Invent. math. 137, 449–460 (1999)
Digital Object Identifier (DOI) 10.1007/s002229900934

Inventiones mathematicae

The variety of circular complexes and F-splitting

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Oblatum 13-X-1998 & 17-II-1999 / Published online: 21 May 1999

1. Introduction

In this paper we study the variety of circular complexes (see Theorem 4.1 for the definition) in positive characteristic. Our methods are similar to those used to study the variety of complexes, in our earlier paper [MT]. However, we will make use of new Frobenius splittings, obtained using the methods of [MVr] and [LT].

Let V_0 and V_1 be finite dimensional vector spaces over an algebraically closed field k of characteristic $p > 0$. Let

$$L = \mathrm{Hom}(V_0, V_1) \times \mathrm{Hom}(V_1, V_0), \quad \text{and let}$$

$$H = GL(V_0) \times GL(V_1) = G_0 \times G_1,$$

Recall that a *circular complex* is an element $f = (f_1, f_2) \in L$ such that $f_1 \circ f_2 = f_2 \circ f_1 = 0$. Given a circular complex $f \in L$, we consider the orbit closure $\overline{O}_f := \overline{H(f)} \subseteq L$, where the action of H on L is given by $g \cdot f = (g_1 f_1 g_0^{-1}, g_0 f_2 g_1^{-1})$ for $g = (g_0, g_1) \in H$. Each $\overline{O}_f$ is an irreducible closed subset of the variety of circular complexes, and the variety of circular complexes is the union of such $\overline{O}_f$. The Cohen-Macaulay and normality properties for each component of this variety was first proved by Strickland [St], using Hodge algebras. There also seems to be some overlap between the results in the present paper and those in [F]. Here we give generators of the ideal of $\overline{O}_f \subseteq L$ (see Theorem 4.1), using a result of [MuSe], but our main result in this paper is the following.

Theorem 1.1. *For a circular complex $f \in L$, the orbit closure $\overline{O}_f$ is normal, Cohen-Macaulay with rational singularities (with respect to the natural resolution given by the map ϕ defined in Section 3).*

Remark. The case $\dim V_0 = \dim V_1 = 2$ was proved by Cowsik ([Se], Chapter 8, Theorem 30).

2. Preliminaries

We recall some basic facts about F-splitting, compatible F-splitting, the relation between F-splitting and normality, and the Grauert-Riemenschneider vanishing theorem in characteristic p. For complete proofs and more detailed discussion we refer to notes by Ramanathan [R] and references given there.

Let X be a variety (reduced but not necessarily irreducible) over an algebraically closed field of characteristic $p > 0$ and let $F : X \longrightarrow X$ be the Frobenius morphism. We say that X is *Frobenius-split* or just *F-split* if there exists a splitting $\sigma : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ of the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X/\mathcal{O}_X \longrightarrow 0.$$

Let Y be a closed subvariety of X with ideal sheaf $\mathcal{I}_Y$. If there exists a splitting section $\sigma : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ of X such that $\sigma(F_*\mathcal{I}_Y) \subseteq \mathcal{I}_Y$, then we say that Y is *compatibly split in* X. In this case σ induces a splitting, say $\overline{\sigma} : F_*\mathcal{O}_Y \longrightarrow \mathcal{O}_Y$, of the sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow F_*(\mathcal{O}_Y) \longrightarrow F_*(\mathcal{O}_Y)/\mathcal{O}_Y \longrightarrow 0,$$

where we denote the Frobenius map on Y by the same letter F.

Remarks.

1. More generally, one can define F-splitting for any scheme of finite type over k, but existence of such a splitting implies that X is reduced. If $Y_1, Y_2 \subseteq X$ are compatibly F-split by the same splitting σ then σ gives compatible splitting of $Y_1 \cap Y_2$, $Y_1 \cup Y_2$ and any irreducible component of these. In particular, if $\mathcal{I}_{Y_1}$ and $\mathcal{I}_{Y_2} \subseteq \mathcal{O}_X$ denote the ideal sheaves of Y_1 and Y_2 respectively, then the scheme theoretic ideal of $Y_1 \cap Y_2$, namely $\mathcal{I}_{Y_1} + \mathcal{I}_{Y_2}$ is reduced.
2. Restriction of a splitting section of X to any open set U gives a splitting section of U, hence if X is F-split then so is any open subset.
3. For any smooth (or Gorenstein) variety X/k there exists an isomorphism

$$[\ \ ]_X : H^0\left(X, \omega_X^{\otimes 1-p}\right) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}\left(F_*\mathcal{O}_X, \mathcal{O}_X\right),$$

   obtained using duality for finite flat maps. A divisor in the linear system $|\omega_X^{\otimes 1-p}|$ which is associated to a splitting section $\sigma : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ is called *a splitting divisor* on X.
4. If Y is compatibly split in X, then for any ample line bundle L on X, the restriction map $H^0(X, L) \longrightarrow H^0(Y, L\mid_Y)$ is surjective.

