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Nef line bundles which are not ample

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1 Introduction

In [6], there is a construction of a line bundle on a complex projective nonsingular variety which is ample on every proper subvariety but which is nonample on the ambient variety. The example is obtained as the projective bundle associated to a "general" stable vector bundle of degree zero on a compact Riemann surface of genus $g \geq 2$. The proof in [6] constructs the vector bundle using a unitary representation of the fundamental group with the required property (of denseness). Here, we give an algebraic argument which is characteristic free, to show the existence of a variety of dimension ≤ 3 with a line bundle as above (see Theorem (3.1) and Remark (3.2)).

2 The vector bundle on the curve

Let C be a complete nonsingular curve defined over an uncountable algebraically closed field (of any characteristic). Let M_r^s denote the moduli space of stable bundles of rank r and degree zero on C and M_r^{ss} the moduli of semistable bundles of rank r and degree zero. We further assume that the curve C is ordinary. We first show

Proposition 2.1 *Let the characteristic of the ground field be positive and F the Frobenius morphism on C. There is a proper closed subset of M_r^s such that for any stable bundle V in the complement of this closed set, F^*V is also stable.*

Proof. We prove this Proposition in a sequence of Lemmas:

Lemma A *M_r^s is non-empty, irreducible, and non-singular and dim $M_r^s = r^2(g-1)+1$*

Proof. This follows from the computations made in [7, Sec. 5]. Those computations were made for vector bundles with parabolic structure, but they also go through for bundles without any extra structure.

Let L be a fixed line bundle of sufficiently large degree such that for all semistable bundles V on C of rank r and degree $0, H^0(C, V \otimes L)$ generates $V \otimes L$ and $H^1(C, V \otimes L) = 0$. Consider the quotient scheme Q of quotients

$$H^0(C, V_0 \otimes L) \otimes L^* \to V \to 0$$

where V_0 is a fixed semistable vector bundle of rank r and degree 0. Let R^{ss} denote the open subset of points $q \in Q$ such that a) the corresponding bundle V_q is semistable, b) $H^0(C, V_0 \otimes L) \to H^0(C, V_q \otimes L)$ is an isomorphism and c) $H^1(C, V_q \otimes L) = 0$. We observe that R^{ss} is irreducible and non-singular [8, Remark 5.5].

Lemma B *The Frobenius map $F : C \to C$ defines a rational map $R^{ss} \to R^{ss}$.*

Proof. If $q_0 \in Q$ corresponds to the trivial bundle $\mathcal{O}^{\oplus r} = V_{q_0}$, then clearly $F^{\#}(V_{q_0})$ is semistable. Hence there exists a nonempty open subset , say $\mathcal{U}$, of R^{ss} such that if $q \in \mathcal{U}$, then $F^{\#}(V_q)$ is also semistable, as $q_0 \in R^{ss}$. This defines a rational map $R^{ss} \to R^{ss}$ whose domain of definition is $\mathcal{U}$ and sends q_0 to itself. Call this rational map $\tilde{f}$.

Lemma C *There exists a point $q_1 \in \mathcal{U}$ such that the corresponding bundle V_{q_1} is stable.*

Proof. Let $\mathcal{U}_1$ denote the subset of R^{ss} consisting of stable bundles. Clearly, $\mathcal{U}_1$ is open and *non-empty* since M_r^s is non-empty, by Lemma A. Therefore $\mathcal{U} \cap \mathcal{U}_1$ is non-empty as R^{ss} is irreducible and we can choose $q_1 \in \mathcal{U} \cap \mathcal{U}_1$.

