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PERTURBATION OF THE GROUND VARIETIES OF $C=1$ STRING THEORY

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ABSTRACT

We discuss the effect of perturbations on the ground rings of $c = 1$ string theory at the various compactification radii defining the A_N points of the moduli space. We argue that perturbations by plus-type moduli define ground varieties which are equivalent to the unperturbed ones under redefinitions of the coordinates and hence cannot smoothen the singularity. Perturbations by the minus-type moduli, on the other hand, lead to semi-universal deformations of the singular varieties that can smoothen the singularity under certain conditions. To first order, the cosmological perturbation by itself can remove the singularity only at the self-dual (A_1) point.

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Introduction

The study of polynomial rings associated to systems with a BRS cohomology is a useful way to understand their physical properties[1,2]. For noncritical $c = 1$ string theory, the ring associated to cohomology states of ghost number zero[3] is particularly interesting, as it gives an insight into the structure of the unbroken gauge symmetries of this backgrounds[2,4]. It turns out that the polynomial ground ring defines a singular variety, and the unbroken symmetries have a natural action on this variety as volume-preserving diffeomorphisms.

The $c = 1$ string has a large collection of marginal deformations generated by its various moduli. The best known of these moduli are the cosmological operator, the radius-changing operator, and the operator which deforms the background into a black hole[5,6]. The effects of the cosmological perturbation at the self-dual point and in the uncompactified theory have been examined in Refs.[2,7–9]. The moduli space of the $c = 1$ string generated by the radius-changing operator, and the nature of the polynomial ground ring and its associated variety at various points of this moduli space, have been analyzed in[10] and found to be related to some beautiful mathematical structures, the Kleinian singular varieties. One can explicitly see how the symmetries vary as a function of the compactification radius. An equally explicit understanding of the symmetry-breaking pattern along general directions in the moduli space is, however, lacking so far.

In this note we uncover some aspects of the nature of ground rings and their associated varieties when the $c = 1$ string is perturbed by various generic moduli. For this purpose, we start with the theory defined at some integer multiple of the self-dual radius, corresponding to the A_N points. Possible perturbations fall into two classes, generated by the plus- and minus-type moduli. We will argue that generic minus-type perturbations smoothen the singularities of the ground varieties, but plus-type perturbations cannot do so. Also, the cosmological perturbation alone does not effect a smoothening of the singularity. We will also find an intriguing relation between the minus-type perturbations and the theory of semi-universal deformations of Kleinian singularities.

Let us now briefly recall some facts about $c = 1$ string theory. The matter sector is described by the CFT of a compact free boson X. The moduli space of $c = 1$ CFT is well known[11]. At the self-dual radius, the theory has an enhanced $SU(2) \otimes SU(2)$ symmetry, and the (chiral) operators are labelled by their $SU(2)$ quantum numbers $s = 0, \frac{1}{2}, 1, \cdots$ and $-s \leq n \leq s$: $V_{s,n}$. When this CFT is coupled to gravity, the operators $V_{s,n}$ are "dressed" by Liouville vertex operators which can have two possible momenta $p^{\varphi}_{\pm} = i\sqrt{2}(-1 \pm s)$. The former is called the *plus-type* dressing and the latter *minus-type*.

The BRS analysis[3] shows that apart from the operators $Y^{\pm}_{s,n} = cV_{s,n}e^{\sqrt{2}(1\mp s)\varphi}$ of standard ghost number 1 (which exist for either dressing) there exist an infinite number of operators $\mathcal{O}^{(+)}_{s,n}$ and $\mathcal{P}^{(-)}_{s,n}$ at ghost numbers 0 and 2 respectively. The operators relevant for the closed string theory are constructed by combining the chiral and anti-chiral operators. Since the Liouville field is non-compact, its left and right momenta must be matched. This results in plus-type operators of ghost numbers 0, 1 and 2 and minus-type of ghost numbers 2, 3 and 4.

There also exist additional operators ("new moduli") which are in the relative cohomology of $b_0 - \bar{b}_0$ but not of b_0 and $\bar{b}_0$ separately[12,4]. These can be expressed as BRS commutators of terms explicitly involving the Liouville field ϕ, hence it has been argued that they are not genuinely BRS-exact. However, it has been pointed out[13] that in string field theory such operators are normally taken to be BRS-trivial, as they do not generate new deformations of the theory. In what follows, we will therefore work with the cohomology in which these new moduli are set to zero.

