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On some Frobenius restriction theorems for semistable sheaves

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Abstract. We prove a version of an effective Frobenius restriction theorem for semistable bundles in characteristic p. The main novelty is in restricting the bundle to the p-fold thickening of a hypersurface section. The base variety is G/P, an abelian variety or a smooth projective toric variety.

Keywords. Semistable bundles; Frobenius restriction theorem; generic hypersurface; general hypersurface; p-fold thickening.

1. Introduction

Throughout this paper semistability is understood in the sense of Mumford–Takemoto (see Definition 2.2). Semistable sheaves on higher dimensional varieties have been studied for a long time. In particular restriction theorems for semistable sheaves are of interest. The first result in this direction is due to Maruyama [Ma], with a restriction on the rank of the sheaf. In [MR2], Mehta and Ramanathan proved a similar theorem, without any restriction on the rank, but not fixing the degree of the hypersurface. Flenner [F] proved an effective restriction theorem but was restricted to characteristic zero. Later on, Bogomolov [B] and Langer [L1] proved effective restriction theorems in char 0 and p, respectively.

In this paper, we prove a version of an effective Frobenius restriction theorem in characteristic p. The base variety X is either (1) G/P, where G is a semisimple algebraic group and P is a maximal parabolic subgroup, (2) an abelian variety or (3) a nonsingular projective toric variety. In all these three cases, the Frobenius map $F_X : X \to X$ preserves semistability under pull back (Remark 2.4 and Lemma 2.5). Hence it is natural to ask the following question: If X is as above, and $\mathcal{E}$ is a semistable sheaf on X (with respect to some polarization $\mathcal{H}$), can one find an integer d such that $\mathcal{E}$ as well as all its Frobenius pull backs $\{F_X^{t*}(\mathcal{E})\}_{t\geq 1}$, are again semistable on the generic hypersurface of degree d?

This question was answered affirmatively (1) when X is a smooth projective variety and $\mathcal{E}$ is a strongly semistable bundle of rank $< \dim(X)$, by Maruyama [Ma] with $d \geq 1$, (2) when $\mathcal{E}$ is a homogeneous bundle on $\mathbf{P}^n$ induced by an irreducible representation of P, by [MT] with $d \geq 2$, where P is a maximal parabolic subgroup of $GL(n+1)$ such that $GL(n+1)/P = \mathbf{P}^n$.

We give a partial answer to this question. More precisely we prove:

Let X be as in (1), (2) or (3) above and $\mathcal{E}$ be a semistable sheaf on X with respect to a polarization $\mathcal{H}$. Assume that there exists d such that $\{\wedge^i \mathcal{E}\}$, $1 \leq i \leq$ rank $\mathcal{E}$ are all semistable on Y_d, the generic degree d hypersurface of X. Then $F_X^{t*}(\mathcal{E})$ is semistable on the generic degree dp^t hypersurface Y_{dp^t} of X, for all $t \geq 1$. Note that by [MR2], an integer d satisfying the hypothesis of the above theorem can always be found.

The proof, in the case $X = G/P$, uses the vanishing theorems of Anderson [A] and Haboush [H]. When X is a toric variety, we can use the description of $F_{X*}(\mathcal{L})$, for any $\mathcal{L} \in \mathrm{Pic}\,(X)$, due to results of Bøgvad [Bo] and Thomsen [T]. When X is an abelian variety, we use the result of Mukai [Mu1] for semi-homogeneous bundles on abelian varieties. The basic idea is to translate the semistability of $\{\wedge^i \mathcal{E}\}$, where $1 \leq i \leq$ rank $\mathcal{E}$, on the generic deg d hypersurface Y_d of X into a vanishing result. Then applying $F_{X*}F_X^*$, using the projection formula and the relevant vanishing theorems in each case, we prove a vanishing theorem for $F_X^{t*}(\mathcal{E})$ on the generic deg dp^t hypersurface Y_{dp^t}. This gives the main result. In fact the result can be stated and proved for general complete intersection in X (Remark 2.8) and principal G-bundles on X (Remark 2.9). However this method do not give a generic restriction theorem for strong semistability, as the degree of the generic hypersurface grows with each Frobenius pull back. As a consequence of his proof of boundedness in characteristic p, Langer [L1] has proved a similar result, valid for any X, but the degree of the hypersurface gets multiplied by p^2, for each application of the Frobenius F_X. In his survey article (Thm 2.20 of [L2]) Langer has also proved an effective generic restriction theorem for strong semistability, but with an assumption on the characteristic of the ground field (it should be greater than the degree of the hypersurface, where degree of the hypersurface should be greater than a function depending on the invariants of X and $\mathcal{E}$). The question whether our results remain valid for any X is open.

