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A note on Schubert varieties in G/B

V. B. Mehta and V. Srinivas

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Introduction

Let G be a semi-simple simply connected algebraic group over an algebraically closed field of characteristic $p>0$. Let $T\subset G$ be a maximal torus, $B\supset T$ a Borel subgroup of G, $W=N(T)/T$ the Weyl group and $Q\supset B$ a parabolic subgroup of G. The Schubert varieties in G/Q are defined to be the closures of the B-orbits in G/Q. It was proved by Hochster, Kempf, Laksov and Musili that Schubert varieties in Grassmannians are Cohen-Macaulay (cf. [3, 5, 7, 10]). Seshadri and Musili proved that Schubert varieties in $SL(n)/B$ are Cohen-Macaulay [11], see also [1, 4, 6]. Finally Ramanathan proved the result in general [12, 13]. His methods made use of the ideas of Frobenius-splitting and compatible splitting introduced in [8]. The other ingredients were the calculations of the canonical bundle of the standard resolutions of Schubert varieties introduced by Kempf, as well as the fact that there is a canonical splitting of G/B which compatibly splits all the Schubert varieties in G/B. He also needed the fact that if X is a union of a finite collection of Schubert varieties in G/B and L is any ample line bundle on G/B, then $H^i(X,L)=0\ \forall i>0$ and $H^0(G/B,L)\to H^0(X,L)$ is surjective.

In this note we want to give a short proof that Schubert varieties in G/B are Cohen-Macaulay. Our proof is similar in spirit to the proof of normality of Schubert varieties given by us in [9]. We also make use of two simple lemmas. The first states that if $\pi:X\to Y$ is a birational map with a) X Frobenius-split and b) $H^i(X_y,\mathcal{O}_{X_y})=0\ \forall i>0\ \forall y\in Y$, then $R^i\pi_*\mathcal{O}_X=0\ \forall i>0$ (here X_y denotes the scheme-theoretic fibre over $y\in Y$). The second one states that if $\pi:X\to Y$ is a morphism of Frobenius-split schemes, with a) π surjective with connected fibres and b) each component of Y is birationally dominated by some component of X, then $\pi_*\mathcal{O}_X=\mathcal{O}_Y$. This is very similar to Lemma 1 of [9]. Finally, we make use of the calculation of the canonical bundles of the standard resolutions of Schubert varieties as carried out in [12]. In Sect. I we recall these results and in Sect. II we prove the main result.

I

Let G, B and W be as in the introduction and let $w \in W$. Let $l(w) = i$ and write w as a product of reflections associated to the simple roots, $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_i}$. Let $X_i = \overline{BwB}/B$ be the corresponding Schubert variety in G/B. Put $P_i = B \cup B s_{\alpha_i} B$ and let $g_i: G/B \to G/P_i$ be the corresponding $\mathbb{P}^1$-fibration. Put $w' = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_{i-1}}$, and $X_{i-1} = \overline{Bw'B}/B$, a Schubert variety of codim 1 in X_i. Assume, by induction, that a surjective birational morphism $\pi_{i-1}: Z_{i-1} \to X_{i-1}$ has been constructed with

1) Z_{i-1} smooth projective

2) $\exists\, i-1$ smooth subvarieties of Z_{i-1}, intersecting transversally, whose union is denoted by ∂Z_{i-1}, s.t. $\pi_{i-1}(\partial Z_{i-1}) = \partial X_{i-1}$, where ∂X_{i-1} is the union of the codimension 1 Schubert subvarieties of X_{i-1}. Further, $\partial Z_{i-1} = \pi_{i-1}^{-1}(\partial X_{i-1})$.

