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Localized coherent structures of $(2+1)$ dimensional generalizations of soliton systems

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Abstract. We briefly review the recent progress in obtaining $(2+1)$ dimensional integrable generalizations of soliton equations in $(1+1)$ dimensions. Then, we develop an algorithmic procedure to obtain interesting classes of solutions to these systems. In particular using a Painlevé singularity structure analysis approach, we investigate their integrability properties and obtain their appropriate Hirota bilinearized forms. We identify line solitons and from which we introduce the concept of ghost solitons, which are patently boundary effects characteristic of these $(2+1)$ dimensional integrable systems. Generalizing these solutions, we obtain exponentially localized solutions, namely the dromions which are driven by the boundaries. We also point out the interesting possibility that while the physical field itself may not be localized, either the potential or composite fields may get localized. Finally,the possibility of generating an even wider class of localized solutions is hinted by using curved solitons.

Keywords. Solitons in higher dimensions; integrability; coherent structures; Painlevé analysis; Hirota method.

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1. Introduction

The concept of soliton was born from the now famous numerical experiments of Zabusky and Kruskal [1] on the stability properties of solitary wave solutions of the Korteweg-de Vries (KdV) equation. These results were then analytically confirmed by the seminal work of Gardner, Greene, Kruskal and Miura [2], who essentially developed the method of inverse scattering transform (IST) procedure to solve the Cauchy initial value problem of the KdV equation. Since then,a very large class of $(1+1)$ dimensional nonlinear dispersive wave equations such as the sine-Gordon (sG), nonlinear Schrödinger (NLS), Heisenberg ferromagnetic spin equations and so on have been identified [3, 4] which are also solvable by the IST method exhibiting the so called soliton solutions. Furthermore,these systems are understood to be completely integrable infinite dimensional dynamical systems. These equations encompass a vast number of topics in $(1+1)$ dimensions in many areas of physics and other sciences and engineering [5]. In $(1+1)$ dimensions, the soliton systems admit solitary waves and these waves retain their shape and speed under collisions. Mathematically, the corresponding nonlinear evolution equations are linearizable and linear spectral problems can be identified with each one of them. What are the possible counterparts of such soliton systems in 2 space and 1 time, (2 + 1) dimensions? In this review, we wish to bring out some of the recent developments in this regard and in particular we wish to point out how exponentially localized coherent structures can be identified here, whenever appropriate boundary effects are included in the (2 + 1) dimensional systems. The plan of the review is as follows. In § 2, we give a very brief sketch of the concept of solitons in (1 + 1) dimensions and some of the counterparts in (2 + 1) dimensions. In § 3, we point out how such soliton equations are associated with linear spectral problems and how their initial value problems can be solved using the inverse scattering transform method. In addition, we point out how by treating the IST problem as just a d-bar problem associated with analytic functions, the above procedure can be extended to (2 + 1) dimensions. In addition, we list some of the important (2 + 1) dimensional integrable nonlinear evolution equations which have been considered in the recent literature and possible algorithmic procedures to look for exponentially localized solutions. In § 4, we sketch the salient features associated with the Painlevé analysis and Hirota bilinearization procedure treating KdV as an example. Exponentially localized solutions for the simplest (2 + 1) dimensional generalization of KdV is obtained in § 5 indicating the influence of boundary effects. Section 6 contains brief details of the results for the other (2 + 1) dimensional evolution equations. Finally § 7 gives a summary of the results.

2. The concept of soliton and its generalization

2.1 *Solitons in* (1 + 1) *dimensions*

For linear dispersive systems, it is well known that the associated evolution equation admits elementary wave solutions of trignometric type, satisfying appropriate dispersion relations $\omega = \omega(k)$. The Fourier transform of such elementary solutions over the k-space then gives the general solution, thereby solving the initial value problem [6].

In contrast, often the elementary wave solutions of nonlinear dispersive systems are expressible in terms of elliptic functions (example cnoidal waves) and in particular, one has amplitude-dependent dispersion relations, $\omega = \omega(k, A)$. For appropriate choice of constants, one can often obtain solitary waves as special solutions for these wave equations. However, because of the nonlinearity, Fourier transform of such elementary solutions will no longer be a solution. On the other hand,one may be able to identify a nonlinear superposition principle asympotically.

Example: Korteweg – de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

The solitary wave solution is

$$u = u_1(x,t) = \frac{k^2}{2}\text{sech}^2\left[\frac{k}{2}(x - k^2 t + \delta)\right], \tag{2}$$

where k and δ are constants. One can also have a more general solution

$$u = u_2(x,t) = 2(k_2^2 - k_1^2)\left[\frac{k_2^2\,\text{cosech}^2\chi_2 + k_1^2\,\text{sech}^2\chi_1}{(k_2\coth\chi_2 - k_1\tanh\chi_1)^2}\right], \tag{3}$$

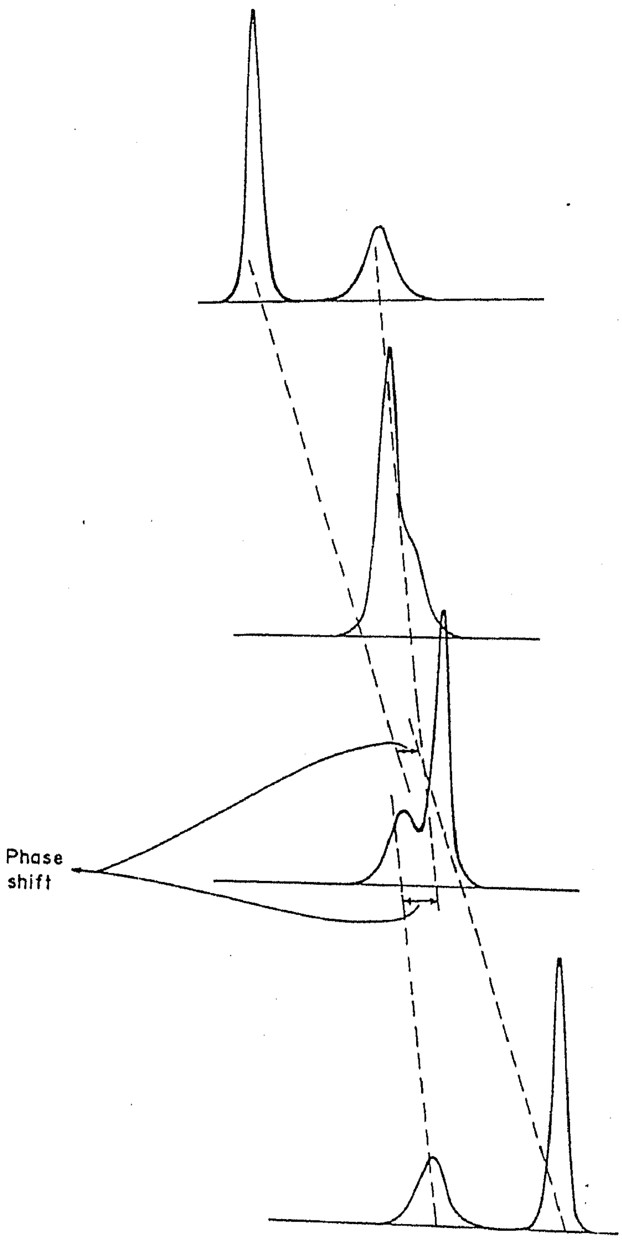

Figure 1. Two soliton interaction in the KdV equation (see (3)).