Lemma D *There exists a simple bundle $V \in \mathcal{U}$ such that $F^{\#}(V)$ is also simple*

Proof. We consider a stable bundle V in Q at which $\tilde{f}$ is defined. Since V is a stable bundle, it is also a simple bundle. Let $\widetilde{V}$ be the universal quotient bundle on $C \times Q$. Let

$$0 \to \mathcal{O}_C \to F_*\mathcal{O}_C \to B^1 \to 0$$

be the exact sequence of vector bundles on C induced by the Frobenius F on C. Since the curve C is ordinary, $H^0(C, B^1) = 0$, and therefore for the trivial bundle $\mathcal{O}^{\oplus r^2}, H^0(C, B^1 \otimes \mathcal{O}^{\oplus r^2}) = 0$. The universal bundle $\widetilde{V}$ gives a family $\mathrm{End}(\widetilde{V})$ on $C \times Q$ and at a point $p \in Q$, corresponding to the trivial bundle $\mathcal{O}^{\oplus r}$, we have $H^0(C \times p, B^1 \otimes \mathrm{End}\,(V) \mid C \times p) = 0$. It follows by semicontinuity that $H^0(C, B^1 \otimes \mathrm{End}\,(V)) = 0$ for the general (Zariski open set) semistable bundle V of rank r and degree zero. From the sequence

$$0 \to \mathrm{End}\,(V) \to \mathrm{End}\,(V) \otimes F_*\mathcal{O} \to \mathrm{End}(V) \otimes B^1 \to 0$$

we obtain

$$\begin{aligned} H^0(C,\ \text{End}\,(V)) &= H^0(C,\ \text{End}\,(V) \otimes F_* O_C) \\ &= H^0(C,\ \text{End}\, F^*V)) \end{aligned}$$

where the first equality is true for the general semistable bundle V in Q. This shows that for the general simple bundle V in Q, F^*V also is simple.

Therefore we can consider a stable bundle V in Q at which $\widetilde{f}$ is defined such that F^*V is also simple. We observe that by Artin's theorem (see [1], Th.5.2) there is a local moduli space (as an algebraic scheme) of simple bundles at any simple bundle of rank r and degree zero. Letting S_1 and S_2 denote the local moduli spaces of simple bundles at V and F^*V respectively, we see that $W \mapsto F^*W$ defines a morphism in the étale topology from S_1 to S_2 (i.e., there are étale neighborhoods N_1 of V in S_1 and N_2 of F^*V in S_2 and a morphism $f_s : N_1 \to N_2$ induced by $W \mapsto F^*W$). We now observe that the tangent space to S_2 (and hence N_1) at V is $H^1(C,\ \text{End}\ V)$ and the tangent space to S_2 at F^*V is $H^1(C,\ \text{End}\ (F^*V))$. The differential of the morphism $f_s : N_1 \to N_2$ at V is the natural map $H^1(C,\ \text{End}\ V) \to H^1(C, \text{End}\ F^*V)$. We have the sequence

$$0 \to \text{End}\ V \to \text{End}\ V \otimes F_*\mathcal{O}_C \to \text{End}\ V \otimes B^1 \to 0$$

and the induced maps

$$H^0(C,\ \text{End}\ V \otimes B^1) \to H^1(C,\ \text{End}\ V) \to H^1(C,\ \text{End}\ V \otimes F_*\mathcal{O}_C).$$

By the observation made earlier, since C is an ordinary curve, $H^0(C,\ \text{End}\ V \otimes B^1) = 0$ for general V, so we have an injection

$$0 \to H^1(C,\ \text{End}\ V) \to H^1(C,\ \text{End}\ V \otimes F_*\mathcal{O}_C) = H^1(C,\ \text{End}\ F^*V)$$

and this is also the differential of the morphism $f_s : N_1 \to N_2$ at V. Since the dimensions of the tangent spaces are the same (both V and F^*V are simple) we find that f_s is differentially isomorphic at V and hence $f_s : N_1 \to N_2$ is an étale morphism (after shrinking N_1 and N_2 if necessary). Since the local moduli scheme at a simple bundle contains stable bundles, it follows from f_s being etale and hence dominant, that there is a stable bundle W (in a "neighborhood" of V) such that F^*W is also stable. From the openness of stability in any family of bundles, the proposition follows. *Q.E.D.*

We can now prove

Corollary 2.1.1 *Let C be an ordinary curve. Then the rational map $f : M_r^{ss} \to M_r^{ss}$ induced by the Frobenius $F : C \to C$, is etale on an open set, and in particular, dominant.*

