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Equivalence of quotient Hilbert modules

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Abstract. Let $\mathscr{M}$ be a Hilbert module of holomorphic functions over a natural function algebra $\mathscr{A}(\Omega)$, where $\Omega \subseteq \mathbb{C}^m$ is a bounded domain. Let $\mathscr{M}_0 \subseteq \mathscr{M}$ be the submodule of functions vanishing to order k on a hypersurface $\mathscr{Z} \subseteq \Omega$. We describe a method, which in principle may be used, to construct a set of complete unitary invariants for quotient modules $\mathscr{Q} = \mathscr{M} \ominus \mathscr{M}_0$. The invariants are given explicitly in the particular case of $k = 2$.

Keywords. Hilbert modules; function algebra; quotient module; longitudinal and transversal curvature; kernel function; jet and angle.

1. Preliminaries

Let Ω be a bounded domain in $\mathbb{C}^m$ and $\mathscr{Z} \subseteq \Omega$ be an analytic hypersurface defined (at least, locally) as the zero set of a single analytic function φ. Let $\mathscr{A}(\Omega)$ be the algebra of functions obtained by taking the closure with respect to the supremum norm on Ω of all functions which are holomorphic on a neighbourhood of Ω. Let $\mathscr{M}$ be a Hilbert space consisting of holomorphic functions on Ω. We assume that the evaluation functionals $h \to h(w),\ h \in \mathscr{M},\ w \in \Omega$ are bounded. This ensures, via the Riesz representation theorem, that there is a unique vector $K(\cdot,w) \in \mathscr{M}$ satisfying the reproducing property

$$h(w) = \langle h, K(\cdot,w)\rangle,\ h \in \mathscr{M},\ w \in \Omega.$$

In this paper, a module $\mathscr{M}$ over the function algebra $\mathscr{A}(\Omega)$ will consist of a Hilbert space $\mathscr{M}$ as above together with a continuous action of the algebra $\mathscr{A}(\Omega)$ in the sense of ([8], Definition 1.2). Suppose, we are given a quotient module $\mathscr{Q}$ over the function algebra $\mathscr{A}(\Omega)$. This amounts to the existence of a resolution of the form

$$0 \longleftarrow \mathscr{Q} \longleftarrow \mathscr{M} \longleftarrow \mathscr{M}_0 \longleftarrow 0, \tag{1}$$

where $\mathscr{M}_0 \subseteq \mathscr{M}$ are both modules over the algebra $\mathscr{A}(\Omega)$. We make the additional assumption that the submodule $\mathscr{M}_0$ consists of functions in $\mathscr{M}$ which vanish to some fixed order k on the hypersurface $\mathscr{Z}$. Then (cf. [7], (1.5)) the module $\mathscr{M}_0$ may be described as

$$\mathscr{M}_0 = \left\{ f \in \mathscr{M} : \frac{\partial^\ell f}{\partial z_1^\ell}(z) = 0,\ z \in U \cap \mathscr{Z},\ 0 \le \ell \le k-1 \right\},$$

where U is some open subset of Ω.

Let ∂ denote the differentiation along the unit normal to the hypersurface $\mathscr{Z}$. Recall (cf. [7]) that the map $J : \mathscr{M} \to \mathscr{M} \otimes \mathbb{C}^k$ defined by

$$h \mapsto (h, \partial h, \partial^2 h, \ldots, \partial^{k-1} h),\ h \in \mathscr{M}$$

plays a crucial role in identifying the quotient module. The requirement that

$$\{(e_n, \partial e_n, \ldots, \partial^{k-1} e_n)_{n\geq 0} : (e_n)_{n\geq 0} \text{ is an orthonormal basis in } \mathscr{M}\ \}$$

is an orthonormal basis in $\operatorname{ran} J$, makes the map J unitary onto its range $J\mathscr{M} \subseteq \mathscr{M} \otimes \mathbb{C}^k$. Thus we obtain a pair of modules $J\mathscr{M}_0$ and $J\mathscr{M}$, where $J\mathscr{M}_0$ is the submodule of all functions in $J\mathscr{M}$ which vanish on $\mathscr{Z}$. In this realisation, the module $J\mathscr{M}$ consists of holomorphic functions taking values in $\mathbb{C}^k$. Let $\mathbb{C}^{k\times k}$ denote the linear space of all $k \times k$ matrices over the field of complex numbers. We recall that a function $K : \Omega \times \Omega \to \mathbb{C}^{k\times k}$ satisfying

$$\sum_{i,j=1}^{n} \Big\langle K(\omega_i, \omega_j)\zeta_j, \zeta_i \Big\rangle_E \ \geq\ 0,\ \ w_1, \ldots, \omega_n \in \Omega,\ \ \zeta_1, \ldots, \zeta_n \in E, n \geq 0 \tag{2}$$

is said to be a *nonnegative definite* (*nnd*) *kernel* on Ω. Given such an nnd kernel K on Ω, it is easy to construct a Hilbert space $\mathscr{M}$ of functions on Ω taking values in $\mathbb{C}^{k\times k}$ with the property

$$\Big\langle f(\omega), \zeta \Big\rangle_{\mathbb{C}^k} = \Big\langle f, K(\cdot, \omega)\zeta \Big\rangle,\ w \in \Omega,\ \zeta \in \mathbb{C}^k,\ f \in \mathscr{M}. \tag{3}$$

