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Model equations from a chaotic time series

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Abstract. We present a method for obtaining a set of dynamical equations for a system that exhibits a chaotic time series. The time series data is first embedded in an appropriate phase space by using the improved time delay technique of Broomhead and King (1986). Next, assuming that the flow in this space is governed by a set of coupled first order nonlinear ordinary differential equations, a least squares fitting method is employed to derive values for the various unknown coefficients. The ability of the resulting model equations to reproduce global properties like the geometry of the attractor and Lyapunov exponents is demonstrated by treating the numerical solution of a single variable of the Lorenz and Rossler systems in the chaotic regime as the test time series. The equations are found to provide good short term prediction (a few cycle times) but display large errors over large prediction times. The source of this shortcoming and some possible improvements are discussed.

Keywords. Time series; chaos; model equations; strange attractor.

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1. Introduction

In recent years, it has become clear that many chaotic time series observed experimentally in nature, owe their stochasticity to the intrinsic nonlinear dynamics of a system evolving on a low dimensional strange attractor in phase space. Methods of phase space reconstruction (Broomhead and King 1986; Packard *et al* 1980; Takens 1981) allow one to study the geometrical structure of the strange attractor and to determine such important global parameters as the fractal dimension, Lyapunov exponents etc. For many such systems it becomes meaningful to write down model nonlinear equations which may reproduce the observed chaotic behaviour. There is a great deal of interest in this direction in recent years (Farmer and Sidorowich 1987; Casdagli 1989; Crutchfield and McNamara 1987; Abarbanel *et al* 1989). Obtaining model equations (preferably a set of simple differential equations) can serve as an important first step in gaining a better understanding of the underlying physical processes responsible for the observed chaos. They can also form good mathematical simulators and be usefully employed for prediction and control of the chaotic system.

In this paper, we present a method for obtaining a set of differential equations from an experimentally measured chaotic time series of a single physical variable. These equations are found to reproduce reasonably well the geometrical and dynamical features of the attractor reconstructed from the measured time series. The theorems due to Takens (1981) and Whitney (1936) form the theoretical basis for the phase space reconstruction and the determination of embedding dimension respectively.

The improved embedding technique of Broomhead and King (1986) which is based on singular value analysis provides a means to determine the topological dimension of the underlying dynamical system and to construct the attractor in an orthogonal phase space. The evaluation of the dimension yields an estimate of the minimum number of dynamical variables which are required to describe the dynamical system. In this analysis we assume that the dynamics on the reconstructed attractor in orthogonal phase space can be described by a set of coupled first order nonlinear ordinary differential equations and choose a simple third order polynomial in dynamical variables as the basis functions to fit the derivatives of the dynamical variables. We also assume that the divergence of the flow in the reconstructed phase space is a constant quantity. This condition is valid for any dynamical system with constant dissipation. To obtain a set of model equations, the unknown coefficients are determined by a linear least square fitting procedure.

We apply the above technique to two test time series obtained by solving the Lorenz (1963) and Rossler (1976) equations in chaotic regimes to see if the method can be used for obtaining dynamical equations for an experimental time series. For this purpose a single variable of the dynamical system (e.g. z coordinate of Lorenz) is treated as experimental data and dynamical equations (a set of coupled nonlinear ordinary differential equations) are obtained. In each test case, the resulting dynamical equations are solved numerically to compare the geometrical and dynamical features of the attractor reconstructed from the input time series.

In §2, we describe the method for obtaining dynamical equations from the observed time series of a single physical variable. The method is applied to two test cases viz z time series of Lorenz and x time series of Rossler systems, in §3. The results and some possible improvements are discussed in §4.

2. Method

Let $x(t_i)$, $(i = 1, 2, 3, \ldots)$ denote a time series of some physical quantity measured at the discrete time interval Δt. From this time series we construct a $N \times n$ trajectory matrix by using the time delay method

$$X = N^{-1/2} \begin{pmatrix} x(t_1) & x(t_1+\tau) & \cdots & x(t_1+(n-1)\tau) \\ x(t_2) & x(t_2+\tau) & \cdots & x(t_2+(n-1)\tau) \\ \vdots & \vdots & \vdots & \vdots \\ x(t_N) & x(t_N+\tau) & \cdots & x(t_N+(n-1)\tau) \end{pmatrix}. \tag{1}$$

Here $\tau = m\Delta t$, is the delay time where m is an integer; $t_i = i\Delta t$, $i = 1, 2, \ldots, N$; Δt is the sampling time; n is the embedding dimension and $N + m(n-1)$ is the total number of samples. The trajectory matrix X describes the attractor in phase space. To determine an orthonormal basis for the embedding and the dimensionality of the underlying dynamical system, the matrix X is singular value decomposed (Broomhead and King 1986) as

$$X = S \Sigma C^T \tag{2}$$

where S is an $N \times n$ orthogonal matrix of eigenvectors of XX^T, Σ is an $n \times n$ diagonal matrix, C is an $n \times n$ orthonormal matrix and the superscript T stands for the transpose of the matrix. The columns of matrix C provides a set of n linearly independent vectors $\{c_1, c_2, \ldots, c_n\}$ in the embedding space. The matrices Σ and C in (2) are determined from the following eigenvalue equation

$$ZC = \Sigma^2 C \tag{3}$$

where $Z = X^T X$ is the covariance matrix of rank $n \times n$. The diagonal elements of Σ, called the singular values, are arranged in descending order, i.e. $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ where $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant 0$. Plotting σ_i vs i gives the singular value spectrum which contains the information about the extent to which the trajectory explores the embedding space. One may think of the trajectory described by the matrix X as exploring an n-dimensional ellipsoid. The vectors $\{c_j\}$ then give the direction and the $\{\sigma_j\}$ the lengths of the principal axes of the ellipsoid. For a stochastic system, the singular value spectrum would be almost flat indicating that the number of degrees of freedom is too large. However for a deterministic chaotic system, number of points above noise floor gives the number of degrees of freedom that can be determined. However this does not give an upper limit for the same since as the noise reduces one gets more points above the noise floor and vice versa. Thus this method provides only an approximate value of the dimension. This is similar to choosing only those modes with significant power.

