Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 5, November 2009, pp. 655–667.

Measure free martingales and martingale measures

M G NADKARNI[1] and B RAJEEV[2]

[1]Department of Mathematics, University of Mumbai, Kalina, Mumbai 400 098, India
[2]Indian Statistical Institute, Bangalore Centre, 8th Mile Mysore Road, RVCE Post, Banglore 560 059, India
E-mail: mgnadkarni@gmail.com; brajeev@isibang.ac.in

MS received 21 October 2008; revised 20 March 2009

Abstract. Let $T \subset \mathbb{R}$ be a countable set, *not necessarily discrete*. Let $f_t, t \in T$, be a family of real-valued functions defined on a set Ω. We discuss conditions which imply that there is a probability measure on Ω under which the family $f_t, t \in T$, is a martingale.

Keywords. Martingale; Kolmogorov consistency theorem; weak convergence; von Neumann selection theorem; filtration.

0. Introduction

The purpose of this paper is to discuss measure free martingales when the indexing set is an arbitrary countable subset of the positive real numbers. Measure free martingales were considered in [4], where the case when the functions take finite number of distinct values and the indexing set is the set of natural numbers was discussed. The case of general Borel measurable functions, but with natural numbers as the indexing set was discussed in [1]. The method of these papers does not carry over to the case of continuous parameter, and, weak convergence has to be considered before a measure is constructed with respect to which the the measure free martingale is a martingale.

Let Ω be a non-empty set, and T a non-empty subset of $[0, \infty)$. Let $f_t, t \in T$, be a family of real-valued functions on Ω indexed by T. For each $t \in T$, the collection $\{f_t^{-1}(a): a \in f_t(\Omega)\}$ is a partition of Ω which we denote by $\mathbb{Q}_t$. For each linearly ordered finite subset $t_1 < t_2 < \cdots < t_k$ of T, let $\mathbb{Q}_{t_1,t_2,\ldots,t_k}$ denote the superposition of the partitions $\mathbb{Q}_{t_i}, i = 1, 2, \ldots k$. If $q \in \mathbb{Q}_{t_1,t_2,\ldots,t_k}$, then the functions $f_{t_1}, f_{t_2}, \ldots, f_{t_k}$ are constant on q. In particular f_{t_k} is constant on q. We will write this constant as $f_{t_k}(q)$. Note that if $t \notin \{t_1, t_2, \ldots, t_k\}$, in particular if $t > t_k$, then f_t need not be constant on q. We say that $f_t, t \in T$, is a measure free martingale if for any ordered subset $t_1 < t_2 < \cdots < t_{k+1}$ of T and for any $q \in \mathbb{Q}_{t_1,t_2,\ldots,t_k}$, the value $f_{t_k}(q)$ is in the convex hull of the values assumed by $f_{t_{k+1}}$ on q, equivalently, if $f_{t_{k+1}}$ does not assume value $f_{t_k}(q)$ as one of its values on q, then on q, $f_{t_{k+1}}$ assumes values both less than and greater than $f_{t_k}(q)$.

The following theorem is a consequence of our considerations in this paper.

Theorem. *Let Ω, $f_t, t \in T$, be a measure free martingale where T is countable and each f_t is bounded. Assume that the functions $f_t, t \in T$, separate points of Ω. Then there is a compact metric space Ω' in which Ω is densely embedded, and there exist continuous functions $f'_t, t \in T$, defined on Ω', such that* (i) *for each t, f'_t extends f_t,* (ii) *$f'_t, t \in T$, is a measure free martingale,* (iii) *there exists a probability measure μ on Borel subsets of Ω'*

with respect to which f'_t, $t \in T$, is a martingale, i.e., for any finite subset $t_1 < t_2 < \cdots < t_k$ of T,

$$E_\mu(f_{t_k} | f_{t_1}, f_{t_2}, \ldots, f_{t_{k-1}}) = f_{t_{k-1}} \ \ a.e.$$

Let M denote the set of probability measures μ on Ω' such that f'_t, $t \in T$, is a martingale with respect to μ. Note that we have not assumed any continuity properties for $f_t(\omega)$, $t \in T$, or $f'_t(\omega)$, $t \in T$, as a function of t. However, it follows as a consequence of martingale property of $f'_t, t \in T$ (with respect to any $\mu \in M$), that there is a set N, null with respect to every $\mu \in M$, such that for all $\omega \in \Omega - N$, $f'_t(\omega), t \in T$, as a function of t, admits left and right limits at all limit points of T where these limits make sense, and at points where left and right limits exist they agree ouside possibly a countable set (see [2,3]).

It is understood here that we speak of left limit of the function, $f'_t(\omega), t \in T$, only at those points in the topological closure $\overline{T}$ of T, which are limit points of T from the left. Similarly the right limit of $f'_t(\omega), t \in T$, makes sense only at those points in $\overline{T}$ which are limit points of T from the right. Also the equality of left and right limit of $f'_t(\omega), t \in T$, makes sense only at those points in $\overline{T}$ which are limit points of T from both left and right. Let Ω_0 denote the set of points $\omega \in \Omega'$ for which the left limit $f'_{t_-}(\omega)$ and right limit $f'_{t_+}(\omega)$ exist at each point $t \in \overline{T}$ where these limits make sense. Then $\mu(\Omega_0) = 1$ for all $\mu \in M$. For each $\omega \in \Omega_0$, for $t \in \overline{T} - T$ which is a limit point of T from the right we define $f'_t(\omega)$ to be equal to $\lim_{s \to t, s > t, s \in T} f'_s(\omega)$. If $t \in T$ is a limit point of T from the right we *redefine* $f'_t(\omega)$ to be equal to the right limit $\lim_{s \to t, s > t, s \in T} f'_s(\omega)$. If $t \in \overline{T} - T$ is a limit point of T from left but not from the right then we define $f'_t(\omega) = \lim_{s \to t, s < t, s \in T} f'_s(\omega)$. This extends function $f'_t(\omega), t \in T$, to $\overline{T}$ and modifies it at those points in T which are limit points of T from the right. The resulting function, which we continue to denote by $f'_t(\omega), t \in \overline{T}$, is right continuous and also admits left limit where they make sense. This modified and extended function is called the right continuous modification of the original function. In case T is the set of rational numbers then $f'_t, t \in \overline{T} = \mathbb{R}$, is a continuous parameter martingale for every $\mu \in M$.

Assume that T is the set of rationals in a bounded closed interval $[a, b]$ with a, b rationals. Let $(\Pi_n)_{n=1}^\infty$ be a refining system of partitions of $[a, b]$, $\Pi_n = \{a = t_{0,n} < t_{1,n} < t_{2,n} < \cdots < t_{k_n,n}\}$, such that $\max_{1 \le i \le k_n}(t_{i,n} - t_{i-1,n}) \to 0$. Let ω be in Ω_0. Write

$$q(\omega, \Pi_n) = \sum_{i=1}^{k_n} (f'_{t_{i,n}}(\omega) - f'_{t_{i-1,n}}(\omega))^2.$$

Although each $q(\omega, \Pi_n)$ is finite, it is not *a priori* obvious that $\limsup_{n \to \infty} q(\omega, \Pi_n)$ is finite for a 'large' set of points $\omega \in \Omega_0$. However, the right continuous modification $f'_t, t \in [a, b]$ discussed above is a square integrable martingale with respect to every $\mu \in M$, and it follows by a theorem of C Doleans-Dade that the sum $q(\cdot, \Pi_n)$ converges in L_1 norm with respect to every $\mu \in M$ (see [2]). Hence the set of points $Q = \{\omega \in \Omega_0\colon \limsup_{n \to \infty} q(\omega, \Pi_n) < \infty\}$ has full measure with respect to every $\mu \in M$.