The Hilbert space $\mathscr{M}$ is simply the completion of the linear span of all vectors of the form $K(\cdot, \omega)\zeta$, $\omega \in \Omega$, $\zeta \in \mathbb{C}^k$, with inner product defined by (3). Conversely, let $\mathscr{M}$ be a Hilbert space of functions on Ω taking values in $\mathbb{C}^k$. Let $e_\omega : \mathscr{M} \to \mathbb{C}^k$ be the evaluation functional defined by $e_\omega(f) = f(\omega)$, $\omega \in \Omega$, $f \in \mathscr{M}$. If e_ω is bounded for each $\omega \in \Omega$, then it is easy to verify that the Hilbert space $\mathscr{M}$ possesses a reproducing kernel $K(z,\omega) = e_z e_\omega^*$, that is, $K(z,\omega)\zeta \in \mathscr{M}$ for each $\omega \in \Omega$ and K has the reproducing property (3). Finally, the reproducing property (3) determines the reproducing kernel K uniquely. If e_n is an orthonormal basis in $\mathscr{M}$ then it is not hard to verify that the reproducing kernel K has the representation

$$K(z, w) = \sum_{n=0}^{\infty} e_n(z) e_n(w)^*,\ \ z, w \in \Omega,$$

where $e_n(z)$ is thought of as a linear map from $\mathbb{C}$ to $\mathbb{C}^k$. Of course, this sum is independent of the choice of the orthonormal basis e_n since K is uniquely determined.

The module $J\mathscr{M}$ possesses a reproducing kernel JK in the sense described above. It is natural to construct this kernel by forming the sum:

$$JK(z, w) = \sum_{n=0}^{\infty} (Je_n)(z)(Je_n)(w)^*,\ z, w \in \Omega.$$

This prescription then allows the identification of the reproducing kernel $JK : \Omega \times \Omega \to \mathbb{C}^{k\times k}$ for the module $J\mathscr{M}$:

$$(JK)_{\ell,j}(z, w) = \big(\partial^\ell \bar{\partial}^j K\big)(z, w),\quad 0 \leq \ell, j \leq k-1. \tag{4}$$

It is then easy to verify, using the unitarity of the map J, that JK has the reproducing property:

$$\langle h, JK(\cdot,w)\zeta\rangle = \langle h(w),\zeta\rangle,\ w\in\Omega,\ \zeta\in\mathbb{C}^k.$$

The module action for $J\mathscr{M}$ is defined in a natural manner. Indeed, let Jf be the array

$$(Jf)_{\ell,j} = \begin{cases} \binom{\ell}{j}(\partial^{\ell-j}f), & 0\le \ell\le j\le k-1 \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

for $f\in\mathscr{A}(\Omega)$. We may now define the module action to be $J_f : h\to Jf\cdot Jh$. Notice that Jf is a $k\times k$ matrix-valued function on Ω while J_f is the module action, that is, it is an operator on $J\mathscr{M}$. The action of the adjoint is then easily seen to be

$$J_f^* JK(\cdot,w)\cdot \mathbf{x} = JK(\cdot,w)(Jf)(w)^*\cdot\mathbf{x},\ \mathbf{x}\in\mathbb{C}^k. \tag{6}$$

We will say that two modules over the algebra $\mathscr{A}(\Omega)$ are *isomorphic* if there exists a unitary module map between them.

It is shown in [7] that the quotient module $\mathscr{Q}$ is isomorphic to $J\mathscr{M}\ominus J\mathscr{M}_0$. Once this is done, we are reduced to the multiplicity free case. Thus our previous results from [6] apply and we conclude that the quotient module $\mathscr{Q}$ is the restriction of $J\mathscr{M}$ to the hypersurface $\mathscr{Z}$.

Let $\mathscr{M}$ be any Hilbert module over the function algebra $\mathscr{A}(\Omega)$. In particular, each of the coordinate functions $z_i,\ 1\le i\le m$ in $\mathbb{C}^m$ acts boundedly as the multiplication operator M_i on $\mathscr{M}$. Let $\mathbf{M}$ denote this commuting m-tuple of multiplication operators. We denote by $\mathbf{M}^*$ the m-tuple $(M_1^*,\dots,M_m^*)$. To each m-tuple $\mathbf{M}$, we associate the operator $D_{\mathbf{M}} : \mathscr{M}\to\mathscr{M}\otimes\mathbb{C}^k$ defined by $D_{\mathbf{M}}h = (M_1h,\dots,M_mh)$, $h\in\mathscr{M}$.

The class $B_n(\Omega)$ was introduced in [3] for a single operator. This definition was then adapted to the general case of an m-tuple of commuting operators (cf. [4]). We let $\Omega^*\subseteq\mathbb{C}^m$ denote the domain $\{w\in\mathbb{C}^m : \bar{w}\in\Omega\}$ and say that $\mathbf{M}^*$ is in $B_k(\Omega^*)$ if

(i) Ran $D_{\mathbf{M}^*-w}$ is closed for all $w\in\Omega^*$,
(ii) span $\{\ker D_{\mathbf{M}^*-w} : w\in\Omega^*\}$ is dense in $\mathscr{M}$,
(iii) dim $\ker D_{\mathbf{M}^*-w} = n$ for all $w\in\Omega^*$,

where $\mathbf{M}^*-w = (M_1^*-w_1,\dots,M_m^*-w_m)$.

If the adjoint of the m-tuple of multiplication operators is in $B_n(\Omega^*)$ (for some $n\in\mathbb{N}$), then we say that $\mathscr{M}$ is in $B_n(\Omega^*)$. The assumption that $\mathscr{M}$ is in $B_1(\Omega^*)$ includes, among other things, (a) the existence of a common eigenvector $\gamma(w)\in\mathscr{M}$, that is, $M_i^*\gamma(w) = \bar{w}_i\gamma(w)$, for $w\in\Omega^*$, (b) the dimension of the common eigenspace at $\bar{w}$ is 1. Furthermore, it is possible to choose $\gamma(w)$ so as to ensure that the map $w\to\gamma(w)$ is anti-holomorphic. Thus we obtain an anti-holomorphic hermitian line bundle E over Ω whose fiber at w is the one-dimensional subspace of $\mathscr{M}$ spanned by the vector $\gamma(w)$, that is, γ is an anti-holomorphic frame for E. In the case of $n>1$, a similar construction of an anti-holomorphic hermitian vector bundle of rank n can be given. In our case, it is easy to verify that $K(\cdot,w)$, the reproducing kernel at w, is a common eigenvector for the m-tuple $(M_1^*,\dots,M_m^*)$. Since $K(\cdot,w)$ is anti-holomorphic in the second variable, it provides a natural frame for the associated bundle E. The metric with respect to this frame is obviously the real analytic function $K(w,w)$.

