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Semistable Sheaves on Projective Varieties and Their Restriction to Curves

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Introduction

Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let H be a very ample line bundle on X. If V is a torsion free coherent sheaf on X we define $\deg V$ to be $c_1(V)\cdot c_1(H)^{n-1}$ and $\mu(V)=\deg V/\operatorname{rk} V$. We call V *semistable* (resp. *stable*) if for all proper subsheaves W of V we have $\mu(W)\leqq\mu(V)$ [resp. $\mu(W)<\mu(V)$] (cf. [7, 14]).

In this paper we prove that if V is semistable on X then its restriction to a general complete intersection curve of sufficiently high degree is semistable (Theorem 6.1).

To give an idea of the proof assume X is a surface and V a vector bundle of rank 2. The restriction of V to a general curve C_m of degree m is not semistable if and only if it is not semistable on the generic curve Y_m defined over the function field of $\mathbb{P}H^0(X,H^m)$. Let $\bar{L}_m$ be the line bundle on Y_m contradicting the semistability of $V|Y_m$ (cf. Sects. 4.1 and 4.2). First we show that $\bar{L}_m$ extends uniquely to a line bundle L_m on X (Proposition 2.1). If we can get L_m as a subbundle of V we are through, for then L_m would contradict the semistability of V. So we would like the restriction map $H^0(X,\mathrm{Hom}(L_m,V))\to H^0(C_m,\mathrm{Hom}(L_m,V))$ to be surjective. Now for fixed L it follows from the lemma of Enriques-Severi (Proposition 3.2; [6, Corollary 7.8]) that $H^0(X,\mathrm{Hom}(L,V))\to H^0(C_m,\mathrm{Hom}(L,V))$ is surjective for large m. Therefore it is enough if the L_m remain the same line bundle L for infinitely many m.

To prove that $L_m=L$ we construct a degenerating family of curves $D\xrightarrow{f}S$, $X\times S\supset D\xrightarrow{p}X$, such that the generic fibre is a curve $C_{(m+1)}$ of degree 2^{m+1} and the special fibre is a reduced curve with two nonsingular components $C^i_{(m)}$ of degree 2^m (cf. Sect. 5). Let (m) denote 2^m. Extending the subbundle $L_{(m+1)}|C_{(m+1)}$ to a subsheaf of $p^*(V)$ on D and restricting the extension to $C^i_{(m)}$ gives a lower bound for the maximal degree of a line subbundle of $V|C^i_{(m)}$ in terms of that for $V|C_{(m+1)}$ (Proposition 4.3). This implies that $\deg L_m$ is bounded (Lemma 6.5.1) so that for an infinite subsequence of m, $\deg L_m$ is constant. If $\deg L_{(m+r)}=\deg L_{(m)}$ by refining the above argument with the degenerating family one can prove that $L_{(m+r)}|C^i_{(m)}$

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$=L_{(m)}|C^i_{(m)}$ (Lemma 6.5.2). Therefore $L_{(m+r)}|Y_{(m)}=L_{(m)}|Y_{(m)}$ so that $L_{(m+r)}=L_{(m)}$ (by Proposition 2.1).

When X is of higher dimension and V is a torsion free sheaf of arbitrary rank the pattern of the proof is the same but the details get a bit more technical.

We have made essential use of an unpublished manuscript of Mumford.

1. Families of Complete Intersection Subvarieties

Let X be a projective nonsingular algebraic variety of dimension $n \geqq 2$ over an algebraically closed field k. Let H be a given very ample line bundle on X corresponding to a projectively normal embedding $X \subset \mathbb{P}^N$.

For a positive integer m let S_m denote the projective space of lines in the vector space $H^0(X, H^m)$. For a sequence of positive integers $\mathbf{m}=(m_1, \ldots, m_t)$, $1 \leqq t \leqq n-1$, let $S_{\mathbf{m}}$ be the product $S_{m_1} \times \ldots \times S_{m_t}$. We have the following diagram

1.1.
$$\begin{array}{ccc} X \times S_{\mathbf{m}} \supset Z_{\mathbf{m}} & \xrightarrow{q_{\mathbf{m}}} & S_{\mathbf{m}} \\ \downarrow{\scriptstyle p_{\mathbf{m}}} & & \\ X & & \end{array}$$

where $Z_{\mathbf{m}}$ is the correspondence variety:

$$Z_{\mathbf{m}}=\{(x, s_1, \ldots, s_t) \in X \times S_{\mathbf{m}} \mid s_i(x)=0,\ 1 \leqq i \leqq t\}$$

and $p_{\mathbf{m}}$ and $q_{\mathbf{m}}$ are the projections.

1.2. The fibre of $q_{\mathbf{m}}$ over $(s_1, \ldots, s_t) \in S_{\mathbf{m}}$ is embedded in X via $p_{\mathbf{m}}$ as the subscheme of X defined by the ideal generated by $s_1, \ldots, s_t$ in the homogeneous coordinate ring of X. So we always think of the fibres of $q_{\mathbf{m}}$ as subschemes of X. The projection $p_{\mathbf{m}}$ is a fibration with the fibre over $x \in X$ being identified by $q_{\mathbf{m}}$ with the product of hyperplanes $H_1 \times \ldots \times H_t$, $H_i=\{s \in S_{m_i} \mid s(x)=0\}$. Therefore $Z_{\mathbf{m}}$ is nonsingular.

1.3. Let $K_{\mathbf{m}}$ be the function field of $S_{\mathbf{m}}$. Let $Y_{\mathbf{m}}$ be the generic fibre of $q_{\mathbf{m}}$, given by the fibre product

$$\begin{array}{ccc} Z_{\mathbf{m}} & \longrightarrow & S_{\mathbf{m}} \\ {\scriptstyle \varphi_{\mathbf{m}}}\uparrow & & \uparrow \\ Y_{\mathbf{m}} & \longrightarrow & \operatorname{Spec} K_{\mathbf{m}} \end{array}$$

By Bertini's theorem (cf. [17, Theorem I.6.3]) $Y_{\mathbf{m}}$ is an absolutely irreducible nonsingular variety and there is a nonempty open subset of $S_{\mathbf{m}}$ over which the geometric fibres of $q_{\mathbf{m}}$ are irreducible and nonsingular (cf. EGA IV/3, Theorem 12.2.4 (viii), p. 183 and [6, Theorem 8.18]).