2. Main theorem

Notation 2.1. We recall and use the following notion of 'generic' and 'general' as given in [MR2], throughout this paper. Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective variety over k. We can further assume that dim $X \geq 2$, as, for dim $X = 1$, there is nothing to prove. Let $\mathcal{H}$ be a given very ample line bundle on X. Let

$$S_d = \mathrm{Proj}(\mathrm{Sym}\; H^0(X, \mathcal{H}^d)^*)$$

be the projective space of lines in $H^0(X, \mathcal{H}^d)$. We note that an element in $H^0(X, \mathcal{H}^d)$ defines an effective divisor. Then we have

$$\begin{array}{ccc} Z_d & \xrightarrow{\pi_d} & S_d \\ \downarrow^{\eta_d} & & \\ X & & \end{array},$$

where

$$Z_d = \{(x, s) \in X \times S_d | s(x) = 0\} \subset X \times S_d,$$

and η_d, π_d are projections. The fiber of π_d over any closed point $s \in S_d$ is embedded in X via η_d, as a hypersurface of X. Moreover there is a nonempty open subset of S_d over which the geometric fibres of π_d are irreducible, as dim $X \geq 2$. Let K_d be the function field of S_d. Let Y_d be the generic fiber of π_d given by the fiber product

$$\begin{array}{ccc} Z_d & \xrightarrow{\pi_d} & S_d \\ \uparrow & & \uparrow \\ Y_d & \longrightarrow & \mathrm{Spec}\; K_d, \end{array}$$

thus Y_d is an absolutely irreducible, nonsingular hypersurface in X_d, where we define

$$X_d = X \times_k \text{Spec } K_d.$$

In particular, we have

$$\begin{array}{ccccc} Y_d & \hookrightarrow & X_d & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ & & \text{Spec } K_d & \longrightarrow & \text{Spec } k \end{array}.$$

We call Y_d *the generic hypersurface* of degree d.

We recall the following definition of semistable sheaves, given in [HL].

DEFINITION 2.2

A coherent sheaf $\mathcal{E}$ of dimension $d = \dim(X)$ is semistable (with respect to a polarization $\mathcal{H}$), if

$$T_{d-1}(\mathcal{E}) = T_{d-2}(\mathcal{E}) \quad \text{and} \quad \mu(\mathcal{F}) \leq \mu(\mathcal{E})$$

for all subsheaves $\mathcal{F} \subset \mathcal{E}$, where $T_i(\mathcal{E})$ denotes the maximal subsheaf of $\mathcal{E}$ of dimension $\leq i$, and where $\mu(\mathcal{E}) = c_1(\mathcal{E}) \cdot \mathcal{H}^{d-1}/\text{rank}\,(\mathcal{E})$ and rank $(\mathcal{F}) <$ rank $(\mathcal{E})$.

We say $\mathcal{E}$ is strongly semistable if, for every t-th iteration of the Frobenius map $F : X \mapsto X$, the coherent sheaf $F^{t*}\mathcal{E}$ is semistable,

Remark 2.3. For a coherent sheaf $\mathcal{E}$ of dimension $d = \dim(X)$ on a smooth projective variety X, if $T_{d-1}(\mathcal{E}) = T_{d-2}(\mathcal{E})$, then

$$T_{\dim \mathcal{F}-1}(\mathcal{F}) = T_{\dim \mathcal{F}-2}(\mathcal{F}), \text{ if } \mathcal{F} = \wedge^i \mathcal{E}, \quad \mathcal{F} = F_X^{t*}\mathcal{E} \text{ or } \mathcal{F} = \mathcal{E}|_{Y_d},$$

where Y_d is the generic degree d hypersurface of X. Moreover $\mathcal{E}$ is semistable if and only if $\mathcal{E}^{DD}$ is semistable, where $\mathcal{E}^{DD}$ denotes the reflexive hull of $\mathcal{E}$ in $\mathcal{O}_X$.

Let $\mathcal{E}$ be a coherent sheaf on X. We say $\mathcal{E}$ is semistable on a subvariety $Y \subseteq X$ if $\mathcal{E}|_Y$ is μ-semistable with respect to the line bundle $\mathcal{H}|_Y$, and, the sheaf $\eta_d^*(\mathcal{E})|_{Y_d}$ is semistable, if it is μ-semistable with respect to $\eta_d^*(\mathcal{H})|_{Y_d}$.

Whenever a property holds for $\pi_d^{-1}(s)$ (a hypersurface in X of degree d given by the equation $s \in S_d$), for s in a nonempty Zariski open subset of S_d, then we say it holds for *a general* s.