Example. For a connected semisimple simply-connected algebraic group over k and a Borel group B, there exists an F-splitting σ of G/B which simultaneously compatibly splits every Schubert subvariety and every opposite Schubert subvariety in G/B. This splitting corresponds to the divisor

$(p-1)(D+\widetilde{D})$, where D and $\widetilde{D}$ respectively denote the union of codimension 1 Schubert varieties, and the union of codimension 1 opposite Schubert varieties, in G/B.

Lemma 2.1. [MS$_3$] *Let $\pi : X \to Y$ be a projective birational map, such that (a) X is F-split, and (b) for all $y \in Y$, we have $H^i(X_y, \mathcal{O}_{X_y}) = 0$ for all $i > 0$, for some choice of scheme structure on the fiber X_y. Then $R^i\pi_*\mathcal{O}_X = 0$ for all $i > 0$.*

The connection between the normality of a variety X and the F-splitting of X is illustrated by the following (see [MS$_1$]):

Theorem 2.2. *Let $f : Y \longrightarrow X$ be a proper surjective morphism between varieties in char p. Assume that 1) Y is normal, 2) the fibres of f are connected and 3) X is F-split. Then X is also normal.*

Definition (Kempf). Let X be a variety and $f : Z \longrightarrow X$ a birational proper morphism with Z nonsingular. We define $f : Z \longrightarrow X$ to be *a rational resolution* if

1. $f_*\mathcal{O}_Z = \mathcal{O}_X$ and
2. for $i > 0$, one has $R^i f_*\mathcal{O}_Z = 0$ and $R^i f_*\omega_Z = 0$.

Kempf has proved, using duality, that if X admits a rational resolution then X is Cohen-Macaulay. Condition 2 is known as the Grauert-Riemenschneider theorem. The Grauert-Riemenschneider theorem for varieties over fields of characteristic p, in case a suitable F-splitting is available, can be obtained using the following result.

Theorem 2.3. [MVk] Let $\pi : X \longrightarrow Y$ be a projective morphism of varieties over an algebraically closed field k of char $p > 0$. Let D be a closed subscheme of X with ideal sheaf I and let E be a closed subscheme of Y and let $i \geq 0$. Assume that

1. D contains $\pi^{-1}(E)$ set theoretically,
2. $R^i\pi_*(I)$ vanishes off E,
3. X is F-split, compatibly with D.

Then $R^i\pi_*(I) = 0$.

Let X/k be a smooth variety and $Y \subseteq X$ be a reduced effective Cartier divisor with ideal sheaf $\mathcal{I}_Y \subseteq \mathcal{O}_X$. Consider the following diagram

$$\begin{array}{ccc}
H^0\left(X, \omega_X^{\otimes 1-p}\right) & \xrightarrow[\cong]{[\ ]_X} & \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \\
\alpha \big\uparrow & & \\
H^0\left(X, \mathcal{O}_X((1-p)Y) \otimes \omega_X^{\otimes 1-p}\right) & & \\
\phi \big\downarrow & & \\
H^0\left(Y, \omega_Y^{\otimes 1-p}\right) & \xrightarrow[{[\ ]_Y}]{\cong} & \mathrm{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y, \mathcal{O}_Y)
\end{array}$$

where α is induced by the canonical map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X\,((1-p)Y)\otimes\omega_X^{\otimes 1-p}\hookrightarrow\omega_X^{\otimes 1-p}$$

and ϕ is the *residue map* induced by adjunction formula. Let $\sigma \in H^0(X, \omega_X^{\otimes 1-p})$ vanish to order at least $p-1$ along Y (*i.e.*, $\sigma \in H^0(X, \mathcal{O}_X((1-p)Y)\otimes\omega_X^{\otimes 1-p})$). Then (see [MS$_2$], Lemma 3)

1. $[\sigma]_X(F_*\mathcal{I}_Y) \subseteq \mathcal{I}_Y$ and therefore it induces a $\mathcal{O}_Y$-linear map say $\sigma' : F_*\mathcal{O}_Y \longrightarrow \mathcal{O}_Y$ canonically. Moreover $[\phi(\sigma)]_Y = \sigma'$, *i.e.*, the following diagram is commutative

$$\begin{array}{ccc} F_*\mathcal{O}_X & \xrightarrow{[\sigma]_X} & \mathcal{O}_X \\ \downarrow & & \downarrow \\ F_*\mathcal{O}_Y & \xrightarrow{[\phi(\sigma)]_Y} & \mathcal{O}_Y \end{array}\ .$$

2. In particular, if X is also a projective variety then $[\sigma]_X$ is an F-splitting for X if and only if $[\phi(\sigma)]_Y$ is an F-splitting of Y.