Before we continue, we make the additional assumption that the module $\mathscr{M}$, which occurs in the resolution (1) of the quotient module $\mathscr{Q}$, lies in the class $B_1(\Omega^*)$. Let $i : \mathscr{Z} \to \Omega$ be the inclusion map and $i^* : \mathscr{A}(\Omega) \to \mathscr{A}(\mathscr{Z})$ be the pullback. Then $\mathscr{Q}$ is clearly also a module over the smaller algebra $i^*\big(\mathscr{A}(\Omega)\big)$. We identify this latter algebra with $\mathscr{A}(\mathscr{Z})$. Let $(\mathscr{Q}, \mathscr{A}(\mathscr{Z}))$ stand for $\mathscr{Q}$ thought of as a module over the smaller algebra $\mathscr{A}(\mathscr{Z})$. Although it is possible that $(\mathscr{Q}, \mathscr{A}(\mathscr{Z}))$ lies in $B_k(\mathscr{Z}^*)$ whenever $\mathscr{M}$ is in $B_1(\Omega^*)$, we were able to prove it only in some special cases ([7], Proposition 3.6). However, in this paper, we assume that the quotient module $(\mathscr{Q}, \mathscr{A}(\mathscr{Z}))$ always lies in $B_k(\mathscr{Z}^*)$. These assumptions make it possible to associate (a) an anti-holomorphic hermitian line bundle E over the domain Ω with the module $\mathscr{M}$ and (b) an anti-holomorphic jet bundle $JE_{|\text{res }\mathscr{Z}}$ of rank k over the domain $\mathscr{Z}$ with the module $(\mathscr{Q}, \mathscr{A}(\mathscr{Z}))$. The details of the *jet* construction are given in ([7], pp. 375–377). One of the main results in [3] states that two modules $\mathscr{M}$ and $\tilde{\mathscr{M}}$ in $B_k(\Omega)$ are isomorphic if and only if the associated bundles are locally equivalent. While the local equivalence of bundles is completely captured in the case of line bundles by the curvature, it is more complicated in the general case (cf. [3]). We recall that the quotient module $\mathscr{Q}$ may be described completely by specifying the action of the algebra $\mathscr{A}_k(\mathscr{Z}) := \mathscr{A}(\mathscr{Z}) \otimes \mathbb{C}^{k\times k}$ (cf. [7], p. 385). The action of the algebra $\mathscr{A}_k(\mathscr{Z})$, in particular, includes the multiplication induced by the local defining function φ, namely,

$$(J\varphi)_{|\text{res }\mathscr{Z}} : J\mathscr{M}_{|\text{res }\mathscr{Z}} \to J\mathscr{M}_{|\text{res }\mathscr{Z}}.$$

To exploit methods of [3], it is better to work with the adjoint action. To describe the adjoint action, we first construct a natural anti-holomorphic frame (*not necessarily* orthonormal) for the jet bundle E on Ω. Let $\{\varepsilon_\ell : 1 \le \ell \le k\}$ be the standard orthonormal basis in $\mathbb{C}^k$. For a fixed $w \in \Omega$, let $e_1 = \sum_{\ell=1}^{k} \partial^{\ell-1} K(z,w) \otimes \varepsilon_\ell$ be simply the image of $K(z,w)$ in $J\mathscr{M}$. It is then clear that $\{e_j(w) : 1 \le j \le k\}$, where $e_j(w) := (\bar{\partial}^{j-1} e_1)(w)$ is a natural anti-holomorphic frame for JE. (Of course, as is to be expected, $e_\ell(w)$, $1 \le \ell \le k$ are the columns of the reproducing kernel JK given in (4).) Thus the fiber of the jet bundle JE at $w \in \Omega$ is spanned by the set of vectors $\{e_\ell(w) \in J\mathscr{M} : 1 \le \ell \le k\}$.

Suppose we start with a resolution of the form (1). Then we have at our disposal the domain $\Omega \subseteq \mathbb{C}^m$ and the hypersurface $\mathscr{Z} \subseteq \Omega$. Let φ be a local defining function for $\mathscr{Z}$ (cf. [7], p. 367). Then φ lies in $\mathscr{A}(\mathscr{Z})$ and induces a nilpotent action on each fiber of the jet bundle $JE_{|\text{res }\mathscr{Z}}$ via the map J_φ^*, that is,

$$(J_\varphi^* e_\ell)(w) = JK(\cdot, w)(J\varphi)(w)^* \varepsilon_\ell. \tag{7}$$

Therefore in this picture, with the assumptions we have made along the way, we see that the *quotient modules* $\mathscr{Q}$ must meet the requirement listed in (i)–(iii) of the following Definition.

DEFINITION.

We will say that the module $\mathscr{Q}$ over the algebra $\mathscr{A}(\Omega)$ is a *quotient module* in the class $B_k(\Omega, \mathscr{Z})$ if

(i) there exists a resolution of the module $\mathscr{Q}$ as in eq. (1), where the module $\mathscr{M}$ appearing in the resolution is required to be in $B_1(\Omega^*)$,
(ii) the module action on $\mathscr{Q}$ translates to the nilpotent action J_φ on $J\mathscr{M}_{|\text{res }\mathscr{Z}}$ which is an isomorphic copy of $\mathscr{Q}$,
(iii) the module $\big(\mathscr{Q}, \mathscr{A}(\mathscr{Z})\big)$ is in $B_k(\mathscr{Z}^*)$.

