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On the existence of automorphisms with simple Lebesgue spectrum

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Abstract. It is shown that if T is a measure preserving automorphism on a probability space $(\Omega, \mathcal{B}, m)$ which admits a random variable X_0 with mean zero such that the stochastic sequence $X_0 \circ T^n, n \in \mathbb{Z}$ is orthonormal and spans $L_0^2(\Omega, \mathcal{B}, m)$, then for any integer $k \neq 0$, the random variables $X \circ T^{nk}, n \in \mathbb{Z}$ generate $\mathcal{B}$ modulo m.

Keywords. Dynamical system; simple Lebesgue spectrum; Walsh function.

1. Introduction

The purpose of this note is to show that if there is an automorphism with simple Lebesgue spectrum, then the probability measure of the corresponding stationary processes must be supported in a small subset of $\mathbb{C}^{\mathbb{Z}}$, in some precise sense explained below.

Let $\mathbb{C}$ denote the set of complex numbers with its Borel σ-algebra. Let $\Omega_0 = \mathbb{C}^{\mathbb{Z}}$ with the product σ-algebra $\mathcal{B}_0$ and let T be the left shift

$$T(\omega_n)_{n=-\infty}^{\infty} = (y_n)_{n=-\infty}^{\infty}, \quad y_n = \omega_{n+1}, \quad n \in \mathbb{Z}.$$

Suppose there exists a probability measure m on Ω_0 invariant under T and such that:

(i) the coordinate functions X_n, $n \in \mathbb{Z}$, $X_n(\omega) = \omega_n$, $\omega \in \Omega_0$, all have mean zero, they are square integrable and mutually orthogonal,

(ii) X_n, $n \in \mathbb{Z}$, span $L_0^2(\Omega_0, \mathcal{B}_0, m)$, the linear subspace of functions in $L^2(\Omega_0, \mathcal{B}_0, m)$ with mean zero.

Write:

$$X = \prod_{n=-\infty}^{\infty} \mathbb{C}_{2n+1}, \quad Y = \prod_{n=-\infty}^{\infty} \mathbb{C}_{2n}, \quad \mathbb{C}_{2n} = \mathbb{C}_{2n+1} = \mathbb{C},$$

and view Ω_0 as the product $X \times Y$. If $\omega = (\omega_n)_{n=-\infty}^{\infty} \in \Omega_0$, then

$$\zeta = (\omega_{2n+1})_{n=-\infty}^{\infty} \in X, \quad \eta = (\omega_{2n})_{n=-\infty}^{\infty} \in Y$$

and we identify (ζ, η) with ω.

Let m_1 and m_2 denote the projections of m on X and Y respectively. Define $T_1 : X \longrightarrow Y$, $T_2 : Y \longrightarrow X$ as follows:

$$T_1(\omega_{2n+1})_{n=-\infty}^{\infty} = (y_{2n})_{n=-\infty}^{\infty}, \quad y_{2n} = \omega_{2n+1},$$
$$T_2(\omega_{2n})_{n=-\infty}^{\infty} = (y_{2n+1})_{n=-\infty}^{\infty}, \quad y_{2n+1} = \omega_{2n+2}.$$

Then

$$T(\zeta,\eta)=(T_2\eta, T_1\zeta); \quad (\zeta,\eta)\in X\times Y.$$

It is shown that m (if one such exists) is supported on a set S in Ω_0 such that the projection maps onto X and Y, when restricted to S, are one-to-one. It follows that T^2, $T_1\circ T_2$, $T_2\circ T_1$ are isomorphic and the σ-algebra generated by the variables X_{2n}, $n\in\mathbb{Z}$, is, modulo m-null sets, equal to the Borel σ-algebra of Ω_0.

For the case when $\Omega_0=\{-1,+1\}^{\mathbb{Z}}$, it is shown that such a measure, if it exists, admits a support which is wandering under the natural action of the subgroup of Ω_0 consisting of sequences $\omega_n, n\in\mathbb{Z}$, with all but finitely many $\omega_n=+1$.

