Prefitem

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KLEINIAN SINGULARITIES AND THE GROUND RING OF C=1 STRING THEORY

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ABSTRACT

We investigate the nature of the ground ring of $c = 1$ string theory at the special A-D-E points in the $c = 1$ moduli space associated to discrete subgroups of $SU(2)$. The chiral ground rings at these points are shown to define the A-D-E series of singular varieties introduced by Klein. The non-chiral ground rings relevant to closed-string theory are 3 real dimensional singular varieties obtained as $U(1)$ quotients of the Kleinian varieties. The unbroken symmetries of the theory at these points are the volume-preserving diffeomorphisms of these varieties. The theory of Kleinian singularities has a close relation to that of complex hyperKähler surfaces, or gravitational instantons. We speculate on the relevance of these instantons and of self-dual gravity in $c = 1$ string theory.

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1. **Introduction**

An important advance in the understanding of non-critical $c = 1$ string theory was the observation [1] that a large class of the physical states in this theory may be identified with the residual modes, in two spacetime dimensions, of the graviton and higher-spin gauge fields that are familiar from the study of critical strings. These states occur for special values of the matter momentum. In addition to these "discrete states", which arise due to the presence of null vectors in the matter CFT, there are other tachyon-like states whose spectrum in the infinite-radius theory is continuous.

From this point of view, it seems worthwhile to use $c = 1$ string theory as a toy model to understand the enormous variety of physical states that appear in the critical case, and to investigate the symmetries associated to them.

A seemingly unrelated development was the discovery $[2-4]$ that the BRS cohomology in the $c = 1$ background has infinitely many extra states of non-standard ghost number, a fact which has essentially no counterpart in critical string theory. A powerful approach to the understanding of these states was presented in Ref.[5], where it was noted that a subclass of them forms a polynomial ring (the "ground ring") which controls the symmetry properties of the theory. It was shown that the symmetries are associated to volume-preserving diffeomorphisms of the variety defined by the polynomial ground ring. One curious consequence of this approach is that this variety, which is three-dimensional (or four-dimensional in a certain sense) for many of the possible $c = 1$ backgrounds, seems to play the role of a target spacetime in $c = 1$ string theory, even though conventionally the target spacetime in this theory is thought to be two-dimensional.

The study of ground rings at $c = 1$ was taken further in [6], where in particular the volume-preserving symmetries were related to the presence of string field theory gauge parameters, of ghost number 1, which are in the BRS cohomology. (Precisely the same suggestion, in the context of $c < 1$ backgrounds, had been made earlier in Ref.[7], where however it could not be made more explicit due to the unavailability of a complete classification of the cohomology at the time.) It was also shown in Ref.[6] that the closed string theory has even more "extra states" in the cohomology than was previously thought.

Virtually all of the analysis above was done in the closed string theory where the matter coordinate is either noncompact or compactified at the self-dual radius. In the latter case, the matter theory acquires an $SU(2) \otimes SU(2)$ symmetry via the Frenkel-Kac-Segal mechanism, and this is evidently the phase of the string theory with the maximum number of unbroken symmetries.

It is known [8, 9] (see also [10]) that the moduli space of $c = 1$ conformal field theory has a very nontrivial and elegant structure, including two critical lines (associated to circle and Z_2-orbifold compactifications of the free scalar fields) intersecting at a multicritical point, and three isolated points disconnected from the above lines. It is of interest to understand how the symmetries of $c = 1$ string theory change as we move in the $c = 1$ CFT moduli space. This is important particularly because we need to understand how symmetry-breaking takes place in string theory. (One approach to this question was proposed recently in Ref.[11].)

It is reasonably straightforward to construct the ground ring and symmetries explicitly at all points of the CFT moduli space, as we will show below. What is remarkable is that, for a special subset of points in this moduli space, the result of this investigation turns out to have a beautiful connection with the theory (studied by Klein [12] in the last century) of polynomials on $\mathbf{C}^2$ invariant under discrete subgroups of $SU(2)$, and the associated algebraic varieties. This theory in turn has, in the last two decades, proved to be intimately related to self-dual gravity in four spacetime dimensions [13 − 19] . This therefore suggests a crucial role for self-dual gravity in $c = 1$ string theory (which has been suggested before on other grounds [20, 21]).

In what follows, we first describe the explicit construction of the chiral ground ring of $c = 1$ string theory at a set of special (A-D-E) points in the moduli space, and show that the corresponding spaces are precisely the Kleinian singular varieties. Next we combine left and right movers in closed string theory and obtain the full non-chiral ground rings (called "quantum ground rings" in Ref.[5]). These are shown to be certain quotients of the Kleinian varieties. For completeness, we also present a classification of polynomial ground rings of $c = 1$ string theory at other points in the moduli space, where they are not related to Kleinian varieties. Finally, we consider the four-dimensional topological action obtained in Ref.[6] at the $SU(2)$ point, and argue that it has nontrivial solutions at the other A-D-E points, corresponding to hyperKähler surfaces or gravitational instantons. It is here that the connection to self-dual gravity emerges.

2. The Ground Ring of $c = 1$ String Theory

The BRS cohomology of $c = 1$ string theory contains, in particular, operators of zero ghost number [2] [3] [4]. It was noted in Ref.[5] that these form a ring under the operator product, because ghost number 0 is obviously invariant under this product, and there are no singularities in the OPE since cohomology elements all have conformal dimension 0.

The structure of this ring was analyzed in detail [5] [6] for the case where the matter field $X(z, \bar{z})$ is compactified on a circle of radius $\frac{1}{\sqrt{2}}$, the self-dual or $SU(2)$ point. In this case, the chiral ground ring, consisting of the purely holomorphic part of the (relative) cohomology elements, is the free ring on two generators

$$\begin{aligned} x &\equiv \mathcal{O}_{\frac{1}{2},\frac{1}{2}}(z) \equiv\, :\big(c(z)b(z) + i\partial X^-(z)\big)\, e^{iX^+(z)}: \\ y &\equiv \mathcal{O}_{\frac{1}{2},-\frac{1}{2}}(z) \equiv\, :\big(c(z)b(z) - i\partial X^+(z)\big)\, e^{-iX^-(z)}: \end{aligned} \tag{1}$$

where $X^{\pm} \equiv \frac{1}{\sqrt{2}}(X \pm i\phi)$ and ϕ is the Liouville field.

