\\ &|\vec{q}+\vec{p}|^{1-s}|\vec{q}\times\vec{p}|^{s} \quad \text{neutral}\end{aligned} \tag{31}$$

and the representation

$$C_s:\ \vec{q}+\vec{p}=\hat{n}/Q\,,\ \vec{q}=\hat{n}s\frac{P}{2}+\underline{\vec{n}}'Q^{1/s},\ \ \hat{n}\cdot\underline{\vec{n}}'=0\,. \tag{32}$$

The representative of D_k, with parameters $(k/2,\ k/2,\ -k/2,\ -k/2,\ 0,\ 0)$, requires a slightly different treatment. The PB's

$$\begin{aligned}&\{q_j,q_k\}=\{p_j,p_k\}=\frac{k}{2}(q_jp_k-q_kp_j)\,,\\ &\{q_j,p_k\}=-\frac{k}{2}(q_jq_k+p_jp_k)\end{aligned} \tag{33}$$

acquire a more transparent form in terms of the complex vector $\vec{a}=\vec{q}+i\vec{p}$:

$$\{a_j,a_k\}=\{a_j^*,a_k^*\}=0,\ \ \{a_j,a_k^*\}=ika_ja_k^*. \tag{34}$$

We now see that the "direction" of $\vec{\underline{a}}$ is neutral:

$$D_k:\ \vec{\underline{a}}/|\vec{\underline{a}}|\ \ \text{neutral}. \tag{35}$$

There are, in fact, four independent real neutrals here. If for the moment we denote the unit vector (with complex components) (35) by $\vec{\underline{c}}$, it is clear that in its place we could consider any fixed complex multiple of it as constituting the neutral elements for the PB's (33). Thus we could replace $\vec{\underline{c}}$ by $\vec{\underline{c}}'=z\vec{\underline{c}}$ with z complex and nonzero. We can now use the phase freedom to make the real and imaginary parts of $\vec{\underline{c}}'$ mutually perpendicular real vectors, and then the magnitude freedom to make the real part of $\vec{\underline{c}}'$ a unit vector. In this way one arrives at the representation

$$D_k:\ \vec{q}+i\vec{p}=(\hat{n}+i\underline{\vec{n}}')e^{Q-ikP/2},\ \ \hat{n}\cdot\underline{\vec{n}}'=0. \tag{36}$$

Finally, the representative of E_k has parameter values (0, 0, 0, 0, $-k$, 0) leading to the PB's

$$\{q_j,q_k\}=\{p_j,p_k\}=0,\ \ \{q_j,p_k\}=-kq_jp_k. \tag{37}$$

The neutral elements are recognized immediately as

$$E_k:\ \vec{q}/|\vec{q}|,\ \vec{p}/|\vec{p}|\ \ \text{neutral} \tag{38}$$

so we get the representation

$$E_k:\ \vec{q}=\hat{n}e^{Q},\ \vec{p}=\hat{n}'e^{-P/k}. \tag{39}$$

It is interesting to note that while the neutral quantities are formed differently from $\vec{q}$ and $\vec{p}$ in each case, there are geometrical similarities as we go from case to case. In all the cases A, B, C_s, and D_k, the neutral quantities constitute two mutually orthogonal vectors, one of them of unit length. In E_k the situation loosens a bit and we have two unit vectors with arbitrary orientations. One must also note the ways in which the continuous parameters s and k enter the constructions in Eqs. (32), (36), and (39); as seen in the last section, these parameters distinguish distinct orbits within generic families.

We now come to the following question: For each choice of GL(3,R)-invariant PB's among q's and p's, which subgroup of GL(3,R) is realized via canonical transformations appropriate to that PB? We shall immediately restrict ourselves to the identity component of GL(3,R). Those infinitesimal elements in this component, which act as inner automorphisms in any given case, will appear as infinitesimal canonical transformations with well-defined functions (of Q and P) as generators, and by a process of integration one builds up that subgroup of GL(3,R), each of whose elements is realized as a finite canonical transformation. One realizes at once that this subgroup of GL(3,R) is constant over each G orbit in S, since the actions of the two groups on $\vec{q}$, $\vec{p}$ commute. We see this in detail as follows. Let us write ξ^a for the six variables $\vec{q}$, $\vec{p}$, collectively; z for a general point in S, g for an element in G, gz for the action of g on z via Eqs. (11) and (13), M for an element in GL(3,R), and $M\xi$ for the action of M on $\vec{q}$, $\vec{p}$ via Eq. (4). The PB (5) can be denoted by