The ghost number 0 operators $\mathcal{O}_{s,n}\overline{\mathcal{O}}_{s,n'}$ form a ring, called the ground ring, with the OPE (modulo BRS exact terms) defining the ring multiplication. The ring at the $SU(2)$ point is generated by the four operators (two electric and two magnetic)

$$a_1 = \mathcal{O}_{\frac{1}{2},\frac{1}{2}}\overline{\mathcal{O}}_{\frac{1}{2},\frac{1}{2}} \qquad a_2 = \mathcal{O}_{\frac{1}{2},-\frac{1}{2}}\overline{\mathcal{O}}_{\frac{1}{2},-\frac{1}{2}} \qquad a_3 = \mathcal{O}_{\frac{1}{2},\frac{1}{2}}\overline{\mathcal{O}}_{\frac{1}{2},-\frac{1}{2}} \qquad a_4 = \mathcal{O}_{\frac{1}{2},-\frac{1}{2}}\overline{\mathcal{O}}_{\frac{1}{2},\frac{1}{2}} \tag{1}$$

satisfying the relation

$$a_1 a_2 - a_3 a_4 = 0. \tag{2}$$

which define a singular conical variety, the "ground cone".

The ghost number 1 operators $J_{s+1,n,n'} \equiv Y^+_{s+1,n}\overline{\mathcal{O}}_{s,n'}$ and $\bar{J}_{s+1,n,n'} \equiv \mathcal{O}_{s,n}\overline{Y}^+_{s+1,n'}$ act on the ground ring as generators of volume-preserving diffeomorphisms of the ground cone. Finally, the ghost number 2 operators $Y^+_{s,n}\overline{Y}^+_{s,n'}$ and $Y^-_{s,n}\overline{Y}^-_{s,n'}$ correspond to the moduli since they give rise, via the descent equation, to ghost number 0 two-forms $W^{\pm}_{s,n,n'}dz\wedge d\bar{z} \equiv V_{s,n}V_{s,n'}e^{\sqrt{2}(1\mp s)\varphi}dz\wedge d\bar{z}$, that correspond to marginal deformations of the action.

All of the above applies to the theory at the self-dual radius. Consider now the special set of A_N points correponding to compactification radius $N/\sqrt{2}$, which arise on modding out the self-dual theory by the discrete subgroups $\Gamma = \mathbf{Z}_{2N}$ of $SU(2)$[11]. The operators at these points satisfy a constraint arising from the matching of the left and right matter momenta

$$n - n' = 0 \quad \mathrm{mod} \quad N. \tag{3}$$

It was shown[10] that the generators of the ground ring at each of these points are the $\mathbf{Z}_{2N}$-invariant polynomials in the generators (1) at the $SU(2)$ point, namely,

$$a_1 = \mathcal{O}_{\frac{1}{2},\frac{1}{2}}\overline{\mathcal{O}}_{\frac{1}{2},\frac{1}{2}} \qquad a_2 = \mathcal{O}_{\frac{1}{2},-\frac{1}{2}}\overline{\mathcal{O}}_{\frac{1}{2},-\frac{1}{2}} \qquad X = \mathcal{O}_{\frac{N}{2},\frac{N}{2}}\overline{\mathcal{O}}_{\frac{N}{2},-\frac{N}{2}} \qquad Y = \mathcal{O}_{\frac{N}{2},-\frac{N}{2}}\overline{\mathcal{O}}_{\frac{N}{2},\frac{N}{2}} \tag{4}$$

satisfying the relation

$$(a_1 a_2)^N - XY = 0. \tag{5}$$

which defines the A_N ground variety. While the ground cone (2) has an isolated singularity at the origin, the singular locus of the A_N variety (5) (except for N=1) is given by the pair of straight lines defined by $a_1 a_2 = 0$.

While eq.(3) forbids some of the moduli in the orbifold theory, one gets additional moduli in the twisted sector $c\bar{c}\mathcal{T}^{\pm}_q = c\bar{c}e^{i\sqrt{2}qX+\sqrt{2}(1\mp q)\varphi}$, $q \in \mathbf{Z}^+ + \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}$. We call these the intermediate tachyons. In the limit of infinite radius ($N \to \infty$), the intermediate tachyons form a continuum of operators labelled by a real number q. There are, however, no states with non-standard ghost number in the twisted sector.

It was argued[7] that the ghost number 1 operators and the plus-type moduli, including the intermediate tachyons, form a module for the ground ring (modulo BRS exact terms) under the OPE. More precisely, the plus-type discrete moduli form a faithful module and the plus-type intermediate tachyons form unfaithful modules for each fractional part of q.

Deformation Of The Ground Varieties

In this section we will study the effect of perturbations by various marginal operators on the ground rings and associated varieties of $c = 1$ string theory. Our arguments are based on momentum counting, and we will restrict ourselves to the A-type points in the moduli space of the $c = 1$ CFT.

We will use the Hilbert space of the unperturbed theory to describe the effects of the perturbations, under the assumption that the theory changes smoothly. This needs some justification as we know that at least for the radius perturbation the cohomology varies in a singular fashion[10]. For the perturbations that we are interested in, our assumption can be made plausible by appealing to their connection to the deformation of the corresponding Kleinian singularities.