The number of singular values d above the noise floor gives the true embedding dimension of the attractor. The corresponding column vectors $\{c_1, c_2, \ldots, c_d\}$ of C form the basis vectors of an orthogonal space of dimension d. The trajectory matrix X is now projected onto the basis $\{c_j\}$ to give the matrix $\xi = XC$. In the new basis, the column vectors $\{\xi_1, \xi_2, \ldots, \xi_d\}$ of the trajectory matrix ξ are uncorrelated

$$\xi_j(t_i) = \sum_{k=1}^{n} x(t_i + (k-1)\tau) C_{kj};\ i = 1, 2, \ldots, N; j = 1, 2, \ldots, d. \tag{4}$$

By Takens theorem (Takens 1981), a reconstructed attractor in this new phase space should preserve the essential topological features of the original attractor. Accordingly we prescribe the evolution of the system on this attractor by a set of coupled nonlinear ordinary differential equations for $\xi_j(t)$

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = F_j(\xi_1, \xi_2, \ldots, \xi_d) \tag{5}$$

where F_j are nonlinear functions, which can be expanded in terms of appropriate basis functions ϕ_k

$$F_j = \sum_k a_k^{(j)} \phi_k(\xi_1, \xi_2, \ldots, \xi_d) \tag{6}$$

where $a_k^{(j)}$ are the unknown coefficients to be determined. To illustrate our method, in the test examples considered below, we have chosen a simple third order polynomial form in d dynamical variables for F_j. The polynomial for $d = 3$ is

$$F_j = \sum_{l_1, l_2, l_3 = 0}^{3} a_{l_1 l_2 l_3}^{(j)} \xi_1^{l_1} \xi_2^{l_2} \xi_3^{l_3};\ (l_1 + l_2 + l_3) \leqslant 3. \tag{7}$$

In order to reduce the number of unknown coefficients $a^{(j)}_{l_1 l_2 l_3}$ we have imposed a constraint that the divergence of the flow in the reconstructed phase space is a constant quantity, i.e.

$$\nabla \cdot F = \text{constant}. \tag{8}$$

Here by flow we mean processing of some physical quantity continuous in time. With this constraint ξ_j will occur only in linear terms in the dynamical equation for $\mathrm{d}\xi_j/\mathrm{d}t$. Thus the constant in (8) will be the sum of coefficients of ξ_j in $\mathrm{d}\xi_j/\mathrm{d}t$ equation. The dynamical equation for the variables ξ_1 can be written as

$$\frac{\mathrm{d}\xi_1}{\mathrm{d}t} = a_1 + a_2\xi_1 + a_3\xi_2 + a_4\xi_3 + a_5\xi_2^2 + a_6\xi_2\xi_3 + a_7\xi_3^2 + a_8\xi_2^3 + a_9\xi_2^2\xi_3 + a_{10}\xi_2\xi_3^2 + a_{11}\xi_3^3. \tag{9}$$

Similarly the equations for the variables ξ_2 and ξ_3 can be written. The derivatives $\mathrm{d}\xi_j/\mathrm{d}t$ are calculated numerically from $\xi_j(t)$ given by eq. (4) by using cubic spline method (Press *et al* 1987). The unknown coefficients in the three equations are determined separately by the χ^2 minimization method. If we denote the numerically calculated derivatives at time t_i by $\dot{\xi}_j(t_i)$ and the fitted value of the derivative by $F_j(a_1, a_2, \ldots, a_m)$ then the function χ^2 can be written as

$$\chi^2 = \sum_{i=1}^{N} [\dot{\xi}_j(t_i) - F_j(a_1, a_2, \ldots, a_m)]^2 \tag{10}$$

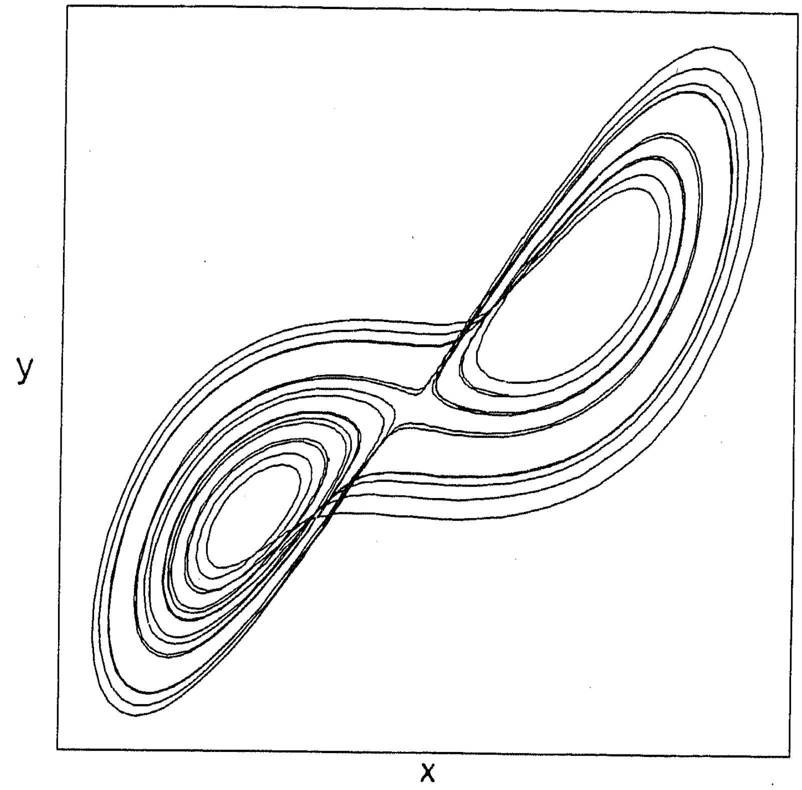
where N is the number of points on the reconstructed attractor. The minimization of χ^2 with respect to the fitting parameters a_k, $k = 1, 2, \ldots, m$ results in m simultaneous linear equations which can be solved to obtain the values of $a_1, a_2, \ldots, a_m$.