This ring should be relevant for open-string theory, while in closed-string theory we need to combine the holomorphic pieces with their anti-holomorphic counterparts, keeping in mind that the Liouville momentum in the left and right sectors must be equal since the Liouville field is non-compact. It follows that, if $\bar{x}$ and

$\bar{y}$ are the anti-holomorphic counterparts of x and y defined above, the non-chiral ground ring is generated by the four operators

$$a_1 \equiv x\bar{x}, \quad a_2 \equiv y\bar{y}, \quad a_3 \equiv x\bar{y}, \quad a_4 \equiv y\bar{x} \tag{2}$$

satisfying the relation

$$a_1 a_2 = a_3 a_4 \tag{3}$$

Thus we have a ring on four generators satisfying one relation. The set of ghost-number 0 operators obtained by taking powers of the generators above is given by

$$\mathcal{O}_{s,n}(z)\overline{\mathcal{O}}_{s,n'}(\bar{z}) \tag{4}$$

where

$$\mathcal{O}_{s,n}(z) \;\sim\; h_{s,n} \equiv x^{s+n} y^{s-n} \tag{5}$$

and similarly for the antiholomorphic part. Here, s and n are half-integers labelling the total isospin and J_3-eigenvalue with respect to $SU(2)$.

In the language of algebraic geometry, a polynomial ring defines a variety. We will take the point of view that the chiral ring defines a complex affine variety in $\mathbf{C}^2$ (in the present case it is just $\mathbf{C}^2$ itself), while the non-chiral ring defines a real affine variety in $\mathbf{R}^4$ (in the present case it is a quadric cone). This is natural as the chiral ring is associated to holomorphic conformal fields, while the non-chiral ring is associated to real fields.

It has been shown in [5] [6] that the existence of the ground ring in $c = 1$ string theory is responsible for the existence of a large algebra of unbroken symmetries. This comes about because symmetry generators in closed-string theory are associated to states of ghost number 1 which are annihilated by the antighost zero mode $b_0^- \equiv b_0 - \bar{b}_0$. These are obtained by combining holomorphic states of ghost number 1 (the chiral version of tachyons and discrete states) with anti-holomorphic states of ghost number 0 (the ground ring elements) and vice versa. At the $SU(2)$ point, the chiral ground ring elements are given by the $\mathcal{O}_{s,n}(z)$ of Eq.(5), while the relevant chiral operators of ghost number 1 are

$$Y^+_{s,n} \equiv \,: c(z) V_{s,n}(z) e^{\sqrt{2}(1-s)\phi(z)} : \tag{6}$$

where $V_{s,n}$ are the $SU(2)$ multiplets which make up the primary fields in the matter sector. Thus, the symmetry generators are

$$Y^+_{s,n}(z)\overline{\mathcal{O}}_{s-1,n'}(\bar{z}) \tag{7}$$

and the conjugate ones.

In addition, there is a set of ghost-number 1 operators whose existence follows from the fact that $a + \bar{a} \equiv [Q_B, \phi]$ is in the cohomology if we restrict to the set of usual conformal fields. This leads to the "new" symmetry generators

$$: (a(z) + \bar{a}(\bar{z}))\, \mathcal{O}_{s,n}(z)\overline{\mathcal{O}}_{s,n'}(\bar{z}) : \tag{8}$$

In what follows, we will investigate the nature of the ground ring of $c = 1$ string theory at various points in the moduli space of $c = 1$ conformal field theories.

3. A-D-E Classification of Chiral Ground Rings

The conformal field theory of a single free boson has a special set of points in its moduli space associated to quotients of $SU(2)$ by its discrete subgroups, which are the binary groups classified as cyclic ($\mathcal{C}_n$), dihedral ($\mathcal{D}_n$), tetrahedral ($\mathcal{T}$), octahedral ($\mathcal{O}$) and icosahedral ($\mathcal{I}$) [8] [9]. These finite groups are in turn associated with the Dynkin diagrams of the A, D and E series of simply-laced Lie algebras [22]. The special points in the moduli space correspond to starting with the $c = 1$ theory compactified on a circle of self-dual radius $R = \frac{1}{\sqrt{2}}$, where the theory has an $SU(2) \otimes SU(2)$ symmetry, and taking the quotient of $SU(2)$ by its various discrete subgroups. The cyclic or A_n-type subgroups lead to theories of a free boson compactified on a circle of radius $n/\sqrt{2}$, while the dihedral or D_n-type subgroups give theories of a free boson compactified on a Z_2-orbifold of the circle, at radius $n/\sqrt{2}$ (in both these cases, replacing n by $1/n$ gives the same theory, by duality). The remaining three discrete subgroups, associated to the Dynkin diagrams of E_6, E_7 and E_8, lead to a set of three models which have no integrable moduli and are disconnected (in the $c = 1$ CFT) from the circle and orbifold compactifications [8] [9].

To find the ground rings at these points, we need to keep those generators of the $SU(2)$ ground ring that survive the modding out procedure. It is clear that the basic generators x, y defined in Eq.(1) form a doublet under $SU(2)$, since they are obtained from each other by the action of the $SU(2)$ raising and lowering operators $J^{\pm}$ of the $c = 1$ CFT. Thus the elements of the ground ring at the A-D-E points are obtained as all polynomials in x, y which are invariant under the corresponding discrete group Γ acting as a subgroup of $SU(2)$.

The mathematical problem of characterising the polynomials in two complex variables which are invariant under any discrete subgroup Γ of $SU(2)$ was addressed and solved in the last century by Klein [12]. To each binary subgroup Γ, Klein associated a polynomial ring in three variables X, Y and Z, with one relation between them. This defines a complex affine variety in $\mathbf{C}^3$, which is smooth everywhere except at the origin, where it is singular in general. These are commonly known as Kleinian singular varieties or simply Kleinian singularities. Using this result of Klein, it is an easy matter to describe the chiral ground rings at each of the special points in the moduli space of $c = 1$ conformal field theory. Note that the problem of finding the *non-chiral* ground ring is more complicated and we will discuss it in some detail in the following section.

Let us start with the the A-series. The corresponding discrete subgroups of $SU(2)$ are the binary cyclic groups $\mathcal{C}_n = Z_{2n}$, whose action on x, y is generated by

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

The smallest set which generates all polynomials invariant under $\mathcal{C}_n$ is clearly

$$X \equiv x^{2n}, \quad Y \equiv y^{2n}, \quad Z \equiv xy \tag{10}$$

and these three generators obey the single relation

$$XY = Z^{2n} \tag{11}$$

This defines the chiral ground ring at the $R = n/\sqrt{2}$ points, as a variety in $\mathbf{C}^3$.