$\chi_i = k_i x - 4k_i^3 t + \delta_i$, $i = 1, 2$ (k_i, δ_i : constants). With a little bit of analysis, one can easily check that

$$u_2(x,t) \underset{t\to-\infty}{\longrightarrow} \begin{cases} 2k_1^2 \operatorname{sech}^2(\chi_1 - \Delta) & \text{as} \quad \chi_1 \to \mathrm{O}(1), \chi_2 \to \infty \\ 2k_2^2 \operatorname{sech}^2(\chi_2 + \Delta) & \text{as} \quad \chi_2 \to \mathrm{O}(1), \chi_1 \to \infty \end{cases} \tag{4}$$

and

$$u_2(x,t) \underset{t\to\infty}{\longrightarrow} \begin{cases} 2k_1^2 \operatorname{sech}^2(\chi_1 + \Delta) & \text{as} \quad \chi_1 \to \mathrm{O}(1), \chi_2 \to \infty \\ 2k_2^2 \operatorname{sech}^2(\chi_2 - \Delta) & \text{as} \quad \chi_2 \to \mathrm{O}(1), \chi_1 \to \infty \end{cases} \tag{5}$$

and so one can interpret the solution (3) as representing two solitary waves of differing amplitudes travelling in the same direction undergoing elastic collisions and coming out without change of shape or velocity except for a phase shift. The actual plot of the solution (3) for different times as given in figure 1 actually confirms this interpretation. Such solitary waves are then essentially called solitons. In fact,one can obtain explicit N-soliton solutions too as is well known [3] and also show that the solution of the Cauchy initial value problem for general initial conditions leads asymptotically to such N-soliton solutions in the background of small oscillatory decaying waves.

The above soliton picture holds good not only for the KdV equation, but for many other ubiquitous systems in $(1+1)$ dimensions; sine-Gordon, nonlinear Schrödinger, modified KdV, Heisenberg continuum spin chain equations are some of the standard examples [3].

2.2 *Excitations in* $(2+1)$ *dimensions*

A question naturally arises as to what are the basic nonlinear excitations in higher dimensions, especially in $(2+1)$ dimensions. In recent years, at least three basic excitations have been identified in $(2+1)$ dimensional nonlinear generalizations of $(1+1)$ dimensional soliton possessing systems. They are

(i) line solitons (decaying everywhere except along certain lines),
(ii) lump solitons (algebraically decaying) and
(iii) dromions (exponentially decaying).

The line solitons are nothing but straightforward generalization of the soliton solutions to $(2+1)$ dimensions. They decay exponentially everywhere in space except along certain lines, where they are bounded. For example, for the case of the so called Kadomtsev–Petviashvilli I (KPI) equation,

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{6}$$

which is obviously a generalization of KdV, one has the line soliton solution (figure 2)

$$u(x, y, t) = \frac{k^2}{2}\,\mathrm{sech}^2[\tfrac{1}{2}(kx + 2k^3 t + k^2 y + \delta)]. \tag{7}$$

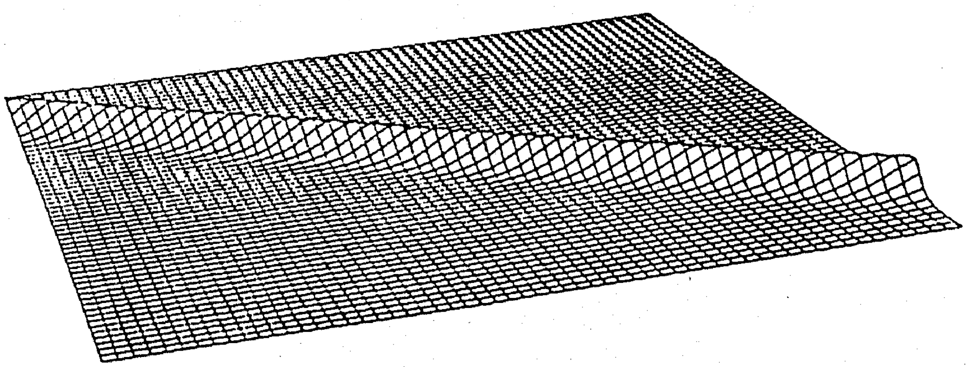

Figure 2. Line soliton solution of the KPI equation (see (7)).

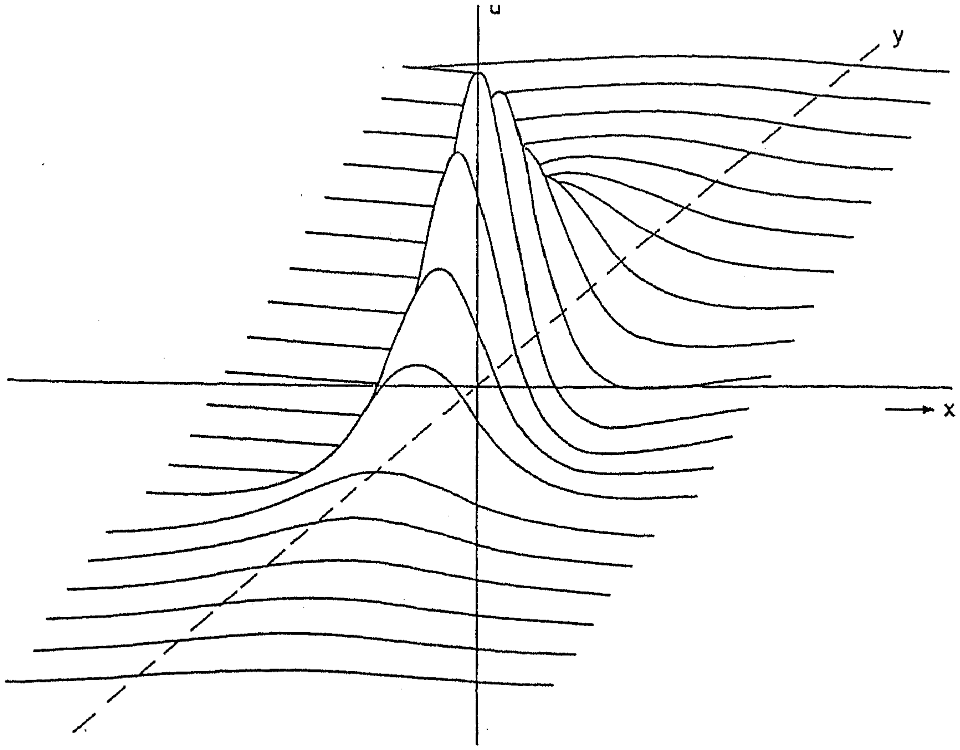

Figure 3. Lump soliton solution of the KPI equation (see (8)).