3. Proof of the Main Theorem

We introduce some notations. Fix $J = (J_0, J_1)$, where $J_i \subseteq V_i$ are subspaces such that $\dim J_0 = \dim\ker f_1$, $\dim J_1 = \dim\ker f_2$. Let

$$\dim J_0 = k_0 \ \text{ and } \ \dim J_1 = k_1.$$

Define $W = W(J)$ to be

$$W = \{a = (a_1, a_2) \in L\ |\mathrm{im}\, a_1 \subseteq J_1 \subseteq \ker a_2, \mathrm{im}\, a_2 \subseteq J_0 \subseteq \ker a_1\}\,.$$

Therefore

$$W = \mathrm{Hom}(V_0/J_0, J_1) \times \mathrm{Hom}(V_1/J_1, J_0) \subseteq L.$$

Let

$$\widetilde{P} = P_0 \times P_1 \subseteq H,$$

where P_i is the stablizer of J_i in $GL(V_i)$. Then W is a $\widetilde{P}$-stable subspace of L. We have the following commutative diagram

$$\begin{tikzcd} H \times^{\widetilde{P}} W \arrow[r] \arrow[dr, "\phi"'] & H \times^{\widetilde{P}} L \cong (H/\widetilde{P}) \times L \arrow[d] \\ & L, \end{tikzcd}$$

where the map ϕ is given by $\phi(h_0, h_1, t_1, t_2) = (h_1t_1h_0^{-1}, h_0t_2h_1^{-1})$, for $(h_0, h_1) \in H$ and $(t_1, t_2) \in W$

Remark. The following facts are easy to check.

1. The image of $\phi : H \times^{\widetilde{P}} W \longrightarrow L$ is $\overline{O}_f$,
2. $H \times^{\widetilde{P}} W$ is smooth, and
3. the map ϕ is proper, hence the image of $H \times^{\widetilde{P}} W$ in L is closed.

Lemma 3.1. *For every $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) \in \overline{O}_f$, there is an isomorphism of varieties*

$$\phi^{-1}(\widetilde{f}) \cong \mathbf{G}\left(k_1 - d_1, \frac{\ker \widetilde{f}_2}{\widetilde{f}_1(V_0)}\right) \times \mathbf{G}\left(k_0 - d_2, \frac{\ker \widetilde{f}_1}{\widetilde{f}_2(V_1)}\right),$$

where $d_i = \operatorname{rank} \widetilde{f}_i$ and $\mathbf{G}(k, V)$ denotes the Grassmannian of k-dimensional subspaces of the vector space V, and where the left hand side is given the reduced scheme structure.

Proof. This follows using the same argument as for Lemma 2 of [MT]. $\square$

We now introduce the following further notation.

$$\begin{aligned}
n_0 &= \dim(V_0), \quad n_1 = \dim(V_1)\\
G &= GL(n_0 + n_1 + n_0)\\
l_0 &= n_0,\ l_1 = n_0 + n_1,\ \text{and } l_2 = n_0 + n_1 + n_0\\
Q &= \left\{ \begin{bmatrix} g_0 & * & * \\ 0 & g_1 & * \\ 0 & 0 & g_2 \end{bmatrix} \in GL(l_2) \,\middle|\, g_0, g_2 \in GL(V_0),\quad g_1 \in GL(V_1) \right\}.
\end{aligned}$$

B = the Borel subgroup consisting of upper triangular matrices in G. Consider the map

$$\frac{GL(n_0)}{B_0} \times \frac{GL(n_1)}{B_1} \longrightarrow \frac{Q}{B},$$

given by

$$(g_0 B_0, g_1 B_1) \mapsto \begin{bmatrix} g_0 & 0 & 0 \\ 0 & g_1 & 0 \\ 0 & 0 & g_0 \end{bmatrix} (\text{mod } B),$$

where $B_i \subseteq GL(n_i)$ is the group of upper triangular matrices in $GL(n_i)$. Let $\widetilde{Y}$ denote image of this map.

Following notation is meant only for Lemma 3.2, given below. Let $\bar{G}$ be a semisimple simply connected algebraic group over an algebraically closed field k of char p > 0 with a Borel subgroup $\bar{B}$. Let $\bar{\rho}$ stand for the sum of fundamental weights. Let $St_1 = H^0(\bar{G}/\bar{B}, L((p-1)\bar{\rho}))$ denote the Steinberg Module, with highest weight $(p-1)\bar{\rho}$. Let $\langle\ \rangle : St_1 \times St_1 \longrightarrow k$ be the $\bar{G}$-invariant bilinear nondegenerate form on St_1 (see [MVr]).