Proof. As seen in the proof of (2.1), there is an open set (nonempty) of M_r^{ss} such that $H^0(C,\ \text{End}\ V \otimes B^1) = 0$ for V in this open set. Also, by Proposition (2.1), there is an open subset of M_r^s such that for V in this open set F^*V is stable. So we can consider a stable bundle V such that f is defined at V, F^*V is stable, and $H^0(C,\ \text{End}\ V \otimes B^1) = 0$. It follows that there is an injection

$$0 \to H^1(C, \text{ End } V) \to H^1(C, \text{ End } V \otimes F_*\mathcal{O}_C) = H^1(C, \text{ End } (F^*V))$$

The tangent spaces at V and F^*V to M_r^s are $H^1(C, \text{ End } V)$ and $H^1(C, \text{ End } F^*V)$ respectively, and the above cohomology map is the differential of f at V. This shows that f is differentially injectively and hence a differential isomorphism (since $\dim H^1(C, \text{ End } V) = \dim H^1(C, \text{End } F^*V)$) at V. Therefore f is etale at V and hence etale on an open set in M_r^s. In particular, f is dominant. *Q.E.D.*

Corollary 2.1.2 *Let C be an ordinary curve. For any positive integer k, there is a nonempty open subset of M_r^s such that for V in this open set, $F^{m*}(V)$ is stable for $1 \leq m \leq k$.*

Proof. We apply Proposition (2.1) and Corollary (2.1.1) successively. *Q.E.D.*

We now show

Proposition 2.2 *Let $\pi : \widetilde{C} \to C$ be a finite Galois morphism of smooth curves. If V is a stable bundle on C such that $F^{k*}V$ is semistable for all positive integers k, and π^*V is not stable, then there exists a etale morphism $p : C_1 \to C$ and a factoring*

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\pi} & C \\ \searrow & & \nearrow p \\ & C_1 & \end{array}$$

such that $p^*(V)$ is not stable.

Proof. We first observe that $\pi^*(V)$ is semistable (otherwise the β-subbundle of π^*V is invariant under Galois automorphisms and descends to a subbundle of V). Suppose π^*V is not stable. Let S be the socle of π^*V (see [5]). Then S is also invariant under Galois automorphisms and descends to a subbundle of V. Since V is stable, this shows that $S = \pi^*V$ and therefore π^*V is a direct sum of stable bundles. Thus π^*V not being stable is equivalent to π^*V not being simple.

Let

$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_{\widetilde{C}} \to W \to 0$$

be the exact sequence of bundles on C induced by π and we consider

$$0 \to \text{ End } (V) \to \text{ End } (V) \otimes \pi_*\mathcal{O}_{\widetilde{C}} \to \text{ End } (V) \otimes W \to 0.$$

This gives

$$0 \to H^0(C, \text{ End } V) \to H^0(C, \text{ End } V \otimes \pi_*\mathcal{O}_{\widetilde{C}}) \to H^0(C, \text{ End } V \otimes W)$$

since $H^0(C, \text{End } V \otimes \pi_*\mathcal{O}_{\widetilde{C}}) = H^0(\widetilde{C}, \text{ End } (\pi^*V))$, π^*V not being simple implies that $H^0(C, \text{ End } V \otimes W) \neq 0$. Let $\text{End } V \to W$ be a nonzero homomorphism induced by a nonzero section of $\text{End } V \otimes W$. Let $S_1 = \text{Image } (\text{End } V \to W)$ and $\widetilde{S}$ be the inverse image of S_1 in $\pi_*\mathcal{O}_{\widetilde{C}}$. Let S be the sheaf of algebras generated by S_1 in $\pi_*\mathcal{O}_{\tilde{C}}$. Then deg $S \leq 0$. On the other hand, since S is a quotient of the tensor algebra of End (V), and End(V) is semistable, we have deg $S \geq 0$. Therefore, deg $S = 0$. We now consider the curve C_1 = Spec (S) and the induced morphism $p : C_1 \to C$. Since deg $S = 0$, the morphism p is etale. Further, from