Equation (6) has also a lump soliton of the form

$$u = \frac{4[-(x' + k_R y')^2 + k_I^2 y'^2 - 3/k_I^2]}{[(x' + k_R y')^2 + k_I^2 y'^2 - 3/k_I^2]^2},$$
$$x' = x + (k_R^2 + k_I^2)t,$$
$$y' = \frac{1}{\sqrt{3}} y - 2k_R t, \tag{8}$$

which decays algebraically and is illustrated in figure 3. Finally, considering the Davey – Stewartson equation I,

$$iq_t + q_{\xi\xi} + q_{\eta\eta} + (\nu_1 + \nu_2)q = 0,$$
$$\nu_{1\xi} = -2\epsilon|q|_\eta^2,$$
$$\nu_{2\eta} = -2\epsilon|q|_\xi^2, \quad \epsilon = \pm 1, \tag{9}$$

it admits exponentially localized solutions [7, 8], namely dromions which are driven along specified tracks determined by the boundaries (dromos in Greek) of the form

$$q(\xi, \eta, t) = \frac{\rho \exp\{p_{1R}\hat{\xi} + s_{1R}\hat{\eta} + i[p_{1I}\hat{\xi} + s_{1I}\hat{\eta} + \{|p_1|^2 + |s_1|^2\}t]\}}{1 + j\exp(2p_{1R}\hat{\xi}) + k\exp(2s_{1R}\hat{\eta}) + l\exp(2(p_{1R}\hat{\xi} + s_{1R}\hat{\eta}))}, \tag{10a}$$

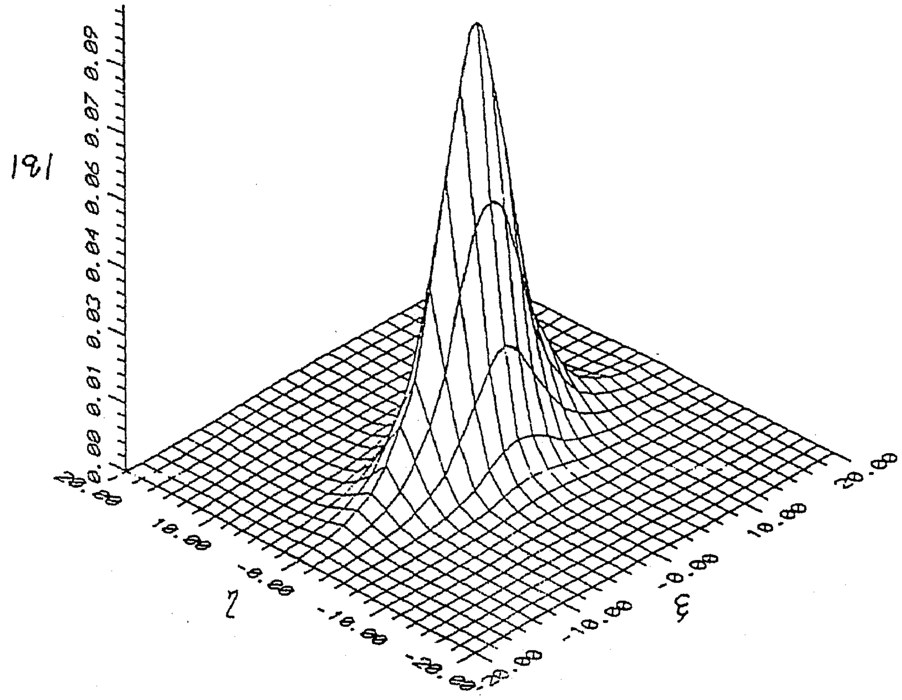

Figure 4. (1,1) dromion solution of the DSI equation (see (10)).

where

$$\hat{\xi} = \xi - 2p_{1l}t, \quad \hat{\eta} = \eta - 2s_{1l}t \tag{10b}$$

and p, s, j, k, l are constants. The form of (10a) is explicitly shown in figure 4.

In the remaining sections, we wish to explain how such exponentially localized dromion possessing equations can be generated and how such solutions may be realized. To start with, we demonstrate this for the KdV type equations.

3. Linearization and IST-(2+1) dimensional KdV

It is well known [3] that the $(1+1)$ dimensional soliton systems are linearizable in the sense that given the nonlinear evolution equation

$$u_t + K(u) = 0, \tag{11}$$

where K is some nonlinear functional, then the compatibility of the two linear systems

$$L_1\psi = 0, \quad L_2\psi = 0, \tag{12}$$

where L_1 and L_2 are linear differential operators, gives rise to the Lax equation

$$[L_1, L_2] = 0, \tag{13}$$

which is equivalent to the evolution equation (11). Example: KdV:

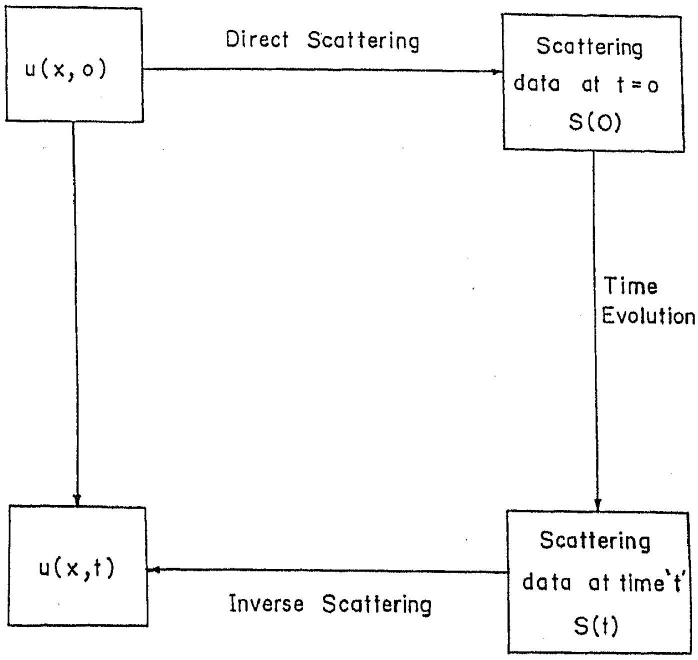

Figure 5. The schematic representation of the IST method.

$$L_1 = -\frac{\partial^2}{\partial x^2} + (u(x,t) - \lambda), \tag{14a}$$

(Schrödinger operator)

$$L_2 = \frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x. \tag{14b}$$

Then given the initial data $u(x,0)$ with $u \underset{|x|\to\infty}{\longrightarrow} 0$, (i) analyzing the scattering problem for $L_1\psi = 0$, the scattering data $S(0)$ at time $t = 0$ can be obtained. (ii) Making use of $L_2\psi = 0$, one can obtain the time evolution of $S(t)$ almost trivially. (iii) Carrying out an inverse scattering transform procedure for the scattering data $S(t)$ and by solving the associated Gelfand–Levitan–Marchenko equation, the 'potential' $u(x,t)$ can be uniquely determined and so the initial value problem can be solved.