Lemma 3.2. *For any non-zero $f \in St_1 = H^0(\bar{G}/\bar{B}, L((p-1)\bar{\rho}))$, there exists $h \in H^0(\bar{G}/\bar{B}, L(\bar{\rho}))$ such that $\langle f, h^{p-1}\rangle \neq 0$.*

Proof. Given $f \in St_1$, there exists $s \in St_1$ such that $\langle f, s\rangle \neq 0$. We have to show that s can be taken to be of the form t^{p-1}, where $t \in H^0(\bar{G}/\bar{B}, L(\bar{\rho}))$. Choose a non-zero $\sigma \in H^0(\bar{G}/\bar{B}, L(\bar{\rho}))$ and consider the $\bar{G}$-span of σ^{p-1} in St_1, that is $W = \{\sum_i \alpha_i g_i(\sigma^{p-1}) \mid \alpha_i \in k \text{ and } g_i \in \bar{G}\}$. It is clear that W is a nonzero $\bar{G}$-submodule of St_1 and hence, by [MVr], $W = St_1$. Hence one can find an element $\tau = \sum_i \alpha_i g_i(\sigma^{p-1})$ such that $\langle f, \tau\rangle \neq 0$. But then there exists some i such that $\langle f, \alpha_i g_i(\sigma^{p-1})\rangle \neq 0$. One can take $t = g_i(\sigma)$, as $g_i(\sigma^{p-1}) = (g_i(\sigma))^{p-1}$. Hence the lemma. $\square$

Lemma 3.3. *There exists an F-splitting on $Q/B \times G/B$ which simultaneously compatibly F-splits $\widetilde{Y} \times G/B$ and $Q \times^B D$, where D denotes the union of all codimension 1 Schubert varieties in G/B.*

Proof. For an arbitrary reductive connected algebraic group G, with Borel subgroup B, we let $L(\rho_G)$ denote the line bundle on G/B such that

$$L(2\rho_G) = L(\rho_G)^{\otimes 2} = \omega_{G/B}^{-1}.$$

We let $L(n\rho_G)$ denote $L(\rho_G)^{\otimes n}$. One then has an identification

$$\mathrm{Hom}(F_*\mathcal{O}_{G/B}, \mathcal{O}_{G/B}) = H^0(G/B, L(2(p-1)\rho_G)).$$

To prove the lemma, we construct a splitting divisor of $Q/B \times G/B$ (*i.e.*, a divisor which is the divisor of zeroes of a splitting section of $Q/B \times G/B$), as follows. By abuse of notation we use the same notation for a section of a line bundle and its divisor of zeroes. Consider the isomorphism

$$G_0/B_0 \times G_1/B_1 \times G_0/B_0 \longrightarrow Q/B$$

given by

$$(g_0, g_1, g_2) \to \begin{bmatrix} g_0 & 0 & 0 \\ 0 & g_1 & 0 \\ 0 & 0 & g_2 \end{bmatrix} (\mathrm{mod}\ B)$$

Here $G_0 = GL(n_0)$ and $G_1 = GL(n_1)$. Define

$$L(\rho_Q) := L(\rho_{G_0}) \boxtimes L(\rho_{G_1}) \boxtimes L(\rho_{G_0}).$$

We note that $L(\rho_Q) = L(\rho_G)\mid_{Q/B}$. Let D_0 and D_1 denote the unions of codimension one Schubert varieties in G_0/B_0 and G_1/B_1 respectively. Let

$$E_1 = p_{13}^*(G_0 \times^{B_0} D_0) + \left(\frac{G_0}{B_0} \times D_1 \times \frac{G_0}{B_0}\right) \in H^0\left(\frac{Q}{B}, L(\rho_Q)\right),$$

where

$$p_{13} : G_0/B_0 \times G_1/B_1 \times G_0/B_0 \longrightarrow G_0/B_0 \times G_0/B_0$$

is the canonical projection map. Now, by the above lemma and Theorem 2.3 of [LT], there exists $\widetilde{E_1} \in H^0(Q/B, L(\rho_Q))$ such that

$$(p-1)(E_1 + \widetilde{E_1}) \in H^0(Q/B, L(2(p-1)\rho_Q))$$

is a splitting divisor for Q/B which compatibly splits E_1. Since Q/B is a Schubert variety in G/B, it compatibly splits in G/B. Therefore, as $L(\rho_G)$ is an ample line bundle on G/B, the canonical map

$$H^0(G/B, L(\rho_G)) \longrightarrow H^0(Q/B, L(\rho_Q))$$

is surjective. Hence one can lift $\widetilde{E_1}$ to a section, say $\widetilde{E} \in H^0(G/B, L(\rho_G))$. Consider