$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_{C_1} = S \to S_1 \to 0$$

we get

$$0 \to \text{End } V \to \text{End } V \otimes S \to \text{End } V \otimes S_1 \to 0$$

and

$$0 \to H^0(C, \text{ End } V) \to H^0(C, \text{ End } V \otimes S) \to H^0(C, \text{ End } V \otimes S_1)$$

and the nonzero section of End $V \otimes S_1$ lifts to a section of End $V \otimes S$ by construction. This shows that p^*V is not simple on C_1. *Q.E.D.*

Lemma 2.3 *Let $p : C_1 \to C$ be a Galois etale morphism of smooth curves, and V a stable vector bundle on C. Then p^*V is stable iff p^*V is simple.*

Proof. The bundle p^*V is semistable (otherwise the β-subbundle of p^*V is invariant under Galois automorphisms and descends to a subbundle of V). Suppose p^*V is not stable. Let S be the socle of p^*V (see [5]). Then S is also invariant under Galois automorphisms and descends to a subbundle of V. Since V is stable, this shows that $S = p^*V$ and therefore p^*V is a direct sum of stable bundles. Thus p^*V not being stable is equivalent to p^*V not being simple. *Q.E.D.*

We now have

Proposition 2.4 *Given a finite etale morphism $p : C_1 \to C$, there exists a proper closed subset of M_r^s such that any vector bundle in the complement of this closed set remains stable on C_1.*

Proof. We first suppose that $p : C_1 \to C$ is Galois etale. Then by Lemma (2.3) above, a stable bundle V on C becomes unstable on C_1 if and only if p^*V is not simple. The condition that p^*V is not simple is $H^0(C_1, \text{End } p^*V) \geq 2$ which is a closed condition by semicontinuity (even though there may be no universal bundle on M_r^s, we can prove this by considering the universal bundle on the quot scheme Q). Therefore the set of semistable bundles on C which remain simple on C_1 is an open subset (possibly empty) of M_r^{ss}. We show that this open set is nonempty as follows. Let

$$0 \to L \to V \to \mathcal{O} \to 0$$

be a nonsplit extension of $\mathcal{O}$ by a nontrivial line bundle L of degree zero such that p^*L is also a nontrivial line bundle on C_1. From

$$0 \to \mathcal{O}_C \to p_*\mathcal{O}_{C_1} \to W \to 0$$

we get

$$0 \to L \to L \otimes p_* \mathcal{O}_{C_1} \to L \otimes W \to 0.$$

We observe that there are only finitely many line bundles of degree zero with a nonzero homomorphism to the given bundle W since $\mu_{\max}(W) \leq 0$ (this can be seen by considering a stable filtration of W). Hence for a general choice of $L, H^0(C, L \otimes W) = 0$. Therefore, there is an injection in the cohomology sequence

$$0 \to H^1(C,L) \to H^1(C, L \otimes p_* \mathcal{O}_{C_1}) = H^1(C_1, p^* L).$$

The extension

$$0 \to p^* L \to p^* V \to \mathcal{O}_{C_1} \to 0$$

is therefore nonsplit. Since any nonsplit extension V of the form

$$0 \to L \to V \to \mathcal{O} \to 0$$

with L a nontrivial line bundle of degree zero is a simple bundle, we see that $p^* V$ is a simple bundle.