The above three steps are schematically illustrated in figure 5. Now the crucial point is that an extension of the above procedure to $(2+1)$ dimensions can also be carried out provided we interpret the IST procedure as (i) a d-bar problem in complex analysis for the direct scattering and (ii) the inverse scattering part as equivalent to obtaining the analytic function (given the d-bar data) using the generalized Cauchy integral formula [9–11].

Thus the extension of the IST analysis crucially depends upon the linearization of $(2+1)$ dimensional equations and identifying appropriate Lax pairs in order to carry out the d-bar analysis. One way of doing this is to generalize the Lax pairs L_1 and L_2 as given in (12) by introducing operators and functions dependent on a second spatial variable y in a consistent way such that the Lax equation (13) gives the appropriate nonlinear evolution equation. We demonstrate this for the case of KdV equation in the following subsection. Similar analysis can be carried out for other ubiquitous soliton equations [12].

3.1 *Generalization of KdV to* $(2+1)$ *dimensions*

There are several ways to generalize the Lax operators L_1 and L_2 but retaining the form of the Lax equation (13). We may identify at least three interesting possibilities in the case of the KdV equation. They are as follows.

Method 1 [13]. One essentially includes first derivative operators of y in L_1 while making compensatory changes in L_2 in order to obtain the evolution equation from Lax condition. For example for the KdV Lax pair (14), one can make the modifications

$$L_1 = \sigma\partial_y + \partial_x^2 + u(x,y,t) \tag{15a}$$

and

$$L_2 = \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x + 3\sigma\partial_x^{-1}u_y \tag{15b}$$

so as to get the KPI and KPII equations (for $\sigma = i$ and 1 respectively).

Method 2 [14]. One keeps the operator L_1 unchanged while introducing first derivatives of y in L_2:

$$L_1 = L(\partial_x) - \lambda, \tag{16a}$$

$$L_2 = \partial_t + f(\lambda)\partial_y + \sum_{m=0}^{\mu} \nu_m(x,y,t)\partial_x^m. \tag{16b}$$

For KdV, this reads

$$L_1 = -\partial_x^2 + u(x,y,t) - \lambda,$$

$$L_2 = \partial_t - 4\beta\partial_x^2\partial_y + 4\beta u_x\partial_y + 3\beta u_{xy} + 2\beta u\partial_x. \tag{17}$$

Then the Lax equation $[L_1, L_2] = 0$ leads to the so called breaking soliton equation:

$$u_t = 4\beta uu_y + 2\beta u_x\partial_x^{-1}u_y - \beta u_{xxy} = 0, \tag{18}$$

provided the eigenvalue λ in (17) evolves as

$$\lambda_t = \lambda\lambda_y \tag{19}$$

leading to the breaking wave nature in the y direction.

Method 3 [15]. One can also look at the possibility of introducing derivatives in the second (y) variable with order greater than one. However such a possibility in general leads to constant coefficients of L_1 or L_2 and so to trivial evolution equations. To circumvent this difficulty in the case of $(2+1)$ dimensional systems, one realizes that it is enough to have only a sufficiently broad class of solutions of the linear equations $L_1\psi = 0$ and $L_2\psi = 0$ and that the spectrum need not be common. As a result, one can impose the weaker Lax condition

$$[L_1, L_2]\psi = 0 \tag{20}$$

in the subspace of the eigenfunction ψ or equivalently

$$[L_1, L_2] = \gamma L_1, \tag{21}$$

where γ is some operator.

Example: KdV generalization

$$L_1 = (\partial_x^2 - \sigma^2\partial_y^2) + u(x,y,t) = \partial_\xi\partial_\eta + u(\xi,\eta,t), \tag{22}$$

$$L_2 = \partial_t + k_1\partial_\xi^3 + k_1\partial_\eta^3 + 3k_1\partial_\xi^{-1}u_\eta\partial_\eta + 3k_1\partial_\eta^{-1}u_\xi\partial_\xi. \tag{23}$$

From the weak Lax equation, one can then obtain the Boiti *et al* generalized KdV equation [15]

$$u_t + k_1 u_{\xi\xi\xi} = 3k_1(u\partial_\eta^{-1}u_\xi)_\xi \tag{24}$$

when $k_2 = 0$ in (23). When $k_2 \neq 0$, the Nizhnik–Novikov–Veselov (NNV) equation is obtained as

$$u_t + k_1 u_{\xi\xi\xi} + k_2 u_{\eta\eta\eta} = 3k_2(u\partial_\xi^{-1}u_\eta)_\eta + 3k_1(u\partial_\eta^{-1}u_\xi)_\xi. \tag{25}$$

For a more general class of equations, see for examaple Ablowitz and Clarkson [3]. In this way,the other soliton equations in $(1+1)$ dimensions can also be generalized to $(2+1)$ dimensions though the exact procedure may vary from system to system [3]. Some of the higher dimensional systems which have been studied extensively in recent times include the following evolution equations apart from the ubiquitous K–P and Davey–Stewartson equations already mentioned:

(1) Generalized KdV [15]

$$u_t + u_{\xi\xi\xi} = 3(u\partial_\eta^{-1}u_\xi)_\xi. \tag{26}$$

(2) Generalized Nizhnik–Novikov–Veselov (NNV) equation[16]

$$u_t + u_{\xi\xi\xi} + u_{\eta\eta\eta} + au_\xi + bu_\eta = 3(u\partial_\eta^{-1}u_\xi)_\xi + 3(u\partial_\xi^{-1}u_\eta)_\eta. \tag{27}$$

(3) Breaking soliton equation [12]

$$u_t + au_{xxx} + 6auu_x + bu_{xxy} + 4buu_y + 4bu_x\partial_x^{-1}u_y = 0. \tag{28}$$