In §1 we define filtration as a refining system of partitions on a set Ω, and neat filtration is one where each decreasing system of non-empty elements from the filtration has non-empty intersection. In §2 we define analytic filtration and the associated system of σ-algebras and show that if such a filtration is also neat then the union σ-algebra is also of analytic type. Every analytic filtration can be embedded in a neat analytic filtration, and there is a related result 'embedding' a finitely additive measure in a countably additive measure. In §3 a natural extension of the domain of measure free martingale is discussed. In §§4 and 5 martingale measures are constructed for a measure free martingale when the indexing parameter set is finite or discrete. Section 6 discusses weak limits of martingale measures and then the main result is proved in §7 where a martingale measure is exhibitted when the parameter set is arbitrary countable subset of $[0, \infty)$.

1. Filtrations

Let Ω be a non-empty set. Any collection $\mathbb{Q}$ of subsets of Ω such that any two distinct elements of $\mathbb{Q}$ are disjoint and the union of elements of $\mathbb{Q}$ is all of Ω is called a partition of Ω.

It is clear that a partition of a set Ω can not contain more non-empty elements than the cardinality of Ω.

Let $\{A_1, A_2, \dots\}$ be a countable number of non-empty sets in Ω. Write $A^0 = A$, $A^1 = \Omega - A$. For each sequence $\epsilon = (\epsilon_1, \epsilon_2, \dots)$ of zeros and ones consider the set

$$A(\epsilon) = \cap_{n=1}^{\infty} A_n^{\epsilon_n}.$$

It is easy to see that if $\epsilon \neq \epsilon'$, then $A(\epsilon) \cap A(\epsilon') = \emptyset$. For a given $\omega \in \Omega$, write $\epsilon_n(\omega) = 0$ or $\epsilon_n(\omega) = 1$ according as $\omega \in A_n$ or $\omega \notin A_n$. Then, clearly $\omega \in A(\epsilon(\omega))$, where $\epsilon(\omega)$ is the sequence $(\epsilon_n(\omega))_{n=1}^{\infty}$. Thus the collection $\{A(\epsilon): \epsilon \in \{0, 1\}^{\mathbb{N}}\}$ forms a partition of Ω.

A partition $\mathbb{Q}$ of Ω is said to be countably generated if there exist countable number of subsets $A_1, A_2, \dots$ of Ω such that $\mathbb{Q} = \{A(\epsilon): \epsilon \in \{0, 1\}^{\mathbb{N}}\}$.

The partition of the real line into singleton sets is countably generated, for example, by intervals with rational end points. Consequently, any partition $\mathbb{Q}$ of a set Ω into $\mathbf{c}$ many or fewer sets is countably generated. Simply define a real-valued function, say f, on Ω which is constant on non-empty members of the partition and assumes distinct values on distinct non-empty elements of $\mathbb{Q}$. Then the collection $\{f^{-1}(I)\}$ where I runs over intervals with rational end points generates the partition $\mathbb{Q}$. It is also clear that if Ω has cardinality $\mathbf{c}$ or less, then any partition of Ω is countably generated.

Given partitions $\mathbb{Q}_1$ and $\mathbb{Q}_2$, $\mathbb{Q}_2$ is said to be a refinement of $\mathbb{Q}_1$ if every element $\mathbb{Q}_2$ is contained in some element of $\mathbb{Q}_1$.

DEFINITION 1.1

Let T be a non-empty subset of $[0, \infty)$. A family $\{\mathbb{Q}_t, t \in T\}$, of countably generated partitions of Ω is called a filtration on Ω if $s, t \in T$, $s < t$, implies that $\mathbb{Q}_t$ is a refinement of $\mathbb{Q}_s$.

Remark. This definition of filtration is different from the usual one where one calls a family $\mathcal{F}_t, t \in T$, of σ-algebras a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$, whenever $s \leq t$. If each $\mathcal{F}_t$ is countably generated and if $\mathbb{Q}_t$ is the partition of Ω given by a countable set of generators of $\mathcal{F}_t$, then $\mathbb{Q}_t, t \in T$, is a filtration in our sense.

DEFINITION 1.2

For each $t \in T$, let $q_t \in \mathbb{Q}_t$. The family $q_t, t \in T$, is said to be decreasing if $s < t$ implies $q_t \subset q_s$ and the filtration is said to be neat if for every such decreasing family of non-empty sets $\cap_{t \in T} q_t \neq \emptyset$.

Given a filtration $\mathbb{Q}_t$, $t \in T$, one can embed Ω in a bigger set Ω' in a certain minimal way, and equip Ω' with a *neat* filtration $\mathbb{Q}'_t$, $t \in T$, such that for each t, $\mathbb{Q}'_t|_\Omega = \mathbb{Q}_t$. One simply adds to Ω as new elements each decreasing family (q_t), $t \in T$, of non-empty sets with empty intersetion and calls the new set Ω'. If we add the new element $(q_t)_{t \in T}$ to each q_t, and call the new set q'_t, then $\mathbb{Q}'_t = \{q'_t\colon q_t \in \mathbb{Q}_t\}$ is a partition of Ω' and the family $\mathbb{Q}'_t$, $t \in T$, is a neat filtration on Ω' which satisfies $\mathbb{Q}'_t|_\Omega = \mathbb{Q}_t$ for all $t \in T$. We will call this embedding the minimal neat embedding of the filtration $\mathbb{Q}_t$, $t \in T$.

2. Filtrations and analytic Borel spaces

Let [0, 1] be equipped with its Borel σ-algebra, denoted by $\mathcal{B}$. A subset A of [0, 1] is called analytic if it is the image of a Borel measurable function from [0, 1] into [0, 1]. Analytic subsets of [0, 1] have the following interesting property: Let $A \subset [0, 1]$ be analytic. Restrict $\mathcal{B}$ to A. Denote it by $\mathcal{B}_A$. Let $A_1, A_2, \ldots$ be a countable collection of sets in $\mathcal{B}_A$ which generates the partition of A into singletons. Then the σ-algebra generated by $A_1, A_2, \ldots,$ is same as $\mathcal{B}_A$. This observation is independently due to David Blackwell and G W Mackey (see [6]).

