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Measure free martingales

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Abstract. We give a necessary and sufficient condition on a sequence of functions on a set Ω under which there is a measure on Ω which renders the given sequence of functions a martingale. Further such a measure is unique if we impose a natural maximum entropy condition on the conditional probabilities.

Keywords. Martingale; Boltzmann distribution; asset pricing.

1. Introduction

The notion of measure free martingale is implicit in the construction of equivalent martingale measures in the theory of asset pricing in financial mathematics [1, 2], but it has not been fully isolated and made free of probability. Rather it has remained hidden by specific processes and terminology of asset pricing theory. We define a martingale purely in terms of sets and functions, called measure free martingale, and show that every martingale is a measure free martingale and conversely that every measure free martingale admits a probability measure, which may be finitely additive, under which it is a martingale. We describe the convex set (together with their extreme points) of all probability measures under which a measure free martingale is a martingale. Among these measures there is one which in some sense is most symmetric or most well spread, and entirely determined by the measure free martingale. Boltzmann's entropy maximizing distribution is needed here. To the best of our knowledge probabilist's have not asked the simple question as to when a sequence of function is a martingale under some measure. The answer is relatively easy but has some pedagogic as well as research value.

2. Means of finite set of points

Let $x_1, x_2, x_3, \ldots, x_k$ be k real numbers, with repetitions allowed. Assume that x_1 and x_k are respectively the smallest and the largest of $x_1, x_2, \ldots, x_k$. Let α be a real number. Then there exists a probability vector $(p_1, p_2, \ldots, p_k)$ such that

$$x_1 p_1 + x_2 p_2 + \cdots + x_k p_k = \alpha,$$

if and only if $x_1 \leq \alpha \leq x_k$. If $k = 2$ and $x_1 \neq x_2$, such a probability vector is unique. If $k > 2$, it is not unique without some additional requirements.

A result of Boltzmann proved using Lagrange's multipliers says that there is a unique probability vector $(p_1, p_2, \dots, p_k)$ which satisfies $x_1 p_1 + x_2 p_2 + \cdots + x_k p_k = \alpha$, and maximizes the entropy

$$-p_1 \log p_1 - p_2 \log p_2 - \cdots - p_k \log p_k.$$

It is given by

$$p_j = \frac{\exp(\lambda x_j)}{\sum_{i=1}^{k} \exp(\lambda x_i)}, \quad i = 1, 2, \dots, k,$$

where λ is a constant.

We will call these probabilities the Boltzmann probabilities for $x_1, x_2, \dots, x_k; \alpha$.

In this connection it should be noted that for a fixed $x_1, x_2, \dots, x_k$ and variable λ, the probabilities

$$p_i(\lambda) = \frac{\exp(\lambda x_i)}{\sum_{i=1}^{k} \exp(\lambda x_i)}, \quad i = 1, 2, \dots, k$$

of $x_1, x_2, \dots, x_k$ respectively have the mean $\sum_{i=1}^{k} x_i p_i(\lambda)$ which we denote by $m(\lambda)$. Since x_1 and x_k are minimum and maximum of $x_1, x_2, \dots, x_k$, we have

$$\lim_{\lambda \to -\infty} p_i(\lambda) = \frac{\delta_{1,i}}{n_i}, \quad \lim_{\lambda \to \infty} p_i(\lambda) = \frac{\delta_{k,i}}{n_i},$$

where n_i is the frequency of occurrence of x_i in $x_1, x_2, \dots, x_k$. As a consequence,

$$\lim_{\lambda \to -\infty} m(\lambda) = x_1, \quad \lim_{\lambda \to \infty} m(\lambda) = x_k.$$

A calculation shows that $\mathrm{d}m/\mathrm{d}\lambda = v(\lambda) > 0$, where $v(\lambda)$ is the variance of the system $x_1, x_2, \dots, x_k$ with probabilities $p_1(\lambda), p_2(\lambda), \dots, p_k(\lambda)$. Thus $m(\lambda)$ is a strictly increasing function of λ which assumes every value between x_1 and x_k. If $m(\lambda) = \alpha$, then $p_1(\lambda), p_2(\lambda), \dots, p_k(\lambda)$ are the probabilities which maximize the entropy for the constraint $\sum_{i=1}^{k} p_k x_k = \alpha$. (See [3], p. 172 for a related discussion of Boltzmann distribution in the continuous case.)