$$\begin{aligned}\sigma &= (p-1)(Q \times^B D + p_1^* E_1 + p_2^* \widetilde{E}) \\ &\in H^0\left(Q/B \times G/B, L(2(p-1)\rho_Q) \boxtimes L(2(p-1)\rho_G)\right),\end{aligned}$$

where $p_1 : Q/B \times G/B \longrightarrow Q/B$ and $p_2 : Q/B \times G/B \longrightarrow G/B$ are the canonical projection maps. Now we prove that σ is the required splitting divisor. By the result of [MS$_2$] (given at the end of Section 2 here), since σ vanishes to order $p-1$ along the divisor $Q \times^B D$, to show that σ compatibly splits $Q \times^B D$ in $Q/B \times G/B$, it is sufficient to show that σ', the residue of σ on $Q \times^B D$, is a splitting of $Q \times^B D$.

But σ' on $Q \times^B D$ is precisely $(p-1)(p_1^* E_1 + p_2^* \widetilde{E}) \mid_{Q \times^B D}$. To prove that σ' is an F-splitting of $Q \times^B D$, it is sufficient to prove that σ'', the (iterated) residue of σ' on $Q \times^B eB$ (eB is the intersection of all Schubert divisors of G/B, where e is the identity element of the group G), is a splitting of $Q \times^B eB = Q/B$ imbedded in $Q/B \times G/B$ diagonally. But σ'' is precisely $(p-1)(E_1 + \widetilde{E_1})$, which is a splitting section of Q/B, by the choice of $\widetilde{E_1}$. Hence, by [MS$_2$], we conclude that σ compatibly splits $Q \times^B D$ in $Q/B \times G/B$.

To see that $\widetilde{Y} \times G/B$ is compatibly splits by σ, we argue as follows: denote the components of D_0 by $\{D_{0i}\}_{i=1}^k$. Then

$$\widetilde{Y} \times G/B = \cap_{i=1}^k p_1^* p_{13}^* \left(G_0 \times^{B_0} D_{0i}\right).$$

By construction each divisor $p_1^* p_{13}^* (G_0 \times^{B_0} D_{0i})$ is compatibly splits in $Q/B \times G/B$ by σ. Hence their intersection, $\widetilde{Y} \times G/B$ is also compatibly split by σ. Hence the lemma. □

Lemma 3.4. *Let*

$$D_{p_{l_0}}(\theta_1)_{k_1 \times k_2} = \left\{ \begin{bmatrix} Id_{n_0} & 0 & 0 \\ A & Id_{n_1} & 0 \\ 0 & 0 & Id_{n_0} \end{bmatrix} \middle| A = \begin{bmatrix} 0 & \{a_{ij}\}_{k_1 \times k_2} \\ 0 & 0 \end{bmatrix}, a_{ij} \in k \right\} \subseteq Z_1,$$

$$D_{p_{l_1}}(\theta_2)_{t_1 \times t_2} = \left\{ \begin{bmatrix} Id_{n_0} & 0 & 0 \\ 0 & Id_{n_1} & 0 \\ 0 & B & Id_{n_0} \end{bmatrix} \middle| B = \begin{bmatrix} 0 & \{b_{ij}\}_{t_1 \times t_2} \\ 0 & 0 \end{bmatrix}, b_{ij} \in k \right\} \subseteq Z_2$$

be Schubert cells in G/P_{l_0} and G/P_{l_1} with Schubert closure Y_1 and Y_2 respectively, where Z_1 and Z_2 denote opposite big cells of G/P_{l_0} and G/P_{l_1} respectively. If $p_1 : G/Q \longrightarrow G/P_{l_0}$ and $p_2 : G/Q \longrightarrow G/P_{l_1}$ denote the canonical projection maps then the variety $X = p_1^{-1}(Y_1) \cap p_2^{-1}(Y_2)$ is an intersection of Schubert varieties in G/Q, and

$$X \cap Z = \left\{ \begin{bmatrix} Id_{n_0} & 0 & 0 \\ A & Id_{n_1} & 0 \\ BA & B & Id_{n_0} \end{bmatrix} \middle| \begin{array}{l} A = \begin{bmatrix} 0 & \{a_{ij}\}_{k_1 \times k_2} \\ 0 & 0 \end{bmatrix} \\ B = \begin{bmatrix} 0 & \{b_{ij}\}_{t_1 \times t_2} \\ 0 & 0 \end{bmatrix} \end{array} \right\},$$

where Z denotes the opposite big cell of G/Q.