Let $\mathbb{Q}$ be a partition of a set Ω into **c**-many or countably many sets. Let f be a function on Ω whose image is an analytic subset A of [0, 1], which is constant on members of $\mathbb{Q}$ and assumes distinct values on distinct members of $\mathbb{Q}$. In other words, f is such that

$$\mathbb{Q} = \{f^{-1}(\{x\})\colon x \in A\}.$$

Consider the σ-algebra $f^{-1}(\mathcal{B}_A)$ on Ω. In view of the result of Blackwell and Mackey mentioned above, it is clear that any countable collection of subsets in $f^{-1}(\mathcal{B}_A)$ which generates $\mathbb{Q}$ also generates $f^{-1}(\mathcal{B}_A)$. The σ-algebra $f^{-1}(\mathcal{B}_A)$ on Ω is called analytic type, where $f\colon \Omega \to [0, 1]$ is such that its image $A = f(\Omega)$ is an analytic subset of [0, 1]. The elements of $\mathbb{Q}$ are called atoms of the σ-algebra $f^{-1}(\mathcal{B}_A)$.

DEFINITION 2.1

A filtration $\mathbb{Q}_t$, $t \in T$, is said to be of analytic type if there is an increasing family $\mathcal{F}_t$, $t \in T$, of σ-algebras on Ω, each of analytic type and such that for each $t \in T$, any countable set of generators for $\mathcal{F}_t$ also generates the partition $\mathbb{Q}_t$.

If we have an increasing family $\mathcal{F}_t$, $t \in T$, of countably generated σ-algebras on a set Ω, and if $\mathbb{Q}_t$ is the partition of Ω into atoms of $\mathcal{F}_t$, then we call the filtration $\mathbb{Q}_t$, $t \in T$, the filtration associated to the family $\mathcal{F}_t$, $t \in T$.

Theorem 2.1. *Let $T \subset [0, \infty)$ be countable. Let $\mathcal{F}_t$, $t \in T$, be an increasing family of σ-algebras on Ω each of analytic type. For each $t \in T$, let $\mathbb{Q}_t$ denote the set of atoms of $\mathcal{F}_t$. If the filtration $\mathbb{Q}_t$, $t \in T$, is neat, then the σ-algebra generated by $\cup_{t \in T} \mathcal{F}_t$ is of analytic type.*

Proof. We will assume that Ω is a subset of $\mathbb{R}^T$ and for each $t \in T$, $\mathcal{F}_t$ is the σ-algebra generated on Ω by the co-ordinate maps π_s, $s \leq t$. The general case can be reduced to this. The assertion that $\mathcal{F}_t$ is of analytic type means that the canonical projection of Ω onto $\mathbb{R}^{[0,t] \cap T}$, say A_t, is analytic. Let J_t denote the canonical projection from $\mathbb{R}^T$ onto $\mathbb{R}^{[0,t] \cap T}$. The hypothesis that the filtration is neat is equivalent to saying that

$$\cap_{t \in T} J_t^{-1} A_t = \Omega.$$

Now each A_t is analytic, hence $J_t^{-1}A_t$ is analytic in $\mathbb{R}^T$. Since countable intersection of analytic sets is analytic we see that the left-hand side of the above equality is analytic, hence the right-hand side is also analytic. This completes the proof of the theorem.

Example 2.1. The requirement in the above theorem that the $\mathbb{Q}_t, t \in T$, be neat can not be dropped as the following example shows. Let Ω be a non-analytic subset of $\{0, 1\}^{\mathbb{N}}$ equipped with the σ-algebra $\mathcal{C}|_\Omega$. Let f_i denote the projection map to the i-th co-ordinate space of $\{0, 1\}^{\mathbb{N}}$ restricted to Ω and let $\mathcal{F}_i$ be the σ-algebra on Ω generated by $f_1, f_2, \ldots, f_i$. Now let $T = \{1, 2, \ldots\}$. Then each $\mathcal{F}_t, t \in T$, is of analytic type since each $\mathcal{F}_t$ has only finitely many elements, while $\cup_{t\in T}\mathcal{F}_t$ generates $\mathcal{C}|_\Omega$ which is not of analytic type.

Let $\mathcal{F}_t, t \in T$, be an increasing family of σ-algebras on a set Ω, each of analytic type. For each $t \in T$, let $\mathbb{Q}_t$ denote the set of atoms of $\mathcal{F}_t$. Since $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s < t$, we have $\mathbb{Q}_t$ refines $\mathbb{Q}_s$ whenever $s < t$. For each $t \in T$, let μ_t be a probability measure on $\mathcal{F}_t$, such that $\mu_t|_{\mathcal{F}_s} = \mu_s$, whenever $s < t$. On the algebra $\mathcal{A}_\infty = \cup_{t\in T}\mathcal{F}_t$ one can define a finitely additive measure μ by setting $\mu(A) = \mu_t(A)$ whenever $A \in \mathcal{F}_t$.

Kolmogorov consistency theorem (Descriptive set theoretic version). *If the filtration $\mathbb{Q}_t, t \in T$, is neat, then μ is countably additive on $\mathcal{A}_\infty$, so it extends to a countably additive measure μ_∞ to the σ-algebra $\mathcal{F}_\infty$ generated by $\mathcal{A}_\infty$.*

We refer the reader to [5] for a proof of this theorem.

If the filtration $\mathbb{Q}_t, t \in T$, associated to $\mathcal{F}_t, t \in T$, is not neat, then we can enlarge Ω to a set Ω' as described above in a certain minimal way and obtain a new filtration $\mathbb{Q}'_t, t \in T$, on Ω', such that for each $t \in T$, $\mathbb{Q}'_t|_\Omega = \mathbb{Q}_t$. Note that for each t, $q_t \subset q'_t$ and the mapping which sends $q_t \in \mathbb{Q}_t$ to $q'_t \in \mathbb{Q}'_t$ is a bijection which defines a natural σ-algebra $\mathcal{F}'_t$ on Ω' which is isomorphic to $\mathcal{F}_t$, hence of analytic type. The measure μ_t gives rise to a measure μ'_t on $\mathcal{F}'_t$ such that $\mu'_t|_\Omega = \mu_t$. We also have $\mathcal{F}'_t|_\Omega = \mathcal{F}_t$. Also $\mu'_t|_{\mathcal{F}'_s} = \mu'_s$ whenever $s < t$. Since the filtration $\mathbb{Q}'_t, t \in T$, is neat, the naturally defined finitely additive measure μ' defined on $\mathcal{A}'_\infty = \cup_{t\in T}\mathcal{F}'_t$ is countably additive and so has a countably additive extension μ'_∞ to the σ-algebra $\mathcal{F}'_\infty$ generated by $\mathcal{A}'_\infty$. For any $A \in \mathcal{A}_\infty$, there is an $\mathcal{F}_t$ to which A belongs. The set A is a union of certain set of elements $q_t \in \mathbb{Q}_t$. The union of the corresponding $q'_t \in \mathbb{Q}'_t$ is the set $A' \in \mathcal{F}'_\infty$ associated in a unique manner to A, and we have $\mu(A) = \mu'(A')$. With these notations in mind, we have proved the following.