Suppose $x_1, x_2, \dots, x_k$ are distinct. The set C of probability vectors $(p_1, p_2, \dots, p_k)$ such that $\sum_{j=1}^{k} x_j p_j = \alpha$ is a convex set. It is easy to see that its extreme points are precisely those $(p_1, p_2, \dots, p_k) \in C$ which have at most two non-zero entries.

3. Measure free martingales

Let Ω be a non-empty set. Let $f_n, n = 1, 2, 3, \dots$ be a sequence of real valued functions such that each f_n has a finite range, say $(x_{n1}, x_{n2}, \dots, x_{nk_n})$, and these values are assumed on the subsets $\Omega_{n1}, \Omega_{n2}, \dots, \Omega_{nk_n}$. These sets form a partition of Ω which we denote by $\mathbb{P}_n$. We denote by $\mathbb{Q}_n$ the partition generated by $\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_n$ and the algebra generated by $\mathbb{Q}_n$ is denoted by $\mathcal{A}_n$. Let $\mathcal{A}_\infty$ denote the algebra $\cup_{n=1}^{\infty} \mathcal{A}_n$.

Define $\mathcal{A}_n$ measurable functions m_n, M_n as follows: For $Q \in \mathbb{Q}_n$ and $\omega \in Q$,

$$m_n(\omega) = \min_{q \in Q} f_{n+1}(q),$$

$$M_n(\omega) = \max_{q \in Q} f_{n+1}(q).$$

DEFINITION

The sequence $(f_n, \mathcal{A}_n)_{n=1}^{\infty}$ is said to be a measure free martingale or probability free martingale if

$$m_n(\omega) \leq f_n(\omega) \leq M_n(\omega) \quad \forall \omega \in \Omega,\ n \geq 1.$$

Clearly, for each $Q \in \mathbb{Q}_n$, the function f_n is constant on Q. We denote this constant by $f_n(Q)$. With this notation, it is easy to see that $(f_n, \mathcal{A}_n)_{n=1}^{\infty}$ is a measure free martingale or probability free martingale if and only if for each n and for each $Q \in \mathbb{Q}_n$, $f_n(Q)$ lies between the minimum and the maximum values of $f_{n+1}(Q')$ as Q' runs over $Q \cap \mathbb{Q}_{n+1}$.

It is easy to see that if there is a probability measure on $\mathcal{A}_\infty$ with respect to which $(f_n, \mathcal{A}_n)_{n=1}^{\infty}$ is a martingale, then $(f_n, \mathcal{A}_n)_{n=1}^{\infty}$ is also a measure free martingale. Indeed, let P be such a measure. Then, for any Q in $\mathbb{Q}_n$, $f_n(Q)$ is equal to

$$\frac{1}{P(Q)} \sum_{\{Q' \in \mathbb{Q}_{n+1}, Q' \subseteq Q\}} f_{n+1}(Q') P(Q'),$$

so that $f_n(Q)$ lies between the minimum and the maximum values $f_{n+1}(Q')$, $Q' \in Q \cap \mathbb{Q}_{n+1}$. The theorem below proves the converse.

Theorem 1. *Given a measure free martingale $(f_n, \mathcal{A}_n)_{n=1}^{\infty}$, there exists for each $n \geq 0$, a measure P_n on $\mathcal{A}_n$ such that*

$$P_{n+1}|_{\mathcal{A}_n} = P_n, \quad E_{n+1}(f_{n+1}|\mathcal{A}_n) = f_n,$$

where E_{n+1} denotes the conditional expectation with respect to the probability measure P_{n+1}. There is a finitely additive probability measure P on the algebra $\mathcal{A}_\infty$, which may be countably additive, such that for each n, $P|_{\mathcal{A}_n} = P_n$.