Henceforth, for a sheaf $\mathcal{E}$ on X, we denote

$$\mathcal{E}|_{Y_d} := \eta_d^*(\mathcal{E})|_{Y_d} \quad \text{and} \quad \mathcal{E}_s := \eta_d^*(\mathcal{E})|_{\pi_d^{-1}(s)}.$$

We also note the fact that the sheaf $\mathcal{E}|_{Y_d}$, for the generic hypersurface Y_d, is semistable if and only if $\mathcal{E}_s$ is semistable for a general hypersurface $\pi_d^{-1}(s)$ of degree d.

Remark 2.4. In the case $X = G/P$, where G is a semisimple algebraic group and P is a maximal parabolic subgroup of G, the tangent bundle is generated by global sections and hence $\mu_{\min}(\mathcal{T}_X) \geq 0$ (with respect to any polarization $\mathcal{H}$). In case X is an abelian variety, the tangent bundle is trivial and hence is semistable of slope 0 (with respect to any polarization $\mathcal{H}$). Therefore, by Theorem 2.1 of [MR1], every semistable bundle $\mathcal{E}$ (with respect to any polarization $\mathcal{H}$) is strongly semistable in both cases.

Lemma 2.5. *Let X be a nonsingular projective toric variety over a perfect field of* char p. *Then for any given polarization $\mathcal{H}$, a semistable bundle on X (with respect to $\mathcal{H}$) is strongly semistable.*

Proof. Let $F_{X*}\Omega^{\bullet}_X$ be the complex obtained by applying F_{X*} to the De Rham complex $\Omega^{\bullet}_X$ of X. Let $B^i_X \subseteq Z^i_X \subseteq F_{X*}\Omega^i_X$ denote the coboundaries and cocycles in degree i. By 3.4 in §3 and the proof of Theorem 2 in [BTLM], the following short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow B^i_X \longrightarrow Z^i_X \longrightarrow \Omega^i_X \longrightarrow 0,$$

given by the Cartier operator, splits, for all $i \geq 0$. Now we prove that $\mu_{\min}(\mathcal{T}_X) \geq 0$, or equivalently, $\mu_{\max}(\Omega^1_X) \leq 0$ (with respect to any fixed polarization $\mathcal{H}$). If not then there is a subsheaf $V \subset \Omega^1_X$ with $\mu(V) > 0$. Let $r = \text{rank } V \leq \dim(X)$. Then $\mathcal{L} = \wedge^r V$ is a line bundle of positive degree. But

$$\mathcal{L} \hookrightarrow \Omega^r_X \implies \mathcal{L} \hookrightarrow Z^r_X \hookrightarrow F_{X*}(\Omega^r_X) \implies \mathcal{L}^p \hookrightarrow \Omega^r_X.$$

Repeating this argument, we get $\mathcal{L}^{p^t} \hookrightarrow \Omega^r_X$, for all $t > 0$, which is a contradiction because $\deg \mathcal{L}^{p^t} \to \infty$ as $t \to \infty$. Therefore we conclude that $\mu_{\min}(\mathcal{T}_X) \geq 0$ and hence the lemma follows by [MR1]. ■

PROPOSITION 2.6

Let X be a nonsingular projective variety over a field of char $p > 0$. *Let $\mathcal{E}$ be a coherent reflexive sheaf on X. Fix t and let $d \in \mathbb{N}$ and $q = p^t$ such that $dq \geq 3$. If $F^{t*}_X\mathcal{E}|_{Y_{dq}}$ is not semistable on the generic hypersurface of degree dq of X then there exists $\mathcal{L} \in \text{Pic } X$ and an integer $1 \leq r <$ rank $\mathcal{E}$ such that*

(1) $H^0(Y_d, (\Lambda^r\mathcal{E})|_{Y_d} \otimes (F^t_{X_d*}\mathcal{L}^{-1})|_{Y_d}) \neq 0$ *and*
(2) $\mu(\mathcal{L}) > \mu(F^{t*}_X(\wedge^r\mathcal{E}))$,

where $\Lambda^r\mathcal{E} = (\wedge^r\mathcal{E})^{DD}$ is the reflexive hull of $\wedge^r\mathcal{E}$ in X.

Proof. Since the sheaf $F^{t*}_X\mathcal{E}|_{Y_{dq}}$ is not semistable, there is a dense open set V_{dq} of S_{dq} such that $(F^{t*}_X\mathcal{E})|_s := \eta^*_{dq}(F^{t*}_X\mathcal{E})|_{\pi^{-1}_{dq}(s)}$ is a torsion-free sheaf of $\mathcal{O}_{\pi^{-1}_{dq}(s)}$-modules and there exists a torsion-free coherent subsheaf $\mathcal{W}_{dq}$ defined over V_{dq} such that

$$\mathcal{W}_{dq} \hookrightarrow \eta^*_{dq}F^{t*}_X\mathcal{E}|_{\pi^{-1}_{dq}(V_{dq})}$$

and such that $\mathcal{W}_{dq}|_{\pi^{-1}_{dq}(s)}$ is the maximal destabilizing subsheaf of $(F^{t*}_X\mathcal{E})|_s$, for every $s \in V_{dq}$. Now $\mathcal{W} = \mathcal{W}_{dq}|_{Y_{dq}}$, being torsion-free, is locally free on an open dense set U of Y_{dq}, where $Y_{dq}\backslash U$ is of codimension ≥ 2 in Y_{dq}.