3. Test examples

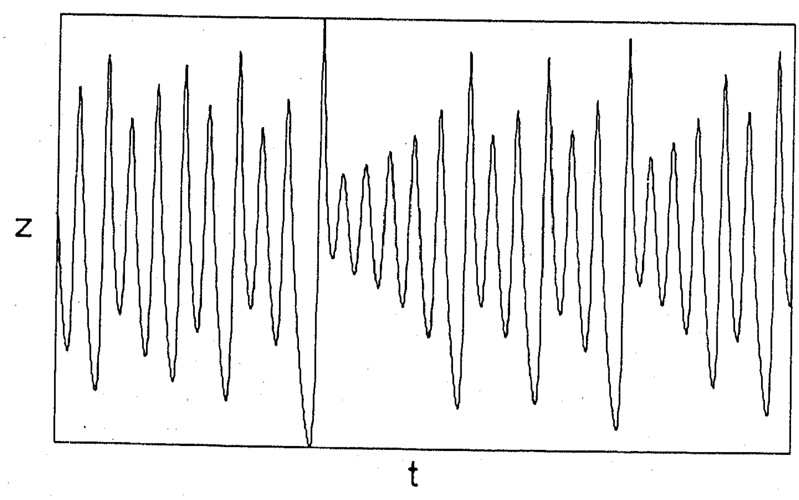
We consider the z time series of Lorenz system as the first test case. The Lorenz equations

$$\begin{aligned} \dot{x} &= -\sigma(x - y) \\ \dot{y} &= -zx + rx - y \\ \dot{z} &= xy - bz \end{aligned} \tag{11}$$

are solved numerically using Bulirsch-Stoer method (Press *et al* 1987) for the parameter values $\sigma = 10$, $r = 38$ and $b = 8/3$ and 6000 solution points are stored with time step $\Delta t = 0{\cdot}009$. The xy projection of the Lorenz attractor is shown in figure 1(a). Using z time series of this solution [figure 1(b)], an embedding is carried out with $\tau = \Delta t$ and $n = 7$. The phase space portraits of the reconstructed attractor in orthogonal phase space are shown in figure 2. The singular value spectrum yields $d = 3$ which means 3 dynamical variables are required to describe the dynamics on the attractor. The coefficients of the three equations $\mathrm{d}\xi_j/\mathrm{d}t$, $j = 1, 2, 3$ as determined by the least square fitting method, are given in table 1. The model equations with the coefficients

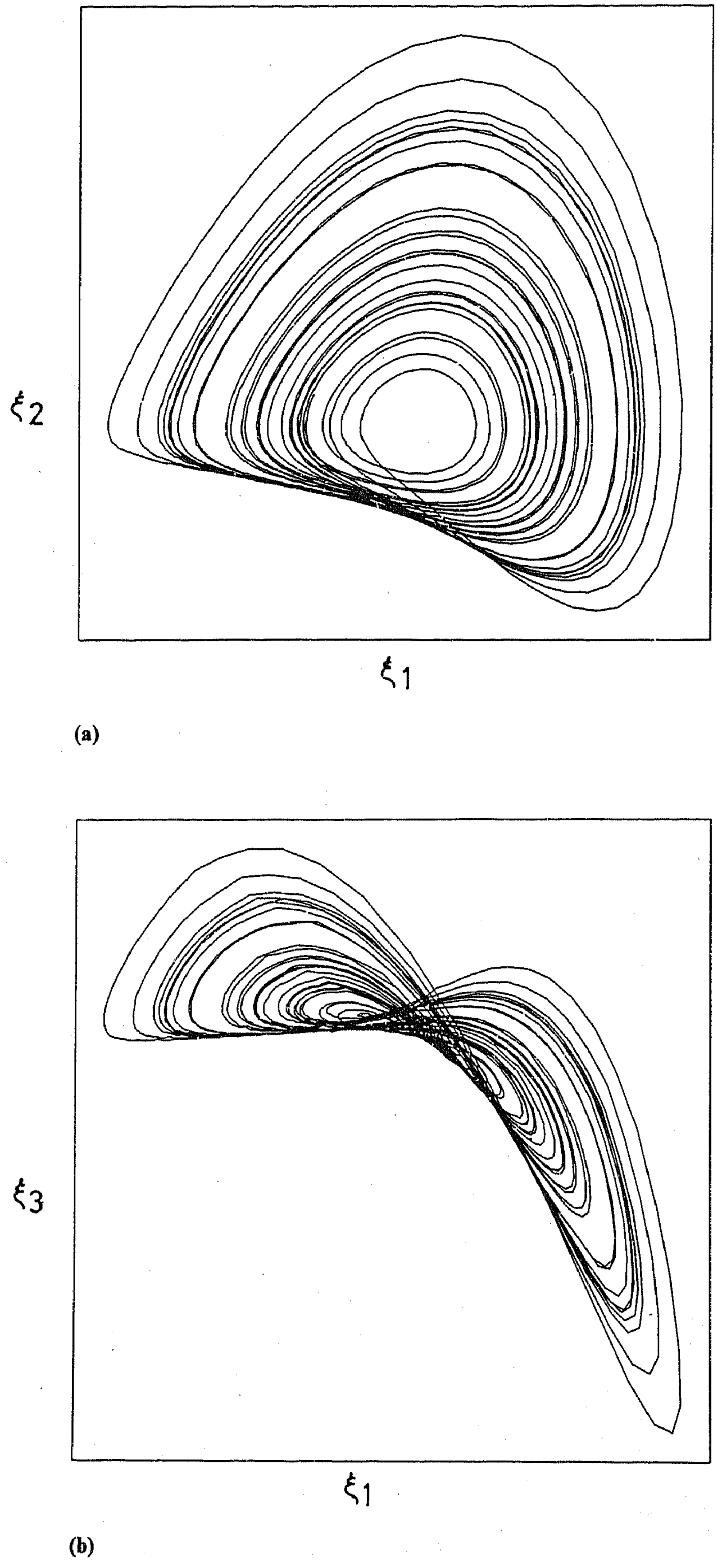


(a)