Proof. As argued in [MT] (see the proof of the Claim after Lemma 3). $\square$

For any $\widetilde{P}$-stable subspace W_0 of L (recall the definition of $\widetilde{P}$ from the beginning of the Section 3), consider the following diagram

$$\begin{tikzcd} H \times^{\widetilde{P}} W_0 \arrow[r] \arrow[dr, "\phi_{W_0}"'] & H \times^{\widetilde{P}} L \arrow[r] \arrow[d] & H \times^{\widetilde{P}} Z \arrow[d, "\widetilde{\phi}"] \\ & L \arrow[r] & Z, \end{tikzcd}$$

where the map ϕ_{W_0} is given by $\phi(h_0, h_1, t_1, t_2) = (h_1 t_1 h_0^{-1}, h_0 t_2 h_1^{-1})$, for $(h_0, h_1) \in H$ and $(t_1, t_2) \in W_0$, and where the P-equivariant map $L \longrightarrow Z$ is given by

$$(t_1, t_2) \to \begin{bmatrix} Id_{n_0} & 0 & 0 \\ t_1 & Id_{n_1} & 0 \\ t_2 \circ t_1 & t_2 & Id_{n_0} \end{bmatrix}.$$

Corollary 3.5. *Let W_0 be a $\widetilde{P}$-stable subspace of L such that $W_0 = X_0 \cap Z$, where $X_0 \subseteq G/Q$ is an intersection of some Schubert varieties of G/Q. Then $\widetilde{\phi}(H \times^{\widetilde{P}} W_0)$ $(= \phi_{W_0}(H \times^{\widetilde{P}} W_0))$ is compatibly F-split in Z. Moreover, one can choose a splitting section on Z which simultaneously compatibly splits all such subvarieties $\widetilde{\phi}(H \times^{\widetilde{P}} W_0)$.*

Proof. Let $\widetilde{X_0} = p^{-1}(X_0)$, where $p : G/B \longrightarrow G/Q$ is the canonical map. By Lemma 3.3, the closed subvariety $(\widetilde{Y} \times G/B) \cap (Q \times^B \widetilde{X_0})$ is compatibly F-split in $Q/B \times G/B$. Now, following the arguments of [MT] (especially the discussion between Lemma 3 and Lemma 4), one can deduce that $\phi(H \times^{\widetilde{P}} W_0)$ is compatibly F-split in Z. Moreover, the splitting section of Z determined by Lemma 3.3 also implies that splitting section is independent of the choice of W_0. $\square$

Proof of Theorem 1.1. We recall the linear subspace W defined in the beginning of Section 3. We can identify W with $X \cap Z$, where as given in Lemma 3.4

$$X = p_1^{-1}(\overline{D_{p_{l_0}}(\theta)_{k_1\times(n_0-k_0)}}) \cap p_2^{-1}(\overline{D_{p_{l_1}}(\theta)_{k_0\times(n_1-k_1)}}),$$

and the overbar denotes the Schubert closure in the appropriate space. Then the above corollary implies that $\phi(H \times^{\widetilde{P}} W) = \overline{O}_f$ is compatibly Frobenius split in Z.

Now we sketch the rest of the proof, as the arguments are very similar to those given after Lemma 4 in [MT]. By Lemma 3.1, each fiber of ϕ is connected, and hence using Theorem 2.2, we see that $\overline{O}_f$ is normal. Moreover, since each fiber is a product of Grassmannians, it has no higher cohomology for the structure sheaf, and hence by Lemma 2.1, we have $R^i\phi_*\mathcal{O}_{H\times^{\widetilde{P}}W} = 0$ for all $i > 0$.

It remains to show that $R^i\phi_*\omega_{H\times^{\widetilde{P}}W} = 0$ for all $i > 0$. Let E be the exceptional locus given by

$$E = \overline{O}_f \setminus O_f = \{(h_1, h_2) \in W \mid \text{ rank } h_1 < \text{rank } f_1 \text{ or rank } h_2 < \text{rank } f_2\}.$$

Let

$$\mathcal{N} = \left(\widetilde{Y} \times G/B\right) \cap \left(Q \times^B \widetilde{X}\right) \cap \left(Q/B \times p^{-1}(Z)\right),$$

where $p : G/B \to G/Q$ is the canonical map and $\widetilde{X} = p^{-1}(X)$. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\pi"] \arrow[dr, "\phi\circ\pi"'] & H \times^{\widetilde{P}} W \arrow[d, "\phi"] \\ & \overline{O}_f, \end{tikzcd}$$