In this paper, we obtain a complete set of unitary invariants for a module $\mathscr{Q}$ in the class $B_2(\Omega, \mathscr{X})$. This means that the module $\mathscr{Q}$ admits a resolution of the form (1) and the module $\mathscr{M}$ that appears in this resolution lies in $B_1(\Omega)$. However, it is possible to considerably weaken this latter hypothesis as explained in the Remark below.

[*Remark.* Although we have assumed the module $\mathscr{M}$ to be in the class $B_1(\Omega)$, it is interesting to note that the proof of our Theorem requires much less. Specifically, the requirement that the 'Ran $D_{\mathbf{M}^*-w}$ is closed' is necessary to associate an anti-holomorphic vector bundle with the module. However, in our case, there is already a natural anti-holomorphic vector bundle which is deteremined by the frame $w \to K(\cdot, w)$. Indeed, if we assume that the module $\mathscr{M}$ contains the linear space $\mathscr{P}$ of all the polynomials and $\mathscr{P}$ is dense in $\mathscr{M}$, then the eigenspace at w is forced to be one dimensional. (To prove this, merely note that for any eigenvector x at w and all polynomials p, we have

$$\langle p, x\rangle = \langle M_p 1, x\rangle = \langle 1, M_p^* x\rangle = p(w)\langle 1, x\rangle = \langle p, cK(\cdot, w)\rangle,$$

where $c = \overline{\langle 1, x\rangle}$. It follows that $x = cK(\cdot, w)$.) Finally, the linear span of the set of eigenvectors $\{K(\cdot, w) : w \in \Omega\}$ is a dense subspace of the module $\mathscr{M}$. Therefore, for our purposes, it is enough to merely assume that

(a) *$\mathscr{M}$ is a Hilbert module consisting of holomorphic functions on Ω,*
(b) *the module $\mathscr{M}$ contains the linear space of all polynomials $\mathscr{P}$ and that $\mathscr{P}$ is dense,*
(c) *$\mathscr{M}$ possesses a reproducing kernel K.*

It is then clear that the same holds for the quotient module $\mathscr{Q}$, where $\mathscr{P}$ consists of $\mathbb{C}^k$-valued polynomials and K takes values in $\mathbb{C}^{k\times k}$. Hence, if x is an eigenvector at w for the module $(\mathscr{Q}, \mathscr{A}(\mathscr{X}))$, we claim that it belongs to the range of $K(., w)$ which is the k-dimensional subspace $\{K(\cdot, w)v \in \mathscr{Q} : v \in \mathbb{C}^k\}$ of $\mathscr{Q}$. As before, for $1 \le j \le k$, let ε_j be the standard unit vector in $\mathbb{C}^k$ and $p = \sum_{j=1}^k p_j \otimes \varepsilon_j$ be a $\mathbb{C}^k$-valued polynomial. Then we have $\langle p, x\rangle = \sum_{j=1}^k \langle M_{p_j}\varepsilon_j, x\rangle = \sum_{j=1}^k \langle \varepsilon_j, M_{p_j}^* x\rangle = \sum_{j=1}^k p_j(w)\langle \varepsilon_j, x\rangle$ $= \sum_{j=1}^k \langle p, K(\cdot, w)\varepsilon_j\rangle\langle \varepsilon_j, x\rangle = \langle p, \sum_{j=1}^k c_j K(., w)\varepsilon_j\rangle$, where $c_j = \overline{\langle \varepsilon_j, x\rangle}$. Thus x is in the range of $K(\cdot, w)$ as claimed. Therefore the dimension of the eigenspace at w equals the dimension of range $K(., w)$ which is k.]

We now raise the issue of adapting the techniques of [3] to find a complete set of unitary invariants for characterizing the quotient modules $\mathscr{Q}$ in the class $B_k(\Omega, \mathscr{X})$. While the methods described below will certainly yield results in the general case, we have chosen to give the details of our results in the case of $k = 2$. The reason for this choice is dictated by the simple nature of these invariants in this case. Furthermore, these are extracted out of the curvature and the canonical metric for the bundle E.

2. Canonical metric and curvature

Let $\mathscr{M}$ be a module in $B_1(\Omega^*)$ and the reproducing kernel $K(\cdot, w)$ be the anti-holomorphic frame for the associated bundle E. If $\tilde{\mathscr{M}}$ is another module in the class $B_1(\Omega^*)$ with reproducing kernel $\tilde{K}(\cdot, w)$, then it is clear that any isomorphism between these modules must map $K(\cdot, w)$ to a multiple $\psi(w)$ of $\tilde{K}(\cdot, w)$, where $\psi(w)$ is a non zero complex number for $w \in \Omega$. Moreover, the map $w \to \psi(w)$ has to be anti-holomorphic. It follows that $\mathscr{M}$

and $\tilde{\mathscr{M}}$ are isomorphic if and only if $\tilde{K}(z,w) = \overline{\psi(z)}K(z,w)\psi(w)$ (cf. [4], Lemma 3.9) for some anti-holomorphic function ψ. There are two ways in which this ambiguity may be eliminated.