Theorem 2.2. *Let $(\Omega, \mathcal{F}_t, \mu_t), t \in T$, be a family of probability measures indexed by a countable set $T \subset [0, \infty)$ such that* (i) *$\mathcal{F}_t, t \in T$, is an increasing family σ-algebras of analytic type*, (ii) *the family of measures $\mu_t, t \in T$, is consistent in the sense that $\mu_t|_{\mathcal{F}_s} = \mu_s$, whenever $s < t$. Then there is a unique countably additive probability measure μ'_∞ on $(\Omega', \mathcal{F}'_\infty)$ such that for each $A \in \mathcal{A}_\infty$, $\mu(A) = \mu'_\infty(A')$. $(\Omega', \mathcal{F}'_\infty)$ is of analytic type.*

Example 2.2. Let $\Omega =$ the set of rational numbers $r, 0 \le r \le 1$, $\mathbb{Q}_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}) \cap \Omega : 0 \le k \le 2^n - 1\}$, $\mu_n([\frac{k}{2^n}, \frac{k+1}{2^n}) \cap \Omega) = \frac{1}{2^n}, 0 \le k \le 2^n - 1$. Let $\mathcal{F}_n$ be the σ-algebra generated by $\mathbb{Q}_n$. Then the sequence $(\Omega, \mathcal{F}_n, \mu_n), n = 1, 2, \ldots$ satisfies the hypothesis of the above theorem. The measure μ on $\mathcal{A}_\infty = \cup_{n=1}^\infty \mathcal{F}_n$ is only finitely additive and has no countably additive extension to Ω. However, $\Omega' = [0, 1]$, $\mathcal{F}'_\infty$ is the Borel σ-algebra of $[0, 1]$, and $\mu'_\infty =$ Lebesgue measure on $[0, 1]$.

Let S be a subset of $\mathbb{R}^T$. Let A be countable ordered set and for each $a \in A$, let T_a be a subset of T such that (i) $T_a \subset T_b$ whenever $a \leq b$, (ii) $\cup_{a\in A} T_a = T$. Let π_a denote the canonical projection from $\mathbb{R}^T$ onto $\mathbb{R}^{T_a}$, and let S_a denote $\pi_a S$.

Theorem 2.3. $\cap_{a\in A}\pi_a^{-1}(\overline{S}_a) = \overline{S}$. *In particular, if S is closed then* $\cap_{a\in A}\pi_a^{-1}(\overline{S}_a) = S$.

Proof. Clearly $\overline{S} \subset \pi_a^{-1}(\overline{S_a}))$, so $\overline{S} \subset \cap_{a\in A}\pi_a^{-1}(\overline{S_a})$. To show the reverse inclusion, let $\omega \in \cap_{a\in A}\pi_a^{-1}(\overline{S_a})$. We show that ω is in S or a limit point of S. Fix $t_1, t_2, \ldots, t_k$ in T. Let U_i be a neighbourhood of w_{t_i}, the t_i-th co-ordinate of ω, $1 \leq i \leq k$. Then $U_1 \times U_2 \times \cdots \times U_k \times \mathbb{R}^{T-\{t_1,t_2,\ldots,t_k\}} = U$ (say) is a neighbourhood of ω. Let a be such that T_a contains $t_1, t_2, \ldots, t_k$. Since $\pi_a\omega \in \overline{S_a}$, there exists $\omega' \in S$ such that $\pi_a\omega' \in \pi_a U$. Since $\pi_a^{-1}\pi_a U = U$, $\omega' \in U$. Since U is an arbitrary basic neighbourhood of ω, we see that $\omega \in \overline{S}$, and the proposition is proved.

Let $\mathbb{Q}_a, \mathcal{F}_a, a \in A$, be respectively the partition and the σ-algebra generated on S by the co-ordinate functions $f_t, t \in T_a$. If S is closed in $\mathbb{R}^T$ then the filtration $\mathbb{Q}_a, a \in A$, is neat. This is a consequence of Proposition 2.3.

Suppose $(\Omega, \mathcal{B})$ is a standard Borel space and $\mathcal{F}_t, t \in T$, is an increasing family of countably generated sub-σ algebras of $\mathcal{B}$. Then the partitions $\mathbb{Q}_t, t \in T$, is a filtration on Ω, $\mathbb{Q}_t$ being the set of atoms of $\mathcal{F}_t, t \in T$. Moreover, for each $t \in T$, the collection of sets $A \in \mathcal{B}$ which are unions of atoms in $\mathbb{Q}_t$ is precisely the σ algebra $\mathcal{F}_t$. So, in such a case, where all the $\mathcal{F}_t, t \in T$, are countably generated and contained in $\mathcal{B}$, they are uniquely determined by partitions their atoms generate, and so there is a one-to-one correspondence between $\mathcal{F}_t \leftrightarrow \mathbb{Q}_t, t \in T$, and one can call $\mathcal{F}_t, t \in T$, a filtration on Ω.

3. Measure free martingales

Let Ω be a non-empty set, and T a non-empty subset of $[0, \infty)$. Let $f_t, t \in T$, be a measure free martingale indexed by T (see Introduction). We will make the simplifying assumption that the measure free martingale $f_t, t \in T$, separates the points of Ω which means that given $\omega_1, \omega_2 \in \Omega$ there is a $t \in T$ such that $f_t(\omega_1) \neq f_t(\omega_2)$. As a consequence the mapping ϕ:

$$\phi(\omega) = (f_t(\omega))_{t\in T}, \quad \omega \in \Omega,$$

is a one-to-one mapping from Ω into $\mathbb{R}^T$. Without loss of generality we can replace Ω by $\phi(\Omega)$ and f_t by $f_t \circ \phi^{-1} = \pi_t|_{\phi(\Omega)}, t \in T$, where π_t is the canonical projection from $\mathbb{R}^T$ onto the t-th co-ordinate space. Unless otherwise mentioned, Ω will be a subset of $\mathbb{R}^T$ and f_t will mean the restriction of π_t to Ω. The system $(\Omega, f_t, t \in T)$ will be assumed to be a measure free martingale.

Let $(\Omega, f_t, t \in T)$ be a measure free martingale, with T assumed to be countable. Let $\mathbb{Q}(t)$ denote the partition of Ω generated by the functions $f_s, s \leq t$. Then $\mathbb{Q}(t), t \in T$, is a filtration on Ω. For each $t \in T$, let J_t denote the canonical projection of $\mathbb{R}^T$ onto $\mathbb{R}^{\{s:\ s\leq t, s\in T\}}$. It is easy to see that if $q_t \in \mathbb{Q}(t), t \in T$, a decreasing family with each q_t non-empty, then $\cap_{t\in T} J_t^{-1}\{q_t\}$ is a point of $\mathbb{R}^T$ which may or may not be in Ω. We enlarge Ω by adding to Ω all such intersections $\cap_{t\in T} J_t^{-1}\{q_t\}$ of decreasing family $q_t \in \mathbb{Q}_t, t \in T$, each q_t being non-empty. Call the new set Ω' which embeds Ω. We define $f'_s(\cap_{t\in T} J_t^{-1}\{q_t\}) = f_s(q_s)$. This extends f_s to Ω'. It is easy to check that $(\Omega', f'_t, t \in T)$ is a measure free martingale whose restriction to Ω is the original measure free martingale, and the filtration associated to $f'_t, t \in T$ is neat.

DEFINITION 3.1

A measure free martingale $f_t, t \in T$, is said to be bounded if each f_t is bounded, although the bound may be vary with t.