Let us begin with the simplest case: the cosmological perturbation at the $SU(2)$ point, induced by adding the operator $\mu \int W_{0,0,0}$ to the action. (This is the unique operator for which the plus and minus dressings are equivalent.) In Ref.[2], it was argued that under this perturbation, the $SU(2)$ ground cone (2) is deformed to the smooth variety defined by

$$a_1a_2 - a_3a_4 = \mu. \tag{6}$$

This observation was motivated by an analysis of the $SU(2) \otimes SU(2)$ content of the moduli. Let us give an alternative motivation for the same result, using just the matching of Liouville and matter momentum on both sides of the ground ring relation. The Liouville and matter momenta of the cosmological operator are given by $(p^{\varphi}, p_L^X, p_R^X) = \sqrt{2}(-i, 0, 0)$. If we associate the negative of this momentum to the parameter μ then the perturbation formally conserves momentum. Now, the momenta carried by a_1a_2 (and a_3a_4) are $(p^{\varphi}, p_L^X, p_R^X) = \sqrt{2}(i, 0, 0)$, precisely the same as those we have identified with the parameter μ, hence the perturbed equation (6) is the unique one consistent with momentum conservation. Of course, it is assumed that the coefficient of the allowed perturbation to the ring relation does not accidentally vanish, since there is no known reason for it to do so. (The above argument of matching of Liouville momenta is just a scaling argument in the spirit of Refs.[14,15].)

Remaining at the $SU(2)$ point, we can now consider more general perturbations. Suppose first that we perturb by a general modulus of plus-type:

$$S \rightarrow S + \sum_{s,n,n'} u^+_{s,n,n'} \int W^+_{s,n,n'}. \tag{7}$$

The momenta associated with the parameter $u^+_{s,n,n'}$ are $\sqrt{2}(i(1-s), -n, -n')$. Thus, each occurrence of this parameter in the ring relation must be compensated by appropriate powers of the a_i. Restricting for simplicity to the case $n = n'$, we find that the equation of the perturbed ground cone consistent with momentum conservation is

$$a_1a_2 - a_3a_4 = \sum_{s,n} \alpha_{s,n} u^+_{s,n,n} a_1^{s+n} a_2^{s-n}. \tag{8}$$

The $\alpha_{s,n}$ are some constants, which are undetermined by the momentum-matching considerations. Assuming again that they are nonzero, they can be absorbed into the definition of the parameters u^+. (The restriction of this equation to $n = 0$ was also conjectured in[2]. There it was argued that among the perturbations by plus-moduli, those with $n = n' = 0$ are the only ones consistent with integrability, since they commute among themselves — the chiral part $W^+_{s,n}$ of these operators lie in the Cartan subalgebra of w_∞.)

Next we consider perturbing the $SU(2)$ theory by minus-moduli:

$$S \to S + \sum_{s,n,n'} u^-_{s,n,n'} \int W^-_{s,n,n'}. \tag{9}$$

The momenta associated to the parameters $u^-_{s,n,n'}$ are $\sqrt{2}(i(1+s), -n, -n')$, as a result of which the analog of eq.(8) above would be obtained by the replacement $s \to -s$. This would mean, however, that the ring generators a_1 and a_2 would be raised to *negative* powers. Let us at this point make the plausible assumption that the perturbed ground ring is also polynomial. In this case the only allowed term is $s = n = 0$, which is precisely the cosmological perturbation. We conclude, therefore, that except for the cosmological operator, the minus-type moduli cannot perturb the ground cone at the $SU(2)$ point.

Things are quite different at the other A_N points. Repeating the same analysis, we easily find that the plus-type moduli can generate perturbations of the form

$$(a_1 a_2)^N - XY = \sum_{s,n} u^+_{s,n,n} a_1^{N+s+n-1} a_2^{N+s-n-1}, \tag{10}$$

while a *finite number* of minus-type moduli can also generate perturbations, of the form

$$(a_1 a_2)^N - XY = \sum_{0 \le s-n, s+n \le N-1} u^-_{s,n,n} a_1^{N-s+n-1} a_2^{N-s-n-1}. \tag{11}$$

Thus the set of minus moduli which can perturb the A_N ground variety fall into the "diamond" pattern displayed in Fig.1. There are precisely N^2 minus-type perturbations at the A_N point.

