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# Experimental investigation of the response of a harmonically excited hard Duffing oscillator

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**Abstract.** A single degree-of-freedom torsional vibratory system, which constitutes a third-order dissipative dynamical system, has been fabricated as a mechanical analogue of hard Duffing equation with strong nonlinearity. The forced response of the system reveals complicated and chaotic motion at low frequency regime. Besides usual jump phenomenon, unpredictable jump phenomenon with two and three coexisting periodic attractors is also observed.

**Keywords.** Duffing oscillator; chaos.

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## 1. Introduction

The equation of motion of a harmonically excited and viscously damped Duffing oscillator is given by

$$\ddot{x} + c\dot{x} + \alpha x + \beta x^3 = F\cos\omega t; \quad c > 0. \tag{1}$$

The approximate periodic response for the above equation, obtained by various analytical methods, is discussed in almost all textbooks on nonlinear vibration.

If the parameters $\alpha$ and $\beta$ are of opposite signs, then the corresponding undamped and unforced system has three equilibrium states at $x = 0$ and $x = \pm\sqrt{-\beta/\alpha}$. Complicated and chaotic dynamics of such systems possessing heteroclinic and homoclinic orbits have received a lot of attention and have been studied, analytically, numerically and experimentally. In this paper, we refer only to some of the experimental works. With $\alpha < 0$ and $\beta > 0$, i.e. with a double-well potential, famous work of Moon and Holmes [1] revealed chaotic motion of the tip of a cantilever. Large oscillation of a pendulum can be approximated by $\alpha > 0$ and $\beta < 0$. Experimentally obtained chaotic motions of a pendulum were reported by Baker and Gollub [2] and Hatwal *et al* [3].

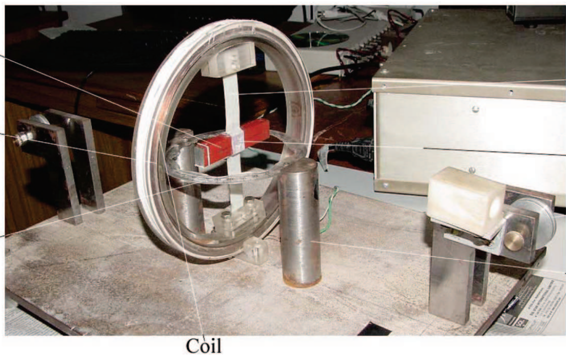


**Figure 1.** Experimental set-up.

However, with both $\alpha$ and $\beta$ as positive, the system has only one equilibrium position in the undamped and unforced state. A very high level of excitation is required for such a system to exhibit chaotic response. The practical difficulty of reaching such a high level of excitation, for these 'hardening' systems to exhibit chaotic response, has been explicitly mentioned by Virgin [4].

The objective of the present paper is to report on chaotic response obtained experimentally for a hardening system. The design of a torsional oscillator obeying eq. (1) with $\alpha$ and $\beta$ as positive is reported. Besides the usual jump phenomenon, chaotic response is observed. If the forcing frequency is around three times the linearised natural frequency of the system, then the response reveals unpredictable jumps between two and even three possible attractors. In such a situation, the response does not reach a steady state during experimentation. Numerical results confirm the above-mentioned experimental findings.

## 2. Experimental set-up

Figure 1 shows the experimental set-up for a torsional oscillatory system. This system is designed following the idea of Pippard [5]. A permanent magnet is held in the middle by two thin mylar strips pasted securely onto the magnet. The ends of the strips are clamped to a perspex ring onto which a coil of several hundred turns is wound. For rotational motion of the magnet about its central vertical axis, mylar strips act as a nonlinear torsional spring. The torsional stiffness characteristic of the system is measured by applying equal tensions near the ends of the magnet using pulleys and dead weights.

The torque ($T_s$) about the central vertical axis of the magnet can be computed if the weights and the distances involved (moment arms) are known [6]. It may be pointed out that the moment arm of the torque due to tension in the string is dependent on the rotation ($\theta$) of the magnet. It can be shown that the expression for the moment arm is $[a(l\cos\theta - a\sin\theta/\sqrt{(1-a\sin\theta)^2+(a-a\cos\theta)^2}]$, where $a$ is the distance of the string attachment point from the axis of rotation along the length of the magnet and $l$ is the distance of the top of the pulley from the string attachment point when $\theta = 0$.