$$\{\xi^a,\xi^b\}=\eta^{ab}(\xi;z), \tag{40}$$

the right-hand sides being quadratic in ξ. GL(3,R) invariance means

$$\eta^{ab}(M\xi;z)=M^a_cM^b_d\eta^{cd}(\xi;z). \tag{41}$$

The action of G on the set of allowed PB's is expressed by

$$\eta^{ab}(\xi;gz)=g^a_cg^b_d\eta^{cd}(g^{-1}\xi;z). \tag{42}$$

[In this condensed notation it is implicit that the 3×3 matrices M_{jk} of Eq. (4) and the 2×2 matrices g of Eq. (10) have been enlarged in appropriate fashion to 6×6 matrices.] Take now a specific PB, i.e., specific z in S, and an infinitesimal element of GL(3,R),

$$\xi^a\to\xi^a+\delta\xi^a,\ \ \delta\xi^a=\epsilon\mathfrak{M}^a_b\xi^b,\ \ |\epsilon|\ll1, \tag{43}$$

which is canonically generated from the generator $\phi(\xi)$ through the PB,

$$\delta\xi^a=\epsilon\eta^{ab}(\xi,z)\frac{\partial\phi(\xi)}{\partial\xi^b}. \tag{44}$$

Since GL(3,R) and G have commuting actions, we have

$$\mathfrak{M}^a_b\xi^b=g^a_b\mathfrak{M}^b_c\,g^{-1}{}^c_d\,\xi^d=g^a_b\mathfrak{M}^b_c(g^{-1}\xi)^c. \tag{45}$$

Combining this with Eqs. (42) and (44) we produce for $\delta\xi^a$ the expression

$$\begin{aligned}\delta\xi^a&=\epsilon g^a_b\eta^{bc}(\bar{\xi};z)\frac{\partial\phi(\bar{\xi})}{\partial\bar{\xi}^c}\bigg|_{\bar{\xi}=g^{-1}\xi}\\&=\epsilon g^a_b\eta^{bc}(g^{-1}\xi;z)g^d_c\frac{\partial}{\partial\xi^d}\phi(g^{-1}\xi)\\&=\epsilon\eta^{ab}(\xi;gz)\frac{\partial}{\partial\xi^b}\phi(g^{-1}\xi)\end{aligned} \tag{46}$$

showing that this element of GL(3,R) is canonically implementable for all choices of PB belonging to the orbit of z in S. This proves the assertion.

Before proceeding to the actual determination of the canonically implementable subgroup in each orbit, it is useful to summarize our expectations and describe in general terms the pattern of results we shall find. The action (4) of the nine-parameter group GL(3,R) on $\vec{q}$ and $\vec{p}$ is such that no function of $\vec{q}$ and $\vec{p}$ is invariant under all of GL(3,R). On the other hand, each PB under study has four independent neutral elements. The canonically implementable subgroup H in any given case must then consist of all those GL(3,R) elements that preserve the neutral elements of that case, i.e., all those elements that survive when four independent conditions are imposed. This suggests that H will be a five-parameter subgroup, and that its action on $\vec{q}$ and $\vec{p}$ translates into constancy of the neutral elements and changes in Q and P alone. We can be sure that these changes indeed represent *canonical* transformations on Q and P; this is a consequence of the original GL(3,R) invariance of the PB. One can now ask whether the action of the five-parameter group H on the two-dimensional Q-P plane is faithful. It turns out not to be so. Now in any nonfaithful realization of a group the trivially realized elements always form a normal subgroup, and it is the factor group that is faithfully realized. In the present situation it happens that H has a three-parameter normal subgroup H_N whose elements do not alter Q and P at all, while only the elements of the two-parameter factor group H/H_N act effectively (and canonically) on Q and P. As a matter of fact we find that while the details vary from case to case, H always appears as a semidirect product of a normal three-parameter subgroup H_N and another two-parameter subgroup K; in fact each element of H appears uniquely as the product of an element of H_N and an element of K.