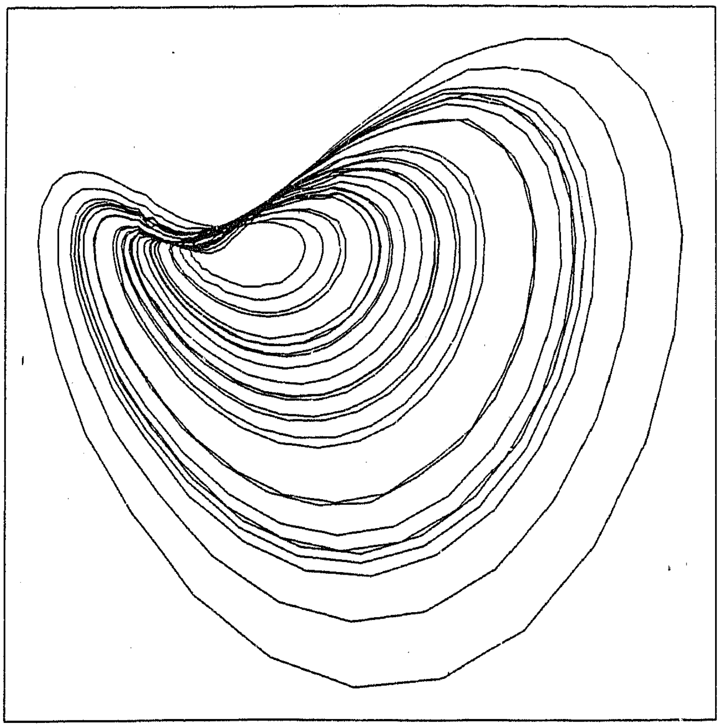


(b)

Figure 1. (**a**) xy projection of the Lorenz attractor for $\sigma = 10$, $b = 8/3$, $r = 38$. (**b**) z time series of the Lorenz system used for time series analysis.



ξ_2
ξ_1
(a)
ξ_3
ξ_1
(b)



(c)

Figure 2. Phase space portraits of the attractor in orthogonal space reconstructed from z time series of the Lorenz system. Figures (a), (b) and (c) show $\xi_1\xi_2$, $\xi_1\xi_3$ and $\xi_2\xi_3$ projections of the attractor respectively.

Table 1. Model equations of the attractor reconstructed from z time series of the Lorenz system.

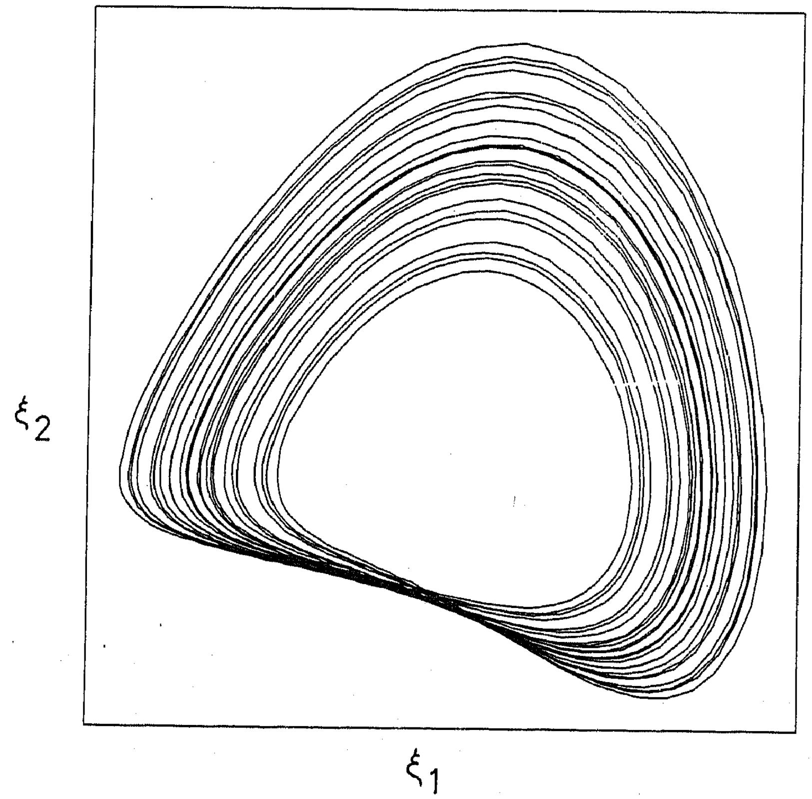
	Term	$d\xi_1/dt$	$d\xi_2/dt$	$d\xi_3/dt$
Constant	1	0·000424	0·020437	−5.743306
Linear	ξ_1	−0·000570	−1·760502	75·773665
	ξ_2	6·784514	−0·029699	83·772072
	ξ_3	−0·005261	15·902427	−24·176651
Quadratic	ξ_1^2	0·0	0·999903	−127·550237
	ξ_2^2	0·001693	0·0	−9·846200
	ξ_3^2	0·024002	−0·225875	0·0
	$\xi_1\xi_2$	0·0	0·0	−105·399142
	$\xi_2\xi_3$	0·072998	0·0	0·0
	$\xi_3\xi_1$	0·0	−1·422202	0·0
Cubic	ξ_1^3	0·0	−0·744035	50·242957
	ξ_2^3	−0·027008	0·0	−0·237277
	ξ_3^3	0·005080	0·027953	0·0
	$\xi_1^2\xi_2$	0·0	0·0	−30·777431
	$\xi_1^2\xi_3$	0·0	0·323328	0·0
	$\xi_2^2\xi_3$	−0·049679	0·0	0·0
	$\xi_2^2\xi_1$	0·0	0·0	0·409609
	$\xi_3^2\xi_1$	0·0	0·798516	0·0
	$\xi_3^2\xi_2$	−0·010660	0·0	0·0
	$\xi_1\xi_2\xi_3$	0·0	0·0	0·0

given in table 1 are solved numerically with the initial conditions as one of the point on the reconstructed attractor and the solution $\xi_1(t), \xi_2(t), \xi_3(t)$ is plotted (figure 3). Trajectories in figure 2 and figure 3 compare well. In other words the model equations are able to reproduce well the geometrical shape of the reconstructed attractor. We also find that the Lyapunov exponents (Wolf *et al* 1985) of the reconstructed equations compare well with those of the original Lorenz equations (table 2). Identical results are obtained when x or y time series of the Lorenz system are used as the input time series.