High values of $\theta$ are measured by the protractor (figure 1) with a least count of 0.5°. For measuring low values of $\theta$, a small mirror is attached to the side of the magnet (figure 1). Such values of $\theta$ are obtained by measuring the deflection of a reflected laser pointer incident on the mirror (at the center of the side of the magnet). This system has a least count of 0.1°. Small frictional torque at the pulley axles is estimated by assuming it to be constant and a linear relationship between the torque and rotation for very small values of $\theta$ [6]. All the torque readings were corrected by subtracting this frictional torque and plotted vs. $\theta$. Fitting the data of this plot as

$$T_s = k_1\theta + k_2\theta^3 \tag{2}$$

the following values are obtained:

$$k_1 = 60.14 \text{ N·mm/rad}$$
$$k_2 = 62.46 \text{ N·mm/rad}^3.$$

A small B&K 4374 accelerometer was attached near the end of the magnet. The accelerometer signal was fed to CF 3200 ONO SOKKI FFT analyzer via a B&K 2635 charge amplifier. Assuming linear viscous damping, the damping factor of the system, measured from the decay rate of free vibration, is estimated to lie between 0.014 and 0.016.

Current sent to the coil wound on the ring provides harmonic excitation to the torsional system. A Philips PM 5132 (0.1 Hz–2 MHz) signal generator and a MB Dynamics SL 6000 VCF power amplifier were used for this purpose. The amplitude ($\Gamma$) and frequency ($\omega$) of the excitation torque are controlled by setting the gains and frequency accordingly in these two instruments. The steady state peak-to-peak response and the associated power spectrum are obtained from the FFT analyzer.

The equation of motion of the driven oscillator, with the measured values of the system parameters, can now be written as

$$\Theta'' + 0.03\Theta' + \Theta + \Theta^3 = T_1\cos(\omega_1\tau), \tag{3}$$

where the prime denotes differentiation with respect to $\tau$ and the other symbols are defined as given below:

$$\Theta = \sqrt{\beta/\alpha}\theta, \quad T_1 = (\sqrt{\beta/\alpha})(T/\alpha), \quad \omega_1 = \omega/\sqrt{\alpha}, \quad \tau = \sqrt{\alpha}t,$$
$$\alpha = k_1/J, \quad \beta = k_2/J, \quad T = \Gamma/J$$

with $J$ representing the centroidal moment of inertia of the bar magnet about the axis of rotation.

To confirm the experimentally obtained features of the response, numerical results [6] are obtained by integrating eq. (3) using a standard solver ode45 in MATLAB 6.5.

## 3. Results and discussion

Experiments were conducted with increasing and decreasing excitation frequency keeping the amplitude of excitation constant. The plot of peak-to-peak response