This proves the proposition when the rank $r = 2$. We now assume the statement of the proposition for rank $(r-1)$. For a stable bundle S of rank $(r-1)$ and degree zero such that $p^* S$ is also stable, we consider a nonsplit extension

$$0 \to S \to V \to \mathcal{O} \to 0$$

(which exists since $h^1(C,S) = (r-1)(g-1) > 0$). We consider the exact sequence

$$0 \to S \to S \otimes p_* \mathcal{O}_{C_1} \to S \otimes W \to 0$$

and since there are only finitely many stable bundles of rank $(r-1)$ and degree zero with a nonzero homomorphism to W (consider a stable filtration of W), we can assume for a general choice of S that $H^0(C, S \otimes W) = 0$. For such an S, the cohomology sequence gives an injection

$$0 \to H^1(C,S) \to H^1(C, S \otimes p_* \mathcal{O}_{C_1}) = H^1(C_1, p^* S),$$

the sequence

$$\mathcal{O} \to p^* S \to p^* V \to \mathcal{O}_{C_1} \to 0$$

is also nonsplit. Any nonsplit extension V,

$$0 \to S \to V \to \mathcal{O} \to 0$$

with S stable of degree zero, is a simple bundle. Therefore $p^* V$ is a simple bundle of rank r. This completes the proof when p is Galois. When $p : C_1 \to C$ is only given to be etale, it can be reduced to the case when p is Galois etale (by taking a further extension which is Galois). *Q.E.D*

We have the following proposition for symmetric powers.

Proposition 2.5 *For a fixed positive integer k, there is a nonempty open subset of M_r^s such that for any stable bundle V in this open set, there is no non-zero homomorphism from a line bundle of degree zero to $S^k(V)$.*

Proof. We first consider the case when rank $r = 2$. Given a nonzero homomorphism

$$L \to S^k(V)$$

with L a line bundle of degree zero, we have the section

$$\mathcal{O} \to S^k(V) \otimes L^*.$$

This defines a curve $\widetilde{C} \subset \mathbf{P}(V)$, with $\widetilde{C}$ a section of $\mathcal{O}_{\mathbf{P}(V)}(k) \otimes \pi^*(L^*)$ where $\pi : \mathbf{P}(V) \to C$ is the projection. If we consider the restriction

$$\pi^*(V) \mid \widetilde{C} \to \mathcal{O}_{\mathbf{P}(V)}(1) \mid \widetilde{C} \to 0$$

then deg $(\mathcal{O}_{\mathbf{P}(V)}(1) \mid C) =$ deg $(L^*) = 0$. The morphism $\pi_1 : \widetilde{C} \to C$ is of degree k and factors as

$$\begin{array}{ccc} \widetilde{C} & \overset{\pi_1}{\to} & C \\ \widetilde{\pi}\searrow & & \nearrow F^m \\ & C & \end{array}$$

where $\widetilde{\pi} : \widetilde{C} \to C$ is a Galois morphism of degree $\leq k$ and $F^m : C \to C$ is a power of the Frobenius on C with $m \leq k$. By Corollary (2.1.2), there is a nonempty open subset of M_r^s such that for any stable bundle V in this open set $F^{m*}(V)$ is stable for $m \leq k$. We can assume that $F^{m*}(V)$ is stable for $m \leq k$ and End (V) is semistable for V in this open set. Then if $W = F^{m^*}(V), \widetilde{\pi}^*(W)$ becomes unstable on $\widetilde{C}$. Let $\widetilde{\pi}' : \widetilde{C}' \to \widetilde{C}$ be a morphism such that $(\widetilde{\pi} \circ \widetilde{\pi}') : \widetilde{C}' \to C$ is a finite Galois morphism, and $(\widetilde{\pi} \circ \widetilde{\pi}')^*(W)$ is unstable on $\widetilde{C}'$, so by Proposition (2.2), there is a Galois etale morphism $p : C_1 \to C$ with degree $p \leq k$ such that $p^*(W)$ is not stable. There are only finitely many Galois etale morphisms $p : C_1 \to C$ with degree $p \leq k$, so by Proposition (2.4), there is a nonempty open subset of M_r^s such that for V in this open set $p^*(V)$ is stable. Such aV would contradict the existence of a $\widetilde{C}$ as above. This completes the proof when $r = 2$. When $r > 2$, we consider a nonsplit extension

$$0 \to W \to V \to \mathcal{O} \to 0$$

where W is stable of degree zero and rank $r - 1$ such that $S^i(W)$ has no line subbundles of degree zero for $1 \leq i \leq k$. Then $S^k(V)$ has a filtration $P_1 \subset P_2 \subset \cdots \subset P_K \subset S^k V$ with $P_i/P_{i-1} \simeq S^{k-i+1}(W)$. The surjection of bundles