(4) $(2+1)$ dimensional generalized NLS equation [17]

$$iq_t - (\alpha-\beta)q_{\xi\xi} + (\alpha+\beta)q_{\eta\eta} - 2\lambda q\left[(\alpha+\beta)\left\{\int_{-\infty}^{\xi}|q|_\eta^2\mathrm{d}\xi' + u_1(\eta,t)\right\}\right.$$
$$\left. - (\alpha-\beta)\left\{\int_{-\infty}^{\eta}|q|_\xi^2\mathrm{d}\eta' + u_2(\xi,t)\right\}\right] = 0. \tag{29}$$

(5) $(2+1)$ dimensional generalized NLS equation [18]

$$iq_t = q_{xy} + Vq,\ V_x = 2\partial_y|q|^2. \tag{30}$$

(6) $(2+1)$ dimensional generalized sine-Gordon equation [19, 20]

$$\theta_{\xi\eta t} + \tfrac{1}{2}\theta_\eta \rho_\xi + \tfrac{1}{2}\theta_\xi \rho_\eta = 0, \tag{31a}$$

$$\rho_{\xi\eta} = \tfrac{1}{2}(\theta_\xi \theta_\eta)_t. \tag{31b}$$

(7) $(2+1)$ dimensional long dispersive wave (2LDW) equation [21]

$$\lambda q_t + q_{xx} - 2qv = 0, \tag{32a}$$

$$\lambda r_t - r_{xx} + 2rv = 0, \tag{32b}$$

$$(qr)_x = v_\eta, \partial_\eta = \partial_x - \lambda\partial_y. \tag{32c}$$

We will consider the existence of exponentially localized solutions in all these systems in the next sections.

4. Painlevé analysis and Hirota bilinearization

4.1 *Painlevé analysis*

In order to investigate the integrability aspects of the above type of nonlinear evolution equations, one can proceed to analyze the singularity structure of the solutions in the complex space of the independent variables, namely the so called Painlevé property. Such a procedure has been found to give considerable wealth of informations for finite dimensional nonlinear systems [22–24]. Application to pdes has also been well developed and the procedure of Weiss, Tabor and Carnevale [25] is algorithmic and identifies the nature of the singularities in the local neighbourhood of a noncharacteristic singular manifold.

Let us consider a NLEE of the form

$$u_t + K(u) = 0, \tag{33}$$

where $K(u)$ is a nonlinear functional of $u(x_1, x_2, \ldots, x_m, t) = u(X, t)$ and its derivatives upto order N so that (33) is an Nth order nonlinear pde. Then one can say that (33) possessess the P-property when the following conditions are satisfied.

The solution of (33) must be single valued about the noncharacteristic movable singular manifold. More precisely, if the singular manifold is determined by

$$\phi(x,t) = 0, \quad \phi_{x_i}(x,t) \neq 0, \quad \phi_t(x,t) \neq 0, \quad i = 1, \ldots, M \tag{34}$$

and $u(x,t)$ is a solution of the pde (33), then we have the Laurent expansion

$$u(x,t) = [\phi(x,t)]^{-m} \sum_{j=0}^{\infty} u_j(x,t)\phi^j(x,t) \tag{35}$$

in a deleted neighbourhood of the singular manifold (34) and m is an integer. By the Cauchy–Kovalevskaya theorem, the solution (35) should contain N arbitrary functions, one of them being the singular manifold ϕ itself and the others coming from the u_j's.

Then the WTC procedure to test the given pde for its P-property essentially consists of the following three steps [25].

(i) Determination of leading order behaviours.
(ii) Identification of powers j (resonance) at which the arbitrary functions can enter into the Laurent series expansion (35) and
(iii) Verifying that at the resonance values j, a sufficient number of arbitrary functions exist without the introduction of a movable critical singular manifold. An important feature of the WTC formalism is that the generalized Laurent series expansion can not only reveal the singularity structure aspects of the solution and integrability nature of a given pde, but can also provide an effective algorithm which in most cases successfully captures all its properties, namely the linearization (Lax pair), the Bäcklund transformation (BT), Hirota bilinearization, symmetries and so on [3]. As a simple application, we illustrate the above aspects with KdV as an example. Any other soliton system can also be likewise analysed [3, 26].

For the KdV equation (1), we substitute the formal Laurent expansion (35) around the singularity manifold $\phi(x,t) = 0$ into (1) and equate the powers of ϕ to zero. One finds that the exponent $m = +2$ and that at $j = -1, 4, 6$ arbitrary functions can enter the Laurent series (35). Identifying the arbitrariness of ϕ with $j = -1$ recursively, one finds

$$j = 0 : u_0 = -2\phi_x^2, \tag{36a}$$

$$j = 1 : u_1 = 2\phi_{xx}, \tag{36b}$$

$$j = 2 : \phi_x\phi_t + 6u_2\phi_{xx} - 2u_3\phi_x^2 + \phi_{xxxx} = 0, \tag{36c}$$

$$j = 3 : \phi_{xt} + 6u_2\phi xx - 2u_3\phi_x^2 + \phi_{xxxx} = 0, \tag{36d}$$

$$j = 4 : \partial_x(\phi_{xt} + 6u_2\phi_{xx} - 2u_3\phi_x^2 + \phi_{xxxx}) = 0. \tag{36e}$$

Now, it is clear that by the condition (36d), (36e) is always satisfied so that $u_4(x,t)$ is arbitrary. Similarly, one can derive the condition at $j = 5$ and prove that at $j = 6$, $u_6(x,t)$ is arbitrary. As the KdV equation is of third order,the Laurent series admits three arbitrary functions and so the Painlevé property is satisfied.

Now, if the arbitrary functions u_4 and u_6 are chosen to be identically zero and if we require $u_3 = 0$, then $u_j = 0, j > 3$ provided u_2 satisfies the KdV equation. Thus, we obtain the Bäcklund transformation for the KdV in the form

$$u = 2(\log\phi)_{xx} + u_2, \tag{37}$$

where u_1 and u_2 solve the KdV and ϕ satisfies (36a–e) with $u_3 = 0$. By a set of transformations, it is possible to show that the defining equations for ϕ are equivalent to linearizing equations (12) and (14). One can apply a similar procedure to any other NLEE in $(1+1)$ or $(2+1)$ dimensions and obtain its integrability property [24]. For recent applications in $(2+1)$ dimensions, see for example Radha and Lakshmanan [27–33].

4.2 *Hirota bilinearization*

We have noted above that the Bäcklund transformation of the form (37) can be used to bilinearize the NLEE. Indeed considering the vaccum solution $u_2 = 0$ in (37) one obtains the transformation

$$u = 2(\log\phi)_{xx} \tag{38}$$

so that the KdV equation can be written in the bilinear form

$$\phi_{xt}\phi - \phi_x\phi_t + \phi_{xxxx}\phi - 4\phi_{xxx}\phi_x + 3\phi_{xx}^2 = 0. \tag{39}$$

Using the so-called Hirota's bilinear operator

$$D_x^m D_t^n(a\cdot b) = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n a(x,t)b(x',t')|_{x'=x,y'=y}, \tag{40}$$

equation (39) can be rewritten in a compact form as

$$(D_xD_t + D_x^4)\phi\cdot\phi = 0. \tag{41}$$

The properties of the bilinear operator can be easily worked out. For example, one has the relations $D_x^m a\cdot 1 = \partial_x^m a$, $D_x^m(a\cdot b) = (-1)^m D_x^m(b\cdot a)$, $D_x^m(a\cdot a) = 0$, m odd, and so on. Using such properties, the calculations can be simplified considerably.