2. Orthogonal decompositions

DEFINITION 2.1

Let $(X,\mathcal{B}_X),(Y,\mathcal{B}_Y)$ be standard Borel spaces in the sense that each is isomorphic to the unit interval equipped with its Borel σ-algebra. We say that a probability measure m on $(X\times Y,\mathcal{B}_X\otimes\mathcal{B}_Y)$ is *good*, if for every complex valued measurable function f on $X\times Y$ there exist complex valued measurable functions u on X and v on Y such that

$$f(x,y)=u(x)+v(y) \quad m\text{–a.e.} \tag{1}$$

The set of measure zero where (1) fails to hold may depend on f.

Let

$$L_0^2(X\times Y,m)=\{f\in L^2(X\times Y,m): E(f)=0\},$$

where $E(f)=\int_{X\times Y}f\,dm$, denotes the expected value of f. We say that m is *very good* if every function $f\in L_0^2(X\times Y,m)$ can be expressed in the form (1) with $u\in L_0^2(X,m_1)$ and $v\in L_0^2(Y,m_2)$ satisfying $E(u\cdot\bar{v})=0$, where m_1, m_2 denote the projections of m on X and Y respectively (also called *marginal measures*).

2.2. In this paper we will be concerned with very good measures although good measures seem relevant for study of measure preserving automorphism whose associated unitary operators have multiplicity one. Theorem 2.4 gives a necessary and sufficient condition on m under which it is a very good measure. The second half of the proof of this theorem is obtained with the assistance of the referee, improving an earlier weaker result.

DEFINITION 2.3

A measurable subset $S\subseteq X\times Y$ is called *a measurable couple*, if $S=G\cup H$, where G is the graph of a measurable function g defined on the measurable subset $A\subseteq X$ into Y, H is the graph of a measurable function h defined on the measurable subset $B\subseteq Y$ into X and

$$g(A)\cap B=h(B)\cap A=\emptyset.$$

Theorem 2.4. *A probability measure m on $(X\times Y,\mathcal{B}_X\otimes\mathcal{B}_Y)$ is very good if and only if it is supported on a measurable couple.*

Proof. Assume that the measure m is supported on a measurable couple $S=G\cup H$. Let $f\in L_0^2\ (X\times Y,\mathcal{B}_X\otimes\mathcal{B}_Y,m)$. Without loss of generality assume that $m(H)\neq 0$. Then, since m is supported on a measurable couple, $m(H) = m_1(X \setminus A) = m_2(B) \neq 0$. Since $E(f) = 0$, we see that $\int_G f + \int_H f = 0$. Write $\int_G f = a$. Define

$$u(x) = \begin{cases} f(x, g(x)) & \text{if } x \in A, \\ -\frac{a}{m_1(X \setminus A)} & \text{if } x \in X \setminus A, \end{cases}$$

$$v(y) = \begin{cases} f(h(y), y) + \frac{a}{m_1(X \setminus A)} & \text{if } y \in B, \\ 0 & \text{if } y \in Y \setminus B. \end{cases}$$

(In case $G = \emptyset$, $u(x) = 0$ for all $x \in X$.) We note that (1) holds for all $(x, y) \in G \cup H$ and

$$Eu = Ev = E(u \cdot \overline{v}) = 0.$$

Assume now that m is very good. Let Π_1, Π_2 denote the projections of $X \times Y$ onto X and Y respectively and let $\mathcal{B}_X, \mathcal{B}_Y$ denote also the σ-algebras $\Pi_1^{-1}(\mathcal{B}_X)$ and $\Pi_2^{-1}(\mathcal{B}_Y)$ respectively. Write

$$E^X f = E(f \mid \mathcal{B}_X), \; E^Y f = E(f \mid \mathcal{B}_Y).$$

If $f \in L^2(X \times Y, m)$ and $E^X f = 0$, then $f(x, y) = v(y)$ a.e.