Theorem 3.1. *If $(\Omega, f_t, t \in T)$ is a bounded measure free martingale, then $(\overline{\Omega}, f_t, t \in T)$ is also a measure free martingale, where $\overline{\Omega}$ is the closure of Ω in $\mathbb{R}^T$ under the Tychonoff topology and where $f_t, t \in T$, are to be regarded as the maps $\pi_t, t \in T$, restricted to $\overline{\Omega}$.*

Proof. Let $t_1 < t_2 < \cdots < t_k < t_{k+1}$ in T, $\overline{q} \in \overline{\mathbb{Q}}_{t_1,t_2,\dots t_k}$ be given, where $\overline{\mathbb{Q}}_{t_1,t_2,\dots t_k}$ is the partition of $\overline{\Omega}$ generated by the functions $f_{t_1}, f_{t_2}, \dots, f_{t_k}$. Let $\overline{\omega}$ be a point in $\overline{q}$. Then there exists two sequences $(\omega_{n,i})_{n=1}^{\infty}, i = 1, 2$ of points in Ω such that

$$f_{t_j}(\omega_{n,i}) \to f_{t_j}(\overline{\omega}), 1 \le j \le k, i = 1, 2,$$

while $f_{t_{k+1}}(\omega_{n,1}) \le f_{t_k}(\omega_{n,1}),\ f_{t_{k+1}}(\omega_{n,2}) \ge f_{t_k}(\omega_{n,2})$.

Any limit point u (which exists, since the measure free martingale $f_t, t \in T$, is bounded) of $(\omega_{n,1})_{n=1}^{\infty}$ will be in $\overline{q}$ and satisfy $f_{t_{k+1}}(u) \le f_{t_k}(u)$, while any limit point v of $(\omega_{n,2})_{n=1}^{\infty}$ will be in $\overline{q}$ and satisfy $f_{t_{k+1}}(v) \ge f_{t_k}(v)$.

Thus $f_{t_k}(\overline{q})$ is in the convex hull of the values assumed by $f_{t_{k+1}}$ on $\overline{q}$. This proves the theorem.

This proves a part of the theorem stated in the Introduction.

4. Measure free martingales and martingale measures (The case of finite T)

Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $T \subset [0, \infty)$ be non-empty. A family $f_t, t \in T$, of random variables defined on this space is said to be a martingale if $E_\mu(f_t) < \infty$ for each $t \in T$ and for each finite subset $t_1 < t_2 < \cdots < t_k$ of T,

$$E_\mu(f_{t_k} | f_{t_1}, f_{t_2}, \dots, f_{t_{k-1}}) = f_{t_{k-1}} \text{ a.e.}$$

This definition of martingale is equivalent to the usual one in which the conditional expectation is taken with respect to an increasing family $\mathcal{L}_t, t \in T$ of σ-algebras.

When T is countable it is easy to see using regular conditional probabilities that there is a μ-null set N such that the martingale $f_t, t \in T$, is a measure free martingale on $X - N$. The rest of the discussion in this paper is devoted to proving a converse of this.

PROPOSITION 4.1

Let $(\Omega, \mathcal{B})$ be a standard Borel space and let $\mathbb{Q}$, $\mathcal{C}$ be respectively the partition and the σ-algebra generated by a countable collection of sets in $\mathcal{B}$. Let $\mathcal{A}$ be the σ-algebra generated by analytic subsets of Ω which are unions of elements of $\mathbb{Q}$. Let f and g be respectively $\mathcal{B}$ and $\mathcal{C}$ measurable real valued functions on Ω such that for each $q \in \mathbb{Q}$, $g(q)$ is in the convex hull of the values assumed by f on q. Then there exists a transition probability $\nu(\cdot, \cdot)$ on $\mathcal{B} \times \mathbb{Q}$ such that for each $A \in \mathcal{B}$, the function $\nu(A, \cdot)$ is $\mathcal{A}$ measurable, while $\nu(\cdot, q)$ is a probability measure on $\mathcal{B}$ supported on at most two points of q satisfying

$$g(q) = \int_q f(\omega)\nu(\mathrm{d}\omega, q), \quad \forall q \in \mathbb{Q}.$$

Proof. The sets

$$S_1 = \{\omega \in \Omega\colon f(\omega) \le g(\omega)\}, \quad S_2 = \{\omega \in \Omega\colon g(\omega) \le f(\omega)\},$$

are in $\mathcal{B}$. For each $q \in \mathbb{Q}$, since $g(q)$ is in the convex hull of the values assumed by f on q, both S_1 and S_2 have non-empty intersections with q. By the von-Neumann selection theorem (see p. 199 of [6]) there exist coanalytic sets $C_1 \subset S_1, C_2 \subset S_2$ which intersect each $q \in \mathbb{Q}$ in exactly one point. For each $q \in \mathbb{Q}$, let

$$\omega_1(q) = C_1 \cap q, \omega_2(q) = C_2 \cap q.$$

Then

$$f(\omega_1(q)) \le g(q) \le f(\omega_2(q)),$$

so that the middle real number $g(q)$ is a unique convex combination of $f(\omega_1(q)), f(\omega_2(q))$. If $f(\omega_1(q)) = f(\omega_2(q)) = g(q)$ write $p_1(q) = 1$, $p_2(q) = 0$, otherwise write

$$p_1(q) = \frac{f(\omega_2(q)) - g(q)}{f(\omega_2(q)) - f(\omega_1(q))}, \quad p_2(q) = \frac{g(q) - f(\omega_1(q))}{f(\omega_2(q)) - f(\omega_1(q))}.$$

Then

$$p_1(q)f(\omega_1(q)) + p_2(q)f(\omega_2(q)) = g(q).$$

For each $q \in \mathbb{Q}$, let $\nu(\cdot, q)$ be the probability measure on q with masses $p_1(q), p_2(q)$ at $\omega_1(q), \omega_2(q)$ respectively. The sets C_1, C_2 are co-analytic and functions $f|_{C_1}, f|_{C_2}$ are $\mathcal{B}|_{C_1}, \mathcal{B}|_{C_2}$ measurable respectively, whence $p_1(\cdot), p_2(\cdot)$ are $\mathcal{A}$ measurable. For any $A \in \mathcal{B}$, and $q \in \mathbb{Q}$,

$$\nu(A, q) = p_1(q)1_A(\omega_1(q)) + p_2(q)1_A(\omega_2(q)),$$

whence, for each $A \in \mathcal{B}$, $\nu(A, \cdot)$ is $\mathcal{A}$ measurable. The proposition is proved.

The above proposition is borrowed from [1]. It is reproduced here with proof since we need the considerations in the sequel.

Let ν be a probability measure on $\mathcal{C}$ and let μ be the measure on $\mathcal{B}$ defined by

$$\mu(A) = \int_\Omega \nu(A, q)\nu(\mathrm{d}q).$$

The expected value of f with respect to μ is the expected value of g with respect to ν which is the same as the expected value of g with respect to μ, since $\mu|_{\mathcal{C}} = \nu$ and g is $\mathcal{C}$ measurable.