The first approach is to note that if the two modules $\tilde{\mathscr{M}}$ and $\mathscr{M}$ are isomorphic, then $\tilde{K}(z,z)/K(z,z) = |\psi(z)|^2$. Since ψ is holomorphic, it follows that

$$\sum_{i,j=1}^{m} \partial_i \bar{\partial}_j \log\big(K(z,z)/\tilde{K}(z,z)\big) dz_i \wedge d\bar{z}_j = 0. \tag{8}$$

On the other hand, if we have two modules for which equation (8) holds, then the preceding argument shows that they must be isomorphic. It is then possible to find, in a small simply connected neighbourhood of some fixed point w_0, a harmonic conjugate $v(w)$ of the harmonic function $u(w) := \log \tilde{K}(w,w)/K(w,w)$. The new kernel defined by $\tilde{\tilde{K}}(z,w) = \exp(u(z)+iv(z))\tilde{K}(z,w)\overline{\exp(u(w)+iv(w))}$ determines a module $\tilde{\tilde{\mathscr{M}}}$ isomorphic to $\tilde{M}$ but with the additional property that the metric $\tilde{\tilde{K}}(w,w) = K(w,w)$. It is then easy to see that the map taking $K(\cdot,w)$ to $\tilde{\tilde{K}}(\cdot,w)$ extends linearly to an isometric module map. Therefore, $\sum_{i,j=1}^{m} \partial_i \bar{\partial}_j \log K(z,z) dz_i \wedge d\bar{z}_j$ is a complete invariant for the module $\mathscr{M}$

The second approach is to normalise the reproducing kernel K, that is, define the kernel $K_0(z,w) = \psi(z)K(z,w)\overline{\psi(w)}$, where $\psi(z) = K(z,w_0)^{-1}K(w_0,w_0)^{1/2}$ for z in some open subset $\Omega_0 \subseteq \Omega$ and some fixed but arbitrary $w_0 \in \Omega_0$. Also, Ω_0 can be chosen so as to ensure $\psi_{|\text{res } \Omega_0} \neq 0$. This reproducing kernel determines a module isomorphic to $\mathscr{M}$ but with the added property that $K_0(z,w_0)$ is the constant function 1. If $\mathscr{M}$ and $\tilde{\mathscr{M}}$ are two modules in $B_1(\Omega^*)$, then it is shown in ([4], Theorem 4.12) that they are isomorphic if and only if the normalisations K_0 and $\tilde{K}_0$ of the respective reproducing kernels at some fixed point are equal. As before, it is then easy to see that the map taking $K(\cdot,w)$ to $\tilde{\tilde{K}}(\cdot,w)$ extends linearly to an isometric module map. The normalised kernel K_0 is therefore a complete unitary invariant for the module $\mathscr{M}$.

Notice that if a module $\mathscr{M}$ is isomorphic to $\tilde{\mathscr{M}}$, then the module map Γ is induced by a nonvanishing function Φ on Ω, that is, $\Gamma = M_\Phi$ ([4], Lemma 3.9). Consequently, if $\mathscr{M}_0$ is the submodule of functions vanishing to order k on $\mathscr{Z}$, then $\Gamma(\mathscr{M}_0)$ is the submodule of functions vanishing to order k in $\tilde{\mathscr{M}}$. It follows that if $\mathscr{M}$ and $\tilde{\mathscr{M}}$ are isomorphic modules, then the corresponding quotient modules must be isomorphic as well. Therefore we can make the following assumption without any loss of generality.

Hypothesis. Now we make a standing hypothesis that the kernel for the module $\mathscr{M}$ appearing in the resolution of the quotient module $\mathscr{Q}$ is normalised.

Recall that if E is a hermitian holomorphic vector bundle of rank k over the domain $\Omega \subseteq \mathbb{C}^m$, then it is possible to find a holomorphic frame $\mathbf{s} = (s_1,\ldots,s_k)$ such that (a) $\langle s_i(w_0), s_j(w_0)\rangle = 1$, (b) $\partial_j \langle \mathbf{s}(w), \mathbf{s}(w)\rangle_{|w=w_0} = 0$ for $1 \leq j \leq m$ (cf. [12], Lemma 2.3). We offer below a variation of this Lemma for the jet bundle JE corresponding to the hypersurface $\mathscr{Z} \subseteq \Omega$ and the Hilbert module $\mathscr{M}$ in the class $B_1(\Omega)$. We state the following Lemma in terms of a frame for the bundle associated with the module $\mathscr{M}$. There is an obvious choice for such a frame in terms of the reproducing kernel of the module. The relationship between the reproducing kernel of the module and the hermitian metric of the associated bundle was explained in ([7], § 2). Let $\langle \mathbf{s}(w), \mathbf{s}(w_0)\rangle$ be the matrix of inner products, that is, $\langle \mathbf{s}(w), \mathbf{s}(w_0)\rangle_{ij} = \langle s_i(w), s_j(w_0)\rangle_{\mathscr{M}}$, $1 \leq i,j \leq k$ for some fixed but arbitrary $w_0 \in \mathscr{Z}$ and all $w \in \mathscr{Z}$.

Lemma. Let $\mathscr{M}$ be Hilbert module in $B_1(\Omega)$ and $\mathscr{M}_0 \subseteq \mathscr{M}$ be the submodule consisting of functions vanishing on the hypersurface $\mathscr{Z} \subseteq \Omega$. Then there exists an anti-holomorphic frame $\mathbf{s}$ *for the jet bundle JE satisfying*

$$\langle \mathbf{s}(w), \mathbf{s}(w_0)\rangle_{|\text{res } \mathscr{Z}} = \begin{pmatrix} 1 & 0 \\ 0 & S(w) \end{pmatrix},$$

for $w \in \mathscr{Z}$ and some anti-holomorphic function S on $\mathscr{Z}$.