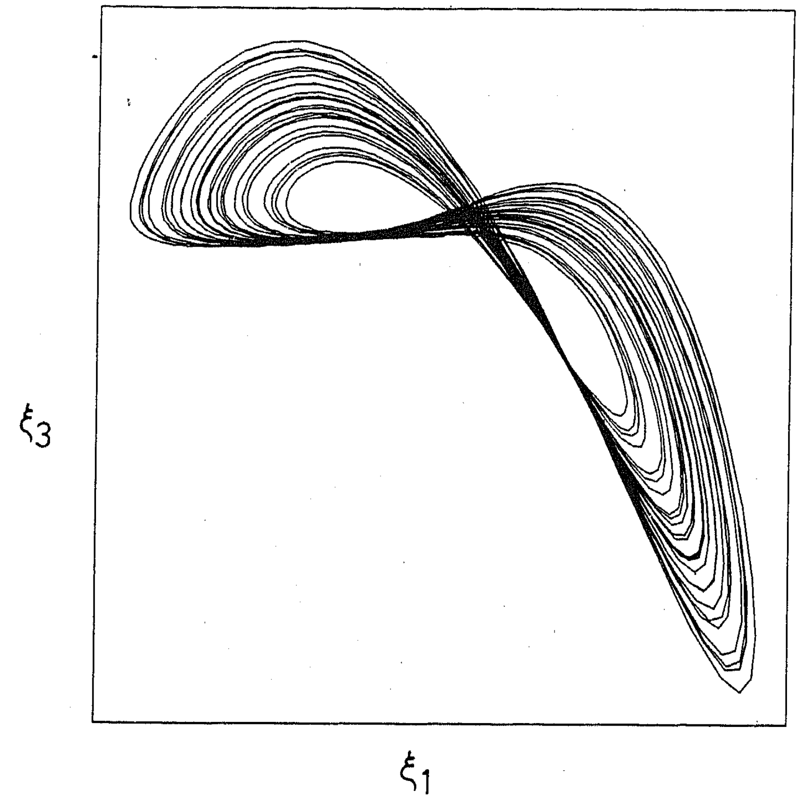
As another test example, the same analysis is carried out using the x time series of the Rossler system (Rossler 1976). The Rossler equations

$$\begin{aligned} \dot{x} &= -(y+z) \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x-c) \end{aligned} \tag{12}$$

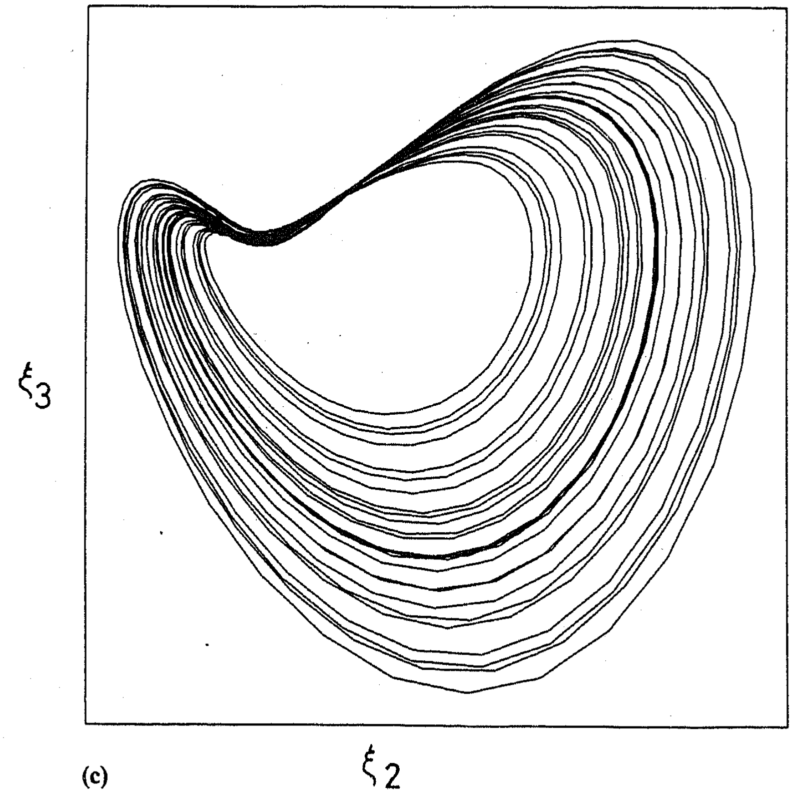
are solved for the parameter values $a = 0{\cdot}2$, $b = 0{\cdot}2$, $c = 5{\cdot}7$ and time step $\Delta t = 0{\cdot}05$. The xy projection of the Rossler attractor is shown in figure 4(a). 6000 points of the x solution [figure 4(b)] are used to construct trajectory matrix X with the delay time $\tau = \Delta t$ and $n = 3$. The phase space portraits of the reconstructed attractor in the orthogonal phase space are shown in figure 5. Table 3 contains the coefficients of the model equations as determined from the χ^2 minimization method. In this case however the $\nabla \cdot F =$ constant constraint does not work and we use an unconstrained fit. Figure 6 shows the phase space portraits obtained by solving the model equations given by



(a)



(b)



(c)

Figure 3a–c. Corresponding portraits obtained by solving the model equations given by table 1. These figures are to be compared with figure 2.

Table 2. Lyapunov exponents.

Lyap. exp. no.	Lorenz system	Model equations
1	1·58	1·3
2	0·0	0·0
3	−21·35	−34·24