Let $D = ((\widetilde{Y} \times G/B) \cap Q \times^B \delta(\widetilde{X})) \mid_{\mathcal{N}}$ be the divisor on $\mathcal{N}$, where $\delta(\widetilde{X})$ denotes the union of codimension 1 Schubert varieties in $\widetilde{X}$. Then $D \supseteq \pi^{-1}\circ\phi^{-1}(E)$. By the proof of Lemma 3.3 there exists a splitting section $\sigma = \tau^{p-1}$ of $\mathcal{N}$, where $\tau \in H^0(\mathcal{N}, \omega_{\mathcal{N}}^{-1})$, which compatibly splits D. Then $\mathcal{I} := \tau^{\vee}(\omega_{\mathcal{N}})$ is the ideal sheaf of D, where $\tau^{\vee} : \omega_{\mathcal{N}} \to \mathcal{O}_{\mathcal{N}}$ is induced by $\tau \in H^0(\mathcal{N}, \omega_{\mathcal{N}}^{-1})$. Using the fact that π is a smooth proper fibre bundle of relative dimension d (say), a Leray spectral sequence argument gives $R^{i+d}(\phi \circ \pi)_*\omega_{\mathcal{N}} \cong R^i\phi_*\omega_{H\times^{\widetilde{P}}W}$ for every $i > 0$. Now applying the result of [MVk] stated earlier (Theorem 2.3), we obtain that $R^{i+d}(\phi\circ\pi)_*\omega_{\mathcal{N}} = 0$ for all $i > 0$.

We conclude that ϕ is a rational resolution of $\overline{O}_f$, and in particular, $\overline{O}_f$ is Cohen-Macaulay. This completes the proof of Theorem 1.1. □

Remark. We note that in case k has characteristic 0, the action of $\widetilde{P}$ on W is completely reducible. Therefore by the theorem of Kempf [K], one can immediately conclude that $\overline{O}_f$ is normal, Cohen-Macaulay and ϕ is a rational resolution for $\overline{O}_f$.

4. Defining equations

Theorem 4.1. *Let $X = [X_{ij}]$ and $Y = [Y_{ij}]$ denote matrices of indeterminates of size $n_0 \times n_1$ and $n_1 \times n_0$ respectively, then*

1. *the variety of circular complexes* Spec $k[X, Y]/(XY, YX)$ *is F-split, and in particular it is seminormal (here $k[X, Y]$ denotes the polynomial ring in the entries of X, Y and (XY, YX) stands for the ideal generated by the entries of the product matrices XY and YX).*
2. *The ideal in $k[X, Y]$ of any orbit closure $\overline{O}_f \subseteq L$, where $f = (f_1, f_2)$ is a circular complex, is given by*

$$I(\overline{O}_f) = (XY, YX, I_{t_0}(X), I_{t_1}(Y)) \subseteq k[X, Y],$$

where $t_0 = \operatorname{rank} f_1 + 1$, $t_1 = \operatorname{rank} f_2 + 1$, and $I_{t_0}(X)$ denotes the set of t_0-minors of X (similarly $I_{t_1}(Y)$).

3. *For any ideal $I(l_0, l_1) := (XY, YX, I_{l_0+1}(X), I_{l_1+1}(Y))$, where l_0, l_1 are non negative integers with $l_0+l_1 \leq \min\{n_0, n_1\}$, the ring $k[X, Y]/I(l_0, l_1)$ is a normal Cohen-Macaulay domain with a rational resolution.*

Remark. Theorems 1.1 and 4.1 together give alternate proofs of the results of [St] and [T].

Proof. Proof of (2): First we give generators for the ideal of $\overline{O}_f \subseteq L$. We assume that $\dim \ker f_1 = k_0$ and $\dim \ker f_2 = k_1$, and W is defined as in the begining of Section 3. By Lemma 3.4, one can find X_1, X_2, which are intersections of Schubert varieties in G/Q such that

$$X_1 \cap Z =: W_1 = \left\{ \begin{bmatrix} Id_{n_0} & 0 & 0 \\ A & Id_{n_1} & 0 \\ BA & B & Id_{n_0} \end{bmatrix} \middle| \begin{array}{c} A = \begin{bmatrix} 0 \ \{a_{ij}\}_{n_1\times(n_0-k_0)} \end{bmatrix} \\ B = \begin{bmatrix} \{b_{ij}\}_{k_0\times n_1} \\ 0 \end{bmatrix} \end{array} \right\},$$

$$X_2 \cap Z =: W_2 = \left\{ \begin{bmatrix} Id_{n_0} & 0 & 0 \\ A & Id_{n_1} & 0 \\ 0 & B & Id_{n_0} \end{bmatrix} \middle| \begin{array}{c} A = \begin{bmatrix} \{a_{ij}\}_{k_1\times n_0} \\ 0 \end{bmatrix} \\ B = \begin{bmatrix} 0 \ \{b_{ij}\}_{n_0\times(n_1-k_1)} \end{bmatrix} \end{array} \right\}.$$

Now, by Corollary 3.5 the closed subvarieties $H \cdot W := \widetilde{\phi}(H \times^{\widetilde{P}} W)$, $H \cdot W_1 := \widetilde{\phi}(H \times^{\widetilde{P}} W_1)$ and $H \cdot W_2 := \widetilde{\phi}(H \times^{\widetilde{P}} W_2)$ are simultaneously compatibly F-split (via the splitting section induced by σ, as chosen in Lemma 3.3) in Z.