$$V \to \mathcal{O} \to 0$$

induces an inclusion of bundles

$$0 \to \mathcal{O} \to V^*$$

and this nowhere vanishing section of V^* defines a composite of surjective contractions

$$S^kV \to S^{k-1}V \to S^{k-2}V \cdots \to V.$$

Under the composite surjection $S^kV \to V, P_k =$ Ker $(S^kV \to \mathcal{O})$ gets mapped onto $W =$ ker $(V \to \mathcal{O})$ (in fact, at every stage, Ker $(S^kV \to \mathcal{O})$ gets mapped onto Ker $(S^{k-1}V \to \mathcal{O})$). We thus have a diagram

$$\begin{array}{ccccccccc}
0 & \to & P_k & \to & S^kV & \to & \mathcal{O} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & W & \to & V & \to & \mathcal{O} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and the map $H^1(C, P_k) \to H^1(C, W)$ sends the extension class of S^kV to the extension class of V. It follows that since V is a nonsplit extension, the extension

$$0 \to P_k \to S^kV \to \mathcal{O} \to 0$$

is also nonsplit. Since P_k has a filtration $(P_1 \subset\subset P_k)$ where subquotients are $S^j(W), j > 0$, there is no nonzero homomorphism from a line bundle of degree zero to P_k. Since $S^kV \to \mathcal{O} \to 0$ is nonsplit, there is no nonzero homomorphism from a line bundle of degree zero to S^kV either. We have constructed a semistable bundle V of rank r and degree zero such that $S^k(V)$ has no line subbundles of degree zero. On the other hand, the set of bundles V such that there is no nonzero homomorphism from a line bundle of degree zero to $S^k(V)$ is an open subset of M_r^{ss}. (This can be seen as follows: if Q is the Quote scheme with universal bundle $\widetilde{V}$ and J the Jacobian of degree zero line bundles with Poincare bundle P, we are considering the condition $H^0(C \times j \times q, P \otimes S^k(\widetilde{V})) = 0$ on $C \times J \times Q$, which is an open condition. This descends to an open set in M_r^{ss}). Our construction has shown that this open set is nonempty. *Q.E.D.*

We have finally

Theorem 2.6 *Let C be a nonsingular ordinary curve of genus ≥ 2 over an uncountable algebraically closed field (of any characteristic). There is a dense subset of M_r^s such that for any stable bundle V in this dense set, we have*

1) *$F^{k*}(V)$ is stable for all $k \geq 1$.*
2) *For any Galois finite morphism $\pi : \widetilde{C} \to C, \pi^*(V)$ is stable.*
3) *There is no non-zero homomorphism from a line bundle of degree zero to the symmetric power $S^k(V)$ for any $k \geq 1$.*

Proof. From Corollary (2.1.2), Proposition (2.2) and Proposition (2.4), and the fact that the set of etale coverings of C is countable, we obtained a countable union of proper closed subsets of M_r^s such that for V in the complement of this union 1) and 2) above are satisfied.

From Proposition (2.5), we obtain a countable union of proper closed subsets of M_r^s such that for V in the complement of this union, $H^0(C, L \otimes V) = 0$ for any line bundle L of degree zero.

We therefore obtain a countable union of proper closed subset of M_r^s such that any bundle V in the complement satisfies 1), 2) and 3) of the Theorem. Over an uncountable algebraically closed field, the complement of a countable union of proper closed sets is nonempty and dense. *Q.E.D.*

Remark 1) If C is a smooth curve defined over a finite field (of characteristic p) then any continuous irreducible representation $\rho : \pi_1^{\text{alg}}(C) \to SL(r, \overline{\mathbf{F}}_p)$ of the algebraic fundamental group of C of rank r over the finite field defines a stable vector bundle V on C such that $F^{m*}V \simeq V$ for some $m \geq 1$, (see [4]). For such a bundle $V, F^{k^*}(V)$ is stable for all $k \geq 1$. It is possible to construct such representations for any curve C of genus ≥ 2 when r is coprime to p, and for an ordinary curve C when p divides r.

