Quasi-parabolic Siegel Formula

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Abstract

The result of Siegel that the Tamagawa number of SL_r over a function field is 1 has an expression purely in terms of vector bundles on a curve, which is known as the Siegel formula. We prove an analogous formula for vector bundles with quasi-parabolic structures. This formula can be used to calculate the betti numbers of the moduli of parabolic vector bundles using the Weil conjucture.

1 Introduction

The Betti numbers of the moduli of stable vector bundles on a complex curve, in all the cases where the rank and degree are coprime, were first determined by Harder and Narasimhan [H-N] as an application of the Weil conjuctures. For this, they made use of the result of Siegel that the Tamagawa number of the special linear group over a function field is 1. In their refinement of the same Betti number calculation in [D-R], Desale and Ramanan expressed the result of Siegel in purely vector bundle terms. This result about the Tamagawa number, called the Siegel formula, was later given a simple proof in the language of vector bundles by Ghione and Letizia [G-L], by introducing a notion of effective divisors of higher rank on a curve, and counting the number of effective divisors which correspond to a given vector bundle. This purpose of this note is to introduce the notion of a quasi-parabolic divisor of higher rank on a curve (Definition 3.1 below), and to prove a quasi-parabolic analogue (Theorem 3.4 below) of the Siegel formula, which is done here by suitable generalizing the method of [G-L]. In a note to follow, this formula is used to calculate the Zeta function and thereby the Betti numbers of the moduli of parabolic bundles in the case 'stable = semistable' (these Betti numbers have already been calculated by a guage theoretic method for genus ≥ 2 in [N] and for genus 0 and 1 by Furuta and Steer in [F-S]).

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2 Divisors supported on $X - S$

Let X be an absolutely irreducible, smooth projective curve over the finite field $k = \mathbf{F}_q$, and let S be any closed subset of X whose points are k-rational. Let K denote the function field of X, and let K_X denote the constant sheaf K on X. Let g denote the genus of X. Let r be a positive integer. Recall that (see [G-L]) a coherent subsheaf $D \subset K_X^r$ of generic rank r is called an r-divisor, and the r-divisor is called effective (or positive) if $\mathcal{O}_X^r \subset D$. The support of the divisor is by definition the support of the quotient $D/\mathcal{O}_X^r$, which is a torsion sheaf. The lenght n of $D/\mathcal{O}_X^r$ is called the degree of the divisor. Note that D is a locally free sheaf of rank r and degree n.

Remark 2.1 Let $Z_X(t)$ be the zeta function of X. Then as S consists of k-rational points, it can be seen that the zeta function Z_{X-S} of $X - S$ is given by the formula

$$Z_{X-S}(t) = (1-t)^s Z_X(t) \tag{1}$$

where s is the cardinality of S.

Note that an effective r-divisor on $X - S$ is the same as an effective r-divisor on X whose support is disjoint from S. The part (1) of the proposition 1 of [G-L] gives the following, with $X - S$ in place of X.

Proposition 2.2 *Let $b_n^{(r)}$ be the number of effective r-divisors of degree n on X whose support is disjoint from S. Let $Z_{X-S}^{(r)}(t) = \sum_{n\geq 0} b_n^{(r)} t^n$. Then we have*

$$Z_{X-S}^{(r)}(t) = \prod_{1\leq j\leq r} Z_{X-S}(q^{j-1}t) \tag{2}$$

In order to have the analogue of the part (2) of the proposition 1 of [G-L], we need the following lemmas.

Lemma 2.3 *Let V be a finite dimensional vector space over $k = \mathbf{F}_q$, and s a positive integer. For any $1 \leq i \leq s$, let $\pi_i : k^s \to k$ be the linear projection. For any surjective linear map $\phi : V \to k^s$, let V_i be the kernel of $\pi_i\phi : V \to k$, which is a hyperplane in V as ϕ is surjective. Let $P = P(V)$, and $P_i = P(V_i)$ denote the corresponding projective spaces. Let $N(\phi)$ denote the number of k-rational points of $P - \cup_{1\leq i\leq s} P_i$. Then for any other surjective $\psi : V \to k^s$, we have $N(\phi) = N(\psi)$. In other words, given s, this number depends only on dim(V).*

Proof Given any two surjective maps $\phi, \psi : V \to k^s$, there exists an $\eta \in GL(V)$ such that $\phi\eta = \psi$. From this, the result follows.