Then the standard resolution Z_i of X_i is defined by the following Cartesian diagram

$$\begin{array}{ccccc}
Z_i & \xrightarrow{\alpha_i} & M_i & \xrightarrow{\beta_i} & X_i \subset G/B \\
\sigma_i \uparrow\downarrow f_i & & \downarrow & & \downarrow g_i \\
Z_{i-1} & \xrightarrow[\pi_{i-1}]{} & X_{i-1} & \xrightarrow[t_{i-1}]{} & g_i(X_i) \subset G/P_i,
\end{array}$$

where $M_i \to X_{i-1}$ is the base-change of $g_i: X_i \to g_i(X_i)$ and $Z_i \to Z_{i-1}$ is the base-change of $M_i \to X_{i-1}$. It is known (cf [5, 12]) that

1) $t_{i-1}: X_{i-1} \to g_i(X_i)$ is birational with fibres either $\mathbb{P}^1$ or a single point.

2) $\exists$ a section $\sigma_i: Z_{i-1} \to Z_i$ such that $f_i^{-1}(\partial Z_{i-1}) \cup \sigma_i(Z_{i-1})$ is a collection of i smooth divisors in Z_i intersecting transversally.

3) Put $\beta_i: M_i \to X_i$ and $\alpha_i: Z_i \to M_i$ and $\pi_i = \beta_i \circ \alpha_i$. Then $K_{Z_i} = \mathcal{O}_{Z_i}(-\partial Z_i) \otimes \pi_\varrho^*$, where L_ϱ is the line bundle on G/B associated to the character $= 1/2$ sum of positive roots, and $\partial Z_i = f_i^{-1}(\partial Z_{i-1}) \cup \sigma_i(Z_{i-1})$. Note that L_ϱ is an ample line bundle on G/B. From this description of K_{Z_i} it follows that Z_i is Frobenius-split and any sub-intersection of the divisors in ∂Z_i is compatibly split in Z_i (cf. [8]).

II

We begin by proving the two Lemmas alluded to in the Introduction.

Lemma 1. *Let $\pi: X \to Y$ be a projective birational map such that*

a) *X is Frobenius-split.*

b) *$H^i(X_y, \mathcal{O}_{X_y}) = 0$ $\forall i > 0$ $\forall y \in Y$, where X_y is the scheme-theoretic fibre over $y \in Y$. Then $R^i \pi_* \mathcal{O}_X = 0$ $\forall i > 0$.*

Proof. Fix $i > 0$. As the question is local on Y, we may assume that $Y = \operatorname{Spec} A$, where A is a Noetherian local ring and $R^i \pi_* \mathcal{O}_X$ is concentrated at the closed point $\underline{m} \in \operatorname{Spec} A$, where $\underline{m}$ is the maximal ideal of A. Then $R^i \pi_* \mathcal{O}_X = H^i(X, \mathcal{O}_X)$ is a A-module of finite length, call it M. Then by the theorem on formal functions, $M \simeq \varprojlim_n H^i(X_n, \mathcal{O}_{X_n})$, where $X_n = X \otimes_A A/\underline{m}^n$ for $n > 0$. The canonical map $H^i(X, \mathcal{O}_X)$

$\to H^i(X_n, \mathcal{O}_{X_n})$ is injective for $n \gg 0$. To see this, put $K_n = \ker H^i(X, \mathcal{O}_X) \to H^i(X_n, \mathcal{O}_{X_n})$. Then K_n is a decreasing sequence of submodules of the Artinian A-module M, hence ultimately stationary. Hence $K_n = 0\ \forall n \gg 0$, as M is isomorphic to $\varprojlim_n H^i(X_n, \mathcal{O}_{X_n})$. Fix $n \gg 0$ such that $K_n = 0$. Then $\exists t \gg 0$, such that F^t, the t-iterate of the Frobenius map $F: X \to X$, has a factorization:

$$\begin{array}{ccc} X_n & \longrightarrow & X \\ \downarrow & & \\ X_1 & & \Big\downarrow F^t \\ \downarrow & & \\ X_n & \longrightarrow & X. \end{array}$$

But F^t is injective on $H^i(X, \mathcal{O}_X)$ as X is Frobenius-split and $H^i(X_1, \mathcal{O}_{X_1}) = 0$ by hypothesis, hence $R^i\pi_* \mathcal{O}_X = H^i(X, \mathcal{O}_X) = 0$.