To exhibit the above structures we use the representations of $\vec{q}$ and $\vec{p}$ in terms of neutral elements and the active canonical pair Q, P. For the representative of orbit A, when Eq. (26) obtains, let us choose without loss of generality the configuration $\hat{n}=(0,0,1)$ and $\vec{n}'=(0,n',0)$. The subgroup H consists of those GL(3,R) transformations M which, acting on

$$\vec{q}=\tfrac{1}{2}(0,n',P+1/Q),\quad \vec{p}=\tfrac{1}{2}(0,-n',-P+1/Q) \tag{47}$$

reduce to changes in Q and P alone. One easily finds H to consist of the elements

$$\begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & 1 & 0 \\ M_{31} & M_{32} & M_{33} \end{bmatrix},\quad M_{11}>0,\quad M_{33}>0 \tag{48}$$

with the effect on Q and P being the canonical transformation

$$Q'=Q/M_{33},\quad P'=M_{33}P+M_{32}n'\,. \tag{49}$$

Equation (49) does not involve M_{11}, M_{21}, and M_{31} at all. Indeed we find the matrix (48) can be written as the product

$$\begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & 1 & 0 \\ M_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & M_{32} & M_{33} \end{bmatrix}. \tag{50}$$

And one can verify: The first factor belongs to a subgroup H_N of H which is normal; the second factor belongs to a subgroup K of H; and, H is the semidirect product of H_N and K.

For the representative of orbit B, choose $\vec{\underline{n}}=(0,0,n)$ and $\hat{n}'=(0,1,0)$ in Eq. (29). Then H consists of matrices

$$\begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{32} & 1 \end{bmatrix},\quad M_{11}>0,\quad M_{22}>0 \tag{51}$$

with the following action on Q and P:

$$Q'=Q+M_{32}\frac{e^{-P/2}}{n}\,,\quad P'=P-2\ln M_{22}\,. \tag{52}$$

Expressing the matrix (51) as the product

$$\begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & 1 & 0 \\ M_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & M_{32} & 1 \end{bmatrix} \tag{53}$$

exhibits H_N and K for this case.

For the representative of orbit C_s, choose $\hat{n}=(0,0,1)$, $\vec{\underline{n}}'=(0,n',0)$ in Eq. (32). We then find elements of H are

$$\begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{32} & M_{22}^{-s} \end{bmatrix},\quad M_{11}>0,\ M_{22}>0\,, \tag{54}$$

the action on Q and P is

$$Q'=QM_{22}^{s},\quad P'=PM_{22}^{-s}+\frac{2n'}{s}M_{32}Q^{1/s}\,, \tag{55}$$

and H_N and K are exhibited by expressing the matrix (54) as the product

$$\begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & 1 & 0 \\ M_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & M_{32} & M_{22}^{-s} \end{bmatrix}. \tag{56}$$

For the representative of orbit D_k, again choose $\hat{n}=(0,0,1)$, $\vec{\underline{n}}'=(0,n',0)$ in Eq. (36). Then elements of H are

$$\begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & -n'^2M_{32} \\ M_{31} & M_{32} & M_{22} \end{bmatrix},\quad M_{11}>0 \tag{57}$$

action on Q and P is determined by

$$e^{Q'-ikP'/2}=(M_{22}+in'M_{32})e^{Q-ikP/2}, \tag{58}$$

and H_N and K arise by expressing (57) as the product

$$\begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & 1 & 0 \\ M_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & M_{22} & -n'^2 M_{32} \\ 0 & M_{32} & M_{22} \end{pmatrix}. \tag{59}$$

Lastly, we come to the representative of E_k. The choice $\hat{n}=(0,0,1)$, $\hat{n}'=(0,\cos\theta,\sin\theta)$ in Eq. (39) leads to H consisting of

$$\begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & (M_{22}-M_{33})\tan\theta & M_{33} \end{pmatrix}, \tag{60}$$

$$M_{11}>0,\ M_{22}>0,\ M_{33}>0;$$

to this action on Q and P,

$$Q'=Q+\ln M_{33},\quad P'=P-\frac{1}{k}\ln M_{22} \tag{61}$$

and to H_N and K determined by the decomposition

$$\begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & 1 & 0 \\ M_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & (M_{22}-M_{33})\tan\theta & M_{33} \end{pmatrix} \tag{62}$$

of the matrix (60).