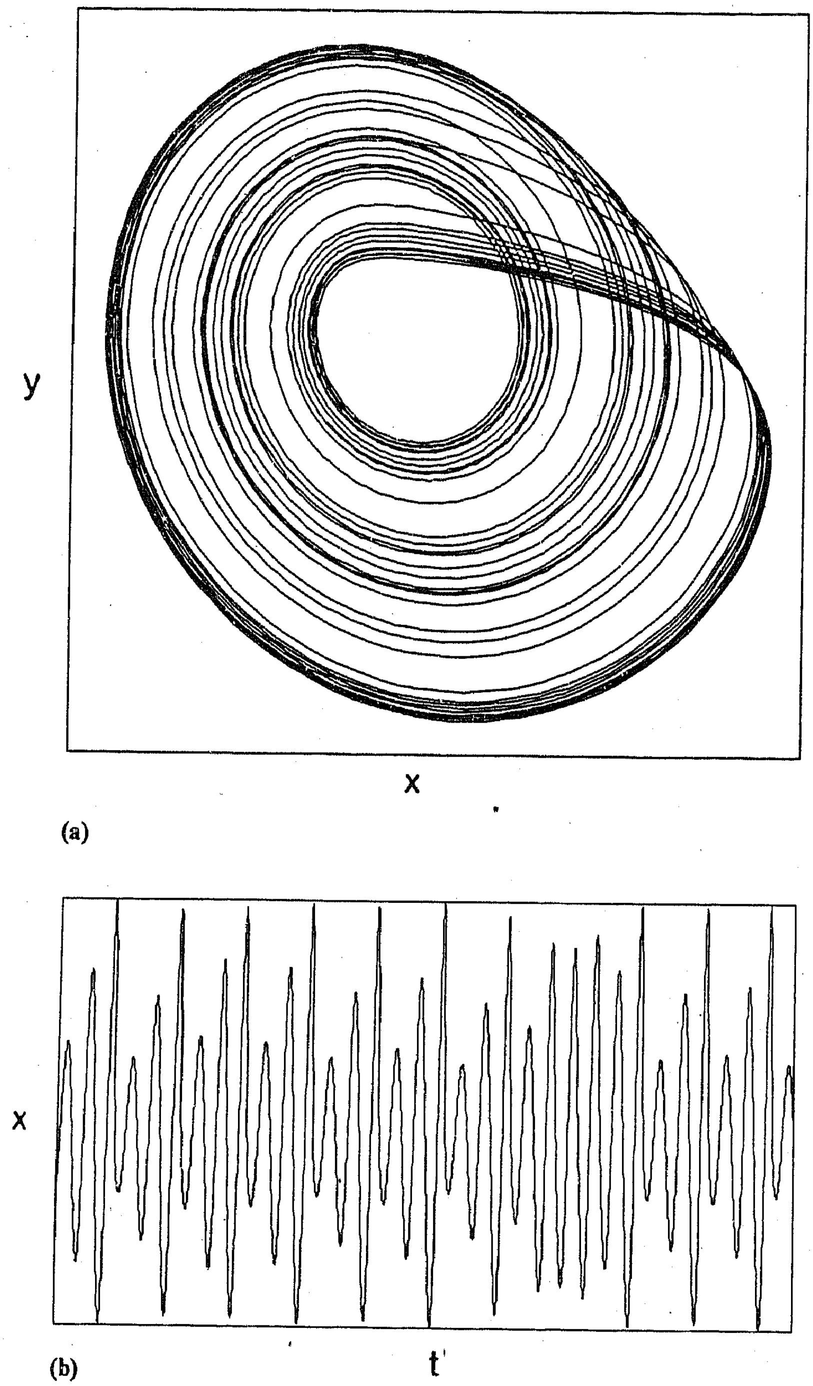
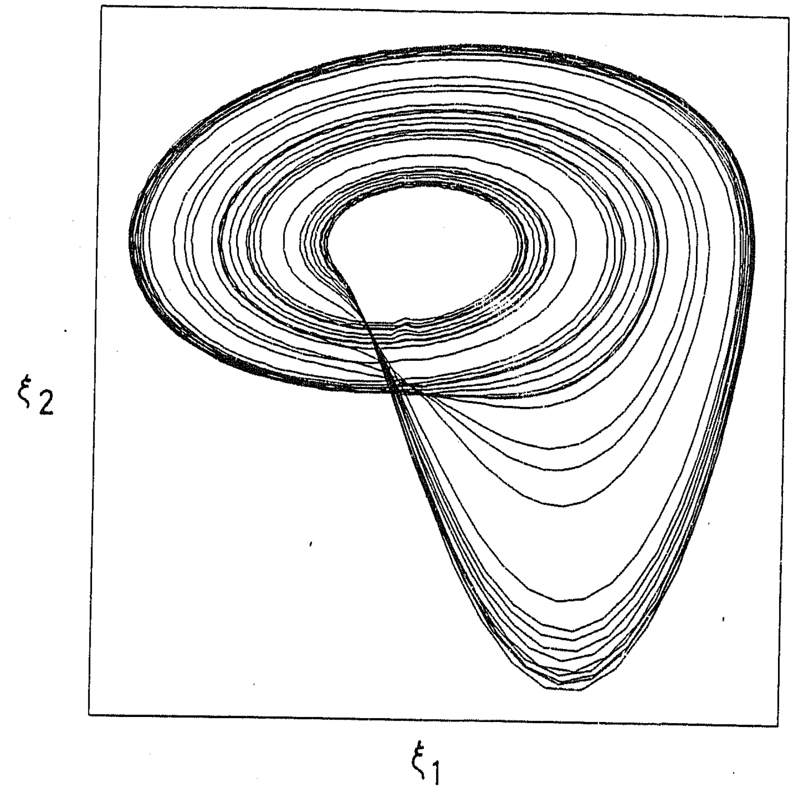
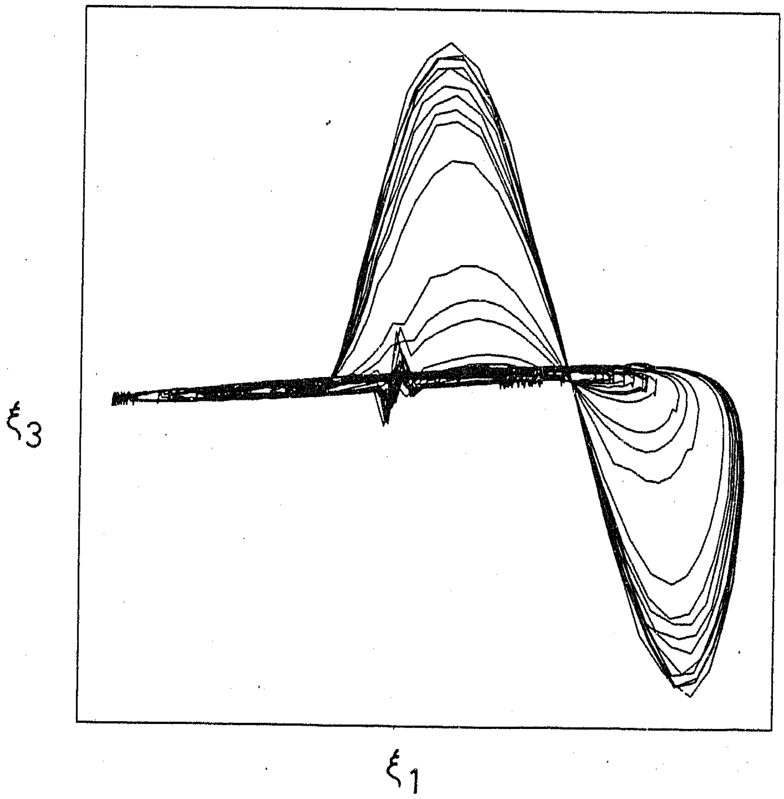