Now

$$\begin{aligned}\text{variance}(f) &= \int_\Omega (f(\omega) - E(g))^2 \mu(\mathrm{d}\omega)\\ &= \int_\Omega \bigg(\int_q ((f(\omega) - g(q))^2 + (g(q) - E(g))^2 \\ &\qquad + 2(f(\omega) - g(q))(g(q) - E(g)) \bigg) \nu(\mathrm{d}\omega, q)\mu(\mathrm{d}q)\end{aligned}$$

Since $\int_q (f(\omega) - g(q))\nu(\mathrm{d}\omega, q) = 0$, for each $q \in \mathbb{Q}$, we see that

$$\text{variance}(f) = \text{variance}(g) + \int_\Omega \left(\int_q (f(\omega) - g(q))^2 \nu(\mathrm{d}\omega, q)\mu(\mathrm{d}q) \right).$$

Thus, the variance of f with respect to μ exists if and only if the variance of g exists and

$$\int_\Omega \left(\int_q (f(\omega) - g(q))^2 \nu(\mathrm{d}\omega, q)\mu(\mathrm{d}q) \right)$$

is finite, which is the case, for example, if $\omega_1(q), \omega_2(q)$ can be chosen such that $g(q) - f(\omega_1(q)), f(\omega_2(q)) - g(q)$ remain bounded as q varies over $\mathbb{Q}$.

Note that with respect to the measure $\nu(\cdot, q)$ constructed in Proposition 4.1, above,

$$\int_q (f(\omega) - g(q))^2 \nu(\mathrm{d}\omega, q) = (f(\omega_2) - g(q))(g(q) - f(\omega_1(q))).$$

If for each $q \in \mathbb{Q}$, the points $\omega_1(q), \omega_2(q)$ can be chosen in such a way that

$$(f(\omega_2(q)) - g(q))(g(q) - f(\omega_1(q)))$$

is a constant, say c, independent of q, then

$$\text{variance}(f) = \text{variance}(g) + c.$$

We will use the above discussion in a while.

Let Ω, $f_t, t \in T$, be a measure free martingale, where $T = \{t_1 < t_2 < \cdots < t_{k+1}\}$ is a finite set. Assume that Ω is a Borel subset of $\mathbb{R}^T$. For $1 \le i \le k$, let $\mathbb{Q}_i, \mathcal{C}_i$ denote the partition and the σ-algebra generated by $f_{t_1}, f_{t_2}, \ldots, f_{t_i}$. These are of analytic type since $f_t, t \in T$ are Borel measurable functions on a standard Borel space. Fix a probability measure μ_1 on $\mathcal{C}_1$, and let $\nu_1(\cdot, \cdot)$ be a transition probability on $\mathcal{C}_2 \times \mathbb{Q}_1$ such that for each $q \in \mathbb{Q}_1$,

$$\int_q f_{t_2}(\omega)\nu_1(\mathrm{d}\omega, q) = f_{t_1}(q) \ \text{ a.e.}$$

This is possible by Proposition 4.1, since, by hypothesis, the value $f_{t_1}(q)$ lies in the convex hull of the values assumed by f_{t_2} on q. For $A \in \mathcal{C}_2$, write

$$\mu_2(A) = \int_\Omega \nu_1(A, q)\mu_1(\mathrm{d}q).$$

Then μ_2 is a measure on $\mathcal{C}_2$, such that $\mu_2|_{\mathcal{C}_1} = \mu_1$, and

$$E_{\mu_2}(f_{t_2}|\mathcal{C}_1) = f_{t_1}.$$

Again, since $f_t, t \in T$, is a measure free martingale, for any $q \in \mathbb{Q}_2$, $f_{t_2}(q)$ lies in the convex hull of the values assumed by f_{t_3} on q. Hence by Proposition 4.1, there exists a transition probability $\nu_2(\cdot, \cdot)$ on $\mathcal{C}_3 \times \mathbb{Q}_2$ such that if

$$\mu_3(\cdot) = \int_\Omega \nu_2(\cdot, q)\mu_2(\mathrm{d}q),$$

then μ_3 is defined on $\mathcal{C}_3$ and

$$E_{\mu_3}(f_{t_3}|\mathcal{C}_2) = f_{t_2} \text{ a.e., } \mu_3|_{\mathcal{C}_2} = \mu_2.$$

Proceeding thus, after k steps we get a measure μ_{k+1} defined on $\mathcal{C}_{k+1}$ with respect to which the measure free martingale $f_t, t \in T$, is a martingale.

Let us continue with the setup of the above paragraph. Let μ_i denote the measure defined on $\mathcal{C}_i$ with respect to which the functions $f_{t_1}, f_{t_2}, \ldots, f_{t_i}$ form a martingale and let $\nu_i(\cdot, \cdot)$ be the transition probability on $\mathcal{C}_{i+1} \times \mathbb{Q}_i$ satisfying

$$\int_q f_{t_{i+1}}(\omega)\nu_i(\mathrm{d}\omega, q) = f_{t_i}(q), q \in \mathbb{Q}_i.$$

Suppose for each i, $1 \leq i \leq k$, and for each $q \in \mathbb{Q}_i$,

$$\int_q (f_{t_{i+1}}(\omega) - f_{t_i}(q))^2 \nu_i(\mathrm{d}\omega, q) = t_{i+1} - t_i.$$

Such is the case, for example, if for each i and for each $q \in \mathbb{Q}_i$, $f_{t_{i+1}}$ assumes two values on q, say a and b, such that $a < f_{t_i}(q) < b$,

$$(f_{t_i}(q) - a)(b - f_{t_i}(q)) = t_{i+1} - t_i,$$

and $\nu_i(\cdot, q)$ is defined using these values. With such a choice of $\nu_i(\cdot, \cdot)$, $1 \leq i \leq k$, and if the variance of f_{t_1} exists with respect to μ_1, then the variance of f_{t_i} exists for each i and

$$\text{variance}(f_{t_{i+1}} - f_{t_i}) = t_{i+1} - t_i, \text{variance}(f_i) = \text{variance}(f_{t_1}) + t_i - t_1.$$

DEFINITION 4.1

Let $f_t, t \in T$, be a measure free martingale, where $T \subset [0, \infty)$, not necessarily finite or countable. We will say that $f_t, t \in T$, is nice if for each k tuple $t_1 < t_2 < \cdots < t_k$ of T, for each i, $1 \leq i \leq k - 1$, for each $q \in \mathbb{Q}_i$ (= partition given by $f_{t_1}, f_{t_2}, \ldots, f_{t_i}$,) $f_{t_{i+1}}$ assumes two values a_i, b_i on q, $a_i \leq f_{t_i}(q) \leq b_i$ such that

$$(f_{t_i}(q) - a_i)(b_i - f_{t_i}(q)) \leq \phi(t_{i+1} - t_i),$$

where ϕ is a function on positive real numbers such that

$$\sum_{i=2}^{k} \phi(t_i - t_{i-1})$$

admits a finite upper bound depending only on $t_k - t_1$ and not on k.