Proof. Define P_1 on $\mathcal{A}_1$ arbitrarily. Having defined $P_1, P_2, \ldots, P_n$ on $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ such that

$$P_j|\mathcal{A}_{j-1} = P_{j-1}, \quad E_j(f_j|\mathcal{A}_{j-1}) = f_{j-1}, \quad j = 2, 3, \ldots, n,$$

we define P_{n+1} on $\mathcal{A}_{n+1}$ as follows: Choose an element Q in $\mathbb{Q}_n$. Let $A_1, A_2, \ldots, A_l$ be the partition of Q induced by f_{n+1} so that f_{n+1} assumes l distinct values, say $a_1, a_2, \ldots, a_l$, on $A_1, A_2, \ldots, A_l$ respectively. Let $a = f_n(Q)$ (the value assumed by f_n on Q). Since $(f_n, \mathcal{A}_n)_{n=1}^{\infty}$ is a measure free martingale, a lies between the minimum and the maximum values of f_{n+1} on Q, so there is a probability vector $(p_1, p_2, \ldots, p_l)$ such that

$$a_1 p_1 + a_2 p_2 + \cdots + a_l p_l = a.$$

We define

$$P_{n+1}(Q_i) = p_i P_n(Q), \quad i = 1, 2, \ldots, l.$$

Carrying out this procedure for all $Q \in \mathbb{Q}_n$ we get a probability measure P_{n+1} on $\mathcal{A}_{n+1}$ for which it is easy to check that

$$P_{n+1}|\mathcal{A}_n = P_n, \quad E_{n+1}(f_{n+1}|\mathcal{A}_n) = f_n.$$

Induction completes the proof of the existence of the measures P_n. Define P by setting, for $A \in \mathcal{A}_\infty$, $P(A) = P_n(A)$, if $A \in \mathcal{A}_n$. Thus the theorem stands proved. □

Remarks. The measure P on $\mathcal{A}_\infty$ may be called a martingale measure associated to the measure free martingale $(f_n, \mathcal{A}_n)_{n=1}^\infty$. The totality of such measures forms a convex set whose extreme points are precisely those P which have the property that for any n and for any $Q \in \mathbb{Q}_n$, P (hence P_{n+1}) assigns positive probability to at most two elements in the partition of Q induced by f_{n+1}. If, for each n and for each $Q \in \mathbb{Q}_n$, $Q \cap \mathbb{Q}_{n+1}$ has two or less elements, then there is only one martingale measure for the measure free martingale $(f_n, \mathcal{A}_n)_{n=1}^\infty$.

Let Q be an element in $\mathbb{Q}_n$. If we assign Boltzmann probabilities of the values of f_{n+1} on Q to the corresponding elements of the partition of Q induced by f_{n+1}, then we have the following theorem.

Theorem 2. *Let $(f_n, \mathcal{A}_n)_{n=1}^\infty$ be a measure free martingale. Then there is a unique probability measure P on $\mathcal{A}_\infty$ such that*

(1) *$(f_n, \mathcal{A}_n)_{n=1}^\infty$ is a martingale with respect to P.*

(2) *For each n and for each $Q \in \mathbb{Q}_n$ if $Q_1, Q_2, \ldots, Q_l$ are the elements of $Q \cap \mathbb{Q}_{n+1}$, then $P(Q_1)/P(Q), P(Q_2)/P(Q), \ldots, P(Q_l)/P(Q)$ are the unique probabilities which maximize*

$$-\sum_{i=1}^{l} p_i \log p_i,$$

subject to the condition $\sum_{i=1}^{l} a_i p_i = a$, where a is the value of f_n on Q and $a_1, a_2, \ldots, a_l$ are the values assumed by f_{n+1} on Q.

(3) *The probabilities $P(Q_i), i = 1, 2, \ldots, l$ are given by the formula*:

$$P(Q_i) = P(Q) \cdot \frac{\exp(\lambda a_i)}{\sum_{i=1}^{l} \exp(\lambda a_i)},$$

where λ is a constant depending on $a, a_1, a_2, \ldots, a_l$.

In a certain sense this distribution P of Theorem 2 may be viewed as most symmetric or most well spread for the given measure free martingale. It is determined entirely by the measure free martingale. One may call P the Boltzmann measure associated to the measure free martingale $(f_n, \mathcal{A}_n)_{n=1}^\infty$, and the resulting measure theoretic martingale, the Boltzmann martingale.

In the theory of asset pricing in financial mathematics there is an important point of existence of equivalent martingale. Here, as a consequence of Theorem 2, we have the following:

COROLLARY

With the notation of Theorem 2 above, if m is a probability measure on $\mathcal{A}_\infty$ for which there exist two positive constants C and D such that for all $A \in \cup_{n=1}^\infty \mathbb{Q}_n$,

$$C \le m(A)/P(A) \le D,$$

then there is measure on $\mathcal{A}_\infty$, (e.g., P), which is equivalent to m and with respect to which $(f_n, \mathcal{A}_n)_{n=1}^\infty$ is a martingale. This martingale measure is unique, and equal to P, if we *require, for each n and for each $Q \in \mathbb{Q}_n$, the conditional distribution on $Q \cap \mathbb{Q}_{n+1}$ to have maximum entropy.*

A question arises. Note that we can associate the number λ to the set Q in Theorem 2. When we do this for all $Q \in \mathbb{Q}_n$ we have a function g_n defined on Ω. Is g_n, $n = 1, 2, 3, \ldots$ a measure free martingale?