1.4. Definition. We call $Y_{\mathbf{m}}$ the generic complete intersection subvariety of type $\mathbf{m}$. In particular when $t=n-1$ we call $Y_{\mathbf{m}}$ the generic curve of type $\mathbf{m}$.

1.4.1. Remark. When a property holds for $q_{\mathbf{m}}^{-1}(s)$ for s in a nonempty open subset of $S_{\mathbf{m}}$ we say that it holds for *a general s*.

1.5. Proposition. *Let* $S'_{\mathbf{m}}=\{s\in S_{\mathbf{m}}\mid \dim q_{\mathbf{m}}^{-1}(s)=n-t\}$. *Let* F *be a coherent sheaf on* X. *For* $s=(s_1,\dots,s_t)\in S'_{\mathbf{m}}$ *let* X_r *be the subscheme of* X *defined by the ideal* I_r *generated by* $s_1,\dots,s_r$ *for* $1\leqq r\leqq t$ *and* $X_0=X$. *Let* $0\to I_r\to\mathcal{O}_{X_{r-1}}\to\mathcal{O}_{X_r}\to 0$ *be the natural exact sequence. Let* $S''_{\mathbf{m}}=\{s\in S'_{\mathbf{m}}\mid 0\to I_r\otimes F\to\mathcal{O}_{X_{r-1}}\otimes F$ *is exact*$\}$. *Then*

i) $S'_{\mathbf{m}}$ *is a nonempty open subset of* $S_{\mathbf{m}}$ *and*

$$q_{\mathbf{m}}^{-1}(S'_{\mathbf{m}}) \text{ is flat over } S'_{\mathbf{m}}.$$

ii) $S''_{\mathbf{m}}$ *is a nonempty open subset of* $S'_{\mathbf{m}}$ *and*

$$p_{\mathbf{m}}^{*}F \text{ is flat over } S''_{\mathbf{m}}.$$

Proof. i) It follows from (EGA IV/3, Theorem 12.2.4) that $S'_{\mathbf{m}}$ is open and by Bertini's theorem it is nonempty. If $\dim q_{\mathbf{m}}^{-1}(s)=n-t$ then its Hilbert polynomial (w.r.t. the given polarisation H on X) depends only on $\mathbf{m}$ as can be seen easily from the cohomology sequence corresponding to the exact sequence $0\to I_r\to\mathcal{O}_{X_{r-1}}\to\mathcal{O}_{X_r}\to 0$ tensored with H^l, using induction on r. Therefore the fibres of $q_{\mathbf{m}}$ over $S'_{\mathbf{m}}$ have the same Hilbert polynomial and hence $q_{\mathbf{m}}^{-1}(S'_{\mathbf{m}})$ is flat over $S'_{\mathbf{m}}$ [6, Theorem 9.9].

ii) We use induction on t. Assume ii) holds for $t-1$. Let $\mathbf{l}=(m_1,\dots,m_{t-1})$ and $T=(S''_{\mathbf{l}}\times S_{m_t})\cap S'_{\mathbf{m}}\subset S_{\mathbf{m}}$. Note that $S''_{\mathbf{m}}\subset T$ and T is open in $S_{\mathbf{m}}$ by the induction assumption. We have the diagram

$$\begin{array}{ccc} Z''_{\mathbf{m}}\subset\pi^*(Z'_{\mathbf{l}}) & \longrightarrow & Z'_{\mathbf{l}} \\ \downarrow & & \downarrow{\scriptstyle p_{\mathbf{l}}} \\ T & \xrightarrow[\pi]{} & S''_{\mathbf{l}} \end{array}$$

where π is the projection and $Z'_{\mathbf{l}}=q_{\mathbf{l}}^{-1}(S''_{\mathbf{m}})$ and $Z''_{\mathbf{m}}=q_{\mathbf{m}}^{-1}(T)$. Note that $Z''_{\mathbf{m}}$ sits in $\pi^*(Z'_{\mathbf{l}})$ as the natural correspondence variety.

By the induction assumption $p_{\mathbf{l}}^*(F)$ on $Z'_{\mathbf{l}}$ is flat over $S''_{\mathbf{l}}$. Therefore $\pi^*p_{\mathbf{l}}^*(F)$ on $\pi^*(Z'_{\mathbf{l}})$ is flat over T and moreover, since $T\subset S'_{\mathbf{m}}$, $Z''_{\mathbf{m}}$ is flat over T. In this situation one can deduce from the openness of flatness that $S''_{\mathbf{m}}$ is open in T [for example, by taking for the $\mathcal{F}'$ of Corollary 11.1.2 in EGA IV/3, the sheaf $I\otimes\pi^*p_{\mathbf{l}}^*(F)$ where I is the ideal sheaf of $Z''_{\mathbf{m}}$ in $\pi^*(Z'_{\mathbf{l}})$ and using the properness of $\pi^*(Z'_{\mathbf{l}})\to T$]. That $S''_{\mathbf{m}}$ is not empty follows by noting that for the sequence

$$0\to I_r\otimes F\to\mathcal{O}_{X_{r-1}}\otimes F\to\mathcal{O}_{X_r}\otimes F\to 0$$

to be exact it is sufficient that s_r is not in any of the associated primes of $\mathcal{O}_{X_{r-1}}\otimes F$ in the homogeneous coordinate ring of X_{r-1}.