Now, expanding ϕ in a formal power series in ϵ,

$$\phi = \sum_{n=0}^{\infty}\epsilon^n\phi^{(n)}, \quad \phi^{(0)} = 1, \tag{42a}$$

where

$$\phi^{(1)} = \sum_{i=1}^{N}\exp(\eta_i), \eta_i = k_ix + \omega_it + \eta_i^{(0)} \tag{42b}$$

and k_i, ω_i, $\eta_i^{(0)}$ are constants, the N soliton solution of KdV can be obtained. To see this, one substitutes (42) in (41) and equates each power of ϵ separately to zero to obtain the following set of equations:

$$O(1): 0 = 0, \tag{43a}$$

$$O(\epsilon): \phi_{xt}^{(1)} + \phi_{xxxx}^{(1)} = 0, \tag{43b}$$

$$O(\epsilon^2): \phi_{xt}^{(2)} + \phi_{xxxx}^{(2)} = -\tfrac{1}{2}(D_xD_t + D_x^4)\phi^{(1)}\cdot\phi^{(1)}, \tag{43c}$$

and so on. The procedure is then to use (43b) in (43c) and succesively solve the remaining equations. In practice, one finds the solution for $N = 1,2,3$ and then hypothesize it for arbitrary N which is to be proved by induction.

For example, for $N = 1$, $\phi^{(1)} = \exp(\eta_1)$ and from (43b), $\omega_1 = -k_1^3$ and $(\partial_x\partial_t + \partial_x^4)$ $\phi^{(2)} = 0$ so that $\phi^{(2)} = 0$ and $\phi^{(i)} = 0, i \geq 2$. Thus, the solution of (43b) can be written as

$$\phi^{(1)} = \exp(\eta_1), \quad \eta_1 = k_1x - k_1^3t + \eta_1^{(0)}. \tag{44}$$

Making use of (38) and (42a), it is straightforward to obtain the one soliton solutions of KdV given by (2).

Similarly for the case $N = 2$,

$$\phi^{(1)} = \exp(\eta_1 + \eta_2), \quad \eta_i = k_ix - k_i^3t + \eta_i^{(0)}, \quad i = 1,2 \tag{45a}$$

so that the solution (42a) becomes

$$\phi^{(2)} = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}), \tag{45b}$$

where A_{12} is a constant. Again this leads to the 2-soliton solution (3) of the KdV equation. In an analogous fashion, one can proceed to find the N-soliton solution also. It is also useful to note that with the solution of the Hirota equation in a form such as (45b), it is quite easy to understand the elastic nature of the soliton interaction as discussed in § 2.

As noted above, all the other known soliton equations in $(1+1)$ dimensions can also be bilinearized and the soliton solution obtained through the Hirota method. In the next sections, we will apply this method to generate interesting coherent structures in $(2+1)$ dimensions.

5. Localized coherent structures in the (2 + 1) dimensional KdV equation

As an example,we consider the generalized KdV equation (26) of Boiti *et al* and obtain its localized solutions [28]. We rewrite it as

$$u_t + u_{\xi\xi\xi} = 3(uv)_\xi, \tag{46a}$$

$$u_\xi = v_\eta. \tag{46b}$$

Note that the potential v itself satisfies the equation

$$v_t + v_{\xi\xi\xi} = \tfrac{3}{2}(v^2)_\xi + 3\partial_\eta^{-1}[v_{\xi\xi}\partial_\xi^{-1}v_\eta + v_\xi v_\eta] \tag{47}$$

5.1 *P-property*

Considering the Laurent expansion

$$u = \sum_{j=0}^{\infty} u_j(x,y,t)\phi^{j+\alpha}, \tag{48a}$$

$$v = \sum_{j=0}^{\infty} v_j(x,y,t)\phi^{j+\beta}, \tag{48b}$$

around the noncharacteristic singular manifold $\phi(x,y,t) = 0$, ϕ_x, ϕ_y, , $\phi_t \neq 0$, we can algorithmically check that

$$\alpha = \beta = -2, \quad u_0 = 2\phi_\xi\phi_\eta, \quad v_0 = 2\phi_\xi^2 \tag{49}$$

and that one member each of (u_2, v_2), (u_4, v_4) and (u_6, v_6) are (the coefficient functions) arbitrary in addition to the manifold ϕ itself without the introduction of a critical singular manifold. Thus, the P-property is satisfied here.

5.2 *Hirota bilinear form*

One can easily check that the Laurent series (48) can be cut off at constant level terms to consider special solutions of the form

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \tag{50a}$$

$$v = \frac{v_0}{\phi^2} + \frac{v_1}{\phi} + v_2, \tag{50b}$$

so that if (u_2, v_2) solve (46), so also (u, v). Thus, (50) can be effectively considered as an auto-Bäcklund transformation and its further properties can be worked out. Particularly useful to us is the vacuum solution

$$u_2 = v_2 = 0. \tag{51}$$

Then with the expressions (49) for (u_0, v_0) and (u_1, v_1) (from a knowledge of the P-analysis), the transformations (50) can be simplified to

$$u = -2\partial_{\xi\eta} \log \phi, \tag{52a}$$

$$v = -2\partial_{\xi\xi} \log \phi. \tag{52b}$$

On substituting the above into the original evolution equation (46), one obtains the bilinear version

$$2\phi\phi_{\eta t} - 2\phi_\eta\phi_t + 2\phi\phi_{\xi\xi\xi\eta} - 2\phi_{\xi\xi\xi}\phi_\eta + 6\phi_{\xi\xi}\phi_{\xi\eta} - 6\phi_\eta\phi_{\xi\xi\eta} = 0. \tag{53}$$

In the Hirota operator notation, (53) can be rewritten as

$$(D_\eta D_t + D_\xi^3 D_\eta)\phi\cdot\phi = 0. \tag{54}$$

Expanding ϕ as a power series,

$$\phi = 1 + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \cdots, \tag{55}$$

equation (54) can be written as a set of coupled linear pdes:

$$\mathrm{O}(\varepsilon): \quad \phi^{(1)}_{\eta t} + \phi^{(1)}_{\xi\xi\xi\eta} = 0, \tag{56a}$$

$$\mathrm{O}(\varepsilon^2): \quad \phi^{(2)}_{\eta t} + \phi^{(2)}_{\xi\xi\xi\eta} = -\tfrac{1}{2}(D_\eta D_t + D_\xi^3 D_\eta)\phi^{(1)}\cdot\phi^{(1)}, \tag{56b}$$

etc.