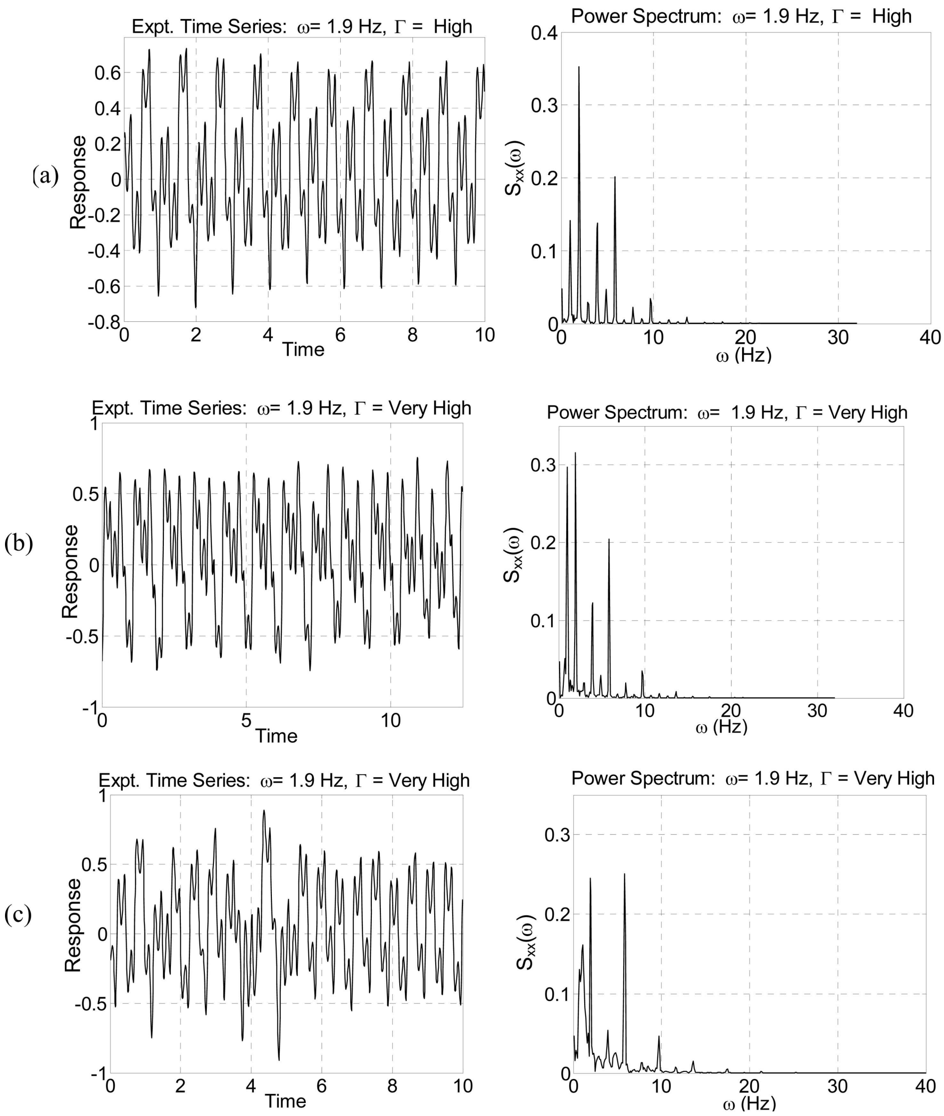


**Figure 2.** Chaotic response obtained experimentally at 1.9 Hz with increasing level of excitation.

versus frequency of excitation, for three different levels of excitation, clearly revealed the usual jump phenomenon. The size of the hysteresis loop, while jumping to and from the resonance branch, was seen to increase with increasing level of excitation. It may be pointed out that with the level of excitation below a critical limit, no jump was seen in the response.

The peak-to-peak response in the low frequency regime exhibited non-smooth variation of the response curve with increasing level of excitation. This irregular

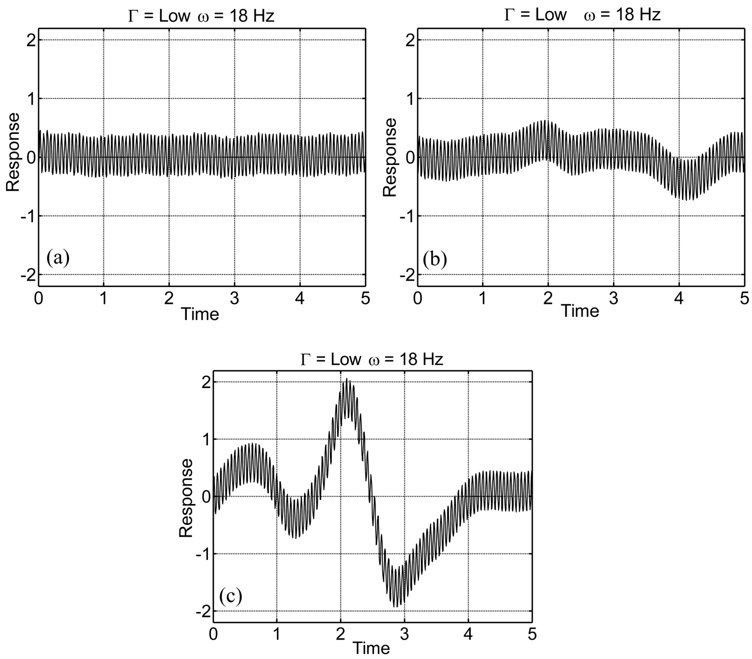


**Figure 3.** Experimentally obtained response at 18 Hz showing unpredictable jumps preventing attainment of steady state.

behavior may be attributed to the complex interaction of higher order unstable regions of the harmonic response. The non-smooth variation of the peak-to-peak response with frequency, obtained experimentally, is confirmed by numerical simulation results [6].

Now the results at a particular frequency in this low frequency regime are discussed. Figures 2a–2c show (i) the time response and (ii) the corresponding power spectrum obtained experimentally at 1.9 Hz for three different levels of excitations. Figure 2a shows a periodic response with a number of superharmonics. With increasing level of excitation, subharmonics start appearing (figure 2b) with consequent increase in the period. The appearance of distinct super- and sub-harmonics is clearly indicated by sharp peaks in the power spectrum. With a still higher level of excitation, all the low frequencies starting from zero appear in the response and the absence of any distinct sharp peaks in this low frequency regime confirms that the response is aperiodic or chaotic (figure 2c). Thus chaotic motion at this frequency is observed experimentally. Numerical simulation confirmed such chaotic response at 1.8 Hz with increasing level of excitation [7].