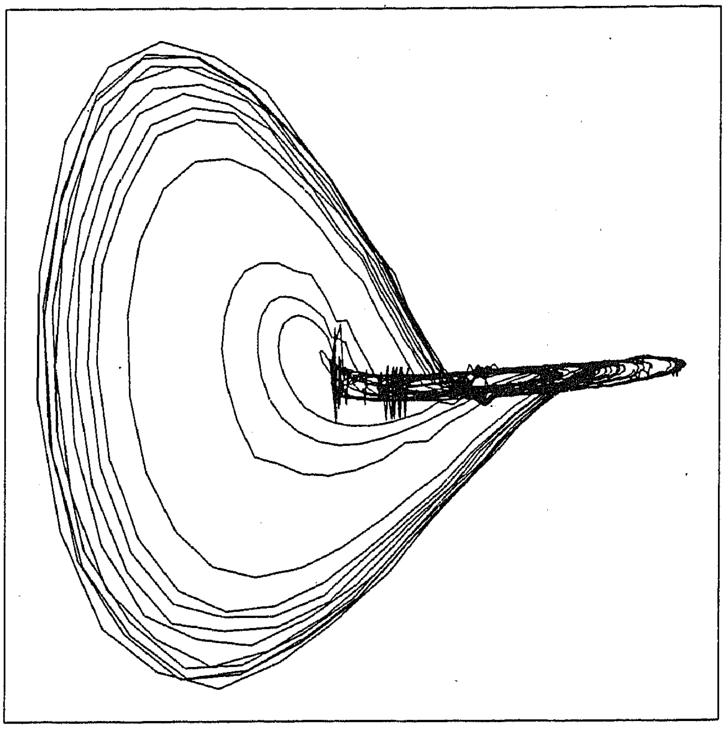
Figure 4. (**a**) xy projection of the Rossler attractor for $a = 0{\cdot}2$, $b = 0{\cdot}2$, and $c = 5{\cdot}7$. (**b**) x time series of the Rossler system used for the analysis.



5(a)



5(b)

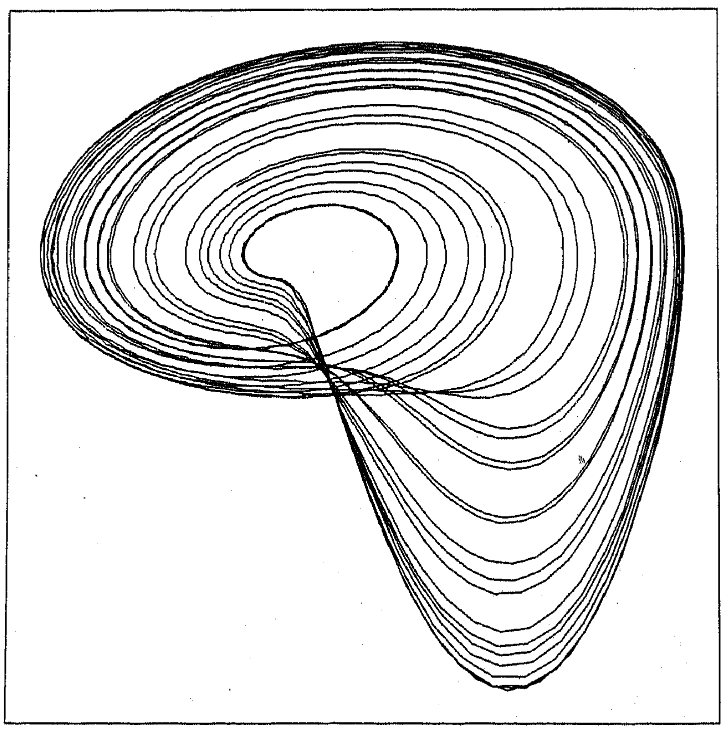


ξ_3

ξ_2

5(c)

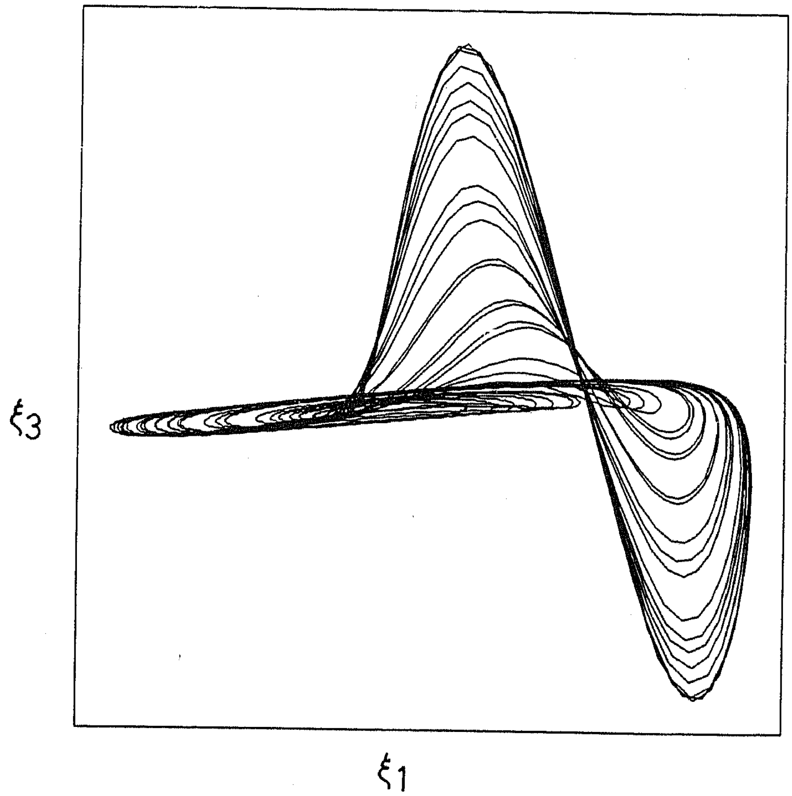
Figure 5. Phase space portraits of the attractor reconstructed from x time series of the Rossler system. Figures (**a**), (**b**) and (**c**) show $\xi_1\xi_2$, $\xi_1\xi_3$ and $\xi_2\xi_3$ projections of the attractor respectively.