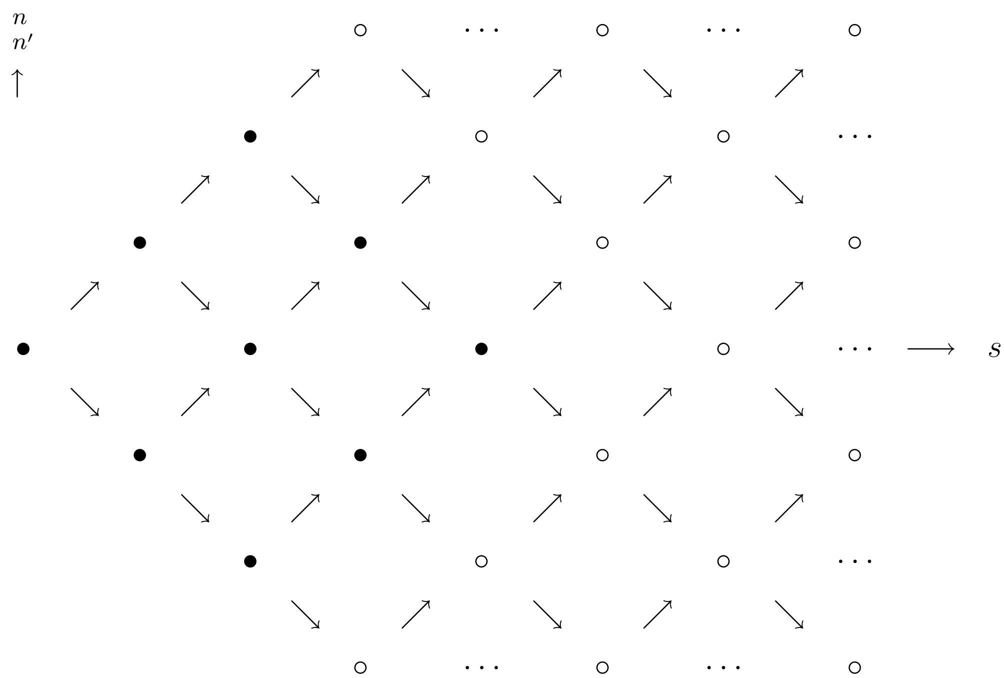

Fig.1: The "diamond" at the A_3 point

(The perturbations corresponding to the semi-universal deformation are denoted by $\bullet$ and the rest by $\circ$)

We notice here that in the case of minus-type perturbations, the restriction to the electric moduli with $n = n'$, involves no loss of generality. This is because of the fact that at the A_N point, we have the condition (3). Momentum counting leaves us with operators in the "diamond", that is operators that satisfy the conditions $0 \leq (s-n), (s+n) \leq N-1$. It is easy to see that there are no magnetic operators in the "diamond" satisfying Eq.(3).

Finally, one may consider perturbing the theory by the intermediate tachyons $T_q^{\pm}$. In this case, momentum counting shows that the perturbation to the ground variety can only be by fractional powers of the a_i. We conclude that in first order, the intermediate tachyon perturbations do not affect the ground variety.

Use of the w_∞ symmetries

We now show how the symmetries $J_{s,n,n'} = W^+_{s,n}\overline{\mathcal{O}}_{s,n'}$ act on the moduli

$W^-_{s,n}\overline{W}^-_{s,n'}$ and relate the different moduli of the theory. In particular, all the operators in the "diamond" with non-zero matter momenta are related to those with zero matter momenta. Furthermore, starting from the negative moduli in the "diamond" with zero matter momenta, one generates all the moduli in the "diamond" and no others. This symmetry can be used to calculate the effect of perturbation by moduli with non-zero matter momenta, from the knowledge of those with zero matter momenta.

Recall that the OPE between the chiral operators $Y^+ = cW^+$ and $Y^- = cW^-$ gives rise to the following commutator[2]

$$[W^+_{s_1,n_1}, W^-_{s_2,n_2}] = \begin{cases} (s_1 n_2 + (s_2+1)n_1)\, W^-_{s_2-s_1+1,n_1+n_2}, & \text{if } s_2 > s_1 - 1 \text{ and} \\ & |n_1+n_2| \le s_2 - s_1 + 1; \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Since at the A_N point, the modulus $W^-_{N-1,0,0}$ at the apex of the "diamond" has the maximum spin $s_2 = N-1$, we can restrict our considerations to symmetries with spin $s_1 \le N - \frac{1}{2}$. Let us for definiteness, illustrate this with the example of the A_2 point. Here the zero momentum operators in the "diamond" are the cosmological operator $W^-_{0,0,0}$ and the operator $W^-_{1,0,0}$ that correspond to the black-hole perturbation[5,6]. Since the maximum value of s_2 is 1, the relevant symmetries for this theory are $J_{\frac{3}{2},\pm\frac{1}{2},\pm\frac{1}{2}}$ and $J_{\frac{3}{2},\pm\frac{3}{2},\mp\frac{1}{2}}$. The second set of operators annihilate $W^-_{1,0,0}$, while the first set produce the states $W^-_{\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}}$ to fill up the "diamond".