For explicitness and clarity, simple choices of the neutral vectors were made in each case, and H, H_N, and K then determined. It is, in principle, possible to leave the neutral elements in a general configuration, then H, H_N, K become explicitly dependent on $\vec{q}$ and $\vec{p}$ (through the neutral elements) but remain conjugate to the subgroups we have found. In this form these subgroups will display invariance along G orbits in S as demanded by Eq. (46).

IV. STABILITY UNDER EQUATIONS OF MOTION: FREE PARTICLE

To examine this question we must first make a finer classification of the set S of all GL(3, R)-invariant PB's than was given in Sec. II. That classification was based on a rather "large" group G, some of whose elements mixed $\vec{q}$ and $\vec{p}$; this enabled us to display the relatively "small" number of distinct G orbits in S, and choose representative points from each, in a compact way. When dealing with the specific equations of motion (1), (2) for a free particle or oscillator, $\vec{q}$ and $\vec{p}$ are, respectively, identified with position and momentum: preservation of this identification suggests that we do not consider any longer the entire group G but only that subgroup G_0 that does not mix $\vec{q}$ and $\vec{p}$. (We emphasize that use of G is perfectly justified for classifying elements of S and for examining the canonical structure as in the previous section.) G_0 then consists of reciprocal scale changes in $\vec{q}$ and $\vec{p}$,

$$\vec{q}\to\vec{q}e^{\theta/2},\quad \vec{p}\to\vec{p}e^{-\theta/2} \tag{63}$$

and the physical time-reversal operation T, Eq. (12). According to Eq. (15), the transformation (63) induces on x (and similarly on y) the special Lorentz transformation

$$\begin{aligned} &x_0\to x_0\cosh\theta+x_1\sinh\theta,\\ &x_1\to x_1\cosh\theta+x_0\sinh\theta,\quad x_2\to x_2, \end{aligned} \tag{64}$$

while T changes the signs of x_0 and x_1, leaving x_2 unaffected [cf. Eq. (16)]. Each of the G orbits A, B, C_s, D_k, E_k in S splits now into many distinct orbits with respect to G_0, and a representative element can be chosen from each of these finer orbits.

We list now one representative element from each G_0 orbit contained in each G orbit in turn, giving both the vectors x, y, and the values of $\alpha, \beta, \ldots$.

From A:

$$\begin{aligned} &x=0,\quad y=(1,1,0);\quad (0,0,0,1,0,0),\\ &x=0,\quad y=(1,-1,0);\quad (0,0,1,0,0,0),\\ &x=0,\quad y=(|y_2|,0,y_2),\quad y_2\neq 0,\quad (0,0,\tfrac12|y_2|,\tfrac12|y_2|,\tfrac12 y_2,\tfrac12 y_2). \end{aligned} \tag{65}$$

From B:

$$\begin{aligned} &x=(1,1,0),\quad y=0;\quad (1,0,0,0,0,0),\\ &x=(1,-1,0),\quad y=0;\quad (0,1,0,0,0,0),\\ &x=(|x_2|,0,x_2),\quad y=0,\quad x_2\neq 0;\quad (\tfrac12|x_2|,\tfrac12|x_2|,0,0,-\tfrac12 x_2,\tfrac12 x_2). \end{aligned} \tag{66}$$

From C_s:

$$x=(1,1,0),\quad y=sx;\quad (1,0,0,s,0,0)\,,$$
$$x=(1,-1,0),\quad y=sx;\quad (0,1,s,0,0,0)\,,\tag{67}$$
$$x=(|x_2|,0,x_2),\quad y=sx\,,\quad x_2\neq 0;\quad \left(\tfrac{1}{2}|x_2|,\tfrac{1}{2}|x_2|,\frac{s}{2}|x_2|,\frac{s}{2}|x_2|,\frac{s-1}{2}x_2,\frac{s+1}{2}x_2\right).$$

From D_k:

$$x=((k^2+x_2{}^2)^{1/2},0,x_2),\quad y=-x,\quad -\infty<x_2<\infty\,;$$
$$(\tfrac{1}{2}(k^2+x_2{}^2)^{1/2},\tfrac{1}{2}(k^2+x_2{}^2)^{1/2},-\tfrac{1}{2}(k^2+x_2{}^2)^{1/2},-\tfrac{1}{2}(k^2+x_2{}^2)^{1/2},-x_2,0)\,.\tag{68}$$