Once we have the exact sequences

$$0\to I_r\otimes F\to\mathcal{O}_{X_{r-1}}\otimes F\to\mathcal{O}_{X_r}\otimes F\to 0,\qquad 1\leqq r\leqq t,$$

it follows from the exact cohomology sequences, using induction on r, that the Hilbert polynomial of F restricted to $q_{\mathbf{m}}^{-1}(s)$ is independent of $s\in S''_{\mathbf{m}}$. Therefore $p_{\mathbf{m}}^*(F)$ is flat over $S''_{\mathbf{m}}$ (cf. [10, Lectures 7 and 8] and [6, Theorem 9.9]).

2. Picard Group of the Generic Curve

For any scheme S we denote by $\mathrm{Pic}(S)$ the (abstract) group of invertible sheaves on S. We then have the following proposition (cf. [16]).

2.1. Proposition. *Let* $\dim X = n \geqq 2$. *For* $\mathbf{m}=(m_1, \dots, m_t)$, $1 \leqq t \leqq n-1$ *with each* $m_i \geqq 3$ *the natural map* $\mathrm{Pic}(X) \to \mathrm{Pic}(Y_{\mathbf{m}})$, *induced by* $Y_{\mathbf{m}} \xrightarrow{\varphi_{\mathbf{m}}} Z_{\mathbf{m}} \xrightarrow{p_{\mathbf{m}}} X$ (cf. 1.3), *is a bijection.*

2.1.1. Remark. In fact one can show that if $\dim Y_{\mathbf{m}} \geqq 2$, then $\mathrm{Pic}(X) \to \mathrm{Pic}(Y_{\mathbf{m}})$ is bijective for all $\mathbf{m}$ with $m_i \geqq 1$ and if $\dim Y_{\mathbf{m}} = 1$ then $\mathrm{Pic}(X) \to \mathrm{Pic}(Y_{\mathbf{m}})$ is bijective if just one of the m_i's is $\geqq 3$ (see Remark 2.1.4 below). If X is a surface then $\mathrm{Pic}(X) \to \mathrm{Pic}(Y_m)$ need not be injective for $m=1$ as shown by the example of the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.

Proof of Proposition 2.1. First we prove the surjectivity of the map. Any subscheme of $Y_{\mathbf{m}}$ (defined over $K_{\mathbf{m}}$) can be extended to an open set $q_{\mathbf{m}}^{-1}(U)$, U open in $S_{\mathbf{m}}$ (by inverting the finitely many elements which occur in the denominator in a set of generators of the ideal defining the subscheme). Therefore if $L \in \mathrm{Pic}(Y_{\mathbf{m}})$ corresponds to the divisor D in $Y_{\mathbf{m}}$ we can extend the divisor to an open subset of $Z_{\mathbf{m}}$ and hence to the whole of $Z_{\mathbf{m}}$. Thus L can be extended to a line bundle on $Z_{\mathbf{m}}$ i.e. $\mathrm{Pic}(Z_{\mathbf{m}}) \to \mathrm{Pic}(Y_{\mathbf{m}})$ is surjective. Now we claim that

$$\mathrm{Pic}(Z_{\mathbf{m}}) = p_{\mathbf{m}}^*(\mathrm{Pic}(X)) \oplus q_{\mathbf{m}}^*(\mathrm{Pic}(S_{\mathbf{m}}))\,.$$

Since $q_{\mathbf{m}}^*(\mathrm{Pic}(S_{\mathbf{m}}))$ is in the kernel of $\mathrm{Pic}(Z_{\mathbf{m}}) \to \mathrm{Pic}(Y_{\mathbf{m}})$ it would then follow that $\mathrm{Pic}(X) \to \mathrm{Pic}(Y_{\mathbf{m}})$ is surjective. To prove the above direct sum decomposition we have only to note that since the fibres of $p_{\mathbf{m}}$ are products of projective spaces embedded in $S_{\mathbf{m}}$ by $q_{\mathbf{m}}$ (Sect. 1.2), given a line bundle L on $Z_{\mathbf{m}}$ we can find a unique line bundle M on $S_{\mathbf{m}}$ such that $L \otimes q_{\mathbf{m}}^*(M)$ is trivial on one and hence all the fibres of $p_{\mathbf{m}}$ so that it comes from X (cf. Lemma 2.1.2 below).

For proving injectivity we need the following lemmas.

2.1.2. Lemma. *Let* $q: Z \to S$ *be a proper flat morphism of irreducible varieties with fibres integral. Let* $L \in \mathrm{Pic}(Z)$. *Then the following are equivalent:*

a) *L is trivial on the generic fibre of q.*

b) *L is trivial on all geometric fibres of q over a nonempty open subset of S.*

c) *L is trivial on all the geometric fibres of q.*

d) $L \approx q^*(M)$, $M \in \mathrm{Pic}(S)$.

Proof. This is a consequence of semicontinuity and the remark that a line bundle L on an integral complete scheme F is trivial if and only if $H^0(F, L) \neq 0 \neq H^0(F, L^{-1})$ (see [11, Corollary 6]).

2.1.3. Lemma. *Let* $\dim X \geqq 2$.

i) *For any point* $P \in X$ *and* $m \geqq 3$ *the rational map given by the linear system* $V = \{s \in H^0(X, H^m) \mid s=0$ *passes through P and is singular at* $P\}$ *gives an isomorphism of* $X - P$ *onto its image.*

ii) *For a nonempty open set of points* $s \in V$ *the divisor* $s=0$ *is integral (and is singular at P).*

iii) *Let* $A = \{s \in S_m \mid q_m^{-1}(s)$ *is not integral}. Then A is a closed set and if* $m \geqq 3$, *A has codimension* $\geqq 2$ *in* S_m, *as does* $q_m^{-1}(A)$ *in* Z_m.