To see this note that if $E^X(f) = 0$, then $E(f) = 0$, and since m is very good, we can write

$$f(x, y) = u(x) + v(y) \; m - \text{a.e.}$$

with

$$E(u) = E(v) = E(u \cdot \overline{v}) = 0.$$

Since $E^X(f) = 0$ and u is $\mathcal{B}_X$-measurable

$$E(\overline{u} \cdot f) = E(\overline{u} E^X(f)) = 0 = E(| u |^2) + E(\overline{u} \cdot v),$$

which implies that $u(x) = 0 \; m_1$ – almost everywhere and then $f(x, y) = v(y) \; m$ – a.e.. So f is $\mathcal{B}_Y$-measurable. Similarly, if $g \in L^2(X \times Y, m)$ satisfies $E^Y g = 0$, then $g(x, y) = u(x)$ m – almost everywhere.

If $f, g \in L^\infty(X \times Y, m)$ and $E^X(f) = E^Y(g) = 0$, then $f \cdot g = 0 \; m$ – a.e.

To see this, let $h = f \cdot g$. Since $E^Y g = 0$ and f is $\mathcal{B}_Y$-measurable we have

$$E(h) = E(f \cdot E^Y(g)) = 0.$$

Since h is bounded, it is square integrable. Since m is very good we can write

$$h(x, y) = t(x) + s(y) \; m - \text{a.e.}$$

with $E(t) = E(s) = E(t \cdot \overline{s}) = 0$.

Again, since $E^X(f) = 0$ and t, g are $\mathcal{B}_X$-measurable,

$$E(\overline{t} \cdot h) = E(\overline{t} \cdot g \cdot E^X f) = 0,$$

so that $E(| t |^2) = 0$. Similarly $E(| s |^2) = 0$ and $h(x, y) = 0 \; m$ – a.e.

We conclude that f vanishes where g does not and that g vanishes where f does not.

Now take $\phi \in L^\infty(Y, m_2)$ and $\psi \in L^\infty(X, m_1)$, identified with $\phi \circ \Pi_2$ and $\psi \circ \Pi_1$. Letting $f = \phi - E^X(\phi)$, $g = \psi - E^Y(\psi)$, it follows that there exists a measurable set K in $X \times Y$ such that

$$\phi(y) = (E^X \phi)(x) \text{ for } m - \text{almost all } (x, y) \in K,$$

$$\psi(x) = (E^Y \psi)(y) \text{ for } m - \text{almost all } (x, y) \in X \times Y \setminus K.$$

Since $m|_K$ is supported on $G=\{(x,y):\phi(y)=E^X\phi(x)\}$ and $m|_{X\times Y\setminus K}$ is supported on $H=\{(x,y):\psi(x)=E^Y\psi(y)\}$, if we choose for ϕ,ψ one-to-one Borel maps of Y and X onto $[0,1]$, then G and H are measurable graphs of functions defined on subsets of X and Y respectively. Thus m is supported by a union of measurable graphs.

Moreover, it is supported by a couple. Indeed, fix a one-to-one bounded function ψ and consider a sequence of bounded functions $\phi_n,\ n\in\mathbb{N}$, which is dense in $L^2(Y,m_2)$. Let K be the intersection of the sets K_n corresponding to the pair ϕ_n,ψ. Each $f_n=\phi_n-E^X\phi_n$ satisfies $E^Xf_n=0$ and thus is equal a.e. to a function of y. Hence $K_n=\{(x,y):f_n(y)=0\}$, $(\operatorname{mod} m)$ and $K=X\times B, (\operatorname{mod} m)$ with $B=\bigcap_{i=1}^{\infty}\{y:f_n(y)=0\}$. Since $\phi_n, n\in\mathbb{N}$, are dense in $L^2(Y,m_2)$,

$$\phi(y)=E^X(\phi)(x) \text{ for } m - \text{ almost all } (x,y)\in X\times B$$

holds for every $\phi\in L^2(Y,m_2)$.