From E_k:

$$x=(0,(k^2-x_2{}^2)^{1/2},x_2)\,,\quad y=-x,\quad x_2{}^2<k^2\,;$$
$$(\tfrac{1}{2}(k^2-x_2{}^2)^{1/2},-\tfrac{1}{2}(k^2-x_2{}^2)^{1/2},\tfrac{1}{2}(k^2-x_2{}^2)^{1/2},-\tfrac{1}{2}(k^2-x_2{}^2)^{1/2},-x_2,0)\,,$$
$$x=(1,1,k)\,,\quad y=-x\,;\quad (1,0,0,-1,-k,0)\,,$$
$$x=(1,-1,k)\,,\quad y=-x\,;\quad (0,1,-1,0,-k,0)\,,$$
$$x=(0,0,k)\,,\quad y=-x\,;\quad (0,0,0,0,-k,0)\,,$$
$$x=(1,1,-k)\,,\quad y=-x\,;\quad (1,0,0,-1,k,0)\,,\tag{69}$$
$$x=(1,-1,-k)\,,\quad y=-x;\quad (0,1,-1,0,k,0)\,,$$
$$x=(0,0,-k)\,,\quad y=-x\,;\quad (0,0,0,0,k,0)\,,$$
$$x=((x_2{}^2-k^2)^{1/2},0,x_2),\quad y=-x\,,\quad x_2{}^2>k^2\,;$$
$$(\tfrac{1}{2}(x_2{}^2-k^2)^{1/2},\tfrac{1}{2}(x_2{}^2-k^2)^{1/2},-\tfrac{1}{2}(x_2{}^2-k^2)^{1/2},-\tfrac{1}{2}(x_2{}^2-k^2)^{1/2},-x_2,0)\,.$$

One sees that the breakup of S into G_0 orbits is quite complex. Some of these orbits, the first two entries in Eq. (65) and the first two in Eq. (66), stand isolated. The third entries in Eqs. (65) and (66), the first two entries in (67), and entries two to seven in (69), belong to generic one-parameter families; the rest belong to generic two-parameter families. GL(3,R)-invariant PB's among functions of $\vec{q}$ and $\vec{p}$, corresponding to points on distinct G_0 orbits in S, must be thought of as being physically different.

The question of stability can now be posed: Starting with the free-particle equations of motion (1) and an allowed PB structure (5), if we take the time derivatives of the brackets and insert the equations of motion, will the equalities hold? We can get general conditions on $\alpha,\beta,\ldots$ to guarantee stability and then go down the list of G_0-orbit representatives to pick out the cases that survive. [The action of G_0 in effect changes only the sign and scale of m in the equations of motion (1). Thus the results below cover the entire G_0 orbit if m is regarded as variable (and $\neq 0$).] The stability of the PB between two q's requires that $\{q_j,p_k\}$ be symmetric in j and k,

$$\frac{d}{dt}[\{q_j,q_k\}-\alpha(q_jp_k-q_kp_j)]=0\Rightarrow\rho=\nu\,.\tag{70}$$

Note, in particular, that α is unconstrained. The PB $\{p_j,p_k\}$ is automatically stable. Stability of $\{q_j,p_k\}$ entails

$$\beta(q_jp_k-q_kp_j)=\lambda(q_jp_k+q_kp_j)+(\nu+\rho)p_jp_k\,.\tag{71}$$

From Eqs. (70) and (71) we immediately see that the necessary and sufficient conditions for stability for the free-particle case are

$$\beta=\lambda=\nu=\rho=0\,.\tag{72}$$

Going down the list of representatives of G_0 orbits in Eqs. (65)–(69) we see that the first of set (65), of set (66), and of set (67) are the only survivors. We can now exhibit quite easily the general GL(3,R)-invariant PB's on these orbits by applying a general element of G_0 to the appropriate representative elements:

$$\{q_j,q_k\}=0\,,\quad \{p_j,p_k\}=0\,,\quad \{q_j,p_k\}=\mu p_jp_k,\quad \mu\neq 0\tag{73a}$$
$$\{q_j,q_k\}=\alpha(q_jp_k-q_kp_j)\,,\quad \{p_j,p_k\}=0\,,\quad \{q_j,p_k\}=0\,,\quad \alpha\neq 0\tag{73b}$$
$$\{q_j,q_k\}=\alpha(q_jp_k-q_kp_j)\,,\quad \{p_j,p_k\}=0\,,\quad \{q_j,p_k\}=\mu p_jp_k\,,\quad \alpha\neq 0\,,\quad \mu\neq 0\,.\tag{73c}$$

To repeat, these are the only possible GL(3, R)-invariant PB's that are stable under the free-particle equations of motion (for any value of the mass m). Because these three possibilities arose from distinct G orbits and make up three distinct G_0 orbits we keep them separate rather than gather them all in one form, (73c), by permitting α or μ to vanish.