Experiments were also conducted at a frequency, which is close to three times the linearised natural frequency. It is well-known that at such frequencies, one-third subharmonic response can be sustained. With a high enough excitation, the normal jump from the resonance to the non-resonance branch can also be shifted to such a high frequency. All these can give rise to a complicated behavior. The linearised natural frequency of the system was found to be 5.7 Hz. Accordingly, experiments were conducted at 18 Hz. It was observed that the response never settles down to a steady state. Figures 3a–3c show three equal time windows within the same run of the experimentally obtained response. All these windows are in a time zone by when the response was supposed to settle down to a steady state. The response clearly depicts unpredictable jumps between different solutions resulting in the disappearance of the steady state. Of course, this response is non-chaotic.

In the frequency range 18.01–18.03 Hz, the response obtained by numerical integration showed intense sensitivity to small variation in frequency [7]. It was found that while at 18.03 Hz, the response has a single peak at the forcing frequency, at 18.02 Hz the response with the same initial conditions has a large one-third subharmonic peak ($n = 3$). The response at 18.01 Hz seems to consist almost entirely of one-half subharmonic ($n = 2$). The disappearance of steady state at 18 Hz, observed in the experimental response, may be attributed to this intense sensitivity of the response to frequency. During experimentation, there is inevitable small fluctuation in the forcing frequency.

## 4. Conclusions

The major conclusions of this paper are listed below:

(a) A torsional, hard Duffing oscillator with strong nonlinearity has been designed and fabricated.

(b) Non-smooth variation (with frequency) of the peak-to-peak periodic response in the low-frequency regime ultimately leading to chaotic motion of a hardening oscillator possessing one equilibrium point has been confirmed experimentally.

(c) Over and above the usual jump phenomenon, the response at excitation frequency nearing three times the linearised natural frequency exhibits unpredictable jumps between multiple periodic attractors. This, in turn, results in the disappearance of steady-state motion during experimentation.

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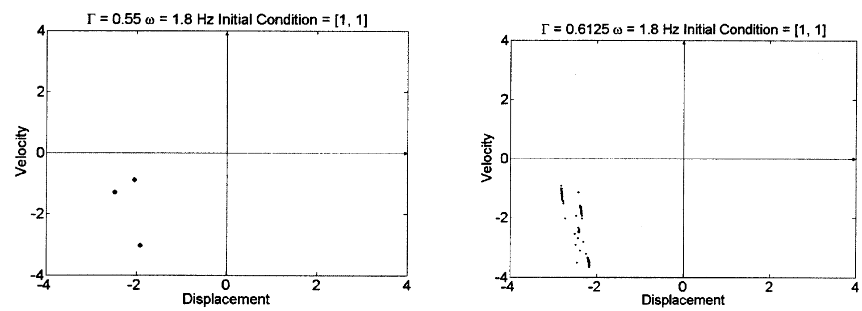
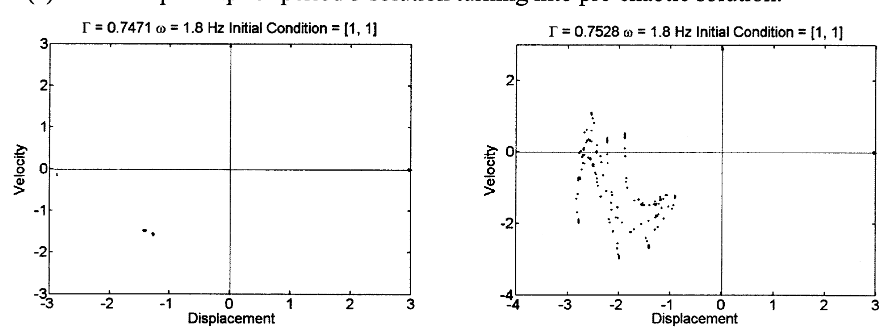
[7] Supplementary material of this paper is available through link in the web version. See http://www.ias.ac.in/pramana/v68/p.99/fulltext.pdf

## Supplementary figures

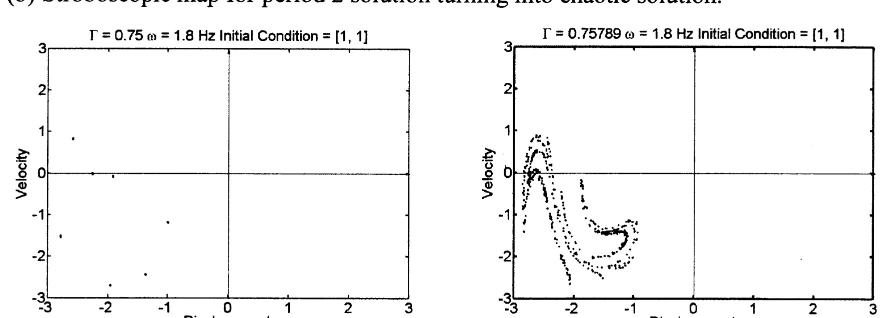
Figure S1a-S1c show stroboscopic maps of the response obtained numerically for different levels of excitation at 1.8 Hz. It may be mentioned that for $\Gamma$ in the range 0.7471 to 0.7578, with changing excitation level, the solution between the one having period 2 and the other having period 7 and eventually both turn chaotic. For some excitation level in this range, attractors having period 2 and period 7 are found to coexist (with different initial conditions). No period doubling route to chaos could be detected.