At this point there is a slight puzzle, since the case $n = 1$ in the above equation gives a quadric in $\mathbf{C}^3$, rather than just $\mathbf{C}^2$ as was found in Ref.[5] for the chiral ground ring at the $SU(2)$ point. This really has to do with the meaning of chiral ground ring. As we noted earlier, this ring is just an auxiliary construction for closed-string theory, which acquires a direct physical interpretation only after combining left and right movers. In this process, the transformation $(x, y) \rightarrow -(x, y)$ is an invariance, from Eq.(2). Thus if we are interested in constructing the building blocks from which to make the non-chiral ground ring, then we need not impose this Z_2 invariance separately on the chiral ring elements. This means that we should quotient not by the binary but rather by the ordinary subgroups of $SU(2)$, which descend to subgroups of $SO(3)$. We will not take this point of view in the present work, since it is the chiral ground ring constructed from binary subgroups which is most closely related to the Kleinian singularities. In a subsequent section, we will construct the correct non-chiral ground ring following a procedure analogous to the one above.

Next we look at the D-series. The binary dihedral group $\mathcal{D}_n$ is the set of transformations generated by the generators of $\mathcal{C}_n$ together with

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{12}$$

The generators of invariant polynomials are

$$X = \frac{1}{2}(x^{2n} + (-1)^n y^{2n}), \quad Y = \frac{1}{2} xy \; (x^{2n} - (-1)^n y^{2n}), \quad Z = (xy)^2 \tag{13}$$

These obey the relation

$$Y^2 = Z(X^2 - Z^n) \tag{14}$$

which therefore defines the chiral ground ring at the orbifold points at radius $R = n/\sqrt{2}$.

Finally, for the exceptional points we build up the chiral ground ring as follows. For E_6, corresponding to the binary tetrahedral group $\mathcal{T}$, the polynomial invariants are obtained by starting with the binary dihedral group $\mathcal{D}_2$ and adjoining the action of the single generator

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^7 & \epsilon^7 \\ \epsilon^5 & \epsilon \end{pmatrix} \tag{15}$$

where ϵ is a primitive eighth root of unity. It is easy to deduce that the invariants of $\mathcal{T}$ are given by suitable combinations of $\mathcal{D}_2$ invariants, whose explicit expressions in terms of x and y are

$$\begin{aligned} X &= \frac{1}{4}(x^8 + y^8 + 14\, x^4 y^4) \\ Y &= \frac{1}{2} xy \; (x^4 - y^4) \\ Z &= \frac{1}{8}(x^4 + y^4)(x^8 + y^8 - 34\, x^4 y^4) \end{aligned} \tag{16}$$

These generators obey the relation

$$Z^2 = X^3 - 27\, Y^4 \tag{17}$$

which defines the chiral E_6 ground ring.

To get the binary octahedral group $\mathcal{O}$ one must adjoin the matrix

$$O = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^7 \end{pmatrix} \tag{18}$$

to the generators of $\mathcal{T}$, where ϵ is again a primitive eighth root of unity. The ring elements invariant under this matrix as well are generated by

$$\begin{aligned} X &= \frac{1}{4}(x^8 + y^8 + 14\, x^4 y^4) \\ Y &= \frac{1}{4} x^2 y^2 (x^4 - y^4)^2 \\ Z &= \frac{1}{16} xy\, (x^8 - y^8)(x^8 + y^8 - 34\, x^4 y^4) \end{aligned} \tag{19}$$

with the relation

$$Z^2 = Y(X^3 - 27\, Y^2) \tag{20}$$

Finally, for the binary icosahedral group $\mathcal{I}$, related to E_8, we need to go back to the ground ring at the $SU(2)$ point, generated by x and y, and impose invariance under the two matrices

$$I_1 = \begin{pmatrix} -\eta^3 & 0 \\ 0 & -\eta^2 \end{pmatrix} \qquad I_2 = \frac{1}{\eta^2 - \eta^3} \begin{pmatrix} \eta + \eta^4 & 1 \\ 1 & -(\eta + \eta^4) \end{pmatrix} \tag{21}$$

Technically this is the most complicated case to analyse. It turns out that the three basic polynomials invariant under this set of transformations are

$$\begin{aligned} X &= xy\, (x^{10} + 11\, x^5 y^5 - y^{10}) \\ Y &= -(x^{20} + y^{20}) + 228\, x^5 y^5 (x^{10} - y^{10}) - 494\, x^{10} y^{10} \\ Z &= (x^{30} + y^{30}) + 522\, x^5 y^5 (x^{20} - y^{20}) - 10005\, x^{10} y^{10} (x^{10} + y^{10}) \end{aligned} \tag{22}$$

and they satisfy the relation

$$Z^2 = -Y^3 + 1728\, X^5 \tag{23}$$

This completes the classification of chiral ground rings in $c = 1$ string theory, in terms of the basic generators x and y at the $SU(2)$ point, for the A-D-E series of

$c = 1$ CFT. In each case, we find that the ground ring defines a Kleinian singularity, which is a variety in $\mathbf{C}^3$ with a singularity at the origin.

One may ask what happens for other points in the moduli space of this CFT, namely, circle and orbifold compactifications where the radius is not an integer multiple of $\frac{1}{\sqrt{2}}$. These cases cannot be obtained by modding out the $SU(2)$ theory by a discrete subgroup, hence they are not related to the Kleinian singularities, but it is straightforward to analyse them separately. We will consider them below when we construct the full non-chiral ground ring at various points in the $c = 1$ CFT moduli space.

4. *A*-*D*-*E* Classification of Non-Chiral Ground Rings

In this section we find the invariant polynomials, and the relations between them, which characterise the ground ring of closed-string theory. First we consider the A-D-E series of special points, as before, and show that the ground rings and their associated varieties can be thought of as suitable $U(1)$ quotients of the Kleinian singularities. We construct these explicitly in all the cases except E_8, which is technically very complicated.