Suppose Ω is a compact metric space and that sets in $\mathcal{A}_\infty$ form a clopen base for its topology. Then any martingale measure for the measure free martingale $(f_n, \mathcal{A}_n)_{n=1}^\infty$ extends to a countably additive measure on the Borel field $\mathcal{B}$ of Ω. The collection C of all martingale measures for $(f_n, \mathcal{A}_n)_{n=1}^\infty$ defined on $\mathcal{B}$ forms a compact convex set under weak topology, whose extreme points are already described above.

4. A result on convergence

Let Ω be a compact metric space and let $(f_n)_{n=1}^\infty$ be a sequence of continuous real valued functions on Ω. Let $\mathbb{Q}_n$ be the partition of Ω generated by $f_1, f_2, \ldots, f_n$. Elements of $\mathbb{Q}_n$ are closed sets. Say that $(f_n, \mathbb{Q}_n)_{n=1}^\infty$ is a martingale of continuous functions if for each n and for each $C \in \mathbb{Q}_n$ the value of f_n on C lies between the minimum and the maximum value of f_{n+1} on C. We have the following theorem.

Theorem 3. *If the martingale $(f_n, \mathbb{Q}_n)_{n=1}^\infty$ of continuous functions is also an equicontinuous sequence, i.e., the sequence of functions $(f_n)_{n=1}^\infty$ is equicontinuous, then $(f_n)_{n=1}^\infty$ converges pointwise.*

Proof. Let $\mathbb{Q}_\infty$ denote the common refinement of all the $\mathbb{Q}_n, n = 1, 2, \ldots$ and assume that $\mathbb{Q}_\infty$ is made of singleton sets. Let ω be a point of Ω and let C_n be the element of $\mathbb{Q}_n$ to which ω belongs. Then $\cap_{n=1}^\infty C_n = \{\omega\}$, and since C_n's are closed, we see that the diameter of C_n tends to zero as n tends to ∞. By martingale and equicontinuity property of the sequence $(f_n)_{n=1}^\infty$ we conclude that given any $\epsilon > 0$ there is an n_0 such that for $n \geq n_0$, $|f_n(\omega) - f_{n_0}(\omega)| < \epsilon$. So $(f_n)_{n=1}^\infty$ converges pointwise.

If $\mathbb{Q}_\infty$ is not made of singletons, then we consider $\overline{\Omega} = \Omega/\mathbb{Q}_\infty$ equipped with the quotient topology. Define for $c \in \mathbb{Q}_\infty$, $\overline{f}_n(c) =$ the constant value of f_n on c. We can view $\mathbb{Q}_n$ also as a partition of $\overline{\Omega}$. The sequence $(\overline{f}_n, \mathbb{Q}_n)_{n=1}^\infty$ forms a martingale of continuous functions on the compact set $\overline{\Omega}$ and the functions $\overline{f}_n, n = 1, 2, \ldots$ form an equicontinuous sequence of functions. The common refinement $\mathbb{Q}_\infty$ of the partitions $\mathbb{Q}_n, n = 1, 2, \ldots$ when considered as partition of $\overline{\Omega}$ is the partition of $\overline{\Omega}$ into singleton sets. By considerations of the previous paragraph we see that the sequence $(\overline{f}_n)_{n=1}^\infty$ converges pointwise, whence the sequence $(f_n)_{n=1}^\infty$ converges pointwise. The theorem is proved. □

We conclude by raising a question about Boltzmann distribution. Let C be a compact subset of the real line and let α be strictly between maximum and minimum points of C. Let $x_1, x_2, \ldots, x_{k_\epsilon}$ be an ϵ-net in C. Let μ_ϵ denote the Boltzmann distribution on this ϵ-net and α. Can one say that μ_ϵ converges weakly to a unique probability measure on C as $\epsilon \to 0$, independent of the choice of the ϵ-nets?

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