Let then ϕ be the indicator function of $X\times B$. We find $1=E^X\phi\ m$ - a.e. on $X\times B$. Let $A=\{x\in X\colon (E^X\phi)(x)=1\}$. Since $E^X(\phi)=1$ on $X\times B$, we see that the part of m on $A\times Y$ is concentrated on $A\times B$, whence $m(A\times(Y\setminus B))=0$. Since for $(x,y)\in(X\setminus A)\times B$, $E^X\phi(x)\neq 1=\phi(x,y)$, we see that $m((X\setminus A)\times B)=0$. Since the restrictions of m to $A\times B(\subset K)$ and $(X\setminus A)\times(Y\setminus B)(\subset X\times Y\setminus K)$ are supported on graphs, the theorem follows.

3. Connection with dynamics, twisted joining

3.1. Let T_1 be a Borel isomorphism from X onto Y and let T_2 be a Borel isomorphism from Y onto X. Define the Borel automorphism T of $X\times Y$ by

$$T(x,y)=(T_2y,T_1x),\quad (x,y)\in X\times Y.$$

We assume that T preserves the measure m on $\mathcal{B}_X\otimes\mathcal{B}_Y$ and that T^2 is ergodic. It follows that $m_2\circ T_1=m_1$ and $m_1\circ T_2=m_2$. Further $T_2\circ T_1:X\longrightarrow X$ preserves the measure m_1 and $T_1\circ T_2:Y\longrightarrow Y$ preserves the measure m_2.

It is obvious that

$$T^2(x,y)=(T_2\circ T_1(x),T_1\circ T_2(y)),\quad (x,y)\in X\times Y,$$

and the automorphisms $T,\ T_1\circ T_2,\ T_2\circ T_1$ are ergodic on the respective spaces.

Theorem 3.2. *Let $(X,\mathcal{B}_X)$ and $(Y,\mathcal{B}_Y)$ be standard Borel spaces in the sense that each is Borel isomorphic to the unit interval with its Borel σ-algebra. Let $(\Omega,\mathcal{B})=(X\times Y, \mathcal{B}_X\otimes\mathcal{B}_Y)$ and m be a probability measure on $\mathcal{B}$. Let $T_1:X\longrightarrow Y$, $T_2:Y\longrightarrow X$ be Borel automorphisms such that $T:(x,y)\longrightarrow(T_2y,T_1x)$ is measure preserving and T^2 is ergodic.*

Assume that m is very good. Then m is supported on the graph of a one-to-one measurable function on a subset of X (and hence also on a subset of Y). The automorphisms T^2, $T_1\circ T_2$ and $T_2\circ T_1$ are isomorphic.

Proof. Since m is very good, it is supported on a measurable couple, say $S=G\cup H$, where G is the graph of a measurable function g defined on a measurable subset $A\subset X$, and H is the graph of a measurable function h defined on a measurable subset $B\subset Y$. Without loss of generality assume that $m(G)>0$. Let $P_x(\cdot), x\in X$, denote the regular conditional probability measure with respect the σ-algebra $\Pi_1^{-1}(\mathcal{B}_X)$, also written as $\mathcal{B}_X$ (by abuse of notation). For each $x\in X$, $P_x(\cdot)$ is a probability measure supported on

$\{x\} \times Y$, such that for any $A \in \mathcal{B}_X \otimes \mathcal{B}_Y$, $P_{(\cdot)}(A)$ is $\mathcal{B}_X$-measurable and