(a) Stroboscopic map for period 3 solution turning into pre-chaotic solution.




(b) Stroboscopic map for period 2 solution turning into chaotic solution.




(c) Stroboscopic map for period 7 solution turning into chaotic solution.

**Figure S1.** Nature of response at different levels of excitation obtained numerically at 1.8 Hz.

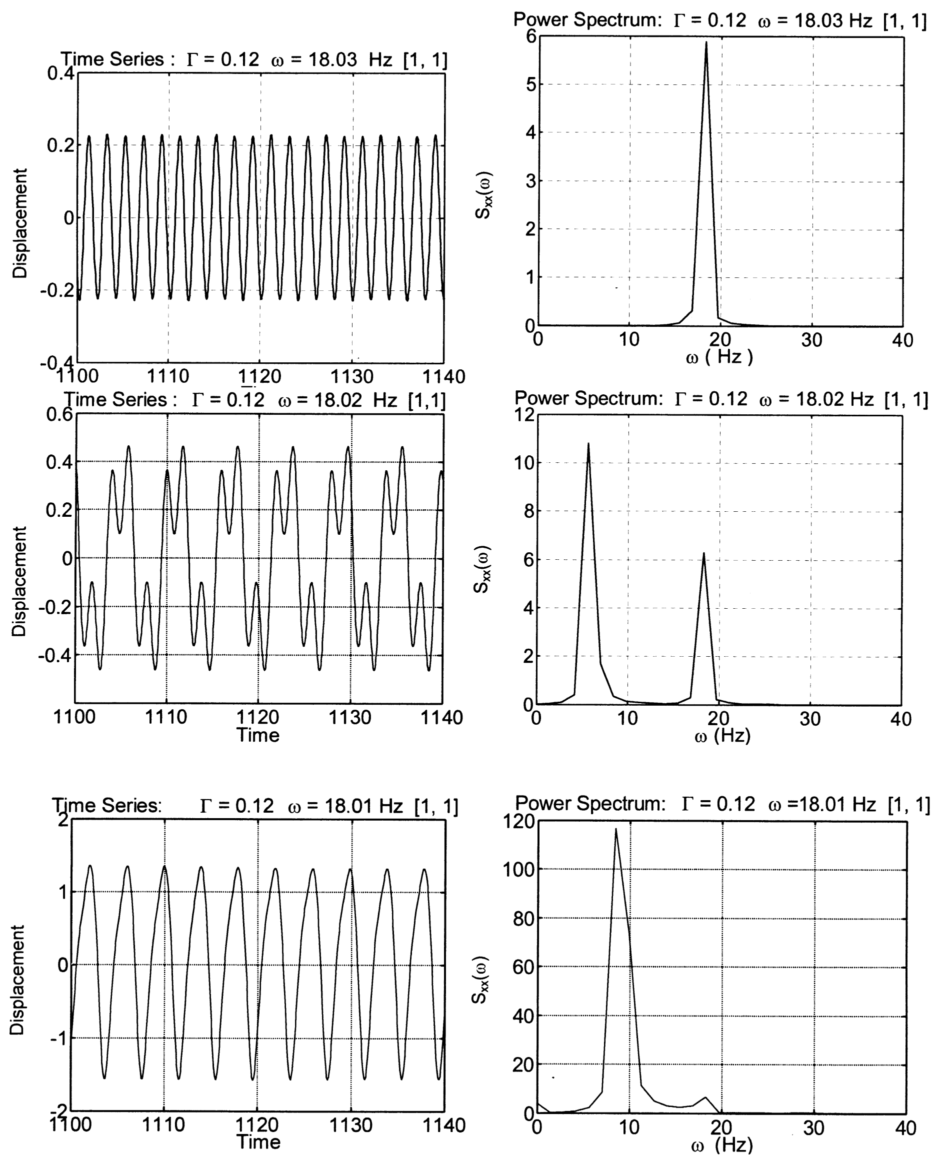



**Figure S2.** Intense sensitivity of the nature of the numerically obtained response to excitation frequency near 18 Hz.