5.3 *Line solitons and ghost solitons*

Equation (56a) obviously admits special classes of solutions of the form

$$\phi^{(1)} = \sum_{j=1}^{N} \exp(\chi_j), \quad \chi_j = k_j\xi + l_j\eta - k_j^3 t + \chi_j^{(0)}, \tag{57}$$

where k_j, l_j and $\chi_j^{(0)}$'s are constants. One can use these solutions for various choices of N to obtain the so called line solitons as in the case of KdV equation as discussed in § 4.2. For example, with the choice $N = 1$, we have

$$\phi^{(1)} = \exp(\chi_1), \quad \chi_1 = k_1\xi + l_1\eta - k_1^3 t + \chi_j^{(0)} \tag{58}$$

and the corresponding solution to the evolution equation becomes

$$u = -2\partial_{\xi\eta} \log \phi = -2\partial_{\xi\eta} \log(1 + \phi^{(1)}) = -\frac{k_1 l_1}{2} \operatorname{sech}^2\left(\frac{\chi_1}{2}\right), \tag{59a}$$

$$v = -2\partial_{\xi\xi} \log \phi = -2\partial_{\xi\xi} \log(1 + \phi^{(1)}) = -\frac{k_1^2}{2} \operatorname{sech}^2\left(\frac{\chi_1}{2}\right), \tag{59b}$$

where ε has been scaled out in the final form of the solution. These structures in $(2+1)$ dimensions are line solitons, because eventhough they have soliton-like properties, they are localized in the two dimensional space except along the lines $\chi_1 = 0$, where the solution is though bounded does not go to zero as $x, y \to \infty$. N-line solitons can also be constructed following the procedure suggested in § 4 and their interaction properties can be studied in the usual way.

But more interestingly, we observe the fact that in (59), k_1 and l_1 are arbitrary parameters. One finds that

$$\text{as} \quad k_1 \to 0, \quad \text{both} \quad u, v \to 0, \tag{60a}$$

whereas

$$\text{as} \quad l_1 \to 0,\ u \to 0,\ v \to \frac{-k_1^2}{2}\,\mathrm{sech}^2(k_1\xi - k_1^3 t + \chi_1^{(0)}). \tag{60b}$$

Thus eventhough the physical field vanishes, the potential

$$v = \int_{-\infty}^{\eta} u_\xi d\eta' + v_1(\xi, t) \tag{61}$$

survives. Comparing (60b) and (61), one concludes that

$$\lim_{\eta \to -\infty} v(\xi, \eta, t) = v_1(\xi, t) = -\frac{k_1^2}{2}\,\mathrm{sech}(k_1\xi - k_1^3 t + \chi_1^{(0)}) \tag{62}$$

is the ghost or background soliton at the boundary $\eta = -\infty$.

5.4 *Localized solution: Dromions*

The existence of ghost solitons at the boundary $\eta = -\infty$ prompts one to look for a more general ansatz than (58) so that we may choose

$$\phi = 1 + \mathrm{e}^{\chi_1} + \mathrm{e}^{\chi_2} + K\mathrm{e}^{\chi_1 + \chi_2}, \quad K > 0, \tag{63a}$$

where

$$\chi_1 = k_1\xi - k_1^3 t + c_1, \quad \chi_2 = l_1\eta + c_2 \tag{63b}$$

so that

$$u = -2\partial_{\xi\eta}\log\phi = \frac{2k_1 l_1(1-K)\exp(\chi_1+\chi_2)}{[1+\exp(\chi_1)+\exp(\chi_2)+K\exp(\chi_1+\chi_2)]^2}, \tag{64}$$

while

$$v = -2\partial_{\xi\xi}\log\phi = -\frac{2k_1^2[\exp(\chi_1)+(1-K)\exp(\chi_1+\chi_2)+\exp(\chi_1+2\chi_2)]}{[1+\exp(\chi_1)+\exp(\chi_2)+K\exp(\chi_1+\chi_2)]^2}. \tag{65}$$

One observes that while the physical field is now exponentially localized in the entire

(ξ, η) plane, the potential v is not. In this case, when

$$u \to 0 \quad \text{as} \quad \eta \to -\infty, \tag{66a}$$

$$v \to -\frac{k_1}{2}\,\mathrm{sech}^2(k_1\xi - k_1^3 t + c_1) \tag{66b}$$

so that we can interpret that the energy is continuously injected at the boundary $\eta \to -\infty$ in the form of a soliton (with $k \neq 0$) so that the physical field can overcome the tendency to disperse so as to get a fully localized structure. Solution (64) is then called the (1,1) dromion solution to the NLEE (46).

Multidromion solutions can also be obtained. For example a (2, 1) dromion solution is obtained as

$$u = \frac{2(1-K)[k_1 l_1 \exp(\chi_1 + \chi_2) + k_2 l_1 \exp(\chi_2 + \chi_3)]}{[1 + \exp(\chi_1) + \exp(\chi_2) + K(\exp(\chi_1 + \chi_2) + \exp(\chi_2 + \chi_3))]^2}, \tag{67a}$$

where

$$\chi_1 = k_1\xi - k_1^3 t + \chi_1^{(0)}, \tag{67b}$$

$$\chi_2 = l_1\eta + \chi_2^{(0)}, \tag{67c}$$

$$\chi_3 = k_2\xi - k_2^3 t + \chi_1^{(0)}. \tag{67d}$$

One may obtain (N, M) dromions also proceeding in this way.

5.5 *Further general localized structures*

Unlike the case of $(1+1)$ dimensional soliton equations, the linear set of Hirota equations (56) can also have more general solutions than the one given by (57). For example, (56a) admits general solutions of the form

$$\phi = \sum_{j=1}^{N} \exp[k_j\xi - k_j^3 t + m_j(\eta)], \tag{68}$$

where $m_j(\eta)$ is an arbitrary function of η. For example with the choice $N = 1$ in (68), we can obtain

$$u(\xi, \eta, t) = -\tfrac{1}{2}k_1 m_1'(\eta)\,\mathrm{sech}^2\tfrac{1}{2}[k_1\xi - k_1^3 t + m_1(\eta)] \tag{69}$$

where $m_1'(\eta) = (\mathrm{d}m_1/\mathrm{d}\eta)$ is arbitrary (as $m_1(\eta)$ is arbitrary). Choosing $m_1'(\eta)$ suitably, one can obtain various types of localized solutions. For example, with the choice,

$$m_1'(\eta) = \mathrm{sech}^2(\alpha\eta + \beta), \quad \alpha, \beta : \text{constants} \tag{70a}$$

we have the exponentially localized solution

$$u = \gamma\,\mathrm{sech}^2(\alpha\eta + \beta)\,\mathrm{sech}^2\tfrac{1}{2}[k_1\xi - k_1^3 t + m_1(\eta)] \tag{70b}$$

Many other interesting classes of exponentially localized solutions including oscillatory and rational functions can be similarly constructed for suitable choices of $m_j(\eta)$ [35].