We have noted earlier that if Γ is a discrete subgroup of $SU(2)$, then the chiral ground ring for the $SU(2)/\Gamma$ theory is given by imposing Γ-invariance on the polynomials in x and y, the analytic polynomials in $\mathbf{C}^2$. Consider now the antiholomorphic generators of the $SU(2)$ ground ring, $\bar{x}$ and $\bar{y}$, and let us associate these with the complex conjugate coordinates of $\mathbf{C}^2$. To find the non-chiral ground ring, we must consider polynomials in $x, y, \bar{x}, \bar{y}$ which have the same Liouville momenta in the holomorphic and antiholomorphic parts [5], and are in addition Γ-invariant. The constraint of Liouville momentum matching amounts to taking a $U(1)$ quotient, where the action is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to e^{i\theta} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \to e^{-i\theta} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \tag{24}$$

Requiring polynomials to be invariant under this action is equivalent to requiring that they be annihilated by the vector field

$$H = x\partial_x + y\partial_y - \bar{x}\partial_{\bar{x}} - \bar{y}\partial_{\bar{y}} \tag{25}$$

Now the quotient by Γ must be taken on the space of $U(1)$-invariant polynomials on $\mathbf{C}^2$. Clearly these polynomials cannot be analytic in x and y but rather must be built out of the combinations defined in Eq.(2), subject to the relation (3). Formally, we may say that while the (complexified) chiral ground ring is the Kleinian variety $\mathbf{C}^2/\Gamma$, the non-chiral ring is $\mathbf{C}^2/(\Gamma \otimes U(1))$, which we view as a real variety in $\mathbf{R}^4$.

To obtain the A_n series, we note that the polynomials built out of $x\bar{x}$, $y\bar{y}$, $x\bar{y}$ and $y\bar{x}$ which are invariant under $\mathcal{C}_n$ are generated by

$$\begin{aligned} W &= \frac{1}{2}(x\bar{x} + y\bar{y}) \\ X &= (x\bar{y})^n \\ Y &= (y\bar{x})^n \\ Z &= \frac{1}{2}(x\bar{x} - y\bar{y}) \end{aligned} \tag{26}$$

satisfying the relation

$$(W^2 - Z^2)^n = XY \tag{27}$$

At this stage we observe that the generator W is present at every A-D-E point, since it is in fact invariant under the whole of $SU(2)$. The other three generators have some similarity with the three generators of the corresponding chiral ground ring. Finally, the relation between the four generators reduces to the Kleinian relation when W is set to zero.

Let us also examine the nature of the singularities of the variety defined by Eq.(27) above. Rewriting this equation in the form $f(X,Y,Z,W) = 0$, one finds that the tangent to the variety at any point is given by the 4-vector (in $\mathbf{R}^4$)

$$(\partial_X f,\ \partial_Y f,\ \partial_Z f,\ \partial_W f) = (Y,\ X,\ 2n\, Z\, (W^2 - Z^2)^{n-1},\ -2n\, W\, (W^2 - Z^2)^{n-1}) \tag{28}$$

This vector vanishes at the set of points

$$X = 0, \qquad Y = 0, \qquad Z = \pm W \tag{29}$$

which defines two intersecting straight lines in $\mathbf{R}^4$. Thus, unlike the chiral ground ring where the corresponding variety is singular only at the origin, the non-chiral ring at the A_n points defines a variety with lines of singular points. The only exception is the $SU(2)$ point ($n = 1$), where the tangent vector vanishes only at the origin, from Eq.(28). At the D and E points to be discussed below, we will find that the varieties have (real) curves of singular points.

For the D_n series, the invariants are easily seen to be

$$\begin{aligned} W &= \frac{1}{2}(x\bar{x} + y\bar{y}) \\ X &= \frac{1}{2}\left((x\bar{y})^n + (y\bar{x})^n\right) \\ Y &= \frac{1}{4}(x\bar{x} - y\bar{y})\left((x\bar{y})^n - (y\bar{x})^n\right) \\ Z &= x\bar{x}y\bar{y} \end{aligned} \tag{30}$$

with the relation

$$Y^2 = (W^2 - Z)(X^2 - Z^n) \tag{31}$$

It is easy to check that for $n = 1$, this is the same as Eq.(27) for $n = 2$, under an invertible polynomial redefinition of the generators. This corresponds to the equivalence of the binary cyclic group $\mathcal{C}_2 = Z_4$ and the binary dihedral group $\mathcal{D}_1$. The associated point in the moduli space is the multicritical point at which the lines of circle and orbifold compactifications intersect.

The singular points for the D_n series are parametrized by the curves

$$Y = 0, \qquad Z = W^2, \qquad X = \pm W^n \tag{32}$$

The E_6 case, as before, is obtained by starting with $\mathcal{D}_2$ invariants and modding out by the action of the matrix T in Eq.(15). One finds the invariants

$$\begin{aligned}
W &= \frac{1}{2}(x\bar{x} + y\bar{y}) \\
X &= \frac{1}{4}\left((x\bar{y})^4 + (y\bar{x})^4 + 14\,(x\bar{x}y\bar{y})^2\right) - x\bar{x}y\bar{y}\,(x\bar{x} + y\bar{y})^2 \\
Y &= \frac{1}{4}(x\bar{x} - y\bar{y})\left((x\bar{y})^2 - (y\bar{x})^2\right) \\
Z &= \frac{1}{8}\left((x\bar{y})^2 + (y\bar{x})^2\right)\left((x\bar{y})^4 + (y\bar{x})^4 - 34\,(x\bar{x}y\bar{y})^2 + 12\,x\bar{x}y\bar{y}(x\bar{x} + y\bar{y})^2 - (x\bar{x} + y\bar{y})^4\right)
\end{aligned} \tag{33}$$

satisfying the relation

$$Z^2 = X^3 - 27\,Y^4 + W^2(18\,XY^2 + X^2W^2 + 16\,Y^2W^4) \tag{34}$$

This is the same as Eq.(17) when W is set to zero. The singular points can be shown to be parametrized by the curves

$$Z = 0, \qquad X = -\frac{4}{3}\,W^4, \qquad Y = \pm\frac{2i}{3\sqrt{3}}\,W^3 \tag{35}$$