We study the three surviving stable cases briefly. In each of them a Hamiltonian can be exhibited:

$$H=\frac{1}{m\mu}\ln|\vec{p}| \quad \text{for cases (73a) and (73c)}$$
$$=-\frac{1}{m\alpha}\ln|\vec{q}\times\vec{p}| \quad \text{for case (73b)}. \tag{74}$$

Stability of the PB's already implies that the time derivative of a neutral quantity, computed with direct use of the equations of motion, is also neutral. The further information that a Hamiltonian exists, to generate the equations of motion via the PB, then means every neutral element is, in fact, a constant of motion. Thus in the cases under discussion, each of the neutrals can be expressed entirely in terms of the known constants of motion, the linear momentum $\vec{p}$, and the angular momentum $\vec{J}=\vec{q}\times\vec{p}$. Now, for the PB's (73a) for instance, one can set up a definite linear transformation of the form (10) which carries the previously chosen representative (18a) of orbit A into the element of A present in Eq. (73a). Use of this transformation in conjunction with Eqs. (25) and (26) will give us the neutral elements and a representation of $\vec{q}$ and $\vec{p}$ appropriate to the PB's (73a), and similarly in the other two cases. [This explains why the neutral elements listed below in Eq. (75) are different functions of $\vec{q}$ and $\vec{p}$ than were given in Eqs. (25) and (26).] Proceeding in this way, or working directly with Eq. (73), one finds the neutral elements

$$\vec{p}/|\vec{p}|,\ \vec{J}/|\vec{p}| \quad \text{for case (73a)},$$
$$\vec{p},\ \vec{J}/|\vec{J}| \quad \text{for case (73b)}, \tag{75}$$
$$\vec{p}/|\vec{p}|,\ \vec{J}/|\vec{J}|,\ |\vec{p}|^{\alpha-\mu}|\vec{J}|^{\mu} \quad \text{for case (73c)}.$$

Suitable representations for $\vec{q},\vec{p}$ in the three cases are

$$\vec{q}=\hat{n}\mu QP^2+\vec{\underline{n}}',\quad \vec{p}=\hat{n}P,\quad \hat{n}\cdot\vec{\underline{n}}'=0, \tag{76a}$$
$$\vec{q}=\vec{\underline{n}}\alpha Q+\hat{n}'e^{-P},\quad \vec{p}=\vec{\underline{n}},\quad \vec{\underline{n}}\cdot\hat{n}'=0, \tag{76b}$$
$$\vec{q}=\hat{n}\mu QP^2+\vec{\underline{n}}'P^{-\alpha/\mu},\quad \vec{p}=\hat{n}P,\quad \hat{n}\cdot\vec{\underline{n}}'=0. \tag{76c}$$

[These differ somewhat from the results of directly applying suitable linear transformations to Eqs. (26), (29), and (32), respectively.] The Hamiltonians (74) reduce to $(\ln P)/m\mu$, $P/m\alpha$, and $(\ln P)/m\mu$ in the three cases. In the conventional treatment the linear and angular momenta serve as generators for a canonical realization of the Euclidean group. They now act quite differently. They reduce to

$$\vec{p}=\hat{n}P,\quad \vec{J}=\vec{\underline{n}}'\times\hat{n}P \quad \text{in case (73a)},$$
$$\vec{p}=\vec{\underline{n}},\quad \vec{J}=\hat{n}'\times\vec{\underline{n}}e^{-P} \quad \text{in case (73b)}, \tag{77}$$
$$\vec{p}=\hat{n}P,\quad \vec{J}=\vec{\underline{n}}'\times\hat{n}P^{(\mu-\alpha)/\mu} \quad \text{in case (73c)}.$$

So, in the first stable bracket just one component each of $\vec{p}$ and $\vec{J}$ survives, they are equal, and so generate a one-parameter group of canonical transformations: translations in Q. Through Eq. (76a) one can see the effect of these transformations on $\vec{q}$. In the second stable case, all that survives is one component of $\vec{J}$, and this generates a one-parameter group of transformations affecting only $\vec{q}$. And in the last stable case, one component each of $\vec{p}$ and $\vec{J}$ survives (unless $\mu=\alpha$ when, by Eq. (75), $\vec{J}$ becomes neutral), and together they generate a two-parameter Abelian group of canonical transformations.