Lemma 2. *Let $\pi: X \to Y$ be a morphism between Frobenius-split schemes such that*
 a) *π is proper and surjective with connected fibres*
 b) *$\forall$ components T of Y, $\exists$ a component S of X such that $\pi|_S: S \to T$ is birational.*
 Then $\pi_* \mathcal{O}_X = \mathcal{O}_Y$.

Proof. By Stein-factorization, $\exists$ a commutative diagram

$$\begin{array}{ccc} \pi: X & \longrightarrow & Y \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & Z & \end{array}$$

with $f_* \mathcal{O}_X = \mathcal{O}_Z$ and g finite. This implies that Z is Frobenius-split and that g is a bijection. Hence Z and Y have the same number of components. Further, it follows from the hypothesis that each component of Z is birational to the corresponding component of Y. Since Y, Z are Frobenius split, we get g is an isomorphism, hence $\pi_* \mathcal{O}_X = \mathcal{O}_Y$.

Now let $X_i \subset G/B$ be a Schubert variety of dim i, $X_i = \overline{BwB}/B$ with $l(w) = i$. Let $\pi_i: Z_i \to X_i$ be the standard resolution of X_i. We have

Proposition 3. $R^j\pi_{i*} \mathcal{O}_{Z_i} = 0\ \forall j > 0$.

Proof. We may assume, by induction on i, that the conclusion holds for smaller values of i. Hence $R^j\pi_{i-1*} \mathcal{O}_{Z_{i-1}} = 0\ \forall j > 0$. As the map $t_{i-1} \to g_i(X_i)$ has fibres either $\mathbb{P}^1$ or a single point, we have $R^j(t_{i-1} \circ \pi_{i-1})_* \mathcal{O}_{Z_{i-1}} = 0\ \forall j > 0$. As $Z_i \to X_i$ is a flat base charge of $Z_{i-1} \to g_i(X_i)$, we get $R^j\pi_{i*} \mathcal{O}_{Z_i} = 0\ \forall j > 0$.

Proposition 4. *On ∂Z_i, we have $H^j(\partial Z_i, \pi_i^* L_\varrho^k) = 0\ \forall j > 0, \forall k > 0$, where L_ϱ is the line-bundle on $X_i \subset G/B$ associated to the half-sum of positive roots.*

Proof. Put $\partial Z_i = R + S$, where $R = f_i^{-1}(\partial Z_{i-1})$ and $S = \sigma_i(Z_{i-1})$. We may assume, by induction, that $H^j(Z_t, \pi_i^* L_\varrho^k) = 0 \ \forall j > 0$, $Vk > 0$ and $\forall t \leqq i-1$. We prove the proposition in a series of claims:

Claim 1. $R^j \pi_{i-1*} \mathcal{O}_{\partial Z_{i-1}} = 0 \ \forall j > 0$ and $\pi_{i-1*} \mathcal{O}_{\partial Z_{i-1}} = \mathcal{O}_{\partial X_{i-1}}$.

Proof. The first assertion follows from the Leray spectral sequence for the morphism $\pi_{i-1} : \partial Z_{i-1} \to \partial X_{i-1}$ along with the induction hypothesis. The second assertion follows from Lemma 2.

Now put $T = t_{i-1}(\partial X_{i-1})$ and $\bar{T} = g_i^{-1}(T)$.

Claim 2. $R^j t_{i-1*}(\mathcal{O}_{\partial X_{i-1}}) = 0 \ \forall j > 0$ and $t_{i-1*}(\mathcal{O}_{\partial X_{i-1}}) = \mathcal{O}_T$.

Proof. The first assertion follows from Lemma 1 and the second from Lemma 2.