ξ_2

ξ_1

(**a**)



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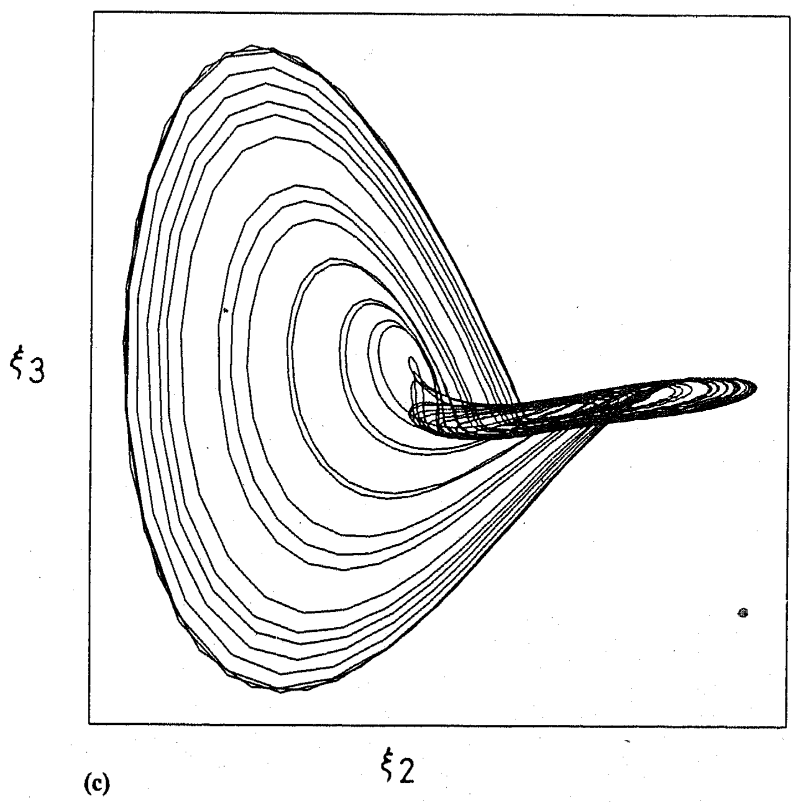

(c)

Figure 6a–c. Corresponding portraits obtained by solving the model equations given by table 3. These figures are to be compared with figure 5.

eqs (5) and (7) with the coefficients given in table 3. These figures compare well with figure 5. Lyapunov exponents of the Rossler equations and the model equations are given in table 4 which also compare well.

4. Discussion

We have developed a method to obtain model dynamical equations from single time series data that exhibit chaotic behaviour. The procedure we have adopted incorporates the improved time delay embedding technique of Broomhead and King to determine the dimensionality of the attractor and to obtain a set of linearly independent vectors along the direction of the principal axes in the embedded space. The evolution of the dynamics on this attractor is then expressed in terms of these new basis vectors. A simple global description is sought by prescribing the evolution to obey a set of coupled first order nonlinear ordinary differential equations in the new variables. The nonlinear flow function is expanded in terms of appropriate basis functions and the

Table 3. Model equations of the attractor reconstructed from x time series of the Rossler system.

	Term	$d\xi_1/dt$	$d\xi_2/dt$	$d\xi_3/dt$
Constant	1	−0·000030	0·000404	−0.012725
Linear	ξ_1	0·002504	−0·520284	0·245215
	ξ_2	2·451089	0·002553	1·245353
	ξ_3	−0·014097	2·450000	−6·010692
Quadratic	ξ_1^2	−0·000568	−0·003848	−0·236573
	ξ_2^2	0·025044	−0·172964	10·676128
	ξ_3^2	−0·044666	3·591581	−19·287406
	$\xi_1\xi_2$	0·000910	−0·152087	0·428880
	$\xi_2\xi_3$	−0·118156	2·599341	−50·674237
	$\xi_3\xi_1$	0·027082	−0·736909	11·366149
Cubic	ξ_1^3	−0·019505	−0·453966	−8·346129
	ξ_2^3	−0·317533	0·308504	−134·679460
	ξ_3^3	0·071346	−4·770292	30·145698
	$\xi_1^2\xi_2$	−0·112489	1·717551	−47·886660
	$\xi_1^2\xi_3$	0·366788	4·923404	156·107332
	$\xi_2^2\xi_3$	−0·221880	1·786956	−95·153484
	$\xi_2^2\xi_1$	0·128040	2·295998	54·379256
	$\xi_3^2\xi_1$	0·329147	−11·952868	−139·729950
	$\xi_3^2\xi_2$	−0·010594	2·812778	−6·235948
	$\xi_1\xi_2\xi_3$	0·324331	−25·067014	139·277139

Table 4. Lyapunov exponents.

Lyap. exp. no.	Rossler system	Model equations
1	0·06	0·14
2	0·0	0·0
3	−7·65	−5·80

unknown coefficients are determined by χ^2 minimization. We have tested this procedure on two sample problems *viz* by treating the numerical solution of a single variable of the Lorenz and Rossler systems in the chaotic regime as experimental data. The resulting dynamical equations reproduce the geometrical features of the original Lorenz (Rossler) attractor quite well. This is evidenced in the visual display of phase space portraits as well as the quantitative comparison of Lyapunov exponents. The equations are also found to provide good short term prediction (a few cycle times) but display large errors over large prediction times. A possible source of this shortcoming is the inadequacy of the global description of the attractor and may be improved upon by adopting a local description as one moves to different parts of the embedding space (Farmer and Sidorowich 1987, 1988). However this would considerably increase the computational complexity of the method. A recent work which is closely related to our work is that of Abarbanel *et al* (1989). These authors set up model nonlinear maps which are then used for system identification and prediction for systems exhibiting chaotic time series. Our method for setting up model differential equations is complementary to theirs and might have some advantage in relating to physical models. An important element in their method is the use of invariants of the dynamical systems e.g. Lyapunov exponents as constraints on the choice of mapping parameters. An unconstrained least squares fit gives them poor values of the invariants like the Lyapunov exponents. Such a constraint does not seem to be necessary in our approach. An alternative constraint that we have found useful is $\nabla \cdot F = \text{constant}$. This reduces the number of coefficients to be determined and helps with convergence of the least squares fitting. The major advantage of the constrained method of Abarbanel *et al* (1989) is in improving the predictive power of the maps and we are exploring such a procedure for our method. These techniques are now being tried out on time series obtained from plasma physics experiments, and will be reported shortly.

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