V. STABILITY: OSCILLATOR CASE

For the isotropic harmonic oscillator (2) the analysis can be carried out in an analogous way and we confine it to the stability of the structure and the search for a Hamiltonian. Stability gives the following relations among the parameters:

$$\alpha,\beta,\lambda \ \text{arbitrary},$$
$$\mu=\lambda/m^2\omega^2, \tag{78}$$
$$\nu=\rho=0,$$

and going through the list of representatives of G_0 orbits we see that the surviving structures are

$$\{q_j,q_k\}=\alpha(q_jp_k-p_jq_k),\quad \{p_j,p_k\}=\{p_j,q_k\}=0,\quad \alpha\neq 0 \tag{79}$$

from the first of (66),

$$\{q_j,q_k\}=\{q_j,p_k\}=0,\quad \{p_j,p_k\}=\beta(q_jp_k-q_kp_j),\quad \beta\neq 0 \tag{80}$$

from the second of (66),

$$\{q_j,q_k\}=\frac{k}{2m\omega}(q_jp_k-q_kp_j),$$
$$\{p_j,p_k\}=\frac{km\omega}{2}(q_jp_k-p_jq_k), \tag{81}$$
$$\{q_j,p_k\}=-\frac{k}{2}\left(m\omega q_jq_k+\frac{1}{m\omega}p_jp_k\right)$$

from (68).

The existence of a Hamiltonian for cases (79) and (80) is soon excluded. In fact

$$\dot{\vec{p}}=\sum_j\left(\frac{\partial H}{\partial q_j}\{\vec{p},q_j\}+\frac{\partial H}{\partial p_j}\{\vec{p},p_j\}\right),$$
$$\dot{\vec{q}}=\sum_j\left(\frac{\partial H}{\partial q_j}\{\vec{q},q_j\}+\frac{\partial H}{\partial p_j}\{\vec{q},p_j\}\right),\tag{82}$$

and $\dot{\vec{p}}$ vanishes for case (79), while $\dot{\vec{q}}$ does for case (80). For case (81), from (82) the following equations have to be solved:

$$\sum_j\left(\frac{\partial H}{\partial q_j}q_j+\frac{\partial H}{\partial p_j}p_j\right)=-\frac{2\omega}{k},$$
$$\sum_j\left(\frac{\partial H}{\partial p_j}\frac{p_j}{m\omega}+\frac{\partial H}{\partial p_j}(-m\omega q_j)\right)=0.\tag{83}$$

A Hamiltonian is then

$$H=-\frac{\omega}{k}\ln(|\vec{p}|^2+m^2\omega^2|\vec{q}|^2).\tag{84}$$

VI. CONCLUDING REMARKS

We have given in this paper a fairly complete analysis of the simplest and yet nontrivial dynamical systems, leading to new perspectives on questions of symmetry, symmetry breakdown, and quantization. It is appropriate to discuss briefly the relation of this work to other approaches.

Briefly stated, we have studied the consequences of forcing a system with a certain superficial number of degrees of freedom to admit a canonical structure which will possess a given group $\mathcal{G}$, acting in a specified way, as a group of automorphisms. We find that, for suitably large $\mathcal{G}$ in relation to the number of variables, the response of the canonical structure can be that it sacrifices some canonical degrees of freedom in order to accommodate itself to the imposed formal invariance under $\mathcal{G}$. Thus the achieving of this invariance can entail the emergence of neutral dynamical variables and superselection.

Let us contrast this for a moment with the converse situation. If one has a finite-dimensional phase space with a nondegenerate PB structure (i.e., no neutral elements other than constants), then every (infinitesimal) transformation preserving the PB structure is necessarily a (infinitesimal) canonical transformation and definitely possesses a (infinitesimal) generator: nondegeneracy of the PB ensures that every automorphism is an inner one. Now one can ask, "What are the (semisimple) Lie groups that can possibly be faithfully realized via canonical transformations on this phase space?" Here the dimension of the phase space places a definite upper limit on the ranks of the groups that can be canonically realized; and moreover, one may not have much freedom in the choice of group action.[5,6] In the present work, the roles of the group and of the PB structure have been precisely reversed: the former and its action are specified, and the latter must then adjust itself accordingly.