By definition

$$H \cdot W_1 = \left\{ \begin{bmatrix} Id_{n_0} & 0 & 0 \\ h_1 & Id_{n_1} & 0 \\ h_2 \circ h_1 & h_2 & Id_{n_0} \end{bmatrix} \middle| \begin{array}{c} h_1 \circ h_2 = 0, \operatorname{rank} h_2 \leq k_0 \\ \operatorname{rank} h_1 \leq n_0 - k_0 \end{array} \right\},$$

$$H\cdot W_2=\left\{\begin{bmatrix} Id_{n_0} & 0 & 0\\ h_1 & Id_{n_1} & 0\\ h_2\circ h_1 & h_2 & Id_{n_0}\end{bmatrix}\;\middle|\; \begin{matrix} h_2\circ h_1=0, \text{rank } h_2\le n_1-k_1\\ \text{rank } h_1\le k_1\end{matrix}\right\}.$$

Therefore $H\cdot W\subseteq H\cdot W_1\cap H\cdot W_2\subseteq H\cdot W$. Hence, by Remark 1 of Section 2,

$$\begin{aligned} &I(H\cdot W)=I(H\cdot W_1)+I(H\cdot W_2)\\ &\quad=\sqrt{(XY, I_{k_0+1}(Y), I_{n_0-k_0+1}(X))}+\sqrt{(YX, I_{k_1+1}(X), I_{n_1-k_1+1}(Y))}.\end{aligned}$$

But $(XY, I_{k_0+1}(Y), I_{n_0-k_0+1}(X))$ and $(YX, I_{k_1+1}(X), I_{n_1-k_1+1}(Y))$ are prime ideals, by [MuSe]. Therefore

$$I(\overline{O_f})=I(H\cdot W)=(XY, YX, I_{t_0}(X), I_{t_1}(Y))$$

is the ideal of $\overline{O_f}$.

Proof of (1): Now we prove that the variety of circular complexes is F-split. For any $0\le k_0\le n_0$ and $0\le k_1\le n_1$ we take $W_{k_0}=W_1$ and $W_{k_1}=W_2$ in the above argument. Then, by [MuSe] we have

$$\bigcap_{0\le k_0\le n_0} I(H\cdot W_{k_0})=(XY)\quad\text{and}\quad \bigcap_{0\le k_1\le n_1} I(H\cdot W_{k_1})=(YX).$$

But all $H\cdot W_{k_0}$ and $H\cdot W_{k_1}$ are simultanously compatibly F-split in Z, therefore, by Remark 1 of Section 2,

$$\bigcap_{0\le k_0\le n_0} I(H\cdot W_{k_0})+\bigcap_{0\le k_1\le n_1} I(H\cdot W_{k_1})=(XY, YX)$$

is a radical ideal and $\operatorname{Spec} k[X, Y]/(XY, YX)$ is F-split.

Proof of (3): Given non-negative integers l_0, l_1 such that $l_0+l_1\le\min\{n_0, n_1\}$, one can construct a circular complex $h=(h_1, h_2)\in L$ such that rank $h_1=l_0$, rank $h_2=l_1$. Now, by statement 2,

$$I(l_0, l_1)=(XY, YX, I_{l_0+1}(X), I_{l_1+1}(Y))=I(\overline{O}_h)$$

and therefore, by Theorem 1.1, the quotient ring $k[X, Y]/I(l_0, l_1)$ is a normal Cohen-Macaulay domain with a rational resolution. This completes the proof of the theorem. □

Remark. It follows from the proof of Theorem 1.1 that all the orbit closures $\overline{O}_f$, where $f=(f_1, f_2)\in L$ denotes an arbitrary circular complex, are simultaneously compatibly F-split in Z. Since $L=X'\cap Z$, where X' is an intersection of Schubert varieties in G/Q (*e.g.*, as described in Lemma 3.4, one can take $D_{p_{l_0}}(\theta_1)_{n_1\times n_0}$ and $D_{p_{l_1}}(\theta_2)_{n_0\times n_1}$) the same section compatibly splits L in Z. Therefore all the orbit closures $\overline{O}_f$, for circular complexes f, are also compatibly F-split in L itself.

Remark. It is easy to check that natural generalizations of all the results stated here for the variety of circular complexes $k[X, Y]/(XY, YX)$ are valid for the variety of circular complexes of arbitrary length, *i.e.*, for $\mathrm{Spec}\, k[X_1, \dots, X_n]/(X_1X_2, X_2X_3, \dots, X_nX_1)$, where X_j's are matrices of indeterminates of compatible size.

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