$$m(A) = \int_X P_x(A)\, \mathrm{d}m_1.$$

From the construction of $P_x(\cdot), x \in X$, it is easy to see that the $G_1 = \{(x,y): P_x\{(x,y)\} = 1\}$ is T^2 invariant. It is also the graph of a measurable function on $\Pi_1 G_1$. Clearly, $G \subset G_1$ and $m(G) > 0$ by assumption, so, by ergodicity of T^2, $m(G_1) = 1$. Moreover $m(TG_1) = 1$ and TG_1 is the graph of a measurable function on a measurable subset Y. Clearly m is supported on $G_1 \cap TG_1$, the graph of a one-to-one measurable function on a measurable subset of X (hence also the graph of a measurable function on measurable subset of Y). The projection map Π_1 is a measure preserving isomorphism of $(X \times Y, \mathcal{B}_X \otimes \mathcal{B}_Y, m)$ and $(X, \mathcal{B}_X, m_1)$. Further $T^2 = \Pi_1^{-1} \circ T_2 \circ T_1 \circ \Pi_1$. Similarly, T^2 and $T_1 \circ T_2$ are isomorphic. The theorem is proved.

4. Application to the problem of simple Lebesgue spectrum

4.1. Now we apply Theorem 3.2 to the problem stated in the introduction. With the notation therein, every function in $L_0^2(\Omega_0, m)$ is an orthogonal sum of functions in $L_0^2(X, m_1)$ and $L_0^2(Y, m_2)$. Indeed, if $f \in L_0^2(\Omega_0, m)$ has the expansion $f = \sum_{n=-\infty}^{\infty} c_n X_n$, then we can set $u = \sum_{n=-\infty}^{\infty} c_{2n+1} X_{2n+1}$ and $v = \sum_{n=-\infty}^{\infty} c_{2n} X_{2n}$. So m is very good. By Theorem 3.2 m is supported on a set S in Ω_0 on which the projections maps in X and Y are one-to-one and so T^2, $T_1 \circ T_2$, $T_2 \circ T_1$ are isomorphic.

F Parreau has asked if σ-algebra generated by $X_{kn}, n \in \mathbb{Z}$, is equal to the Borel σ-algebra of Ω_0 (modulo m-null sets). This is indeed the case, as Theorem 3.2 has the following generalisation (Theorem 4.3), proved by a similar method.

4.2. Let $(X_i, \mathcal{B}_i)$, $1 \le i \le k$ be, as before, standard Borel spaces. Let

$$\Omega_0 = \prod_{i=1}^{k} X_i, \quad \mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \cdots \otimes \mathcal{B}_k.$$

Call a probability measure m on $\mathcal{B}$ k – *very good*, if every $f \in L_0^2(X, m)$ can be written in the form

$$f(x_1, x_2, \ldots, x_k) = u_1(x_1) + u_2(x_2) + \cdots + u_k(x_k)$$

with $E(u_i) = 0$ for all $1 \le i \le k$ and $E(u_i \cdot \overline{u_j}) = 0$ for all $i \neq j$.

Let $T_i : X_i \longleftrightarrow X_{i+1}$, $1 \le i \le k$, be Borel automorphisms, where $X_{k+1} = X_1$. Define T on X by

$$T(x_1, x_2, \ldots, x_k) = (T_k x_k, T_1 x_1, T_2 x_2, \ldots, T_{k-1} x_{k-1}).$$

Theorem 4.3. *Assume that m is a T-invariant and k-very good probability measure on $\mathcal{B}$ such that T^k is ergodic with respect to m. Then m is supported on a set $S \subset \Omega_0$ on which the projection maps $\Pi_1, \Pi_2, \ldots, \Pi_k$ into $X_1, X_2, \ldots, X_k$ respectively are one-to-one.*

Remark 4.4. It is also clear that if A and B are disjoint subsets of $\mathbb{Z}$, $A \cup B = \mathbb{Z}$ and if $X = \prod_{i \in A} \mathbb{C}_i$, $Y = \prod_{i \in B} \mathbb{C}_i$, where $\mathbb{C}_i = \mathbb{C}$ for all $i \in \mathbb{Z}$, then m, if one such exists, is supported on a measurable couple in $X \times Y$. It seems plausible that such an m is supported on a set in Ω_0 on which projections into the coordinate spaces is one-to-one, in which case the projection of m on any of the coordinate spaces of Ω_0 cannot be discrete, or even admit a discrete component.