In previous work on this subject it was shown that it is possible to break a subset of the classical symmetries (i.e., symmetries of the classical equations of motion) at the quantum level by choosing a PB structure relative to which these symmetires are not canonical, so after quantization they are not unitarily represented.[2] However, in these discussions it was implicitly assumed that the PB's were nondegenerate, and so every broken symmetry had the feature that it was not an automorphism of the canonical structure. This paper exhibits a new and novel way in which symmetry breakdown can occur. There are many possible canonical structures for each of which GL$(3,R)$ acts as a group of automorphisms, but in any given case it is far from true that all of GL$(3,R)$ is represented by canonical tranformations. There is in each case a subgroup H of GL$(3,R)$ which alone leaves the neutral elements fixed, a prerequisite to be a group of inner automorphisms. And, furthermore, it is not even H but only the factor H/H_N that is faithfully realized by canonical transformations on the system. Thus one has gone all the way from the nine-parameter automorphism group GL$(3,R)$ to a few possible two-parameter groups H/H_N realized faithfully by canonical transformations. But this is not all. Stability under the equations of motion makes a further selection out of the possible groups H/H_N, and so finally only these survive in the quantum theory. All these various stages are conceivable only because the PB has become degenerate.

In passing we mention that another possibility exists for systems with an infinite number of degrees of freedom.[7] Here it may happen that even with a nondegenerate PB on the system phase space, not every automorphism is an inner one. This contrasts with the situation for finite numbers of degrees of freedom. Following the pattern of calculation in the latter case, one may *formally* construct a generator for an infinitesimal canonical automorphism in the infinite-dimensional case (and nondegeneracy implies one always can), but questions of convergence connected solely with the infinity of independent variables can arise. There can be some states in which a certain generator exists and makes sense, and others in which it does not; thus the states of the system may break up into sectors only in some of which a given automorphism is inner. But this breakup is not based on the assignment of different numerical values to nontrivial neutral quantities: there need be none of these. In this paper we have considered only systems with finitely many degrees of freedom.

Taking the free-particle case as an example it is interesting to see how the stable GL(3, R)-invariant PB's organize the collection of all possible states of motion of the system. At the level of the equations of motion (1) one views the $\vec{q}$ and $\vec{p}$ as coordinates in a certain six-dimensional space. The solutions of the equations of motion are trajectories in this space. If one imposes the usual Euclidean-invariant canonical structure (3) on the space, one knows that, as a matter of fact, the entire Galilei group acts canonically and irreducibly on the system; in fact, one can map any state of motion onto any other with a suitable Galilei transformation. In contrast we see that a stable GL(3, R)-invariant canonical structure breaks up the six-dimensional space into two-dimensional "sheets", there being a four-parameter family of these sheets; each of the three structures in Eq. (73) does this in its own way, of course. Stability guarantees that each sheet contains entire trajectories, i.e., solutions of the equations of motion. But each sheet, specified by the values of the neutral elements on it, is superselected and cannot be connected to any other sheet; only states within a sheet are canonically transformable into one another. Thus there is nothing that can alter the values of the neutral elements, except our doing it by hand. Each choice of stable brackets and then each sheet yield a different physical system—the states of the one particle have been broken into many, though they all share the same equations of motion. Additional problems are posed by the harmonic-oscillator analysis. Here out of the three classes of invariant Poisson structures exhibiting stability only one could admit a compatible Hamiltonian function.

This discussion shows the possibility of spontaneous breakdown of symmetry already at the classical level, not in terms of the symmetry being broken in certain classical "ground-state" configurations, but rather in the canonical structure. Previous work and the present one suggest that the choice of canonical framework for classical theory (which is then quantized by Dirac's prescription) can have profound consequences for the quantum theory. This way of breaking classical symmetry at the quantum level strikes us as among the more interesting of such consequences in view of its possible implications for solid-state physics and particle theory. The origin of spontaneous symmetry breakdown in these domains (cf. the Goldstone-Higgs mechanism) thus need not necessarily lie in the specific quantum aspects of the problem.

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