If for each i, $1 \leq i \leq k$, the values assumed by $f_{t_{i+1}}$ on distinct elements q of $\mathbb{Q}_i$ are disjoint, then any $q \in \mathbb{Q}_i$ is completely determined by $f_{t_i}(q)$ and so every martingale measure $\mu_{t_{k+1}}$ for $f_{t_1}, f_{t_2}, \ldots, f_{t_{k+1}}$ is also a Markov measure in the sense that functions $f_{t_1}, f_{t_2}, \ldots, f_{t_{k+1}}$ form a Markov chain. (Indeed, f_{t_i}, $1 \leq i \leq k+1$, forms a Markov chain with respect to any probability measure on $\mathcal{C}_{k+1}$.)

We now give a less trivial condition under which a martingale measure for $f_{t_1}, f_{t_2}, \ldots, f_{t_{k+1}}$ can be chosen to be Markov. Fix i, fix $q \in \mathbb{Q}_i$, and write $a = f_{t_i}(q)$. Let

$$A_q = \{p \in \mathbb{Q}_i \colon f_{t_i}(p) = f_{t_i}(q) = a\},$$

$$S_{A_q} = \cap_{p \in A_q}\{f_{t_{i+1}}(p) \cap (-\infty, a]\},\ S^{A_q} = \cap_{p \in A_q}\{f_{t_{i+1}}(p) \cap [a, \infty)\}\}.$$

If for each i, $1 \le i \le k$, for each $q \in \mathbb{Q}_i$, S_{A_q} and S^{A_q} are non-empty, then one can choose transition probability $\nu_i(\cdot, p)$ in such a way that $f_{t_{i+1}}|_p$ induces the same distribution on $\mathbb{R}$ for all $p \in A_q$, so that this distribution depends only on $f_{t_i}(q)$, and not on q. With such a choice of $\nu_i(\cdot, q)$, $1 \le i \le k$, $q \in \mathbb{Q}_i$, $f_{t_1}, f_{t_2}, \ldots, f_{t_{k+1}}$ is a Markov chain with respect to μ_{k+1}, apart from being a martingale. In particular, if for each i, there is an open interval I containing $a = f_{t_i}(q)$ such that $I \subset f_{t_{i+1}}(p)$ for all $p \in A_q$, the above condition will be satisfied.

5. Measure free martingales and martingale measures (Discrete parameter case)

Let $(\Omega, f_t, t \in T)$, be a measure free martingale, where $T = \{t_1 < t_2 < \ldots\}$ is an infinite subset of $[0, \infty)$ without any limit points. Assume that Ω is a Borel subset of $\mathbb{R}^T$. For $1 < i < \infty$, let $\mathbb{Q}_i$, $\mathcal{C}_i$ denote the partition and the σ-algebra generated by $f_{t_1}, f_{t_2}, \ldots, f_{t_i}$. The filtration $\mathbb{Q}_i$, $i = 1, 2, \ldots$ and the σ-algebras $\mathcal{C}_i$, $i = 1, 2, \ldots$ are of analytic type. By the discussion of the last section, for each i we get a probability measure μ_i on $\mathcal{C}_i$ such that $f_{t_1}, f_{t_2}, \ldots, f_{t_i}$ is a martingale with respect to μ_i. Moreover, $\mu_{i+1}|_{\mathcal{C}_i} = \mu_i$, so that the family of measures μ_i, $1 \le i < \infty$ is consistent. Assume any one of the following: (i) filtration $\mathbb{Q}_t$, $t \in T$, is neat (otherwise we could replace Ω by its minimal neat embedding, and modify f_t, $t \in T$, accordingly), (ii) Ω is a closed subset of $\mathbb{R}^T$ (so that by Theorem 2.3, the filtration $\mathbb{Q}_i$, $i = 1, 2, \ldots$ is neat). Then by Kolmogorov's consistency theorem there is a probability measure μ_∞ on σ-algebra $\mathcal{C}_\infty$ generated by $\cup_{t \in T}\mathcal{C}_t$, such that $\mu_\infty|_{\mathcal{C}_i} = \mu_i$. Thus the measure free martingale f_t, $t \in T$, with T discrete, admits a measure with respect to which it is martingale. Further the discussion of the last section as to when the variances exist, or when the process is a Markov process carries over.

6. Weak convergence of martingale measures

If μ_a, $a \in A$, be a family of probability measures on Borel subsets of $\mathbb{R}^k$ such that means and variances of the co-ordinate random variables exist with respect to each μ_a and remain bounded as functions of a, then the family μ_a, $a \in A$, is tight. This is an easy consequence of Chebyshev's inequality. Further if X_a, $a \in A$, is a system of random variables with values in $\mathbb{R}^k$ defined on a probability space $(Y, \mathcal{B}, \mu)$ such that for each $a \in A$, X_a has distribution μ_a, then, since the means and variances of μ_a, $a \in A$, remain bounded, the family X_a, $a \in A$, is uniformly integrable (p. 629 of [3]).

Let $(\mu_n)_{n=1}^\infty$ be a weakly convergent sequence from this family whose limit we denote by μ. Then the sequence of marginal measures of μ_n on a given subspace of $\mathbb{R}^k$ converges to the corresponding marginal of μ. Assume that the co-ordinate functions

$$f_i(x_1, x_2, \ldots, x_k) = x_i,\ (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k,\ i = 1, 2, \ldots, k$$

form a martingale with respect to each μ_n. Then they form a martingale with respect to μ. We see this as follows: Since for each i, the expectation and variance of f_i with respect to

μ_n remain bounded as a function of n, and since μ_n converges weakly to μ, using uniform integrability mentioned above it can be shown that

$$\int_{\mathbb{R}^k} f_i \mathrm{d}\mu_n \to \int_{\mathbb{R}^k} f_i \mathrm{d}\mu.$$

So the expectation with respect to μ exists for each co-ordinate random variable f_i. To verify the martingale property of the co-ordinate random variables with respect to μ, let $\mathcal{C}_i$ be the σ-algebra generated by $f_1, f_2, \ldots, f_i$, $1 \le i < k$. Let B be a Borel set of positive μ measure in this σ-algebra whose boundary has μ measure zero. Then, $\mu_n(B) \to \mu(B)$, and $\frac{1_B \mathrm{d}\mu_n}{\mu_n(B)}$ converges weakly to $\frac{1_B \mathrm{d}\mu}{\mu(B)}$. Since the co-ordinate functions obviously have bounded means and variances with respect to these measures also, we see that

$$\frac{1}{\mu_n(B)} \int_{\mathbb{R}^k} f_{i+1} 1_B \mathrm{d}\mu_n \to \frac{1}{\mu(B)} \int_{\mathbb{R}^k} f_{i+1} 1_B \mathrm{d}\mu, \tag{1}$$

$$\frac{1}{\mu_n(B)} \int_{\mathbb{R}^k} f_i 1_B \mathrm{d}\mu_n \to \frac{1}{\mu(B)} \int_{\mathbb{R}^k} f_i 1_B \mathrm{d}\mu. \tag{2}$$

Since the co-ordinate functions form a martingale with respect to each μ_n, we have

$$\int_B f_{i+1} \mathrm{d}\mu_n = \int_B f_i \mathrm{d}\mu_n = \int_{\mathbb{R}^k} 1_B f_i \mathrm{d}\mu_n \to \int_{\mathbb{R}^k} 1_B f_i \mathrm{d}\mu = \int_B f_i \mathrm{d}\mu. \tag{3}$$