Therefore there exists $\bar{\mathcal{L}} \in \text{Pic } Y_{dq}$ such that $\bar{\mathcal{L}}|_U = \wedge^r\mathcal{W}|_U$, where $r = \text{rank } \mathcal{W}$. Since $dq \geq 3$, by Proposition 2.1 of [MR2], we have $\bar{\mathcal{L}} = \mathcal{L}|_{Y_{dq}}$, for some $\mathcal{L} \in \text{Pic } X$. Hence the induced map

$$\bar{\mathcal{L}}|_U \hookrightarrow \wedge^r(F^{t*}_X\mathcal{E})|_U = F^{t*}_X(\wedge^r\mathcal{E})|_U$$

extends to a map

$$\mathcal{L}|_{Y_{dq}} \hookrightarrow (F^{t*}_X(\wedge^r\mathcal{E})|_{Y_{dq}})^{DD},$$

where $\{-\}^{DD}$ denotes the reflexive hull of $\{-\}$ in $\mathcal{O}_{Y_{dq}}$. In particular,

$$H^0(Y_{dq}, (F_X^{t*}(\wedge^r \mathcal{E})|_{Y_{dq}})^{DD} \otimes \mathcal{L}^{-1}|_{Y_{dq}}) \neq 0.$$

Now we have the following.

Claim. $(F_X^{t*}(\wedge^r \mathcal{E})|_{Y_{dq}})^{DD} \simeq (F_X^{t*} \Lambda^r \mathcal{E})|_{Y_{dq}}$.

We note that the canonical map

$$F_X^{t*}(\wedge^r \mathcal{E}) \longrightarrow F_X^{t*}(\Lambda^r \mathcal{E}) \tag{2.1}$$

gives the map

$$F_X^{t*}(\wedge^r \mathcal{E})|_{Y_{dq}} \longrightarrow F_X^{t*}(\Lambda^r \mathcal{E})|_{Y_{dq}}.$$

But, since $F_X^{t*}(\Lambda^r \mathcal{E})$ is a reflexive sheaf on Z_{dq}, the sheaf $F_X^{t*}(\Lambda^r \mathcal{E})|_{Y_{dq}}$ is a reflexive sheaf on Y_{dq}. This gives

$$(F_X^{t*}(\wedge^r \mathcal{E})|_{Y_{dq}})^{DD} \longrightarrow (F_X^{t*} \Lambda^r \mathcal{E})|_{Y_{dq}}. \tag{2.2}$$

Now, as the map (2.1) is an isomorphism outside a codimension ≥ 2 closed subset of Z_{dq}, the map (2.2) is an isomorphism outside a codimension ≥ 2 closed subset of Y_{dq}. Hence, being a map between two reflexive sheaves, the map (2.2) is an isomorphism. This proves the claim.

In particular

$$H^0(Y_{dq}, (F_X^{t*} \Lambda^r \mathcal{E})|_{Y_{dq}} \otimes \mathcal{L}^{-1}|_{Y_{dq}}) \neq 0.$$

Recall that $S_d = \mathbb{P}(V)$, where $V = H^0(X, \mathcal{H}^d)^*$. Let $V^{(q)} = F^{t*}(V)$. Consider the following canonical maps:

$$V^{(q)} \longrightarrow \mathrm{Sym}^q(V) \longrightarrow H^0(X, \mathcal{H}^{qd})^*,$$

the composite map induces a map $\phi_1 : S_d \longrightarrow S_{dq}$. Let $\phi = \phi_1 \circ F_{S_d}$, where $F_{S_d} : S_d \longrightarrow S_d$ is the absolute Frobenius map on S_d. This gives a map

$$\begin{array}{ccc} X \times S_d & \stackrel{\mathrm{Id}\times\phi}{\longrightarrow} & X \times S_{dq} \\ \uparrow & & \uparrow \\ X_d & \stackrel{\mathrm{Id}\times\phi}{\longrightarrow} & X_{dq} \end{array}.$$