Proof. i) It is easy to see that if i) holds for X and H then it holds for Y, H/Y for any nonsingular subvariety Y of X. Therefore to prove i) we can assume $X=\mathbb{P}^N$ and $H=\mathcal{O}(1)$. Let $P\neq Q\in\mathbb{P}^N$. We can find a linear form l such that $l(P)=0$ and $l(Q)\neq 0$. Then $l^m\in V$ so that P is the only base point of V. If $Q\neq R$ are two points of $\mathbb{P}^N$ different from P (and v a tangent at Q) we can find a nonsingular quadric $f=0$ passing through P and Q and not passing through R (or not having v as a tangent at Q). Choose a linear form l such that $l(P)=0$ and $l(Q)\neq 0\neq l(R)$ [or $l(v)\neq 0$]. Then $l^{m-2}\cdot f\in V$ showing that V separates points and infinitesmally separates points. This proves i).

ii) This follows from Bertini's theorem as in [17, Theorem I.6.3] supplemented in characteristic p by [17, Proposition I.6.4] whose condition for $p^e=1$ is satisfied because of i).

iii) That A is a closed set follows from [EGA IV/3 Theorem 12.2.4 (viii)]. From [6, Proof of Theorem 8.18] the closed set $B=\{s\in S_m | q_m^{-1}(s)$ has a singularity$\}$ is irreducible and is not the whole of S_m. Now $A\subset B$, and hence to show that $\operatorname{codim} A\geqq 2$ it is enough to show that for $m\geqq 3$ $A\neq B$, i.e. there is at least one point in S_m such that $q_m^{-1}(s)$ is integral and singular. But that is ii). Since q_m is equidimensional $\operatorname{codim} q_m^{-1}(A)$ is then $\geqq 2$.

Now we return to the proof of the injectivity of $\operatorname{Pic}(X)\to\operatorname{Pic}(Y_{\mathbf{m}})$. We use induction on t. First assume that $t=1$ so that $\mathbf{m}$ is the single integer $m\geqq 3$. Let $L\in\operatorname{Pic}(X)$ be such that its image $\varphi_m^* p_m^* L$ in $\operatorname{Pic}(Y_m)$ is trivial. Then by Lemmas 2.1.2 and 2.1.3iii), on an open subset of Z_m whose complement has codimension $\geqq 2$, $p_m^*(L)$ is isomorphic to $q_m^*(M)$, $M\in\operatorname{Pic}(S_m)$. But then $p_m^*(L)\approx q_m^*(M)$ on the whole of Z_m. As we have seen $\operatorname{Pic}(Z_m)=p_m^*(\operatorname{Pic}(X))\oplus q_m^*(\operatorname{Pic}(S_m))$. Therefore $p_m^*(L)\approx q_m^*(M)$ implies $p_m^* L$ is trivial on Z_m and hence L is trivial (since p_m^* is injective).

Now let $\mathbf{m}=(m_1,\dots,m_t)$. Assume injectivity for $\mathbf{l}=(m_1,\dots,m_{t-1})$. Let $L\in\operatorname{Pic}(X)$ be such that $\varphi_{\mathbf{m}}^* p_{\mathbf{m}}^*(L)$ is trivial on $Y_{\mathbf{m}}$. By Lemma 2.1.2a)$\Rightarrow$c) it follows that L is trivial on $q_{\mathbf{m}}^{-1}(s)$ whenever $q_{\mathbf{m}}^{-1}(s)$ is smooth. For a general $s'\in S_{\mathbf{l}}$, L being trivial on all the smooth $q_{\mathbf{m}}^{-1}(s)$ contained in $q_{\mathbf{l}}^{-1}(s')$, by Lemma 2.1.2b)$\Rightarrow$a) and the above case $t=1$ applied to the smooth variety $q_{\mathbf{l}}^{-1}(s')$ it follows that L is trivial on $q_{\mathbf{l}}^{-1}(s')$. Again by Lemma 2.1.2b)$\Rightarrow$a) and the inductive assumption, L is trivial on X.

2.1.4. Remark. If $\dim X\geqq 3$ Weil proves Lemma 2.1.3iii) for all $m\geqq 1$ [16, Lemmas 3 and 4]. Assuming this fact the above proof then gives the result in the sharper form as in Remark 2.1.1 (cf. [16, No. 12, Theorem 2]).

3. A General Form of the Lemma of Enriques-Severi

A coherent sheaf F on X is called reflexive if the natural map $F\to F^{**}$ of F into its double dual is an isomorphism.

3.1. Lemma. *Let F be a coherent sheaf on X. The following are equivalent.*

a) *F is reflexive.*

b) *Locally, i.e. on each of the open sets U of some covering of X, there is an exact sequence*

$$0\to F|U\to F_1\to T\to 0$$

with F_1 free and T torsion free.

c) *Locally there is an exact sequence*

$$0 \to F|U \to F_1 \to F_0 \to Q \to 0,$$

where F_1, F_0 *are free, i.e.* F *is a 2nd syzygy.*

3.1.1. Corollary. *Let* F *be reflexive. Then*

i) *it satisfies the condition* S_2 *(cf. [1, Definition 2.1]),*

ii) *its restriction to* $q_{\mathbf{m}}^{-1}(s)$ *is reflexive for a general* $s \in S_{\mathbf{m}}$.

Proof. b)⇒c): Any torsion free module over a domain is a submodule of a free module.

c)⇒b): Trivial.

b)⇒a): Follows from [2, Sect. 4, Theorem 7ii)].

a)⇒b): Write F^* as a quotient of a free module:

$$0 \to K \to F_1 \to F^* \to 0.$$

Taking duals and using $F = F^{**}$ we get sequences as in b).

Part i) of the corollary follows from c).

Part ii) follows by noting that the restriction of the sequence in b) to any hyperplane section which does not pass through any of the associated primes of T remains exact.

3.2. Proposition (General Enriques-Severi Lemma, cf. [6, Corollary 7.8]). *Let* $X \subset \mathbb{P}^N$ *be a nonsingular projective variety of dimension* $n \geqq 2$. *Let* F *be a coherent reflexive sheaf on* X. *Then there is an* m_0 *such that if* $\mathbf{m} = (m_1, \ldots, m_t)$, $1 \leqq t \leqq n-1$, *with each* $m_i \geqq m_0$, *then there is a nonempty open subset* U *of* $S_{\mathbf{m}}$ *such that for* $s = (s_1, \ldots, s_t) \in U$ *the restriction map* $H^0(X, F) \to H^0(X_s, F/X_s)$ *is surjective where* $X_s = q_{\mathbf{m}}^{-1}(s)$ *is the subscheme of* X *defined by the ideal generated by* $s_1, \ldots, s_t$.