Lemma 2.4 *Let n be a positive integer, such that $n > 2g - 2 + s$ where g is the genus of X and s is the cardinality of s. Let b_n is the total number of effective 1-divisors of degree n supported on $X - S$. Then for any line bundle L on X of* *degree n, the number of effective 1-divisors supported on $X-S$ which define L is $b_n/P_X(1)$, where $P_X(1)$ is the number of isomorphism classes of line bundles of any fixed degree on X.*

(Here, $P_X(t)$ is the polynomial $(1-t)(1-qt)Z_X(t)$.)

Proof Let L be any line bundle on X of degree n, where $n > 2g-2+s$. Then $H^1(X, L(-S)) = 0$, so the natural map $\phi : H^0(X, L) \to H^0(X, L|S)$ is surjective. Let $V = H^0(X, L)$. Then $dim(V) = n+1-g$. Choose a basis for each fiber L_P for $P \in S$. This gives an identification of $H^0(X, L|S)$ with k^s. Now it follows that the number $N(\phi)$ defined in the preceeding lemma depends only on n, and is independent of the choice of L as long as it has degree n. But $N(\phi)$ is precisely the number of effective 1-divisors supported on $X-S$, which define the line bundle L on X.

Using the above lemma, the following proposition follows, by an argument similar to the proof of part (2) of proposition 1 in [G-L]. The proof in [G-L] expresses the number of r-divisors in terms of the number of 1-divisors, and the above lemma tells us the number of 1-divisors with support in $X-S$ corresponding to a given line bundle on X.

Proposition 2.5 *For L a line bundle of degree n, let $b_n^{(r,L)}$ be the number of effective r-divisors on X supported on $X-S$, having determinant isomorphic to L. Then provided that $n > 2g-2+s$, we have*

$$b_n^{(r,L)} = b_n^{(r)}/P_X(1) \tag{3}$$

Proposition 2.6

$$\lim_{n\to\infty} \frac{b_n^{(r)}}{q^{rn}} = P_X(1)\frac{(q-1)^{s-1}}{q^{g-1+s}} Z_{X-S}(q^{-2}) \cdots Z_{X-S}(q^{-r}) \tag{4}$$

Proof The above statement is the analogue of proposition 2 of [G-L], with the following changes. Instead of all r-divisors on X in [G-L], we consider only those which are supported over $X-S$, and instead of $Z_X(t)$, we use $Z_{X-S}(t)$. As $Z_{X-S}(t) = (1-t)^s Z_X(t)$, the property of $Z_X(t)$ that it has a simple pole at $t = q^{-1}$ and is regular at $1/q^j$ for $j \geq 2$ is shared by $Z_{X-S}(t)$. Hence the proof in [G-L] works also in our case, proving the proposition.

Remark 2.7 There is a minor misprint in the equation labeled (1) in [G-L] (page 149); the factor q^{g-1} should be read as q^{1-g}.

Let L be any given line bundle on X. Choose any closed point $P \in X-S$, and let l denote its degree. For any $\mathcal{O}_X$ module E, set $E(m) = E \otimes \mathcal{O}_X(mP)$. If a vector bundle E of rank r degree n has determinant L, then $E(m)$ has determinant $L(rm)$, degree $n + rml$ and Euler characteristic $\chi(m) = n + rml + r(1-g)$.