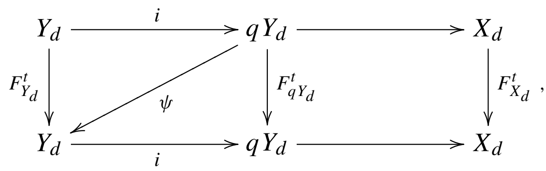
If Y_d is the hypersurface in X_d defined by an equation $f = 0$, where $f \in H^0(X_d, \mathcal{H}^d)$, then let qY_d be the hypersurface given by $f^q = 0$. Then, under the embedding $\mathrm{Id} \times \phi$, the image of Y_d is isomorphic to qY_d and is a fiber over a point in S_{dq}. Let $U_{dq} \subseteq S_{dq}$ be the maximal open dense set such that $\eta_{dq}^* F_X^{t*}(\Lambda^r \mathcal{E})$ is flat over U_{dq}. Then the universal property of the flattening stratification [Mu2], for the projective map $\pi_{dq} : Z_{dq} \to S_{dq}$ and for the sheaf $\eta_{dq}^* F_X^{t*}(\Lambda^r \mathcal{E})$, implies that qY_d is a fiber over a point in U_{dq}. Now, by the semicontinuity theorem for the map

$$\pi_{dq}|_{\pi_{dq}^{-1}(U_{dq})} : Z_{dq}|_{\pi_{dq}^{-1}(U_{dq})} \to U_{dq},$$

we have

$$H^0(qY_d, (F_X^{t*}\Lambda^r\mathcal{E})|_{qY_d} \otimes \mathcal{L}^{-1}|_{qY_d}) \neq 0. \tag{2.3}$$

Consider the following diagram



where $F_Z^t : Z \to Z$ denotes the t-th iterated Frobenius over a given variety Z, and $i : Y_d \to qY_d$ is the canonical inclusion of the reduced subscheme. The Frobenius morphism on the non-reduced scheme qY_d factors through i, giving an induced morphism $\psi : qY_d \longrightarrow Y_d$.

Hence we have

$$\begin{aligned}
&H^0(qY_d, (F_{X_d}^{t*}\Lambda^r\mathcal{E})|_{qY_d} \otimes \mathcal{L}^{-1}|_{qY_d}) \neq 0\\
&\implies H^0(qY_d, F_{qY_d}^{t*}(\Lambda^r\mathcal{E}|_{qY_d}) \otimes \mathcal{L}^{-1}|_{qY_d}) \neq 0\\
&\implies H^0(qY_d, (i \circ \psi)^*(\Lambda^r\mathcal{E}|_{qY_d}) \otimes \mathcal{L}^{-1}|_{qY_d}) \neq 0\\
&\implies H^0(qY_d, \psi^* \circ i^*(\Lambda^r\mathcal{E}|_{qY_d}) \otimes \mathcal{L}^{-1}|_{qY_d}) \neq 0\\
&\implies H^0(qY_d, \psi^*(\Lambda^r\mathcal{E}|_{Y_d}) \otimes \mathcal{L}^{-1}|_{qY_d}) \neq 0\\
&\implies H^0(Y_d, (\Lambda^r\mathcal{E})|_{Y_d} \otimes \psi_*(\mathcal{L}^{-1}|_{qY_d})) \neq 0\\
&\implies H^0(Y_d, (\Lambda^r\mathcal{E})|_{Y_d} \otimes (F_{X_d*}^t\mathcal{L}^{-1})|_{Y_d}) \neq 0.
\end{aligned}$$

This proves part (1) of the proposition.

Since $\mathcal{W}_{dq}|_s$ is the maximal de-stabilizing subsheaf of $F_X^{t*}\mathcal{E}|_s$, for general $s \in S_{dq}$, we have

$$\begin{aligned}
&\mu(\mathcal{W}_{dq}|_{\pi_{dq}^{-1}(s)}) > \mu(F_X^{t*}\mathcal{E}|_s), \quad \text{for general } s\\
&\implies \mu(\wedge^r\mathcal{W}_{dq}|_{\pi_{dq}^{-1}(s)}) > \mu(\wedge^r(F_X^{t*}\mathcal{E})|_s), \quad \text{for general } s\\
&\implies \mu(\wedge^r\mathcal{W}_{dq}|_{Y_{dq}}) > \mu(\wedge^r F_X^{t*}\mathcal{E}|_{Y_{dq}})\\
&\implies \mu(\mathcal{L}|_{Y_{dq}}) > \mu(\wedge^r F_X^{t*}\mathcal{E}|_{Y_{dq}})\\
&\implies \mu(\mathcal{L}) > \mu(\wedge^r F_X^{t*}\mathcal{E}) = \mu(F_X^{t*}\Lambda^r\mathcal{E}).
\end{aligned}$$

This proves part (2) and hence the proposition. ■

Theorem 2.7. *Let X be one of the following projective varieties*:

(1) *G/P, where G is a semisimple algebraic group and P is a maximal parabolic subgroup of G*,

(2) *a nonsingular projective toric variety, or*
(3) *an abelian variety.*

Let $\mathcal{E}$ be a coherent semistable sheaf on X such that, for every $1 \leq i < \operatorname{rank} \mathcal{E}$, the sheaf $(\wedge^i \mathcal{E})_s$ is semistable, for general $s \in S_d$. Fix $t \geq 0$ and let $q = p^t$ such that $dq \geq 3$. Then $F_X^{t*}\mathcal{E}_s$ is semistable for general $s \in S_{dq}$, where F_X^t is the t-th iterated Frobenius map.