Proof. Let us assume, without loss of generality, that $w_0 = 0$. We first observe that if we replace the module $\mathscr{M}$ by an isomorphic copy, then the class of the associated bundle JE does not change. Indeed, if $\mathscr{M}$ and $\tilde{\mathscr{M}}$ are isomorphic modules, then there is an anti-holomorphic map φ which induces a metric preserving bundle map of the associated bundles E and $\tilde{E}$. It is then clear that the map J_φ^* induces a bundle map of the corresponding jet bundles. Therefore, we may assume that the reproducing kernel K for the module $\mathscr{M}$ is normalised, that is, $K(z,0) = 1$. Let $(\tilde{z}, \tilde{w})$ denote (temporarily) the normal coordinates in $\Omega \times \Omega$. From the expansion

$$K(z,w) = \sum_{\ell,n=0}^{\infty} K_{\ell,n}(z,w)\tilde{z}^{\ell}\bar{\tilde{w}}^{n}, \quad z,w \in \mathscr{Z}$$

it is clear that $K_{\ell n}(z,0) = 0$ for $\ell \neq 0$ and $n = 0$. Since $K(z,w) = \overline{K(w,z)}$, it follows that $K_{\ell n}(0,w) = 0$ for $\ell = 0$ and $n \neq 0$. However, $K_{\ell n}(z,w) = (\partial^{\ell}\bar{\partial}^{n}K)_{|\tilde{z}=0,\tilde{w}=0}(z,w)$. Hence $((K_{\ell n}(z,w)))_{\ell,n=0}^{k-1} = JK_{|\text{res } \mathscr{Z}}(z,w)$ for $z,w \in \mathscr{Z}$ by definition (4). Recall that $e_\ell(w) = \sum_{j=1}^{k} \bar{\partial}^{\ell-1}\partial^{j-1}K(\cdot,w) \otimes \varepsilon_\ell$, for $1 \leq \ell \leq k$ is an anti-holomorphic frame for the jet bundle JE. It follows that $\langle e_\ell(w), e_n(0)\rangle = (JK)_{\ell n}(0,w)$. But $(JK)_{\ell n}(0,w) = K_{\ell n}(0,w) = 0$ for $\ell = 0$ as long as $n \neq 0$. The proof is completed by taking $\mathbf{s}(w) = \{e_1(w), \dots, e_k(w)\}$. $\square$

There is a canonical connection D on the bundle JE which is compatible with the metric and has the property $D'' = \bar{\partial}$. Let $C^\infty_{1,1}(\Omega, E)$ be the space of C^∞ sections of the bundle $\wedge^{(1,1)}T^*\Omega \otimes E$. The curvature tensor $\mathscr{K}$ associated with the canonical connection D is in $C^\infty_{1,1}(\Omega, \text{herm}(E,E))$. Moreover, if h is a local representation of the metric in some open set, then $i\mathscr{K} = \bar{\partial}(h^{-1}\partial h)$. The holomorphic tangent bundle $T\Omega_{|\text{res } \mathscr{Z}}$ naturally splits as $T\mathscr{Z} \dot{+} N\mathscr{Z}$, where $N\mathscr{Z}$ is the normal bundle and is realised as the quotient $T\Omega_{|\text{res } \mathscr{Z}}/T\mathscr{Z}$. The co-normal bundle $N^*\mathscr{Z}$ is the dual of $N\mathscr{Z}$; it is the sub-bundle of $T\Omega_{|\text{res } \mathscr{Z}}$ consisting of cotangent vectors that vanish on $T\mathscr{Z} \subseteq T\Omega_{|\text{res } \mathscr{Z}}$. Indeed, the class of the conormal bundle $N^*\mathscr{Z}$ coincides with $[-\mathscr{Z}]_{|\text{res } \mathscr{Z}}$ via the adjunction formula I ([10], p. 146). Let P_1 be the projection onto $N^*\mathscr{Z}$ and $P_2 = (1 - P_1)$ be the projection onto $T^*\mathscr{Z}$. Now, we have a splitting of the $(1,1)$ forms as follows:

$$\wedge^{(1,1)}T^*\Omega_{|\text{res } \mathscr{Z}} = \sum_{i,j=1}^{2} P_i\big(\wedge^{(1,0)}T^*\Omega_{|\text{res } \mathscr{Z}}\big) \wedge P_j\big(\wedge^{(0,1)}T^*\Omega_{|\text{res } \mathscr{Z}}\big).$$

Accordingly, we have the component of the curvature along the transversal direction to $\mathscr{Z}$ which we denote by $\mathscr{K}_{\text{trans}}$. Clearly, $\mathscr{K}_{\text{trans}} = (P_1 \otimes I)\mathscr{K}_{|\text{res } \mathscr{Z}}$. Similarly, let the component of the curvature along tangential directions to $\mathscr{Z}$ be $\mathscr{K}_{\tan}$. Again, $\mathscr{K}_{\tan} = (P_2 \otimes I)\mathscr{K}_{|\text{res } \mathscr{Z}}$. (Here I is the identity map on the vector space $\text{herm}(E,E)$.)

Recall that the fiber of the jet bundle $JE_{|\text{res } \mathscr{Z}}$ at $w \in \mathscr{Z}$ is spanned by the set of vectors $\bar{\partial}^{\ell-1}K(\cdot,w)$, $1 \leq \ell \leq k$. Thus the module action J_φ^* can be determined by calculating it on the set $\{\bar{\partial}^{\ell-1}K(\cdot,w) : 1 \le \ell \le k \text{ and } w \in \mathscr{Z}\}$. This calculation is given in eq. (7). We therefore obtain an anti-holomorphic bundle map J_φ^* on the bundle $JE_{|\mathrm{res}\ \mathscr{Z}}$. Thus the isomorphism of two quotient modules in $B_k(\Omega,\mathscr{Z})$ translates to a question of equivalence of the pair $(JE_{|\mathrm{res}\ \mathscr{Z}}, J_\varphi^*)$. This merely amounts to finding an anti-holomorphic bundle map $\theta : JE_{|\mathrm{res}\ \mathscr{Z}} \to JE_{|\mathrm{res}\ \mathscr{Z}}$ which intertwines J_φ^*. It is clear that if we could find such a bundle map θ, then the line sub-bundles corresponding to the frame $K(\cdot,w)$, $w \in \mathscr{Z}$ must be equivalent. From this it is evident that the curvatures $\mathscr{K}_{\mathrm{tan}}$ in the tangential directions must be equal. Also, we can calculate the matrix representation for the nilpotent action at w, as given in (7), with respect to the orthonormal basis obtained via the Gram–Schmidt process applied to the holomorphic frame at w. A computation shows that the matrix entries involve the curvatures $\mathscr{K}_{\mathrm{trans}}$ in the transverse direction and its derivatives. It is not clear if the intertwining condition can be stated precisely in terms of these matrix entries. In the following section we show, as a result of some explicit calculation, that if $k = 2$ then the curvature in the transverse direction must also be equal. We also find that an additional condition must be imposed to determine the isomorphism class of the quotient modules.