3 The line bundle

Let C be a nonsingular ordinary curve of genus ≥ 2, and V a stable vector bundle of rank 3 and degree zero on C satisfying the conditions of Theorem (2.6). Let $\pi : \mathbf{P}(V) \to C$ be the projective bundle associated to V and $L = \mathcal{O}_{\mathbf{P}(V)}(1)$ the universal line bundle on $P(V)$. Then we have

Theorem 3.1 *The line bundle L is ample on every proper subvariety of* $\mathbf{P}(V)$, *but L is not ample on* $\mathbf{P}(V)$.

Proof. Let $\widetilde{C} \subset \mathbf{P}(V)$ be a reduced irreducible curve contained in a fibre of $\pi : \mathbf{P}(V) \to C$. Then $\mathcal{O}_{\mathbf{P}(V)}(1) \mid \widetilde{C}$ is clearly of degree > 0. Let $\widetilde{C}$ surject onto C under π. Then we have the surjection, $\pi^*V \mid \widetilde{C} \to \mathcal{O}_{\mathbf{P}(V)}(1) \mid \widetilde{C} \to 0$. From 1) and 2) of Theorem (2.6), $\pi^*V \mid \widetilde{C}$ is stable of degree zero, so deg $(\mathcal{O}_{\mathbf{P}(V)}(1) \mid \widetilde{C}) > 0$. This shows that $L \mid \widetilde{C}$ is positive for every integral curve $\widetilde{C}$ in $\mathbf{P}(V)$.

Let D be an irreducible divisor in $\mathbf{P}(V)$. Then D defines a section of $\mathcal{O}_{\mathbf{P}(V)}(k) \otimes \pi^*M$ for some integer $k > 0$ and a line bundle M on C. Then we have

$$\begin{aligned} L^2.D &= L^2.(kL + \pi^*M) \\ &= kL^3 + L^2.\pi^*M \\ &= \deg (M) \end{aligned}$$

On the other hand, the effective divisor D defines a section of $\mathcal{O}_{\mathbf{P}(V)}(k) \otimes \pi^*M$ and hence of $S^k(V) \otimes M$ since

$$H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(k) \otimes \pi^*M) = H^0(C, S^k(V) \otimes M).$$

Since deg $(L \mid \widetilde{C}) > 0$ for every integral curve $\widetilde{C}$ in $\mathbf{P}(V)$ (i.e. L is nef), $L^s \cdot Y \geq 0$ for every integral subvariety Y of dimension s (see [3]). In particular, $L^2 \cdot D =$ deg $M \geq 0$. If deg $M = 0$, then the section of $S^k(V) \otimes M$ we obtained above defines a nonzero homomorphism $M^* \rightarrow S^k(V)$ with deg $M^* = 0$, contradicting 3) of Theorem (2.6). Therefore, $L^2 \cdot D = \deg M > 0$. This shows that L is ample on divisors in $\mathbf{P}(V)$ and hence on any proper subvariety of $\mathbf{P}(V)$. Also, L is not ample on $\mathbf{P}(V)$ since $L^3 = \deg V = 0$ *Q.E.D.*

Remark 3.2 The case $r = 2$ is covered by the first part of the proof of Theorem (3.1).

Remark 3.3 If rank $V = r > 3$, we can still consider L on $\mathbf{P}(V)$ as above. It follows that $L.C > 0$ for all integral curves C in $\mathbf{P}(V)$ and $L^{r-1}.D > 0$ for any integral divisor D in $\mathbf{P}(V)$. This shows, by a theorem of Fujita (see [2], Corollary 6.5) that the Kodaira dimension, $\mathcal{K}(D, L)$, of the line bundle L on the divisor D, is $r - 1 (= dimD)$. However, we don't know if L is *ample* on D. We will return to this question in the future.

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