Proof. By Remark 2.3, we can re-phrase the problem as follows: If we are given a coherent semistable sheaf $\mathcal{E}$ on X such that, for every $1 \leq i < \operatorname{rank} \mathcal{E}$, the sheaf $\wedge^i \mathcal{E}|_{Y_d}$ is semistable, for the generic degree d hypersurface Y_d of X and $dq \geq 3$, where we denote $q = p^t$. Then we need to prove that $F_X^{t*}\mathcal{E}|_{Y_{dq}}$ is semistable for the generic degree dq hypersurface Y_{dq} of X.

Note that $\mathcal{E}|_{Y_d}$ is semistable if $\mathcal{E}^{DD}|_{Y_d}$ is semistable as $\mathcal{E}^{DD}|_{Y_d} = (\mathcal{E}|_{Y_d})^{DD}$, where $\mathcal{E}^{DD}$ is the reflexive hull of $\mathcal{E}$ in $\mathcal{O}_X$ and $(\mathcal{E}|_{Y_d})^{DD}$ is the reflexive hull of $\mathcal{E}|_{Y_d}$ in $\mathcal{O}_{Y_d}$. We also denote $(\wedge^i \mathcal{E})^{DD}$ by $\Lambda^i \mathcal{E}$. Since $(\wedge^i \mathcal{E})|_{Y_d} \simeq \Lambda^i \mathcal{E}|_{Y_d}$ outside a co-dimension 2 closed subset of Y_d, the semistablity of $\wedge^i \mathcal{E}|_{Y_d}$ is equivalent to the semistability of $\Lambda^i \mathcal{E}|_{Y_d}$. Hence, we can assume that $\mathcal{E} \simeq \mathcal{E}^{DD}$ and $\Lambda^i \mathcal{E}|_{Y_d}$ is semistable on Y_d, for all i.

Case 1. Let X be a nonsingular projective toric variety.

If $F_X^{t*}\mathcal{E}|_{Y_{dq}}$ is not semistable then choose $\mathcal{L}$ and an integer r as in Proposition 2.6. By Theorem 1 of [Bo] and Theorem 1 of [T], we have

$$F_{X*}^t \mathcal{L}^{-1} = \oplus \mathcal{L}_i,$$

where $\mathcal{L}_i$ are line bundles on X. Therefore we have $\mathcal{L}_i \hookrightarrow F_{X*}^t \mathcal{L}^{-1}$. Since F_X^{t*} is a right adjoint of F_{X*}^t, this induces a nonzero map $F_X^{t*}\mathcal{L}_i \to \mathcal{L}^{-1}$. Therefore $\mu(F_X^{t*}\mathcal{L}_i) \leq \mu(\mathcal{L}^{-1})$ on X. By Proposition 2.6, we also have

$$\begin{aligned} &\mu(\mathcal{L}) > \mu(F_X^{t*} \wedge^r \mathcal{E}) \\ &\implies \mu(F_X^{t*} \wedge^r \mathcal{E}) + \mu(\mathcal{L}^{-1}) < 0 \\ &\implies \mu(F_X^{t*} \wedge^r \mathcal{E}) + \mu(F_X^{t*}\mathcal{L}_i) < 0 \\ &\implies \mu(\wedge^r \mathcal{E}) + \mu(\mathcal{L}_i) < 0 \\ &\implies \mu(\wedge^r \mathcal{E} \otimes \mathcal{L}_i) < 0 \\ &\implies \mu(\Lambda^r \mathcal{E} \otimes \mathcal{L}_i) < 0 \\ &\implies H^0(Y_d, \Lambda^r \mathcal{E}|_{Y_d} \otimes \mathcal{L}_i|_{Y_d}) = 0, \quad \text{for all } i, \end{aligned}$$

where the last assertion follows as $\Lambda^r \mathcal{E}|_{Y_d}$ is semistable on Y_d. This implies that

$$H^0(Y_d, \Lambda^r \mathcal{E}|_{Y_d} \otimes (F_{X*}^t \mathcal{L}^{-1})|_{Y_d}) = 0,$$

which is a contradiction. Hence we conclude that $F_X^{t*}\mathcal{E}|_{Y_{dq}}$ is semistable on the generic degree dq hypersurface of X.