Proof. It is enough to find an m_0, depending only on F, such that $H^1(X_s, F(-l)) = 0$ for $l \geqq m_0$ for a general $s \in S_{\mathbf{m}}$ for all $\mathbf{m} = (m_1, \ldots, m_t)$ with $m_i \geqq 1$ and all $t \leqq n-1$. For, then from the exact sequence

$$0 \to I_r \otimes F \to \mathcal{O}_{X_{r-1}} \otimes F \to \mathcal{O}_{X_r} \otimes F \to 0$$

on X_{r-1} corresponding to a general s (cf. Proposition 1.5), noting that $I_r = \mathcal{O}_{\mathbb{P}^N}(-m_r) \otimes \mathcal{O}_{X_{r-1}}$, it follows that

$$H^0(X_{r-1}, F|X_{r-1}) \to H^0(X_r, F|X_r)$$

is surjective for $m_r \geqq m_0$.

By duality $H^1(X_s, F(-l))^* \approx \mathrm{Ext}^{n'-1}(F(-l), \omega)$ where ω is the canonical line bundle of X_s and $n' = \dim X_s$ [1, Sect. I, 1.3, p. 5; Corollary IV, 5.6, p. 81]. We have a spectral sequence $H^p(\mathcal{E}xt^q(F(-l), \omega)) \Rightarrow \mathrm{Ext}^{p+q}(F(-l), \omega)$ (cf. [1, Proposition 2.4]). Since $\mathcal{E}xt^q(F(-l), \omega) = \mathcal{E}xt^q(F, \omega) \otimes \mathcal{O}(l)$, the spectral sequence degenerates for large l (depending on s). Then

$$\mathrm{Ext}^{n'-1}(F(-l), \omega) = H^0(\mathcal{E}xt^{n'-1}(F, \omega) \otimes \mathcal{O}(l)).$$

But since F is S_2 on X_s (Corollary 3.1.1) $\mathcal{E}xt^{n'-1}(F, \omega) = 0$ (cf. [1, Theorem 5.19 and Proposition 5.20]). Therefore we can find an m_0 (depending only on F) such that $H^1(X_s, F(-l)) = 0$ for $l \geqq m_0$ for a general $s = (s_1, \ldots, s_t)$ in $S_{(1, \ldots, 1)}$ (i.e. $\deg s_i = 1$, for every i) for all $t \leqq n-1$.

Now suppose $H^1(X_s, F(-l))=0$ for $l \geqq m_0$ for a general $s \in S_{\mathbf{m}}$ with $\mathbf{m}=(m_1, \ldots, m_r, 1, \ldots, 1)$. We will prove that $H^1(X_{s'}, F(-l))=0$ for $l \geqq m_0$ for a general $s' \in S_{\mathbf{m}'}$ where $\mathbf{m}'=(m_1, \ldots, m_r, d, 1, \ldots, 1)$. [When $r=0$ $\mathbf{m}=(1, \ldots, 1)$ and the assumption is true by the choice of m_0 so we can start the induction.] It is easy to see that a permutation of the sequence $\mathbf{m}'$ does not affect anything relevant. So we rewrite $\mathbf{m}'$ as $(m_1, \ldots, m_r, 1, \ldots, 1, d)$. For $s=(s_1, \ldots, s_t) \in S_{\mathbf{m}}$ denote by s' the point $(s_1, \ldots, s_t^d)$ of $S_{\mathbf{m}'}$. Let Y be the subscheme of X defined by $s_1, \ldots, s_{t-1}$. Let X_s (resp. $X_{s'}$) be the subscheme of Y defined by $s_1, \ldots, s_t$ (resp. $s_1, \ldots, s_t^d$). On Y we have the exact sequence (cf. Proposition 1.5)

$$0 \to I \to \mathcal{O}_Y \to \mathcal{O}_{X_s} \to 0,$$

where $I \approx \mathcal{O}_{\mathbb{P}^N}(-1)|Y$ is the ideal generated by s_t in $\mathcal{O}_Y$. For a general s the above sequence tensored with F remains exact (Proposition 1.5) and hence $\mathrm{Tor}^1_{\mathcal{O}_Y}(F, \mathcal{O}_Y/I)=0$. Therefore $\mathrm{Tor}^1_{\mathcal{O}_Y}(F, \mathcal{O}_Y/I^d)=0$ and hence $0 \to I^d \to \mathcal{O}_Y \to \mathcal{O}_{X_{s'}} \to 0$ tensored with F remains exact. Therefore by Proposition 1.5 $p^*_{\mathbf{m}'}(F)$ is flat over s' (for a general s). On $X_{s'}$ we have the exact sequence $0 \to I/I^d \to \mathcal{O}_{X_{s'}} \to \mathcal{O}_{X_s} \to 0$. Since $\mathrm{Tor}^1_{\mathcal{O}_Y}(F, \mathcal{O}_{X_s})=0$ tensoring this with $F(-l)$ gives an exact sequence:

$$0 \to F(-l) \otimes (I/I^d) \to F(-l)/X_{s'} \to F(-l)/X_s \to 0.$$

For $l \geqq m_0$ $H^1(X_s, F(-l))=0$ by induction assumption. From the exact sequence $0 \to I^{d-1}/I^d \to I/I^d \to I/I^{d-1} \to 0$ using induction on d one can see that $H^1(X_s, F(-l) \otimes (I/I^d))=0$ for $l \geqq m_0$. Therefore $H^1(X_{s'}, F(-l))=0$ for $l \geqq m_0$. But $p^*_{\mathbf{m}'}(F)$ is flat at s' and hence by semicontinuity $H^1(X_u, F(-l))=0$ for $l \geqq m_0$ for a general $u \in S_{\mathbf{m}'}$. Thus the proof of the proposition is complete.