Relation to semi-universal deformations

The perturbations generated by the minus-moduli are closely related to the concept of *semi-universal deformations* of singular varieties. A *deformation* of a variety K_0 is essentially a bundle over a certain base space U with a marked point u_0, such that the fibre over u_0 is (isomorphic to) the original variety K_0. The total space of this bundle is like a space of varieties, containing the given variety K_0 and others that are continuously connected to it. Precise definitions of this concept may be found, for example, in Refs.[16,17].

A deformation is called *semi-universal* if any other deformation can be related to it by maps between the bundles and the base spaces. In some sense, the semi-universal deformations are the ones which are truly independent, while the others are equivalent to it by a suitable change of variables. The application of this concept to the Kleinian singular varieties has been worked out by Tjurina[17]. We briefly review her results for the A-series, and then find a relation with the perturbations of the ground varieties of $c = 1$ string theory discussed above.

The A_N Kleinian singularities are complex hypersurfaces in $\mathbf{C}^3$ defined by

$$f(X, Y, Z) \equiv Z^{2N} - XY = 0. \tag{13}$$

It has been shown[17] that the semi-universal deformations of this equation are given by the quotient $\mathbf{C}(X, Y, Z)/\mathcal{I}(f, f_X, f_Y, f_Z)$ where $\mathbf{C}(X, Y, Z)$ is the ring of polynomials in X, Y, Z and $\mathcal{I}$ is the ideal generated by the function f above, and its three partial derivatives. This result is easy to understand to first order in the deformation. If we consider any redefinition $\widetilde{X} = X + \delta X(X, Y, Z)$, (and similarly for Y, Z), then we have, to first order,

$$f(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = f(X, Y, Z) + f_X \delta X + f_Y \delta Y + f_Z \delta Z \tag{14}$$

It is clear from this that such a redefinition can absorb any perturbation containing at least one power of f, f_X, f_Y or f_Z. Hence the independent deformations are given by quotienting with the ideal generated by these polynomials. This quotient is easily computed and one finds that the semi-universal deformation of eq.(13) is the $(2N - 1)$-parameter family of varieties

$$Z^{2N} - XY = t_1 Z^{2N-2} + \cdots + t_{2N-2} Z + t_{2N-1}. \tag{15}$$

Returning now to the (non-chiral) ground varieties in eq.(5)(which are the ones related to closed-string theory), we may ask what are the semi-universal deformations of these. Strictly speaking, the above result on deformations of Kleinian varieties with isolated singular points does not apply to the non-chiral ground varieties at the A_N points, which have lines of singular points. However, these spaces arise as $U(1)$ quotients of products of the Kleinian varieties and their defining equations are very similar (compare eqs.(5) and (13)), so we proceed with the assumption that here too, semi-universal deformations are obtained by quotienting with the ideal generated by the defining function and its first derivatives. This is anyway true to first order in the deformation parameters, as one can simply carry over the argument above eq.(15). The quotient is again straightforward to compute, but leads this time to an infinite-dimensional space of deformations (in accordance with the result[18] that the space is finite-dimensional if and only if the variety has an isolated singularity). The result is

$$(a_1 a_2)^N - XY = \sum_{k,l \text{ not both } > (N-1)} t_{kl} a_1^k a_2^l. \qquad (16)$$

Now to make contact with the results of the previous section, one should ask whether this infinite-dimensional space of semi-universal deformations is realized by physical perturbations of the A_N CFT background. Examining eqs.(10) and (11), we find that only a finite subset of the perturbations in eq.(16) above can be realised in string theory. In fact, none of the semi-universal deformations can be generated by plus-type moduli, while the minus-type generate the finite subset of terms in eq.(16) given by $k, l \leq (N-1)$.

Thus we have found that *all perturbations generated by minus-type moduli correspond to semi-universal deformations of the ground variety.*

Let us ask under what conditions these perturbations can smoothen the singularity of the ground varieties. First restrict to the case of perturbations by minus-moduli of zero matter momentum, hence keep only the terms with $k = l$, $k, l \leq N-1$, in eq.(16) above. Rewrite this equation as $f(a_1, a_2, X, Y) \equiv g(a_1 a_2) - XY = 0$, where g is a polynomial function of degree N. The tangent space to the perturbed variety is defined by the normal vector $(f_{a_1}, f_{a_2}, f_X, f_Y) =$

$(a_2 g'(a_1 a_2), a_1 g'(a_1 a_2), -Y, -X)$. The variety will be singular at points where this vector vanishes, and which satisfy $f = 0$. Such points fall into two classes:

$$\begin{array}{llll} \mathrm{I}: & X = Y = 0, & g(a_1 a_2) = 0, & g'(a_1 a_2) = 0 \\ \mathrm{II}: & X = Y = 0, & g(a_1 a_2) = 0, & a_1 = a_2 = 0. \end{array} \tag{17}$$