3. The case of rank 2 bundles

In this case, the adjoint action of φ on $\mathscr{Q} \cong J\mathscr{M}|_{\mathrm{res}\ \mathscr{Z}}$ produces a nilpotent bundle map on JE which, at $w \in \mathscr{Z}$, is described easily:

$$e(w) := \begin{pmatrix} K(\cdot,w) \\ \partial K(\cdot,w) \end{pmatrix} \to 0 \text{ and } (\bar{\partial}e)(w) := \begin{pmatrix} \bar{\partial}K(\cdot,w) \\ \partial\bar{\partial}K(\cdot,w) \end{pmatrix} \to \overline{(\partial\varphi)(w)}\mathrm{e}(w)$$

on the spanning set $\{e(w), (\bar{\partial}e)(w) : w \in \mathscr{Z}\}$ for the fiber $JE(w)$ of the jet bundle JE at $w \in \mathscr{Z}$. Thus the adjoint action induced by φ determines a nilpotent $N(w)$ of order 2 defined by $\begin{pmatrix} 0 & \overline{(\partial\varphi)(w)} \\ 0 & 0 \end{pmatrix}$ on each fiber $JE(w)$, $w \in \mathscr{Z}$ with respect to the basis $\{e(w), (\partial e)(w)\}$. Now, consider the orthonormal basis: $\{\gamma_0(w), \gamma_1(w)\}$, where

$$\gamma_0(w) = \|e(w)\|^{-1}e(w),$$
$$\gamma_1(w) = a(w)e(w) + b(w)(\bar{\partial}e)(w),\ w \in \mathscr{Z}.$$

The coefficients $a(w)$ and $b(w)$ can be easily calculated (cf. [3], p. 195):

$$-a(w)\|e(w)\|^3 = \langle(\partial e)(w), e(w)\rangle(-\mathscr{K}_{\mathrm{trans}}(w))^{-1/2},$$
$$b(w)\|e(w)\| = (-\mathscr{K}_{\mathrm{trans}}(w))^{-1/2},$$

where $\mathscr{K}_{\mathrm{trans}}(w)$ denotes the curvature in the transversal direction. In the case of a line bundle, we have the following explicit formula:

$$\mathscr{K}_{\mathrm{trans}}(w) = P_1\Big(\sum_{i,j=1}^{m} \partial_i\bar{\partial}_j \log\|e(w)\|^2 dz_i \wedge d\bar{z}_j\Big),\ w \in \mathscr{Z}. \tag{9}$$

The nilpotent action $N_{\mathrm{orth}}(w)$ at the fiber $JE(w)$, $w \in \mathscr{Z}$ with respect to the orthonormal basis $\{\gamma_0(w), \gamma_1(w)\}$ is given by

$$\begin{pmatrix} 0 & b(w)\|e(w)\|(\partial\varphi)(w) \\ 0 & 0 \end{pmatrix}.$$

Now, we are ready to prove the main theorem which gives a complete set of invariants for quotient modules in the class $B_2(\Omega,\mathscr{Z})$. At first, it may appear that the condition angle of the theorem stated below depends on the choice of the holomorphic frame. But we remind the reader that the normalisation of the kernel K for the module $\mathscr{M}$ ensures that it is uniquely dtermined. Therefore so is JK.

Theorem. *If $\mathscr{Q}$ and $\tilde{\mathscr{Q}}$ are two quotient modules, over the algebra $\mathscr{A}(\Omega)$, in the class $B_2(\Omega,\mathscr{Z})$, then they are isomorphic if and only if*

tan: $\mathscr{K}_{\text{tan}} = \tilde{\mathscr{K}}_{\text{tan}}$
trans: $\mathscr{K}_{\text{trans}} = \tilde{\mathscr{K}}_{\text{trans}}$
angle: $\langle(\bar\partial e)(w), e(w)\rangle = \langle(\bar\partial \tilde e)(w), \tilde e(w)\rangle$.

Proof. Suppose, we are given two quotient modules $\mathscr{Q}$ and $\tilde{\mathscr{Q}}$ which are isomorphic. Then the module map $\Phi : \mathscr{Q} \to \tilde{\mathscr{Q}}$ induces an anti-holomorphic bundle map $\Phi : JE_{|\text{res}\ \mathscr{Z}} \to J\tilde{E}_{|\text{res}\ \mathscr{Z}}$. For $w \in \mathscr{Z}$, let $JE(w)$ and $J\tilde{E}(w)$ denote the two dimensional space spanned by $\{e(w), (\bar\partial e)(w)\}$ and $\{\tilde e(w), (\bar\partial \tilde e)(w)\}$, respectively. Then the bundle map Φ defines a linear map $\Phi(w) : JE(w) \to J\tilde{E}(w)$. The map $\Phi(w)$ must then intertwine the two nilpotents $N(w)$ and $\tilde N(w)$ which implies that $\Phi(w)$ must be of the form $\Phi(w) = \begin{pmatrix} \alpha(w) & \beta(w) \\ 0 & \alpha(w) \end{pmatrix}$, where α, β are anti-holomorphic functions for w in some small open set in $\mathscr{Z}$. We observe that $\Phi(w)$ maps $\gamma_0(w)$ to $\alpha(w)\|\tilde e(w)\|\|e(w)\|^{-1}\tilde\gamma_0(w)$. Since $\Phi(w)$ is an isometry, it follows that $\alpha(w) = \|e(w)\|\|\tilde e(w)\|^{-1}$. Because we have chosen to work only with normalised kernels, we infer that $\|e(w)\|\|\tilde e(w)\|^{-1} = 1$ for all $w \in \mathscr{Z}$ which is the same as saying that $\alpha(w) = 1$ for $w \in \mathscr{Z}$. The condition 'tan' of the theorem is evident.