5. Walsh functions

5.1. Let $(\Omega, \mathcal{F}, P)$ be a non-atomic probability space. Does there exist a one-to-one and onto measure preserving transformation $T : \Omega \longrightarrow \Omega$ and an event $A \in \mathcal{F}$, $P(A) = \frac{1}{2}$, such that if

$$X(\omega) = \begin{cases} +1 & \text{if } \omega \in A, \\ -1 & \text{if } \omega \in \Omega \setminus A, \end{cases}$$

then the random variables $X \circ T^n$, $n \in \mathbb{Z}$, are pairwise independent and span $L_0^2(\Omega, \mathcal{F}, P)$?

5.2. Let us reformulate the above question differently breaking it into two parts.

Let $\tilde{\Omega} = \prod_{k \in \mathbb{Z}} \{-1, +1\}_k$, $\{-1, +1\}_k = \{-1, +1\}$, equipped with the usual product topology and the resulting Borel structure $\tilde{\mathcal{B}}$; an element $\tilde{\omega} \in \tilde{\Omega}$ is a bilateral sequence $\{\omega_k\}_{k \in \mathbb{Z}}$ of $+1$ and -1.

Does there exist a probability measure μ on $\tilde{\mathcal{B}}$ such that:

(i) the coordinates X_k, $k \in \mathbb{Z}$, are pairwise independent and they span $L_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu)$?
(ii) moreover, can we choose the measure μ to be invariant under the left shift in $\tilde{\Omega}$?

5.3. The first question has a positive answer provided by the family of Walsh functions defined below.

Expand a real number $x \in [0, 1]$ in its binary form $x = 0.x_1 x_2 \ldots x_k \ldots$, which is made unique by insisting that if there are two such expansions, we choose the one with infinitely many ones. Define

$$R_k(x) = 2x_k - 1, \; k \in \mathbb{Z},$$

equivalently

$$R_k(x) = \begin{cases} +1 & \text{if } x_k = 1, \\ -1 & \text{if } x_k = 0. \end{cases}$$

These are called *Rademacher functions*. They are independent, and since $\int_0^1 R_k(x) \mathrm{d}x = 0$, they are also orthogonal, but they do not span $L_0^2[0, 1]$. However, the collection of all distinct finite products

$$W_{i_1, i_2, \ldots, i_k} = R_{i_1} R_{i_2} \cdots R_{i_k}; \quad i_1 < i_2 < \cdots < i_k,$$

called the *Walsh functions*, is mutually orthogonal and span $L_0^2[0, 1]$. Since they assume only two distinct values, they are also pairwise independent.

5.4. There is another way of viewing Walsh functions. Consider $\tilde{\Omega}$ as a compact group with coordinatewise multiplication and let h denote the normalised Haar measure on $\tilde{\Omega}$. If to each coordinate space $\{-1, +1\}$ we give uniform probability distribution, then h is the product of these measures. With respect to the measure h, the coordinate functions X_k, $k \in \mathbb{Z}$, correspond to the Rademacher functions.

The finite products $X_{i_1} X_{i_2} \cdots X_{i_k}$, $i_1 < i_2 < \cdots < i_k$, which is the collection of non-trivial continuous characters of $\tilde{\Omega}$, correspond to the Walsh functions.

Let $f_k,\ k \in \mathbb{Z}$, be an enumeration of Walsh functions $W_{i_1,i_2,\ldots,i_k}$ on $[0,1]$. Write:

$$\psi(x) = \{f_k(x)\}_{k\in\mathbb{Z}},\ x \in [0,1].$$

The unit interval is mapped by ψ in a one-to-one Borel manner into $\tilde{\Omega}$. Let $\mu_W(A) = \lambda \circ \psi^{-1}(A),\ A \in \tilde{\mathcal{B}}$, where λ denotes the Lebesgue measure on $[0,1]$. The coordinate functions $X_k,\ k \in \mathbb{Z}$, are pairwise independent and span $\mathcal{L}_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu_W)$. This gives the affirmative answer to the first question. We shall call μ_W the measure induced by an enumeration of Walsh functions. It was pointed out by F Parreau to one of us that such a μ_W is not invariant under the shift. In the sequel we will give a description of μ_W, from which this will follow.