Condition I is satisfied whenever the polynomial g has a multiple root. For each such root r_i, we have singularities on the hyperbola $a_1 a_2 = r_i, X = Y = 0$. Thus in this situation, the singularities of the unperturbed variety, which lie on the pair of straight lines $a_1 a_2 = 0$[10], remain present but lie on the union of several hyperbolae, one for each multiple root. Condition II is satisfied if the polynomial g has no constant term, which means that the perturbed variety passes through the origin. In this case, the origin is the only singular point. If neither of the conditions is satisfied (which is true for generic perturbations) then the perturbed variety is nonsingular. Note that if we perturb by the cosmological operator alone, then to first order, 0 is an $(N-1)$-fold multiple root, so that (for $N \geq 2$) condition I is satisfied and the singular locus remains the pair of straight lines $a_1 a_2 = 0$. Thus, the fact that to first order, the cosmological perturbation removes the singularity at the $SU(2)$ point[2] seems to be a nongeneric case.

Calculation for perturbed ground ring action

In this section we will calculate explicitly the perturbations of the ground variety by the minus-moduli in some special cases. The computations substantiate the momentum counting argument presented above.

The set of discrete plus-moduli form a faithful module of the ground ring[7] and we can study the effect of perturbations by studying the ring action on this module. Let us start with the unperturbed theory at the $SU(2)$ point. The a_i acting on a state $Y^+_{s,n}\overline{Y}^+_{s,n'}$ produce states of appropriate momenta. Due to the orthonormality of states with different momenta, the resulting state is uniquely determined by taking its inner product with states in the dual module $Y^-\overline{Y}^-$. For example, $a_1 Y^+_{s,n}\overline{Y}^+_{s,n'}$ is a state that has non-zero inner product only with $Y^-_{s+\frac{1}{2},-n-\frac{1}{2}}\overline{Y}^-_{s+\frac{1}{2},-n'-\frac{1}{2}}$. We now use the $SU(2)$ relation $Y^{\pm}_{s,n} = \sqrt{\frac{(s+n)!}{(2s)!(s-n)!}}\left(\oint J_-\right)^{s-n} Y^{\pm}_{s,s}$, and deform the contour of J_- to make it act on a_i and Y^-. This reduces our problem to knowing the ground ring action on the tachyon. One easily finds that

$$
\begin{aligned}
a_1\, Y^+_{s,s}\overline{Y}^+_{s,s} &= (2s)^2 Y^+_{s+\frac{1}{2},s+\frac{1}{2}}\overline{Y}^+_{s+\frac{1}{2},s+\frac{1}{2}} \qquad & a_2\, Y^+_{s,s}\overline{Y}^+_{s,s} &= |A(s)|^2 Y^+_{s+\frac{1}{2},s-\frac{1}{2}}\overline{Y}^+_{s+\frac{1}{2},s-\frac{1}{2}} \\
a_3\, Y^+_{s,s}\overline{Y}^+_{s,s} &= 2s\overline{A}(s) Y^+_{s+\frac{1}{2},s+\frac{1}{2}}\overline{Y}^+_{s+\frac{1}{2},s-\frac{1}{2}} \qquad & a_4\, Y^+_{s,s}\overline{Y}^+_{s,s} &= 2sA(s) Y^+_{s+\frac{1}{2},s-\frac{1}{2}}\overline{Y}^+_{s+\frac{1}{2},s+\frac{1}{2}},
\end{aligned}
\tag{18}
$$

where $A(s)$ is a constant that we will now determine by self-consistency. Carrying out the contour deformation in the inner product, we have

$$
a_2\, Y^+_{s,n}\overline{Y}^+_{s,n'} = \sqrt{(s-n+1)(s-n'+1)}|A(s)|^2 Y^+_{s+\frac{1}{2},n-\frac{1}{2}}\overline{Y}^+_{s+\frac{1}{2},n'-\frac{1}{2}}. \tag{19}
$$

By the $\mathbf{Z}_2$ symmetry under $X \to -X$ of the problem, the above for $n = n' = -s$ should be the same as the a_1 action on $Y^+_{s,s}\overline{Y}^+_{s,s}$. This determines $A(s) = \frac{2s}{\sqrt{2s+1}}$. (We have chosen the positive square root so as to make the ring action on the module commutative.) This finally gives us the ground ring action on discrete states:

$$
\begin{aligned}
a_{1(2)}\, Y^+_{s,n}\overline{Y}^+_{s,n'} &= \frac{(2s)^2}{2s+1}\sqrt{(s\pm n+1)(s\pm n'+1)}Y^+_{s+\frac{1}{2},n\pm\frac{1}{2}}\overline{Y}^+_{s+\frac{1}{2},n'\pm\frac{1}{2}}, \\
a_{3(4)}\, Y^+_{s,n}\overline{Y}^+_{s,n'} &= \frac{(2s)^2}{2s+1}\sqrt{(s\pm n+1)(s\mp n'+1)}Y^+_{s+\frac{1}{2},n\pm\frac{1}{2}}\overline{Y}^+_{s+\frac{1}{2},n'\mp\frac{1}{2}},
\end{aligned}
\tag{20}
$$

From eqs.(20), it is easy to check the commutativity of the ring and the relation (2). Note also that the above equations agree with the explicit calculations of Ref.[19] after setting the new moduli to zero there.