Case 2. Let X be an abelian variety.

Then, by [Mu1], for any line bundle $\mathcal{M}$, the sheaf $F_X^{t*}F_{X*}^t\mathcal{M}$ has a filtration whose associated graded bundle is a direct sum of the *same* line bundle $\mathcal{M}$. Therefore

$$(F_X^{t*}F_{X*}^t\mathcal{M})|_{Y_d} = F_{Y_d}^{t*}((F_{X*}^t\mathcal{M})|_{Y_d})$$

is filtered by line bundle $\mathcal{M}|_{Y_d}$. This implies that $(F^t_{X*}\mathcal{M})|_{Y_d}$ is semistable on Y_d, for any line bundle $\mathcal{M}$ on X.

Now, if $F_X^{t*}\mathcal{E}|_{Y_{dq}}$ is not semistable on Y_{dq} then there exists $\mathcal{L}$ and an integer r as in Proposition 2.6. In particular, there exists a nonzero $\mathcal{O}_{Y_d}$-linear map

$$\mathcal{O}_{Y_d} \longrightarrow (\Lambda^r\mathcal{E})|_{Y_d} \otimes F^t_{X*}(\mathcal{L}^{-1})|_{Y_d}.$$

This gives a nonzero map

$$(\Lambda^r\mathcal{E}|_{Y_d})^D \longrightarrow (F^t_{X*}\mathcal{L}^{-1})|_{Y_d}.$$

Since both the sheaves are semistable on Y_d, we have

$$\begin{aligned}
&\mu((\Lambda^r\mathcal{E}|_{Y_d})^D) \leq \mu((F^t_{X*}\mathcal{L}^{-1})|_{Y_d})\\
&\quad\Longrightarrow \mu((\wedge^r\mathcal{E}|_{Y_d})^D) \leq \mu((F^t_{X*}\mathcal{L}^{-1})|_{Y_d})\\
&\quad\Longrightarrow \mu((\wedge^r\mathcal{E})^D) \leq \mu(F^t_{X*}\mathcal{L}^{-1})\\
&\quad\Longrightarrow \mu(F_X^{t*}(\wedge^r\mathcal{E})^D) \leq \mu(F_X^{t*}F^t_{X*}\mathcal{L}^{-1}) = \mu(\mathcal{L}^{-1})\\
&\quad\Longrightarrow \mu(\mathcal{L}) \leq \mu(F_X^{t*}\wedge^r\mathcal{E}),
\end{aligned}$$

which is a contradiction.

Case 3. Let $X = G/P$, where G and P are as stated in the theorem.

Without loss of generality we can assume that deg $\mathcal{E} \geq 0$. If $F_X^{t*}\mathcal{E}|_{Y_{dq}}$ is not semistable then we choose $\mathcal{L}$ and r as in Proposition 2.6. Since Pic $(X) = \mathbb{Z}$, we have $\mathcal{L} \simeq \mathcal{O}_X(m)$, for some $m \in \mathbb{N}$. Since $\mu(\mathcal{L}) > \mu(F_X^{t*}(\wedge^r\mathcal{E}))$ we have

$$m > q\mu(\wedge^r\mathcal{E}).$$

Let

$$\begin{aligned}
t_0 &= \lceil\mu(\wedge^r\mathcal{E})\rceil, \quad \text{if } \mu(\wedge^r\mathcal{E}) \text{ is not an integer}\\
&= \mu(\wedge^r\mathcal{E}) + 1, \quad \text{otherwise},
\end{aligned}$$

where $\lceil x\rceil$ denotes the smallest integer $\geq x$. In particular, we have

$$\mu(\Lambda^r\mathcal{E}) = \mu(\wedge^r\mathcal{E}) < t_0 \text{ and} \tag{2.4}$$

$$t_0 - 1 \leq \mu(\wedge^r\mathcal{E}) = \mu(\Lambda^r\mathcal{E}). \tag{2.5}$$

Since $\Lambda^r\mathcal{E}|_{Y_d}$ is semistable on Y_d, by eq. (2.4), we have

$$H^0(Y_d, \Lambda^r\mathcal{E}|_{Y_d} \otimes_{\mathcal{O}_{Y_d}} \mathcal{O}_{X_d}(-t_0)|_{Y_d}) = 0.$$