For E_7, we quotient the E_6 system by the matrix O in Eq.(18), and find the invariants

$$\begin{aligned}
W =& \frac{1}{2}(x\bar{x} + y\bar{y}) \\
X =& \frac{1}{4}\left((x\bar{y})^4 + (y\bar{x})^4 + 14\,(x\bar{x}y\bar{y})^2\right) - x\bar{x}y\bar{y}\,(x\bar{x} + y\bar{y})^2 \\
Y =& \frac{1}{16}(x\bar{x} - y\bar{y})^2\left((x\bar{y})^2 - (y\bar{x})^2\right)^2 \\
Z =& \frac{1}{32}(x\bar{x} - y\bar{y})\left((x\bar{y})^4 - (y\bar{x})^4\right)\left((x\bar{y})^4 + (y\bar{x})^4 - 34\,(x\bar{x}y\bar{y})^2\right. \\
& \left. + 12\,x\bar{x}y\bar{y}(x\bar{x} + y\bar{y})^2 - (x\bar{x} + y\bar{y})^4\right)
\end{aligned} \tag{36}$$

satisfying the relation

$$Z^2 = Y\left(X^3 - 27\,Y^2 + W^2(18\,XY + X^2W^2 + 16\,YW^4)\right) \tag{37}$$

The locus of singular points is given by the two intersecting curves

$$\begin{aligned} &Z = 0, \qquad X = -W^4, \qquad Y = 0 \\ &Z = 0, \qquad X = -\frac{4}{3}\,W^4, \qquad Y = -\frac{4}{27}\,W^6 \end{aligned} \tag{38}$$

The E_8 ground ring is in principle obtained in the same way, but it turns out to be rather involved and we will not write out the explicit expressions here.

Thus (except for the E_8 case) we have explicitly constructed the full non-chiral ground rings and their associated varieties at each of the A-D-E points. In each case, the singularities are described by a pair of curves in $\mathbf{R}^4$ intersecting at the origin, except at the $SU(2)$ point where the only singular point is the origin and the variety is a cone.

5. Non-chiral Ground Rings at Arbitrary Radii

We turn now to the case of points in the moduli space of the $c = 1$ free boson which cannot be obtained by quotienting with a discrete subgroup of $SU(2)$. This analysis proceeds via a direct study of allowed momenta at various radii in $c = 1$ CFT, and provides in particular a re-derivation of the results presented above (for the A_n and D_n cases) directly from conformal field theory.

The first case is that of circle compactification at rational radius in units of $\frac{1}{\sqrt{2}}$. Let $R = \frac{p/q}{\sqrt{2}}$ with (p, q) coprime positive integers. The ground ring is obtained by examining the momenta which are allowed at the given radius. The ground ring elements $\mathcal{O}_{s,n}\overline{\mathcal{O}}_{s,n'}$ at the $SU(2)$ point have left and right matter momenta

$$(p_L, p_R) = \sqrt{2}\,(n, n') \tag{39}$$

where $1 - s \leq n, n' \leq s - 1$ and n, n' are both integer or half-integer. Of these, at rational radii we must keep only those operators whose matter momenta are given by

$$(p_L, p_R) = \left(\frac{M}{2R} + NR, \frac{M}{2R} - NR\right) = \frac{1}{\sqrt{2}}\left(\frac{Mq}{p} + \frac{Np}{q}, \frac{Mq}{p} - \frac{Np}{q}\right) \tag{40}$$

which determines all the possible local vertex operators at the given radius, in terms of two independent integers M and N.

It follows that

$$\begin{aligned} n + n' &= \frac{Mq}{p} \\ n - n' &= \frac{Np}{q} \end{aligned} \tag{41}$$

Since the left hand sides of both expressions above are integers, it follows that we have ground ring elements only if M, N are multiples of p, q respectively, and in that case we find the constraint that

$$\begin{aligned} n + n' &= 0 \mod q \\ n - n' &= 0 \mod p \end{aligned} \tag{42}$$

Imposing this requirement is clearly equivalent to keeping polynomials in the $SU(2)$ ground ring generators which are invariant under under the $Z_p \otimes Z_q$ action

$$\begin{aligned} x\bar{x} &\to e^{2\pi i/q}\ x\bar{x} \qquad & y\bar{y} &\to e^{-2\pi i/q}\ y\bar{y} \\ x\bar{y} &\to e^{2\pi i/p}\ x\bar{y} \qquad & y\bar{x} &\to e^{-2\pi i/p}\ y\bar{x} \end{aligned} \tag{43}$$

This in turn can be obtained by defining the following action on the chiral generators:

$$\begin{aligned} x &\to e^{i\pi/p}e^{i\pi/q}\ x \qquad & \bar{x} &\to e^{-i\pi/p}e^{i\pi/q}\ \bar{x} \\ y &\to e^{-i\pi/p}e^{-i\pi/q}\ y \qquad & \bar{y} &\to e^{i\pi/p}e^{-i\pi/q}\ \bar{y} \end{aligned} \tag{44}$$

It is easy to check that in the non-chiral ring there are five basic invariants:

$$V = (x\bar{x})^q \qquad W = (y\bar{y})^q \qquad X = (x\bar{y})^p \qquad Y = (y\bar{x})^p \qquad Z = x\bar{x}y\bar{y} \tag{45}$$

satisfying the two relations

$$\begin{aligned} VW &= Z^q \\ XY &= Z^p \end{aligned} \tag{46}$$

Thus we have a 3-variety given by two polynomial relations in $\mathbf{R}^5$, as long as $p, q \neq 1$. This is qualitatively different from the A-D-E cases. Note also that, as expected, the generator $\frac{1}{2}(x\bar{x} + y\bar{y})$ is absent since it is invariant only under $SU(2)$ transformations for which $(\bar{x}, \bar{y})$ transform as the complex conjugates of (x, y). The transformation in Eq.(44) does not have this property, hence it does not preserve the interpretation that we have been using above of $(\bar{x}, \bar{y})$ as the complex conjugate coordinates of (x, y). That interpretation was necessary only to make contact with the Kleinian theory, which we cannot do anyway at rational radii other than integers.

In the integer-radius case $q = 1$, the first relation above can be solved for Z, and we then have four generators satisfying one relation. A trivial redefinition leads to Eq.(27). In this case, $(\bar{x}, \bar{y})$ transform as the complex conjugates of (x, y). The other case, $p = 1$, gives the dual result with the role of electric and magnetic operators interchanged.

For radii which are irrational multiples of $\frac{1}{\sqrt{2}}$ on the circle line, the only degenerate fields in the matter theory are those with zero momentum. It follows that the only generator of the ground ring is $x\bar{x}y\bar{y}$, and it satisfies no relation. Thus the ground ring here is the free polynomial ring on one generator.