Now consider the perturbation by the cosmological operator. The inner product that we are considering receives corrections due to the presence of the (integrated) cosmological operator. The corrections to the a_1, a_3 and a_4 actions on the discrete tachyon $Y^+_{s,s}\overline{Y}^+_{s,s}$ vanish by simple momentum counting — they would have produced states whose n-value is greater than the s-value, and such a state is not in the cohomology. Only the a_2 action is non-zero:

$$a_2\, Y^+_{s,s}\overline{Y}^+_{s,s}\Big|_{correction} = -\mu a_2 Y^+_{s,s}\overline{Y}^+_{s,s}\int W_{0,0,0} = \mu B(s) Y^+_{s-\frac{1}{2},s-\frac{1}{2}}\overline{Y}^+_{s-\frac{1}{2},s-\frac{1}{2}}, \quad (21)$$

where $B(s)$ is a constant to be determined. we repeat the steps as in the unperturbed case, and fix $B(s)$ by demanding that the perturbed ring is commutative. This determines $B(s)$ uniquely to be $B(s) = 2s/(2s+1)(2s-1)^2$. Hence the corrections to the ground ring action are:

$$\begin{aligned} a_{1(2)}\, Y^+_{s,n}\overline{Y}^+_{s,n'}\Big|_{correction} &= \mu\frac{\sqrt{(s\mp n)(s\mp n')}}{(2s+1)(2s-1)^2}\, Y^+_{s-\frac{1}{2},n\pm\frac{1}{2}}\overline{Y}^+_{s-\frac{1}{2},n'\pm\frac{1}{2}}, \\ a_{3(4)}\, Y^+_{s,n}\overline{Y}^+_{s,n'}\Big|_{correction} &= \mu\frac{\sqrt{(s\mp n)(s\pm n')}}{(2s+1)(2s-1)^2}\, Y^+_{s-\frac{1}{2},n\pm\frac{1}{2}}\overline{Y}^+_{s-\frac{1}{2},n'\mp\frac{1}{2}}, \end{aligned} \quad (22)$$

Combining eqs.(20) and (22), we get $(a_1a_2 - a_3a_4)Y^+_{s,n}\overline{Y}^+_{s,n'} = \mu Y^+_{s,n}\overline{Y}^+_{s,n'}$, verifying the cosmologically perturbed ring relation (6) at the $SU(2)$ point.

It does not appear to be straightforward to extend these calculations to the other A_N points, although that would provide a useful check on our conclusions. We can, however, study the action of the ground ring elements on the intermediate tachyons — ($N-1$ species of these are present in the A_N theory for $N \geq 2$) — and corrections to them. Since these form an unfaithful module (for each species of tachyon)[7], there could be additional relations in the module that are not ring relations. Indeed, the fermi surface equation $a_1a_2 = 0$ is true only for the intermediate tachyons and not a relation in the (unperturbed) ground ring.

Again, start by considering the effect of the cosmological perturbation. Under this perturbation, the action of the ground ring generators on the tachyon receives corrections of the form $-\mu a_i \mathcal{T}_q \int W_{0,0,0}$. A simple momemtum counting shows once again that the corrections to a_1, a_3 and a_4 on the tachyon vanish — the would-be state is not in the cohomology. The correction to a_2 is the tachyon with momentum $q-1$.

Following[8], this correction can be written as:

$$-\mu \lim_{w\to 0} a_2(w,\bar{w})\mathcal{T}_q(0)\int d^2z\, W_{0,0,0}(z,\bar{z}) = -\mu \lim_{w\to 0}\int d^2z |w|^{2-2q}|z|^{2q-4}\mathcal{T}_{q-1}(0)$$
$$= -\mu\pi \frac{\Delta(q-1)}{\Delta(0)\Delta(q)}\mathcal{T}_{q-1}(0), \tag{23}$$

where $\Delta(x) = \Gamma(x)/\Gamma(1-x)$. To simplify the above formula we define the following (singular) normalization of the tachyons and the couplings u^- corresponding to the perturbation by the minus-moduli

$$\begin{aligned} \widetilde{\mathcal{T}}_q &= \Delta(q)\mathcal{T}_q \\ \tilde{u}^-_{s,n} &= \frac{\pi}{\Delta(-2s)}u^-_{s,n}. \end{aligned} \tag{24}$$

With these normalizations, the perturbed action of a_2 on the tachyon is $a_2\, \widetilde{\mathcal{T}}_q = -\tilde{\mu}\widetilde{\mathcal{T}}_{q-1}$. On the other hand, action of a_1 on the normalized tachyon is $a_1\widetilde{\mathcal{T}}_q = -\widetilde{\mathcal{T}}_{q+1}$. Combining these, we find

$$a_1 a_2 \widetilde{\mathcal{T}}_q = \tilde{\mu}\, \widetilde{\mathcal{T}}_q \tag{25}$$

the equation for the perturbed fermi surface[8,9].