Remark. If F is locally free the proof is much simpler. For, in that case $H^i(X, F(-l)) \approx H^{n-i}(X, F^* \otimes \omega(l))^* = 0$, $0 \leqq i \leqq n-1$ for $l \geqq m_0$. Then for any hyperplane section Y of X tensoring the exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ with $F(-l)$ gives $H^1(Y, F(-l))=0$ for $l \geqq m_0$.

4. Vector Bundles on Families of Curves

4.1. Let V be a vector bundle on a nonsingular projective curve C over a field K. If V is not semistable there is a unique proper subbundle V_1 of V such that

1) $\mu(V_1) \geqq \mu(W)$ for all subbundles W of V.
2) If $\mu(W) = \mu(V_1)$ then $\mathrm{rk}\, W \leqq \mathrm{rk}\, V_1$.

We call V_1 the β-subbundle of V (cf. [7, 5]). Because of the uniqueness the β-subbundle is defined over the base field K even when it is not algebraically closed [7].

4.2. Let $f: D \to S$ be a flat family of nonsingular projective curves over an integral scheme S. Let V be a vectorbundle over D. Then $\{s \in S \mid V_s \text{ is semistable on } D_s\}$ is an open subset of S [12, 8]. When all the V_s are not semistable, or equivalently when V_{s_0} is not semistable for s_0 the generic point of S, we have the β-subbundle W_{s_0} of V_{s_0} defined over K_{s_0} the function field of S. Then W_{s_0} gives a section of the corresponding quot scheme over the generic point and hence gives a section over an open subset $U' \subset S$. Thus we have a subbundle W of $V/f^{-1}(U')$ extending W_{s_0}. By semicontinuity W_s remains the β-subbundle of V_s for s in an open subset U of U'.

4.3. Proposition. *Let A be a discrete valuation ring with quotient field K and residue field k. Let $S = \operatorname{Spec} A$. Let $f: D \to S$ be a flat family of projective curves such that D and D_K, the generic fibre, are nonsingular and the special fibre D_k is reduced with nonsingular components $D_k^1, \ldots, D_k^r$. Let V be a vector bundle on D. Let $\mu_K = \max\{\mu(W) \mid W$ subbundle of $V_K \to D_K\}$ and $\mu_k^i = \max\{\mu(W) \mid W$ subbundle of $V/D_k^i \to D_k^i\}$. Then $\mu_K \leqq \sum_{i=1}^{r} \mu_k^i$.*

Proof. Let W_K be a subbundle of V_K with $\mu(W_K) = \mu_K$. By the completeness of the Quot scheme the exact sequence $0 \to W_K \to V_K \to Q_K \to 0$ can be extended to an exact sequence on $D: 0 \to W_A \to V \to Q_A \to 0$ with Q_A torsion free [3, Lemma 3.7]. By Lemma 3.1 W_A is reflexive and is S_2 (Corollary 3.1.1). Therefore D being nonsingular of dimension 2, W_A is locally free (as follows for e.g. from [1, Theorem 5.19]). Let $W_k^i = W_A \mid D_k^i$ and $V_k^i = V \mid D_k^i$. We then have $\deg W_k = \sum_{i=1}^{r} \deg W_k^i$ and $\operatorname{rk} W_k = \operatorname{rk} W_k^i$. (Where $\deg W_k$ is computed on the reducible curve D_k and $\deg W_k^i$ on the irreducible curve D_k^i.) Similarly $\deg V_k = \sum_{i=1}^{r} \deg V_k^i$ and $\operatorname{rk} V_k = \operatorname{rk} V_k^i$.

Since Q_A is not locally free only at finitely many points of D, W_k^i is a subsheaf of V_k^i. It is easy to see that this implies $\mu(W_k^i) \leqq \mu_k^i$ [13, Sect. 4]. By flatness $\deg W_K = \deg W_k$ and $\operatorname{rk} W_K = \operatorname{rk} W_k$. Therefore

$$\mu_K = \sum_{i=1}^{r} \mu(W_k^i) \leqq \sum_{i=1}^{r} \mu_k^i .$$

4.3.1. Corollary. *If $V \mid D_k^i$ is semistable for all i, then V_K is semistable.*

Proof. Let W_K be a subbundle of V_K. Since V_k^i are semistable $\mu_k^i \leqq \mu(V_k^i)$. By the proposition $\mu(W_K) \leqq \sum_{i=1}^{r} \mu_k^i$. But $\sum_{i=1}^{r} \mu(V_k^i) = \left(\sum_{i=1}^{r} \deg V_k^i\right) \Big/ \operatorname{rk} V = \mu(V_k) = \mu(V_K)$. Therefore $\mu_K \leqq \mu(V_K)$ proving the semistability of V_K.

5. A Degenerating Family of Curves

5.1. Notation. We fix a sequence $(\alpha_1, \ldots, \alpha_{n-1})$ of integers with each $\alpha_i \geqq 2$. We let $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_{n-1}$. For a positive integer m we denote by (m) the sequence $(\alpha_1^m, \ldots, \alpha_{n-1}^m)$.