This implies that

$$H^0(X_d, \Lambda^r\mathcal{E}|_{Y_d} \otimes_{\mathcal{O}_{X_d}} \mathcal{O}_{X_d}(-t_0)) = 0$$

which implies

$$H^0(G/B, p_1^*(\Lambda^r\mathcal{E}|_{Y_d} \otimes_{\mathcal{O}_{X_d}} \mathcal{O}_{X_d}(-t_0))) = 0,$$

where $B \subset P$ is a Borel group and there is a canonical map

$$G/B \hookrightarrow G/P_1 \times \cdots \times G/P_n,$$

with $P = P_1$. Note that if we denote

$$p_1^* \mathcal{O}_{G/P_1}(m_1) \otimes \cdots \otimes p_r^* \mathcal{O}_{G/P_n}(m_n) = \mathcal{O}(m_1, \ldots, m_n)$$

then $\mathcal{O}_{G/B}(1) = \mathcal{O}(1, \ldots, 1)$. So far we have proved that

$$H^0(G/B, p_1^*(\Lambda^r \mathcal{E}|_{Y_d}) \otimes \mathcal{O}(-t_0, 0, \ldots, 0)) = 0.$$

Now, by Theorem 2.5 of Andersen [A] and Theorem 2.1 of Haboush [H], this implies that

$$H^0(G/B, F_{G/B}^{t*}(p_1^*(\Lambda^r \mathcal{E}|_{Y_d})) \otimes \mathcal{O}(-qt_0, 0, \ldots, 0) \otimes \mathcal{O}(q-1, q-1, \ldots, q-1)) = 0.$$

Claim. There exists a nonzero map

$$p_1^* \mathcal{L}^{-1} \longrightarrow \mathcal{O}(-qt_0 + q - 1, q-1, \ldots, q-1).$$

We assume the proof of the claim for the moment. The claim implies that

$$\begin{aligned} & H^0(G/B, p_1^*(F_{X_d}^{t*}(\Lambda^r \mathcal{E}|_{Y_d}) \otimes_{\mathcal{O}_{X_d}} \mathcal{L}^{-1})) = 0 \\ \implies & H^0(X_d, F_{X_d}^{t*}(\Lambda^r \mathcal{E}|_{Y_d}) \otimes_{\mathcal{O}_{X_d}} \mathcal{L}^{-1}) = 0 \\ \implies & H^0(X_d, (F_{X_d}^{t*} \Lambda^r \mathcal{E})|_{qY_d} \otimes_{\mathcal{O}_{X_d}} \mathcal{L}^{-1}) = 0 \\ \implies & H^0(qY_d, (F_{X_d}^{t*} \Lambda^r \mathcal{E})|_{qY_d} \otimes_{\mathcal{O}_{qY_d}} \mathcal{L}^{-1}|_{qY_d}) = 0, \end{aligned}$$

where the second last equality follows by the diagram given in the proof of Proposition 2.6. Since the last equation contradicts (2.3), we conclude that $F_X^{t*} \mathcal{E}|_{Y_{dq}}$ is semistable on Y_{dq}. Hence it is enough to prove the claim.

Proof of the Claim. It is enough to prove

$$H^0(G/B, \mathcal{O}(-qt_0 + q - 1, q-1, \ldots, q-1) \otimes p_1^* \mathcal{L}) \neq 0,$$

equivalently

$$H^0(G/B, \mathcal{O}(m - qt_0 + q - 1, q-1, \ldots, q-1)) \neq 0.$$

But

$$m - qt_0 + q - 1 > q\mu(\wedge^r E) - qt_0 + q - 1 = q(\mu(\wedge^r \mathcal{E}) - (t_0 - 1)) - 1.$$

By (2.5), we have

$$\begin{aligned} & \mu(\wedge^r \mathcal{E}) - (t_0 - 1) \geq 0 \\ \implies & m - qt_0 + q - 1 > -1 \\ \implies & m - qt_0 + q - 1 \geq 0. \end{aligned}$$

In particular $\mathcal{O}(m - qt_0 + q - 1, q-1, \ldots, q-1)$ is a dominant line bundle on G/B and hence has a nonzero section. This proves the claim and hence the theorem. ■

Remark 2.8. In the above results (Proposition 2.6 and Theorem 2.7) one can replace the sentence 'a general (the generic) hypersurface' by 'a general (the generic) complete intersection' everywhere (the reader may refer to [MR2], for the relevant notation). The same proofs, given in this paper go through.

Remark 2.9. Let G be a semisimple group with Lie algebra $\mathfrak{g}$ and let $E \to X$ be a semistable G-bundle with X as given in Theorem 2.7. Let $E(\mathfrak{g})$ be the adjoint bundle of E. If, for every $1 \leq i \leq \text{rank } E(\mathfrak{g})$, the bundle $\wedge^i E(\mathfrak{g})$ is semistable on the generic degree d hypersurface of X then, for any integer $t \geq 0$, the bundle $F_X^{t*}(E)$ is semistable when restricted to the generic degree dp^t hypersurface Y_{dp^t}. This follows from Theorem 2.7.

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