The equations (3) and (4) above imply the following.

$$\lim_{m\to\infty} \frac{b_{n+rml}^{(r,L(rm))}}{q^{r\chi(m)}} = (q-1)^{s-1} q^{(r^2-1)(g-1)-s} Z_{X-S}(q^{-2}) \cdots Z_{X-S}(q^{-r}) \tag{5}$$

3 Quasi-parabolic divisors

For basic facts about parabolic bundles, see [S] and [M-S]. We now introduce the notion of a quasi-parabolic effective divisor of rank r. Let $S \subset X$ be a finite subset consisting of k-rational points. For each $P_i \in S$, let there be given positive integers p_i and $r_{i,1}, \ldots, r_{i,p_i}$ with $r_{i,1} + \ldots + r_{i,p_i} = r$. This will be called, as usual, the quasi-parabolic data. Recall that a quasi-parabolic structure on a vector bundle E of rank r on X by definition consists of flags $E_{P_i} = F_{i,1} \supset \ldots \supset F_{i,p_i} \supset F_{i,p_i+1} = 0$ of vector subspaces in the fibers over the points of S such that $dim(F_{i,j}/F_{i,j+1}) = r_{i,j}$ for each j from 1 to p_i.

Definition 3.1 Let X, S, and the numerical data $(r_{i,j})$ be as above. A positive quasi-parabolic divisor (F, D) on X consists of (i) a quasi-parabolic structure F on the trivial bundle $\mathcal{O}_X^r$, consisting of flags F_i in k^r at points $P_i \in S$ of the given numerical type $(r_{i,j})$, together with (ii) an effective r-divisor D on X, supported on $X - S$.

Note that if (F, D) is a quasi-parabolic r-divisor, then the rank r vector bundle D has a parabolic structure given by F. We denote by $P_E^{(r)}$ the set of all effective parabolic r-divisors whose associated parabolic bundle is isomorphic to a given parabolic bundle E. For any vector bundle E of rank r, let $Hom_{inj}^S(\mathcal{O}_X^r, E)$ denote the set of all injective sheaf homomorphisms $\mathcal{O}_X^r \to E$ which are injective when restricted to S. For any quasi-parabolic bundle E, the group of all quasi-parabolic automorphisms of E will be denoted by $ParAut(E)$. Then $ParAut(E)$ acts on $Hom_{inj}^S(\mathcal{O}_X^r, E)$ by composition. This action is free, and $P_E^{(r)}$ has a canonical bijection with the quotient set $Hom_{inj}^S(\mathcal{O}_X^r, E)/ParAut(E)$. Hence the cardinality of $P_E^{(r)}$ is given by

$$|P_E^{(r)}| = \frac{|Hom_{inj}^S(\mathcal{O}_X^r, E)|}{|ParAut(E)|} \tag{6}$$

For $1 \leq i \leq s$, let Flag_i be the variety of flags in k^r of the numerical type $(r_{i,1}, \ldots, r_{i,p_i})$. Let $\mathrm{Flag}_S = \prod_{1\leq i\leq s} \mathrm{Flag}_i$. Let $f(q, r_{i,j})$ denote the number of k-rational points of Flag_S. If $a_n^{(r,L)}$ denotes the number of quasi-parabolic divisors of flag data $(r_{i,j})$ with degree n, rank r and determinant L, then we have

$$a_n^{(r,L)} = f(q, r_{i,j}) b_n^{(r,L)} \tag{7}$$

Now let $J(r, L)$ denote the set of all isomorphism classes of quasi-parabolic vector bundles of rank r, degree n, determinant L having the given quasi-parabolic data

$(r_{i,j})$ over S. Hence the equation (6) above implies the following.

$$a_n^{(r,L)} = \sum_{E\in J(r,L)} \frac{|Hom^S_{inj}(\mathcal{O}^r_X, E)|}{|ParAut(E)|} \tag{8}$$

For any integer m, the map from $J(r,L) \to J(r, L(rm)$ which sends E to $E(m) = E \otimes O_X(mP)$ is a bijection which preserves $|ParAut|$. Hence for each m, we have

$$a_{n+rml}^{(r,L(rm))} = \sum_{E\in J(r,L)} \frac{|Hom^S_{inj}(\mathcal{O}^r_X, E(m))|}{|ParAut(E)|} \tag{9}$$

Lemma 3.2 *With the above notations,*

$$\lim_{m\to\infty} \frac{|Hom^S_{inj}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}} = \frac{(q^r-1)^s(q^r-q)^s\cdots(q^r-q^{r-1})^s}{q^{r^2s}} \tag{10}$$

If S is non-empty, the limit is already attained for all large enough m (where 'large enough' depends on E).