We can now study the effect of perturbations corresponding to other minus moduli. Consider the modulus $W^-_{1,0,0}$, which corresponds to the black hole perturbation[5,6] and produces a constant term in the equation of the perturbed A_2 variety(11). Momentum-counting shows in particular that the correction to the action of a_2 on the tachyon, $-u^-_{1,0}a_2\mathcal{T}_q \int W^-_{1,0,0}$, would result in a state that is not in the cohomology — which implies that this correction vanishes. Despite this, the correction to the action of a_2^2 on this tachyon is expected to be nonzero by our arguments. This is not really a contradiction, since we are considering the products of various fields in the cohomology when their locations on the worldsheet collide, and the decoupling of BRS-trivial fields can fail to hold at coincident points.

Using the explicit expression for $a_2^2 = \mathcal{O}_{1,-1}\overline{\mathcal{O}}_{1,-1}$, a straightforward though somewhat tedious calculation gives

$$
\begin{aligned}
&- u^-_{1,0} \lim_{w\to 0} a_2^2(w,\bar{w})\mathcal{T}_q(0) \int d^2z\, W^-_{1,0,0}(z,\bar{z}) \\
&\qquad\qquad = -u^-_{1,0} \int d^2z |z|^{2q-4}|1-z|^2 \left| \left(\frac{q/\sqrt{2}}{z} + \frac{3/\sqrt{2}}{(1-z)} \right) \right|^2 \mathcal{T}_{q-2}(0) \\
&\qquad\qquad = u^-_{1,0} \frac{\pi\Delta(-2)^{-1}}{8(q-1)^2(q-2)^2} \mathcal{T}_{q-2}(0),
\end{aligned}
$$

which after normalization (24) gives the correction due to the black hole perturbation

$$
a_2^2 \widetilde{\mathcal{T}}_q = \frac{1}{8}\tilde{u}^-_{1,0}\widetilde{\mathcal{T}}_{q-2}, \tag{26}
$$

which gives $a_1^2 a_2^2 \mathcal{T}_q = \frac{1}{8}\tilde{u}^-_{1,0}\mathcal{T}_q$.

Discussion and Conclusions

We have studied the deformation of the A_N ground varieties of the $c = 1$ string theory under perturbations by the physical moduli of the theory using a simple scaling relation. The unperturbed ground varieties were related to the the Kleinian singular varieties[10]. Here we find that a finite set of physical perturbations produce the the analogue of the semi-universal deformation of the corresponding ground variety.

Let us briefly comment on the question of integrability of the perturbations that we have been considering. For the plus-type perturbations, the requirement of integrability implies that eq.(10) should be taken seriously only for $n = 0$, in which case the right hand side is a power series in $(a_1 a_2)$ with powers ranging from $N-1$ to infinity. The first term is associated to the cosmological perturbation. The second one, proportional to $(a_1 a_2)^N$, is associated to the operator $W^+_{1,0,0} = \partial X \bar{\partial} X$ which is the radius-changing perturbation. This term can clearly be absorbed into the unperturbed equation by a simple rescaling of a_1 and a_2. Thus the ground variety is unaffected, to first order, by a radius-changing perturbation. This may appear a little disturbing at first, since the ground variety of the theory clearly depends on the compactification radius. However, the radius-dependence of this variety, which has been analysed in Ref.[10], is highly singular. Even the dimension of the variety depends on whether the radius is rational or irrational. This phenomenon clearly cannot be perturbative in the radius, which explains why we do not see it here. Naturally, this demonstrable limitation of perturbation theory casts some doubt on the validity of any analysis which is perturbative in the $u^{\pm}$.

As for the minus-type perturbations, the requirement of integrability presumably does not impose any limitation on them, since their chiral ingredients $W^{-}_{s,n}$ satisfy an abelian algebra. It has in fact been shown explicitly in Ref.[6] that the minus-moduli of zero matter momentum generate all-orders solutions to the string field theory equations of motion, which amounts to a proof of their integrability. Thus eq.(11) is meaningful for all s, n in the "diamond" of Fig.1.

The special role played by moduli of minus-type, which include the physically interesting black hole perturbation, seems worth understanding in more detail. Recent attempts to find the black hole perturbation in the framework of matrix models[20] may help to illuminate this question.

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