The module map ϕ has to satisfy the relation

$$JK(z,w) = \overline{\Phi(z)}J\tilde K(z,w)\Phi(w), \quad z,w \in \mathscr{Z}.$$

However, $JK(z,0) = \begin{pmatrix} 1 & 0 \\ 0 & S(z) \end{pmatrix}$, and similarly $\tilde K$ at $(z,0)$ has a matrix representation with S replaced by $\tilde S$. Now, evaluate the formula relating JK and $J\tilde K$ at $w = 0$ to conclude that $\beta(z) = 0$ for all $z \in \mathscr{Z}$.

Now, since $\Phi(w)$ has to preserve the inner products, it follows that $\langle(\bar\partial e)(w), e(w)\rangle - \langle(\bar\partial \tilde e)(w), \tilde e(w)\rangle = \beta(w)\|e(w)\|^2$. Hence it follows that $\langle(\bar\partial e)(w), e(w)\rangle = \langle(\bar\partial \tilde e)(w), \tilde e(w)\rangle$ which is the condition 'angle' of the theorem.

Finally, the requirement that the nilpotents $N(w)$ and $\tilde N(w)$ must be unitarily equivalent for each $w \in \mathscr{Z}$ amounts to the equality of the $(1,2)$ entry of $N_{\text{orth}}(w)$ with that of $\tilde N_{\text{orth}}(w)$. Since we have already ensured $\|e(w)\| = \|\tilde e(w)\|$, it follows that $b(w) = \tilde b(w)$. This clearly forces the condition 'trans' of the theorem which completes the proof of necessity.

For the converse, first prove that the natural map from $JE(w)$ to $J\tilde E(w)$, $w \in \mathscr{Z}$, which carries one anti-holomorphic frame to the other is an isometry. It is evident that this map, which we denote by $\Phi(w)$, defines an anti-holomorphic bundle map and that it intertwines the nilpotent action.

To check if $\Phi(w)$ is isometric, all we have to do is see if it automatically maps the orthonormal basis $\{\gamma_0(w), \gamma_1(w)\}$ to the corresponding orthonormal basis $\{\tilde\gamma_0(w), \tilde\gamma_1(w)\}$. Clearly, $\Phi(w)(\gamma_0(w)) = \tilde e(w)\|e(w)\|^{-1} = \tilde\gamma_0(w)\|\tilde e(w)\|\,\|e(w)\|^{-1}$. Suppose that the two curvatures corresponding to the bundles JE and $J\tilde E$ agree on the hypersurface $\mathscr{Z}$. Then it is possible to find sections of these bundles which have the same norm. Or, equivalently, we may assume that $\|\gamma_0(w)\| = \|\tilde\gamma_0(w)\|$. It then follows that $\Phi(w)(\gamma_0(w)) = \tilde\gamma_0(w)$. Notice that

$$\begin{aligned}\Phi(w)(\gamma_1(w)) &= a(w)\tilde{e}(w) + b(w)(\partial\tilde{e})(w)\\ &= a(w)\|\tilde{e}(w)\|\tilde{\gamma}_0(w) + b(w)(\tilde{b}(w))^{-1}(\tilde{\gamma}_1(w)\\ &\quad - \tilde{a}(w)\|\tilde{e}(w))\|\tilde{\gamma}_0(w)\\ &= (a(w)\tilde{b}(w) - \tilde{a}(w)b(w))\|\tilde{e}(w)\|(\tilde{b}(w))^{-1}\tilde{\gamma}_0(w)\\ &\quad + b(w)(\tilde{b}(w))^{-1}\tilde{\gamma}_1(w).\end{aligned}$$

A simple calculation shows that

$$a(w)\tilde{b}(w) - \tilde{a}(w)b(w) = \|e(w)\|^3\|\tilde{e}(w)\|(-\mathscr{K}(w))^{-1/2}(-\tilde{\mathscr{K}}(w))^{-1/2} \Big(\langle(\overline{\partial}e)(w), e(w)\rangle - \langle(\overline{\partial}\tilde{e})(w), \tilde{e}(w)\rangle\Big).$$

It follows that $\Phi(w)$ maps $\gamma_1(w)$ to $\tilde{\gamma}_1(w)$ if and only if $b(w) = \tilde{b}(w)$ and $\langle(\overline{\partial}e)(w), e(w)\rangle = \langle(\overline{\partial}\tilde{e})(w), \tilde{e}(w)\rangle$.

We have therefore shown that the two bundles JE and $J\tilde{E}$ are locally equivalent (via the bundle map $J\varphi$). We now apply the Rigidity Theorem ([3], p. 202) to conclude that the two modules $\mathscr{Q}$ and $\tilde{\mathscr{Q}}$ must be isomorphic. $\square$

It is not clear if the condition 'angle' of the theorem can be reformulated in terms of intrinsic geometric invariants like the second fundamental form etc.

In the case $k > 2$, if we show that the bundle map is the identity transform on each of the fibers, then it will follow that the matrix entries of the two nilpotent actions on each of these fibers must be equal. These entries are expressible in terms of the curvature in the transverse direction and its normal derivatives. So if two quotient modules are isomorphic, then it follows that these quantities must be equal. However, we are not sure what a replacement for the condition 'angle' in the theorem would be.

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