Proof If S is empty, the above lemma reduces to lemma 3 in [G-L]. If S is nonempty, then any morphism of locally free sheaves on X which is injective when restricted to S is injective. Let m be large enough, so that $E(m)$ is generated by global sections, $H^1(X, E(m)) = 0$, and $h^0(X, E(m)) = \chi(E(m)) \geq rs$. Then $H^0(X, E(m))$ has a basis consisting of sections σ_{i,P_j}, τ_ℓ for $i = 1, \ldots, r$, $j = 1, \ldots, s$, and $\ell = 1, \ldots, \chi(E(m)) - rs$, such that

(1) the sections τ_ℓ are zero on S,

(2) the sections σ_{i,P_j} are zero at all other points of S except P_j (and hence σ_{i,P_j} restrict at P_j to a basis of the fiber of $E(m)$ at P_j.

Any element of $Hom_{\mathcal{O}_X}(\mathcal{O}^r_X, E(m)) = Hom_{\mathbf{F}_q}({\mathbf{F_q}}^r, H^0(X, E(m)))$ is given in terms of this basis by a $r \times q^{\chi(E(m))}$ matrix A. The condition that this lies in

$$Hom^S_{inj}(\mathcal{O}^r_X, E(m)) \subset Hom(\mathcal{O}^r_X, E(m))$$

is the condition that each of the s disjoint $r \times r$-minors, corresponding to the part $\sigma_{1,P_j}, \ldots, \sigma_{r,P_j}$ of the basis, has nonzero determinant. This contributes the factor

$$\frac{|GL_r(\mathbf{F}_q)|}{|M_r(\mathbf{F}_q)|} = \frac{(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})}{q^{r^2}}$$

for each P_j, which proves the lemma.

Lemma 3.3 *The following sum and limit can be interchanged to give*

$$\sum_{E\in J(r,L)} \lim_{m\to\infty} \frac{|Hom^S_{inj}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}|ParAut(E)|} = \lim_{m\to\infty} \sum_{E\in J(r,L)} \frac{|Hom^S_{inj}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}|ParAut(E)|}$$

This lemma has a proof entirely analogous to the corresponding statement in [G-L], so we omit the details.

By equation (10), the left hand side in the above lemma equals

$$\frac{(q^r-1)^s(q^r-q)^s\cdots(q^r-q^{r-1})^s}{q^{r^2s}}\sum_{E\in J(r,L)}\frac{1}{|ParAut(E)|}$$

On the other hand, by (9), the right hand side is $\lim_{m\to\infty} a_{n+rml}^{(r,L(rm))}/q^{r\chi(m)}$. By equations (5) and (7), this limit has the following value.

$$f(q,r_{i,j})(q-1)^{s-1}q^{(r^2-1)(g-1)-s}Z_{X-S}(q^{-2})\cdots Z_{X-S}(q^{-r})$$

By putting $Z_{X-S}(t)=(1-t)^sZ_X(t)$ in the above, and cancelling common factors from both sides, we get the following.

Theorem 3.4 (Quasi-parabolic Siegel formula)

$$\sum_{E\in J(r,L)}\frac{1}{|ParAut(E)|}=f(q,r_{i,j})\frac{q^{(r^2-1)(g-1)}}{q-1}Z_X(q^{-2})\cdots Z_X(q^{-r})$$

Remark 3.5 If S is empty or more generally if the quasi-parpbolic structure at each point of S is trivial (that is, each flag consists only of the zero subspace and the whole space), then on one hand $ParAut(E)=Aut(E)$, and on the other hand each flag variety is a point, and so $f(q,r_{i,j})=1$. Hence in this situation the above formula reduces to the original Siegel formula

$$\sum_{E\in J(r,L)}\frac{1}{|Aut(E)|}=\frac{q^{(r^2-1)(g-1)}}{q-1}Z_X(q^{-2})\cdots Z_X(q^{-r})$$

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