5.2. Proposition. *Let $l = m + r$, $r > 0$. Let $U_m \subset S_{(m)}$ and $U_l \subset S_{(l)}$ be nonempty open subsets. Then there is a point $s \in S_{(l)}$ and a nonsingular curve C in $S_{(l)}$ passing through s such that*

i) $C - \{s\} \subset U_l$.

ii) *$q_{(l)}^{-1}(C)$ is nonsingular and $q_{(l)}^{-1}(C) \to C$ is flat.*

iii) *The fibre $q_{(l)}^{-1}(s)$ is a reduced curve with α^r nonsingular components $C_1, C_2, \ldots$ which intersect transversally and at most two of which pass through any point of X, with each C_i a fibre of $q_{(m)}$ over a point of U_m.*

Proof. Let $S^i=(S_{\alpha_i^m})^{\alpha_i^r}$ and $S=S^1\times\ldots\times S^{n-1}$. Let $\pi_i:S^i\to S_{\alpha_i^l}$ be the multiplication map sending $(s_1,s_2,\ldots)$ to $s_1\cdot s_2\cdot\ldots$. Let $\pi:S\to S_{(l)}$ be the product $\pi_1\times\ldots\times\pi_{n-1}$. Let $u_1,u_2,\ldots$ be the α^r projections $S\to S_{(m)}$ corresponding to different choices of one factor (from among α_i^r) from each $S^i(i=1,\ldots,n-1)$.

$$\begin{array}{ccccc} & & S & & \\ & \swarrow^{u_j} & & \searrow^{\pi} & \\ S_{(m)} & & & & S_{(l)} \end{array}$$

For a general $s\in S$ the curve $q_{(l)}^{-1}(\pi(s))$ has the α^r nonsingular irreducible components $q_{(m)}^{-1}(u_j(s))$. We claim that there is a nonempty open subset T of S such that if $s\in T$ then $q_{(l)}^{-1}(\pi(s))$ is a reduced curve in X satisfying the conditions of iii). For, the condition that $q_{(m)}^{-1}(u_j(s))$ is nonsingular is a nonempty open condition on s as are the conditions that for the families $u_j^*(Z'_{(m)})$ [which are flat over $u_j^{-1}(S'_{(m)})$, cf. Proposition 1.5], $q_{(m)}^{-1}(u_j(s))$ intersect transversally [EGA IV/4, Remark 17.13.4(ii)] and at most two pass through a point [EGA IV/3, Theorem 12.2.4(vi)] as j varies. When these conditions are satisfied $q_{(l)}^{-1}(\pi(s))$ is Cohen-Macaulay and generically reduced and hence reduced [1, Lemma 2.3]. To satisfy the last condition of iii) we have only to intersect with the open sets $u_j^{-1}(U_m)$.

Let $s\in T$. By Proposition 1.5 $q_{(l)}$ is flat at $\pi(s)$. Therefore on the fibre over s, $q_{(l)}$ fails to be smooth only at the singular points of the fibre [1, Theorem 1.8]. At these singular points because of the conditions in iii) the differential of $q_{(l)}$ has a two dimensional kernel. Therefore for a general curve C through s defined by regular parameters at s, $q_{(l)}^{-1}(C)$ is nonsingular. Therefore we can find a C satisfying the conditions of the proposition.

6. Restriction to Curves

In this section we will prove the following theorem.

6.1. Theorem. *Let V be a semistable torsion free sheaf on X (with respect to the polarisation H). Let $Y_{(m)}$ be the generic curve of type (m) (Sects. 1.4 and 5.1). Then there is an m_0 such that for $m\geqq m_0$ the restriction of V to $Y_{(m)}$ (i.e. $\varphi_{(m)}^*p_{(m)}^*V$, Sects. 1.1 and 1.3) is semistable, or equivalently for $m\geqq m_0$ and for s in a nonempty open subset of $S_{(m)}$, $V|q_{(m)}^{-1}(s)$ is semistable.*

6.2. Remark. Conversely if $V|q_{(m)}^{-1}(s)$ is semistable then V is semistable as follows from the fact that the degree of a sheaf on X is determined by its restriction to $q_{(m)}^{-1}(s)$. Therefore from the above theorem it follows that V restricted to a general complete intersection subvariety of type $(\alpha_1^m,\ldots,\alpha_t^m)$, $1\leqq t\leqq n-1$ with $m\geqq m_0$ is also semistable (with respect to the induced polarisation).

6.3. Since V is a torsion free sheaf there is an open subset U of X with $\operatorname{codim}(X-U)\geqq 2$ such that V/U is a vector bundle. Therefore $V/Y_{(m)}$ (i.e. $\varphi_{(m)}^*p_{(m)}^*V$) is a vector bundle.

6.4. Proposition. *If $V/Y_{(m)}$ is semistable then $V/Y_{(l)}$ is semistable for any $l \geqq m$.*

Proof. As $V/Y_{(m)}$ is semistable there is an open set U_m of $S_{(m)}$ such that $V/q_{(m)}^{-1}(s)$ is a semistable vector bundle for $s \in U_m$ (cf. Sect. 4.2). Apply Proposition 5.2 with this U_m and $U_l = S''_{(l)}$ to get a degenerating family $q_{(l)}^{-1}(C) \to C$. The lemma now follows from Corollary 4.3.1 applied to the vector bundle $p_{(l)}^*(V)$ on $q_{(l)}^{-1}(C)$.

6.5. Proposition. *If $V/Y_{(m)}$ is not semistable for every m then V is not semistable.*

Proof. If $V|Y_{(m)}$ is not semistable we can find a nonempty open subset U_m of $S''_{(m)}$ such that (i) $p_{(m)}q_{(m)}^{-1}(U_m) \subset U$ so that $p_m^*(V)|q_{(m)}^{-1}(U_m)$ is a vector bundle, (ii) for $s \in U_m$, $V|q_{(m)}^{-1}(s)$ is not semistable, and (iii) there is a subbundle W_m of $p_m^*(V)|q_{(m)}^{-1}(U_m)$ such that for $s \in U_m$, $W_m|q_m^{-1}(s)$ is the β-subbundle of $V|q_{(m)}^{-1}(s)$ (Sects. 4.1 and 4.2).

Let $r_m = \operatorname{rk} W_m$ and $\beta_m = \mu(W_m|q_m^{-1}(s))$, $s \in U_m$. By Proposition 2.1 there is a unique line bundle L_m on X such that $L_m|Y_{(m)}$ i.e. $\varphi_{(m)}^* p_{(m)}^* L_m$ is isomorphic to $(\det W_m)|Y_{(m)}$. Let d_m = degree of L_m on X. We then have the following lemma.