5.5. The second question remains unsolved. A positive answer to it will solve the problem of the simple Lebesgue spectrum affirmatively. At this point we mention that an example (or rather a family of examples) of mixing rank one transformation, due to D Ornstein ([1]), allows us to construct a strictly stationary processes $\{f_k\}_{k\in\mathbb{Z}}$ such that $\int_{\tilde{\Omega}} f_k f_0 dm \longrightarrow 0$ as $k \to \infty$, while $\{f_k;\ k \in \mathbb{Z}\}$ span $L_0^2(\tilde{\Omega}, \mathcal{F}, m)$. Ornstein's example is deep and has not so far been modified or improved to yield a transformation with simple Lebesgue spectrum.

5.6. Let $\tilde{\Omega}_0$ be the subset of $\tilde{\Omega}$ consisting of those $\omega \in \tilde{\Omega}$, which have only finitely many -1; it is a countable dense subgroup of $\tilde{\Omega}$. The action of $\tilde{\Omega}_0$ on $\tilde{\Omega}$, $\omega \longrightarrow \omega\omega_0,\ \omega \in \tilde{\Omega}$, $\omega_0 \in \tilde{\Omega}_0$, is uniquely ergodic, the Haar measure h being the only probability measure invariant under the $\tilde{\Omega}_0$ action. Other product measures are quasi-invariant and ergodic under this action and there are many other measures with respect to which this action is non-singular and ergodic. The theorem below shows that all these measures are singular to any measure μ for which the coordinate functions are orthogonal and span $L_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu)$.

Theorem 5.7. *If μ is a probability measure on $\tilde{\Omega}$ such that the coordinate functions are pairwise independent and span $L_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu)$, then there is a Borel set E which supports μ and which is wandering under the $\tilde{\Omega}_0$ action, i.e., $\omega_0 E, \omega_0 \in \tilde{\Omega}_0$, are pairwise disjoint. In the case when μ is given by an enumeration of Walsh functions, μ is the Haar measure on a closed subgroup of $\tilde{\Omega}$.*

Proof. We begin with an observation of J B Robertson [2]. If the coordinate functions $X_k,\ k \in \mathbb{Z}$, are pairwise independent, then they span $L_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu)$ if and only if for all i, j,

$$X_iX_j = \sum_{k=-\infty}^{\infty} c_k^{i,j} X_k, \qquad \sum_{k=-\infty}^{\infty} | c_k^{i,j} |^2 = 1.$$

Further

$$c_k^{i,j} = \int_{\tilde{\Omega}} X_iX_jX_k \, d\mu \longrightarrow 0,$$

if any one of i, j, k, tends to $+\infty$.

(Note that X_iX_j is of absolute value one, hence its L^2-norm is 1, and the sum over k of the squares of $c_k^{i,j}$ is one.)

The sum $\sum_{k=-\infty}^{\infty} c_k^{i,j} X_k$, converges in L^2, hence (by diagonal method) there exists an increasing sequence $N_l,\ l \in \mathbb{N}$, of natural numbers such that for all i, j,

$$X_iX_j(\omega) = \lim_{l\to\infty} \sum_{k=-N_l}^{N_l} c_k^{i,j} X_k(\omega)$$

for almost all $\omega \in \tilde{\Omega}$ with respect to μ. Let

$$E_{i,j} = \left\{\omega : X_iX_j(\omega) = \lim_{l\longrightarrow\infty} \sum_{k=-N_l}^{N_l} c_k^{i,j} X_k(\omega)\right\},$$