Since the collection of sets B in $\mathcal{C}_i$ whose boundaries have μ measure zero generate $\mathcal{C}_i$, we see that (1), (2) and (3) together imply that the co-ordinate functions form a martingale with respect to μ. We have proved:

Theorem 6.1. *Let μ_a, $a \in A$, be a family of probability measures on $\mathbb{R}^k$ such that the co-ordinate random variables have mean and variance with respect to each μ_a, $a \in A$, and these means and variances remain bounded. Assume further that the co-ordinate functions form a martingale with respect to each μ_a. Then the co-ordinate functions form a martingale with respect to every measure in the weak closure of the family μ_a, $a \in A$.*

7. Measure free martingales and martingale measures (The general case)

Let T be a countable subset of $[0, \infty)$, not necessarily discrete. We will assume that $0 \in T$. Let Ω be a closed subset of $\mathbb{R}^T$ equipped with the Tychonoff topology, and assume that the canonical projection maps f_t, $t \in T$, restricted to Ω form a measure free martingale. Our aim is to obtain a probability measure on Borel subsets of Ω under which f_t, $t \in T$, is a martingale. Since T is no more assumed to be discrete, the method of §5 does not work, and weak convergence of martingale measures has to be brought into consideration. Further, some conditions which guarantee tightness of measures constructed need to be imposed.

Let $T_1, T_2, \ldots$ be an increasing family of finite subsets of T with their union equal to T. For each k, assume that $0 \in T_k$. Let $T_k = \{t_{k,1} < t_{k,2} < \cdots < t_{k,l_k}\}$. Let μ_k be a probability measure on the σ-algebra generated by f_t, $t \in T_k$, such that $f_{t_{k,1}}, f_{t_{k,2}}, \ldots, f_{t_{k,l_k}}$ is a martingale with respect to it. This is possible as discussed in §4. We will assume that the distribution of f_0 is same under all measures μ_k and that its variance exists. Let $\mathcal{C}_k$ denote the σ-algebra generated by f_t, $t \in T_k$. For each k and for each $r > k$, $f_{t_{k,1}}, f_{t_{k,2}}, \ldots, f_{t_{k,l_k}}$ is a martingale with respect to μ_r. Assume that for each k, the functions f_t, $t \in T_k$, have expectation and variance with respect to the measures $\mu_r, r \geq k$, and that these remain bounded as functions of r. This holds true, for example, if each $f_t, t \in T$, is bounded or more generally if the measure free martingale $f_t, t \in T$, is nice (see Definition 4.1). Let $\pi_{k,l}, k \leq l$ denote the canonical projection from $\mathbb{R}^{T_l}$ onto $\mathbb{R}^{T_k}$. Similarly $\pi_{k,\infty}$ denotes the canonical projection from $\mathbb{R}^T$ onto $\mathbb{R}^{T_k}$.

The measures $\mu_r, r \geq 1$, when restricted $\mathcal{C}_1$, are tight, hence they admit a weakly convergent sequence $\mu_{1,1}, \mu_{1,2}, \mu_{1,3}, \ldots$ converging to a measure, say ν_1, which will be supported on $\pi_{1,\infty}^{-1}(\overline{\pi_{1,\infty}(\Omega)})$. From $\mu_{1,1}, \mu_{1,2}, \ldots$ we can extract a further subsequence $\mu_{2,1}, \mu_{2,2}, \mu_{2,3} \ldots$, such that $\mu_{2,k}|_{\mathcal{C}_2}, k = 1, 2, \ldots$ converges weakly to a probability measure ν_2 supported on $\pi_{2,\infty}^{-1}(\overline{\pi_{2,\infty}(\Omega)})$. Proceeding thus we will get a sequence $\mu_{n,1}, \mu_{n,2}, \ldots$, which is a subsequence of $\mu_{n-1,1}, \mu_{n-1,2}, \ldots$ such that $\mu_{n,k}|_{\mathcal{C}_n}, k = 1, 2, \ldots$ converges weakly to a measure ν_n supported on $\pi_{n,\infty}^{-1}(\overline{(\pi_{n,\infty}(\Omega)})$. Following Cantor we consider the diagonal sequence $\mu_{k,k}, k = 1, 2, \ldots$, and observe that $\mu_{k,k}|_{\mathcal{C}_n}, k = n, n+1, \ldots$, being a subsequence of $\mu_{n,k}, k = 1, 2, \ldots$, converges weakly to ν_n which is supported on $\pi_{n,\infty}^{-1}(\overline{\pi_{n,\infty}(\Omega)})$. Clearly $\nu_l = \nu_n \circ \pi_{l,n}^{-1}$ for $l \leq n$. The measures $\nu_n, n = 1, 2, \ldots$ therefore form a consistent family of measures on $\mathbb{R}^T$. By Kolmogorov consistency theorem there is a unique measure ν_∞ on Borel subsets of $\mathbb{R}^T$ such that for each n, $\nu_\infty \circ \pi_{n,\infty}^{-1} = \nu_n$. Clearly, ν_∞ is supported on each of the sets $\pi_{n,\infty}^{-1}(\overline{\pi_{n,\infty}(\Omega)})$, hence on their intersection

$$\cap_{n=1}^{\infty} \pi_{n,\infty}^{-1}(\overline{\pi_{n,\infty}(\Omega)}).$$

Since Ω is assumed to be closed, by Theorem 2.3, the intersection is Ω. By Theorem 6.1, for each n the functions $f_t, t \in T_n$, form a martingale with respect to ν_n. Clearly ν_∞ is a measure on Ω under which $f_t, t \in T$, form a martingale. We have proved:

Theorem 7.1. *Let Ω be a closed subset of $\mathbb{R}^T$, where T is a countable subset of* $[0, \infty)$, *not necessarily discrete. Assume that $f_t, t \in T$, the co-ordinate projections in $\mathbb{R}^T$ restricted to Ω form a measure free martingale. If each $f_t, t \in T$, is bounded, more generally, if the system $f_t, t \in T$, is nice* (*see Definition* 4.1), *then there is a probability measure μ on Ω under which the measure free martingale $f_t, t \in T$, is a martingale.*

This proves the remaining part of the theorem stated in the Introduction.

References

[1] Athreya K B and Nadkarni M G, *Entropy and Martingale*, Perspectives in Mathematical Sciences I, Probability and Mathematics, edited by N S Narasimha Sastry (World Scientific) (2009)

[2] Dellacherie C and Mayer P A, Probabilities and Potential B, Mathematics Studies, North Holland (1982) p. 67

[3] Doob J L, Stochastic Processes (New York: John Wiley) (1953) p. 629, p. 361

[4] Karandikar Rajeeva L and Nadkarni M G, Measure free martingales, *Proc. Ind. Acad. Sci. (Math. Sci.)* **15(1)** (2005) 111–116

[5] Parthasarathy K R, Probability measures on metric spaces (Academic Press) (1967)

[6] Srivastava S M, A course on Borel sets, Graduate Texts in Mathematics, 180 (New York: Springer-Verlag) (1998)