6.5.1. Lemma. *As a function of m, d_m is bounded.*

Proof. By Proposition 5.2 we have a degenerating family of curves $q_{(m+1)}^{-1}(C) \to C$ with all components of the singular fibre in U_m. Applying Proposition 4.3 to this family we get $\beta_{m+1} \leqq \alpha \cdot \beta_m$. But $\beta_m = d_m \alpha^m / r_m$. Therefore $d_{m+1}/r_{m+1} \leqq d_m/r_m$. Since $1 \leqq r_m \leqq \operatorname{rk} V$ this shows that d_m remains bounded above. Since W_m contradicts the semistability of $V|Y_{(m)}$ we have $d_m/r_m > \deg V/\operatorname{rk} V$ which proves d_m is bounded below.

Now d_m being bounded we can find a subsequence Q of the sequence of natural numbers such that $d_q = d$ and $r_q = r$ are constants for $q \in Q$. Then $\beta_l = \alpha^{l-m} \cdot \beta_m$ for all $l > m$ $l, m \in Q$.

6.5.2. Lemma. *The line bundles L_q, $q \in Q$, are all isomorphic on X.*

Proof. Let $m, l \in Q$ with $l > m$. Using Proposition 5.2 we can construct a degenerating family of curves $D \to \operatorname{Spec} A$, A a discrete valuation ring with quotient field K, with all the components of the special fibre D_k in U_m and the generic fibre D_K in U_l. Extend the β-subbundle $W_l|D_K$ of $p_{(l)}^*(V)|D_K$ to a subsheaf (with torsion free quotient) $\tilde{W}_l$ of $p_{(l)}^*(V)|D$ (as in the proof of Proposition 4.3). Then since $\beta_l = \alpha^{l-m}\beta_m$ the restriction of $\tilde{W}_l$ to any component of D_k is the β-subbundle there. Therefore $\det \tilde{W}_l$, which is isomorphic to L_l on D_K, is isomorphic to L_m on each component of D_k. Thus the two line bundles $\det \tilde{W}_l$ and L_l on D are isomorphic on D_K and have the same degree (since $\beta_l = \alpha^{l-m}\beta_m$) on each component of D_k and hence are isomorphic on D. Therefore L_l and L_m are isomorphic on the components of D_k and thus on $q_{(m)}^{-1}(s)$ for a general $s \in S_{(m)}$. Therefore $L_l|Y_{(m)} \approx L_m|Y_{(m)}$ and hence $L_l \approx L_m$ by Proposition 2.1.

6.5.3. Lemma. *When $m \in Q$ is sufficiently large, for a general $s \in U_m$ there is a subsheaf $\tilde{W}$ of V (which depends on s) such that $\tilde{W}|q_{(m)}^{-1}(s) = W_m|q_{(m)}^{-1}(s)$.*

Proof. Let L be the common line fundle L_q, $q \in Q$. Let U, with $\operatorname{codim}(X-U) \geqq 2$, be the open subset on which V is a vector bundle (Sect. 6.3). Extend $\bigwedge^r (V|U)$ on U to a reflexive sheaf F on X. Consider the reflexive sheaf $\operatorname{Hom}(L,F)=L^* \otimes F$. The Grassmann bundle of r dimensional subspaces of the fibres of $V|U$ is embedded in $\mathbb{P}\left(\bigwedge^r (V|U)\right)=\mathbb{P}\left(L^* \otimes \bigwedge^r (V|U)\right)$. Let $\Sigma \subset L^* \otimes \bigwedge^r (V|U)$ be the cone over it. For $\phi \in H^0(X, L^* \otimes F)$ let $\Sigma(\phi)=\{x \in U \mid \phi(x) \in \Sigma\}$. As ϕ varies over the finite dimensional space $H^0(X, L^* \otimes F)$ the corresponding $\Sigma(\phi)$ form a *bounded* family of subvarieties of U. Then it is easy to see that there is an N_0 such that if $\Sigma(\phi) \neq U$, then $\Sigma(\phi)$ does not contain any $q_{(m)}^{-1}(s)$ for $m \geqq N_0$. By Proposition 3.2 there is an N_1 such that for $m \geqq N_1$, $H^0(X, L^* \otimes F) \to H^0(q_{(m)}^{-1}(s), L^* \otimes F)$ is surjective for a general s. Let $m \in Q$ with $m \geqq \max(N_0, N_1)=m_0$. Let $\bar{\phi} \in H^0(q_{(m)}^{-1}(s), L^* \otimes F)$ correspond to the β-subbundle $W_m | q_{(m)}^{-1}(s)$ and ϕ its lift in $H^0(X, L^* \otimes F)$. Then $\Sigma(\phi)=U$ and on the open set U' where ϕ is nonzero ϕ gives a subbundle W of V extending $W_m | q_{(m)}^{-1}(s)$. Extend W on U' to $\tilde{W} \subset V$ on X.

To complete the proof of Proposition 6.5 we have only to note that $U' \supset q_{(m)}^{-1}(s)$ and since $W | q_{(m)}^{-1}(s)$ contradicts the semistability of $V | q_{(m)}^{-1}(s)$, $\tilde{W}$ contradicts the semistability of V.

Now Propositions 6.4 and 6.5 together immediately imply Theorem 6.1.

6.6. Remarks. i) In our proof m_0 depends on V. If m_0 can be chosen to depend only on the Chern classes of V boundedness of the family of semistable bundles with fixed Chern classes would follow.

ii) If m_0 works for a V, clearly it would do for any small deformation of V. Therefore it follows that for any bounded family of sheaves there is a single curve C on which all of them are semistable.

iii) If $\operatorname{char} k=0$ it follows from the result of [13] relating unitary representations of Fuchsian groups and semistable vector bundles on the corresponding curves that on a curve X if V is semistable then any associated bundle (for e.g. the exterior powers, symmetric powers etc.) is also semistable. When $\operatorname{char} k=0$, from this and Theorem 6.1 (and Remark 6.2) it follows immediately that the same result holds for higher dimensional X as well. This result has also been proved in [9] and [15] by other methods.

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