$$E = \bigcap_{-\infty<i,j<\infty} E_{i,j},$$

which is a support of μ. Let $\omega_0 \in \tilde{\Omega}_0$ with -1 at places $i_1, i_2, \dots, i_p$ and $+1$ at the remaining places. Take an $i \notin \{i_1, i_2, \dots, i_p\}$ and $j \in \{i_1, i_2, \dots, i_p\}$, then for all $\omega \in E$

$$X_iX_j(\omega) = c_{i_1}^{i,j} X_{i_1}(\omega) + \dots + c_{i_p}^{i,j} X_{i_p}(\omega) + \lim_{l\to\infty} \sum_{-N_l\le k\le N_l}^{*} c_k^{i,j} X_k(\omega),$$

where $\sum^*$ indicates that the terms $c_{i_1}^{i,j} X_{i_1}, \dots, c_{i_p}^{i,j} X_{i_p}$ are deleted from the sum. Assume that for some $\omega_0 \in \tilde{\Omega}_0$, $\omega_0 E \cap E \neq \emptyset$. Then there exists $\omega = \{\omega_k\}_{k\in\mathbb{Z}} \in E$, such that $\omega_0\omega \in E$. We have

$$X_i(\omega) = \omega_i, \quad X_i(\omega_0\omega) = \omega_i, \quad X_j(\omega) = \omega_j, \quad X_j(\omega_0\omega) = -\omega_j,$$

so that

$$\omega_i\omega_j = c_{i_1}^{i,j}\omega_{i_1} + \dots + c_{i_p}^{i,j}\omega_{i_p} + \lim_{l\to\infty} \sum_{-N_l\le k\le N_l}^{*} c_k^{i,j}\omega_k,$$

$$-\omega_i\omega_j = -c_{i_1}^{i,j}\omega_{i_1} - \dots - c_{i_p}^{i,j}\omega_{i_p} + \lim_{l\to\infty} \sum_{-N_l\le k\le N_l}^{*} c_k^{i,j}\omega_k,$$

whence,

$$\omega_i\omega_j = c_{i_1}^{i,j}\omega_{i_1} + \dots + c_{i_p}^{i,j}\omega_{i_p}.$$

This holds for all $i \notin \{i_1, i_2, \dots, i_p\}$. Letting $i\longrightarrow\infty$, since $c_{i_1}^{i,j}, \dots, c_{i_p}^{i,j} \longrightarrow 0$, the right hand side of the above equality tends to zero, while the left hand side remains one in absolute value. The contradiction shows that $\omega_0\omega \notin E$, whence $\omega_0 E \cap E = \emptyset$.

Suppose now that μ_W is obtained by an enumeration of Walsh functions. In this case X_iX_j, $i \neq j$, is some X_l. Write $l = g(i,j)$. Then

$$X_iX_j = X_{g(i,j)},$$

so that in the expansion

$$X_iX_j = \sum_k c_k^{i,j} X_k$$

all $c_k^{i,j} = 0$ except for $k = g(i,j)$, in which case $c_k^{i,j} = 1$.

Now $E_{i,j} = \{\omega : \omega_i\omega_j = \omega_{g(i,j)}\}$. The sets $E_{i,j}$ are closed subgroups of $\tilde{\Omega}$. The same is true for the set $E = \bigcap_{-\infty<i,j<\infty} E_{i,j}$. The characters $X_{i_1}X_{i_2}, \dots, X_{i_k}$ of $\tilde{\Omega}$ are also the characters of E, but they need not be distinct characters. In particular X_iX_j and $X_{g(i,j)}$ agree on E. Further $X_{i_1}X_{i_2}, \dots, X_{i_k}$ is either equal to some X_p or equal to one. We have

$\int_E X_{i_1} X_{i_2}, \ldots, X_{i_k}\, d\mu_W$ equal to 0 in the first case and equal to 1 in the second, which also holds if μ_W is replaced by the normalised Haar measure on E. Thus μ_W is the normalised Haar measure on E and the theorem is proved.

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