On the orbifold line, at rational radius, we must mod out by the transformation Eq.(44) as well as the orbifold symmetry $x\bar{x} \leftrightarrow$, $x\bar{y} \leftrightarrow y\bar{x}$. This gives us the set of four generators

$$
\begin{aligned}
W &= \frac{1}{2}\left((x\bar{x})^q + (y\bar{y})^q\right) \\
X &= \frac{1}{2}\left((x\bar{y})^p + (y\bar{x})^p\right) \\
Y &= \frac{1}{4}\left((x\bar{x})^q - (y\bar{y})^q\right)\left((x\bar{y})^p - (y\bar{x})^p\right) \\
Z &= x\bar{x}y\bar{y}
\end{aligned} \tag{47}
$$

with the relation

$$
Y^2 = (W^2 - Z^q)(X^2 - Z^p) \tag{48}
$$

which is a nice generalization of Eq.(31), reducing to that equation when $q = 1$.

At irrational radius on the orbifold line, the situation is the same as for irrational radius on the circle. The ground ring has a single generator $x\bar{x}y\bar{y}$ with no relation.

Finally, we consider infinite radius. This has been studied in [5], where the ground ring was found to be generated by $x\bar{x}$ and $y\bar{y}$ with no relation. This can be obtained as a special case of Eq.(26) above in the limit $n \to \infty$, for which the generators X and Y disappear (consistent with the fact that magnetic operators cannot exist for a noncompact boson) and we are left with W and Z of that equation with no relation between them.

This completes the analysis of ground rings for $c = 1$ string theory at all points in the moduli space of the $c = 1$ conformal field theory.

6. Symmetries of $c = 1$ Strings

It is now understood [5] [6] that the unbroken symmetries of $c = 1$ string theory are the volume preserving diffeomorphisms of the variety defined by the ground ring. At the $SU(2)$ point, this is a quadric cone in $\mathbf{R}^4$. The generators of unbroken symmetry transformations have two descriptions, related by descent equations. In one description, appropriate to string field theory, they are local operators in the BRS cohomology of ghost number 1. These are listed in Eq.(7) above, where the index s is the $SU(2)$ isospin and takes all positive integer and half-integer values, while n, n' vary from $-s$ to s and $1-s$ to $s-1$ respectively. The other description (following from descent equations) gives the symmetry generators as integrals of conserved spin-1 currents, but we will not need that here.

Now we may ask which of the symmetry generators survive when we pass to other points in the $c = 1$ CFT moduli space, particularly the A-D-E points. It has been shown at the $SU(2)$ point [5] [6], that the symmetry generators can be thought of as polynomial vector fields acting on the ground ring. For example, the generators in Eq.(7) have the description

$$Y^{+}_{s,n}(z)\overline{\mathcal{O}}_{s-1,n'}(\bar{z}) \;\sim\; H_{s-1,n,n'}(x,y,\bar{x},\bar{y})\left((s-n)x\frac{\partial}{\partial x} - (s+n)y\frac{\partial}{\partial y}\right) \tag{49}$$

where

$$H_{s,n,n'} \equiv x^{s+n}y^{s-n}\bar{x}^{s+n'}\bar{y}^{s-n'} \tag{50}$$

Let us first re-write these in terms of the variables a_i defined in Eq.(2). Since these variables are constrained by the relation(3), there is no unique representation, but one finds that all polynomial vector fields of the form

$$a_1{}^{\alpha_1}a_2{}^{\alpha_2}a_3{}^{\alpha_3}a_4{}^{\alpha_4}\left((1+\alpha_2+\alpha_4)\left(a_1\frac{\partial}{\partial a_1}+a_3\frac{\partial}{\partial a_3}\right)-(1+\alpha_1+\alpha_3)\left(a_2\frac{\partial}{\partial a_2}+a_4\frac{\partial}{\partial a_4}\right)\right) \tag{51}$$

modulo the relation $a_1a_2 = a_3a_4$, are exactly the same set as those in Eq.(50). (The α_i are arbitrary non-negative integers.)

More generally, we construct the vector fields

$$\prod_{i=1}^{4} a_i^{\alpha_i}\sum_{i=1}^{4}\left(c_i a_i\frac{\partial}{\partial a_i}\right) \tag{52}$$

with two requirements: one is that they have vanishing divergence, so that they are volume-preserving, and the other is that the coefficients c_i (which depend on the α_i) should be invariant under the transformation

$$\alpha_1 \to \alpha_1 + 1 \quad \alpha_2 \to \alpha_2 + 1 \quad \alpha_3 \to \alpha_3 - 1 \quad \alpha_4 \to \alpha_4 - 1 \tag{53}$$

It is easy to see that these are all the symmetries of the type in Eq.(7) and their conjugates. The "new" symmetries in Eq.(8) cannot be represented in this way since they act in the 4-dimensional space $\mathbf{C}^2$ in such a way as to be trivial after projection to the $U(1)$ quotient, the true (3-dimensional) ground ring.

With this description of the symmetries at the $SU(2)$ point, it is straightforward to generalise them to the A-D-E points discussed above. For each case, we define polynomial vector fields

$$X^{\alpha_X} Y^{\alpha_Y} Z^{\alpha_Z} W^{\alpha_W} \left(c_X X \frac{\partial}{\partial X} + c_Y Y \frac{\partial}{\partial Y} + c_Z Z \frac{\partial}{\partial Z} + c_W W \frac{\partial}{\partial W} \right) \tag{54}$$

and subject them to the divergence-free condition, along with the polynomial relation between X, Y, Z, W defining the ground ring. This gives all the residual symmetries of the type of Eq.(7) at the relevant point.

It is straightforward to carry this out explicitly at the A_n points. Starting with the generators X, Y, Z, W defined in Eq.(26), we define the linear combinations $\tilde{W} = W + Z$, $\tilde{Z} = W - Z$. The subset of symmetry generators of type $Y^+ \overline{\mathcal{O}}$ which survive at the A_n point is given by

$$\begin{aligned} \tilde{W}^{\alpha_{\tilde{W}}} \tilde{Z}^{\alpha_{\tilde{Z}}} X^{\alpha_X} Y^{\alpha_Y} \Bigg((1 + \alpha_{\tilde{Z}} + n\alpha_Y) \left(\tilde{W} \frac{\partial}{\partial \tilde{W}} + nX \frac{\partial}{\partial X} \right) & \\ - (1 + \alpha_{\tilde{W}} + n\alpha_X) \left(\tilde{Z} \frac{\partial}{\partial \tilde{Z}} + nY \frac{\partial}{\partial Y} \right) \Bigg) & \end{aligned} \tag{55}$$

Thus our classification of ground rings at the A-D-E points in the $c = 1$ moduli space leads naturally to a description of the unbroken symmetry generators at these points. Symmetries of the type Eq.(7) turn out to be the H-invariant polynomial vector fields on the associated Kleinian singularities, which preserve the holomorphic volume form. Equivalently they are the volume-preserving vector fields on the three-variety $\mathbf{C}^2/(\Gamma \otimes U(1))$.