Claim 3. $R^j(t_{i-1} \circ \pi_{i-1})_* \mathcal{O}_{Z_{i-1}} = 0 \ \forall j > 0$ and $(t_{i-1} \circ \pi_{i-1})_* \mathcal{O}_{Z_{i-1}} = \mathcal{O}_T$.

Proof. This follows from Claim 2 and the Leray spectral sequence for $t_{i-1} \circ \pi_{i-1}$.

Now the map $\pi_i : R \to \bar{T}$ is a flat base-change of $t_{i-1} \circ \pi_{i-1} : \partial Z_{i-1} \to T$, hence $R^j(\pi_i/R)_* \mathcal{O}_R = 0 \ \forall j > 0$ and $\pi_{i*} \mathcal{O}_R = \mathcal{O}_{\bar{T}}$. For $\pi_i : S \to X_{i-1}$, we already know $R^j(\pi_i/S)_* \mathcal{O}_S = 0 \ \forall j > 0$ and $\pi_{i*} \mathcal{O}_S = \mathcal{O}_{X_{i-1}}$. Hence we get $H^j(R, L_\varrho^k) = H^j(S, L_\varrho^k) = 0 \ \forall j > 0$, $\forall k > 0$. Consider the sequence

$$\to \mathcal{O}_{\partial Z_i} \to \mathcal{O}_R \oplus \mathcal{O}_S \to \mathcal{O}_{R \cap S} = \mathcal{O}_{\partial Z_{i-1}} \to .$$

Tensoring with L_ϱ^k, we get

$H^j(Z_i, L_\varrho^k) = 0 \ \forall j \geqq z$, $\forall k > 0$. For $j = 1$, we have $H^0(R \cup S, L_\varrho^k) \to H^0(\partial Z_{i-1}, L_\varrho^k)$ surjective, as $H^0(S, L_\varrho^k) \to H^0(\partial Z_{i-1}, L_\varrho^k)$ is already surjective $\forall k > 0$. Hence $H^1(\partial Z_i, L_\varrho^k) = 0 \ \forall k > 0$. This completes the proof of Proposition 4.

Proposition 5. *X_i is Cohen-Macaulay.*

Proof. From [2, III, 7.6], we only have to prove that $H^j(X_i, L_\varrho^{-k}) = 0 \ \forall j < i$, $\forall k \gg 0$. By the map $\pi_i : Z_i \to X_i$ and Proposition 3, we get $H^j(Z_i, \pi_i^* L_\varrho^{-k}) = H^j(X_i, L_\varrho^{-k})$. But $H^j(Z_i, \pi_i^* L_\varrho^{-k}) = H^{i-j}(Z_i, \mathcal{O}_{Z_i}(-\partial Z_i) \otimes L_\varrho^{k-1})$. Tensoring

$$0 \to \mathcal{O}_{Z_i}(-\partial Z_i) \to \mathcal{O}_{Z_i} \to \mathcal{O}_{\partial Z_i} \to 0$$

with L_ϱ^{k-1} and noting that $H^0(Z_i, L_\varrho^{k-1}) \to H^0(\partial Z_i, L_\varrho^{k-1})$ is onto $\forall k > 0$, we get that $H^{i-j}(Z_i, \mathcal{O}_{Z_i}(-\partial Z_i) \otimes L_\varrho^{k-1}) = 0 \ \forall j < i$. Q.E.D.

Remark 1. Let $\pi : Z \to X$ be the standard resolution of a Schubert variety $X \subset G/B$. If one could prove that $H^i(X_y, \mathcal{O}_{X_y}) = 0 \ \forall i > 0 \ \forall y \in X$ (where X_y is the fibre over $y \in X$), then one could prove directly, along with Lemma 1, that X is Cohen-Macaulay.

Remark 2. More generally, if there was a geometrical proof of Grauert-Riemannschneider vanishing for arbitrary smooth Frobenius-split varieties in char p, again it would follow that X is Cohen-Macaulay.

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