7. HyperKähler Manifolds and the Moduli of $c = 1$ String Theory

In this section we attempt to explore the physical consequences of the A-D-E classification of ground rings of the $c = 1$ string via Kleinian singularities.

In Ref.[6], (see also [23]), an effective action for all the discrete modes of the $c = 1$ string at the $SU(2)$ point was written down. This action is built out of a closed 2-form $F = dA$ and a scalar σ which are functions on $\mathbf{C}^2$, with the action

$$\int_{C^2} \sigma \, F \wedge F \tag{56}$$

There is a constraint that F should be a polynomial 2-form, representing the moduli of the theory at the $SU(2)$ point with the positive Liouville dressing, while σ is some derivative of a δ-function at the origin of $\mathbf{C}^2$. Below, we will try to explore what sort of solutions this action has if we relax this constraint.

Recall that both F and σ should respect the left-right matching of Liouville momenta, since the Liouville field is noncompact. This means that they should be invariant under the vector field H defined in Eq.(25). Equivalently, they should be expressible in terms of the a_i, on which this constraint is automatically implemented. This constraint implies that the 4-dimensional action written above projects down to a 3-dimensional action

$$\int_{C^2/U(1)} \sigma \; du \wedge da \tag{57}$$

where the original 4-dimensional gauge field A is dimensionally reduced to a scalar u and a 3-dimensional 1-form a, and the action lives on the quotient of $\mathbf{C}^2$ by the $U(1)$ transformation defined in Eq.(24).

Now the results that we have discussed in this paper provide us with the effective action at any other of the A-D-E points in the moduli space. It is just

$$\int_{C^2/\Gamma} \sigma\; F \wedge F = \int_{C^2/(\Gamma \otimes U(1))} \sigma\; du \wedge da \tag{58}$$

where the 4-dimensional space on the left-hand side is precisely the Kleinian singularity for the relevant subgroup Γ, while the 3-dimensional space on the right is defined by the true non-chiral ground ring.

The equation of motion from this action,

$$F \wedge F = 0, \qquad dF = 0 \tag{59}$$

is to be solved on a non-trivial space, unlike at the $SU(2)$ point. As a result, if the topology of this space is nontrivial, we should expect to find solutions for F which are closed but not exact. (At the $SU(2)$ point, the solutions of this equation are given by the (old and new) moduli of Ref.[6], all of which are closed since they are annihilated by b_0^-, but also exact since they can be expressed as b_0^- on some other operator. Hence they are trivial in the d-cohomology, which is expected since $\mathbf{C}^2$ is a topologically trivial manifold.)

This equation is well-known and has some very interesting solutions (for a recent review, see Ref.[19]). These are, however, not generally H-invariant. Indeed, on a general 4-manifold, one can write a class of solutions of the form

$$F = \omega + i\lambda G + \lambda^2 \bar{\omega} \tag{60}$$

where, with respect to some complex coordinates on this 4-manifold, ω and $\bar{\omega}$ are closed (2,0) and (0,2) forms, G is a closed (1,1) form, and λ is a parameter. The equation $F \wedge F = 0$ turns into

$$G \wedge G = 2\omega \wedge \bar{\omega} \tag{61}$$

If we interpret ω as a holomorphic volume form and G as a $(1,1)$ Kähler form, then this equation defines a Ricci-flat hyperKähler manifold, which in 4 dimensions is equivalent to self-dual gravity. Indeed, the above equation is the Plebanski equation [24], which has in particular been discussed in some detail in Ref.[20] in the context of $N = 2$ strings. We will see below that while the $(1, 1)$ part of F may be chosen consistent with H-invariance, the $(2, 0)$ part, which is associated to the volume form, is not H-invariant.

To study this in more detail, note that on a complex manifold the Kähler form G can be written

$$G = \partial\bar{\partial}\Phi(z^i, \bar{z}^i) \tag{62}$$

where z^i are complex coordinates on the manifold and $\Phi(z^i, \bar{z}^i)$ is the Kähler potential. Inserting this into Eq.(61) leads to

$$\partial\bar{\partial}\Phi \wedge \partial\bar{\partial}\Phi = 2\omega \wedge \bar{\omega} \tag{63}$$

Thus the dynamics of this system can be defined in terms of a single scalar function on the manifold (although this is not globally defined, but changes across coordinate patches).

In $c = 1$ string theory at the $SU(2)$ point, we can use the ground ring generators x, y as the complex coordinates. The conventional moduli have the following description as 2-forms:

$$Y^+_{s,n}\overline{Y}^+_{s,n'} \;\sim\; dh_{s,n} \wedge d\bar{h}_{s,n'} \tag{64}$$

where $h_{s,n}$ is defined in Eq.(5) and $\bar{h}_{s,n'}$ is its antiholomorphic counterpart. Now the above expression can be written as a closed $(1, 1)$ form of the type of Eq.(62) above, where

$$\Phi(x, y, \bar{x}, \bar{y}) = H_{s,n,n'} \equiv h_{s,n}\bar{h}_{s,n'} \tag{65}$$

A simple example is

$$\Phi(x, y, \bar{x}, \bar{y}) = x\bar{x} + y\bar{y} \tag{66}$$

The corresponding modulus is

$$Y^{+}_{\frac{1}{2},\frac{1}{2}}\overline{Y}^{+}_{\frac{1}{2},\frac{1}{2}}+Y^{+}_{\frac{1}{2},-\frac{1}{2}}\overline{Y}^{+}_{\frac{1}{2},-\frac{1}{2}}\ \sim\ dx\wedge d\bar{x}+dy\wedge d\bar{y} \tag{67}$$

which is the Kähler form for the flat metric on $\mathbf{C}^2$. In fact, all the moduli of this type are exact at the $SU(2)$ point, so none of them can be the Kähler form for a nontrivial metric.

We note in passing that the above modulus, corresponding to the flat Kähler metric, exists at all the A-D-E points in the moduli space, but nowhere else. Indeed, the Kähler potential Φ in Eq.(66) is precisely the generator W of the polynomial ring at all these points.

Consider now the "new moduli" of the type

$$\begin{aligned}(a+\bar{a})\mathcal{O}_{s-1,n}\overline{Y}^{+}_{s,n'}\ \sim H_{s-1,n,n'}\big(\bar{x}\bar{y}\,dx\wedge dy+\frac{s+n'}{2s}y\bar{y}\,dx\wedge d\bar{x}+\frac{s-n'}{2s}y\bar{x}\,dx\wedge d\bar{y}\\ -\frac{s+n'}{2s}x\bar{y}\,dy\wedge d\bar{x}-\frac{s-n'}{2s}x\bar{x}\,dy\wedge d\bar{y}\big)\end{aligned} \tag{68}$$

discovered in [6]. These are the sum of a $(2,0)$ and a $(1,1)$ piece. The sum is closed (and exact) but the individual pieces are not. Thus it is not possible to identify any of the new moduli with the closed $(2,0)$ piece ω in Eq.(60) describing the volume form $dx\wedge dy$. This is anyhow obvious since $dx\wedge dy$ is not an H-invariant form.

One may ask which new modulus would correspond to this volume form if we relaxed the condition of H-invariance. It was noted in [6] that in the chiral cohomology, there is a special operator

$$\tilde{\imath}=c\partial c e^{\sqrt{2}\phi} \tag{69}$$

which plays the role of $a\cdot Y^{+}_{0,0}$, and is represented as a constant bivector on the ground ring, or equivalently a constant function in the dual (form) representation. Starting with the antiholomorphic counterpart of this, there is no ground ring element in the holomorphic sector which can be paired with it to form an H-invariant modulus, since the minimum isospin s for new moduli is 1, from Eq.(68). However, if Liouville momentum matching were not a constraint, we could have chosen an element like $(a + \bar{a})\mathcal{O}_{0,0}\overline{Y}^{+}_{0,0}$ which is precisely $dx \wedge dy$. This shows explicitly that, to obtain the geometric interpretation of Eq.(60), one must relax the H-invariance condition, which corresponds to taking solutions of the four-dimensional action which do not descend to solutions of the three-dimensional one.

Thus, even at the $SU(2)$ point, solutions of the equation of motion (59) of the type (60), which have a geometric interpretation, contain an H-noninvariant piece ω. We have exhibited a solution of this type, which corresponds to the (trivial) hyperKähler geometry of flat $\mathbf{C}^2$. At the other A-D-E points, solutions exist which correspond to non-trivial hyperKähler geometry, as we discuss below.

It is tempting to think of H-invariant solutions as time-independent solutions of Eq.(59), while the nontrivial hyperKähler manifolds which also solve the equations of motion could be thought of as time-dependent solutions, rather like instantons.

Recall now that the Abelian gauge field strength F appearing in the $c = 1$ string effective action is really a collection of the degrees of freedom corresponding to all the discrete states in the theory. The discussion above suggests that the dynamics of these modes is that of *self-dual gravity in a target space defined by the ground ring of the theory*, which is four-dimensional in the sense discussed above.

Now to explicitly find nontrivial solutions of the equation of motion for the action(58) for a given subgroup Γ of $SU(2)$, we need a class of hyperKähler metrics consistent with the topology of $\mathbf{C}^2/\Gamma$. Remarkably, this problem has been completely solved and leads to a very beautiful result [16] [17]. To every Kleinian variety $\mathbf{C}^2/\Gamma$, there corresponds a finite-parameter family of hyperKähler manifolds obtained by resolving the singularity at the origin. These manifolds are precisely the gravitational instantons [15]. The first in the series (corresponding to $\Gamma = \mathcal{C}_2$) is the Eguchi-Hanson instanton [13], while $\Gamma = \mathcal{C}_n$ is in correspondence with the multi-center instanton metrics of Gibbons and Hawking [14] [16]. More general instanton solutions are known corresponding to blowing up the singularities of the D and E-series Kleinian varieties [17] (see also [18]). These instantons are all asymptotically locally Euclidean (ALE) spaces, which means that at infinity their topology is that of S^3/Γ. In each such case there is a finite-parameter moduli space of solutions (this is $3n-6$-dimensional in the $\mathcal{C}_n$ case, for example).

Thus we conclude that at each of the A-D-E points in the moduli space of $c=1$ string theory, associated with the discrete subgroups Γ of $SU(2)$, there is a finite-parameter family of nontrivial classical solutions of the four-dimensional string action corresponding precisely to the blown up version of the Kleinian variety $\mathbf{C}^2/\Gamma$, which plays the role of the "target space" of the theory, and these solutions are the various hyperKähler metrics consistent with the topology of this space.

The principal problem with this interpretation, as we have noted above, is that it appears to be inconsistent with H-invariance, which itself is a consequence of the noncompactness of the Liouville field. We need a better understanding of the physical meaning of H-invariance, and possibly a compactified Liouville field, from the ground-ring target-space point of view.

8. Conclusions

We have shown that the A-D-E classification of "special points" in the moduli space of the $c=1$ free boson conformal field theory results in an analogous classification of ground rings at these points when the CFT is coupled to 2D gravity to form a string theory. The ground rings and their corresponding algebraic varieties so obtained are closely related to the singular varieties studied by Klein in connection with the discrete subgroups of $SU(2)$. The unbroken symmetries of the theory at each such point form the algebra of volume-preserving diffeomorphisms of these varieties.

We have studied the theory at zero cosmological constant. It would be interesting to see how the polynomial ground rings at various points in the moduli space are deformed under cosmological perturbation. At the $SU(2)$ point, the singularity at the origin is smoothened out under this perturbation [5] (see also Refs.[25, 26]). However, at the other points we have shown that the singularities are not isolated but lie along curves, so it is no longer clear that a cosmological perturbation will produce a smooth manifold.

Perhaps the most interesting consequence of our results appears in the relation of Kleinian singularities to hyperKähler surfaces, which are solutions of self-dual gravity in four dimensions. Mathematically, resolving the singularity at the origin produces families of hyperKähler metrics for each Kleinian variety. These metrics are known in four-dimensional Einstein gravity, where they appear as gravitational instantons, and are interpreted (as in Yang-Mills theory) as tunneling solutions between distinct vacua of the theory. It is clearly important to understand whether they play an analogous or different role in the context of string theory. In any case it is very suggestive that the collection of discrete states in $c = 1$ string theory has an effective action (in the ground ring space) which has gravity-like solutions, as this seems to be quite different from the role normally played by gravitation in the context of strings. A dynamical understanding of this phenomenon would give us some clues about the underlying structure of string theory.

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