6. Exponentially localized structures in other (2+1) dimensional NLEEs [34]

The procedure elucidated in § 5 for the (2 + 1) dimensional generalized KdV equation can be extended to other (2 + 1) dimensional systems discussed in § 3 as well. In the following, we summarize the main results.

6.1 *NNV equation*

Proceeding as in the case of the generalized KdV equation (46), we rewrite (27) as

$$u_t + u_{\xi\xi\xi} + u_{\eta\eta\eta} + au_\xi + bu_\eta = 3(uv)_\xi + 3(uq)_\eta, \tag{71a}$$

$$u_\xi = v_\eta, \quad u_\eta = q_\xi, \tag{71b}$$

the Painlevé property can be established [28]. Working out the auto-Bäcklund transformation, the bilinearizing transformation becomes

$$u = -2\partial_{\xi\eta} \log \phi, \tag{72a}$$

$$v = -2\partial_{\xi\xi} \log \phi, \tag{72b}$$

$$q = -2\partial_{\eta\eta} \log \phi. \tag{72c}$$

The line solitons work out to be

$$u = -\frac{k_1 l_1}{2} \operatorname{sech}^2 \frac{\chi_1}{2}, \tag{73a}$$

$$v = -\frac{k_1^2}{2} \operatorname{sech}^2 \frac{\chi_1}{2}, \tag{73b}$$

$$q = -\frac{l_1^2}{2} \operatorname{sech}^2 \frac{\chi_1}{2}, \tag{73c}$$

where

$$\chi_1 = k_1\xi + l_1\eta - (ck_1 + k_1^3)t - (\mathrm{d}l_1 + l_1^3)t + \chi_1^{(0)} \tag{73d}$$

so that in the limit $k_1 \to 0$, u, $v \to 0$, while q survives, whereas when $l_1 \to 0$, u, $q \to 0$, but v survives. Thus we have two nonparallel ghost solitons in the present case. Following the procedure discussed in the previous section, we take ϕ to be of the same form as (63a) so that the (1,1) dromion in the present case becomes

$$u(\xi, \eta, t) = \frac{2k_1 l_1 (1 - K) \exp(\chi_1 + \chi_2)}{[1 + \exp(\chi_1) + \exp(\chi_2) + K \exp(\chi_1 + \chi_2)]^2}, \quad K > 0 \tag{74}$$

where now

$$\chi_1 = k_1\xi - (ck_1 + k_1^3)t + \chi_1^{(0)}, \quad \chi_2 = l_1\xi - (\mathrm{d}l_1 + l_1^3)t + \chi_2^{(0)}. \tag{75}$$

Multidromions solutions can be similarly worked out.

6.2 *Breaking soliton equation*

Rewriting (28) as

$$u_t + au_{xxx} + bu_{xxy} + 6auu_x + 4buu_y + 4bu_x v = 0, \tag{76a}$$

$$u_y = v_x, \tag{76b}$$

one can check algorithmically that the Painlevé property is satisfied in the nontrivial case only for the choice $a = 0$, $b \neq 0$ ($b = 0$, $a \neq 0$ is the trivial KdV case) so that we have the equation of motion

$$u_t + bu_{xxy} + 4buu_y + 4bu_x v = 0, \tag{77a}$$

$$u_y = v_x. \tag{77b}$$

Note that v itself satisfies an equation of the form

$$v_t + bv_{xxy} + 4bvv_x + 4bv_y \partial_y^{-1} v_x = 0. \tag{78}$$

One verifies that bilinearization is possible only for the special case (77) and the required transformation works out to be

$$u = \tfrac{3}{2}\partial_{xx} \log \phi, \tag{79a}$$

$$v = \tfrac{3}{2}\partial_{xy} \log \phi. \tag{79b}$$

The bilinearized equation becomes

$$(D_x D_t + bD_x^3 D_y)\phi \cdot \phi = 0. \tag{80}$$

Expanding ϕ in a formal power series as before and truncating, one obtains the line soliton as

$$u = \tfrac{3}{8} k_1^2 \operatorname{sech}^2\left(\frac{\chi_1}{2}\right), \tag{81a}$$

$$v = \tfrac{3}{8} k_1 l_1 \operatorname{sech}^2\left(\frac{\chi_1}{2}\right), \tag{81b}$$

where

$$\chi_1 = k_1 x + l_1 y - bk_1^2 l_1 t + \chi_1^{(0)}. \tag{81c}$$

As $k \to 0$, both u and $v \to 0$, while when $l_1 \to 0$, the potential $v \to 0$, but the field u survives as a static ghost soliton of the form $u \to \frac{3}{8} k_1^2 \operatorname{sech}^2(k_1 x + \chi_1^{(0)})$ at the boundary $x = -\infty$.

The dromion solutions can be obtained in the usual way. For generating the (1,1) dromion, we take

$$\phi = 1 + e^{\chi_1} + e^{\chi_2} + Ke^{(\chi_1+\chi_2)}, \quad K > 0 \tag{82a}$$

$$\chi_1 = k_1 x + \chi_1^{(0)}, \quad \chi_2 = l_1 y - bk_1^2 l_1 t + \chi_2^{(0)} \tag{82b}$$

so that the potential v becomes fully localized exponentially,

$$v(x,y,t) = \frac{3}{2} \frac{k_1 l_1 (K-1) \exp(\chi_1 + \chi_2)}{[1 + \exp(\chi_1) + \exp(\chi_2) + K \exp(\chi_1 + \chi_2)]^2}, \tag{83}$$

whereas the physical field

$$u(x,y,t) = \frac{3}{2} \frac{(1+K) k_1^2 \exp(\chi_1 + \chi_2) + k_1^2 \exp(\chi_1)(1 + K \exp(2\chi_2))}{[1 + \exp(\chi_1) + \exp(\chi_2) + K \exp(\chi_1 + \chi_2)]^2}, \tag{84}$$

which is though bounded does not fall off to zero as $\chi_2 \to \infty$. One can proceed further [29] and obtain an $(1, N)$ dromion of the form

$$v_{(1,N)} = \frac{3}{2} \frac{(K-1) k_1 \exp(\chi_1) \sum_{i=2}^{N+1} l_i \exp(\chi_i)}{[1 + \sum_{j=1}^{N} \exp(\chi_j) + K \exp(\chi_1) \sum_{i=2}^{N+1} \exp(\chi_i)]^2}. \tag{85}$$

Finally, as in the case of the generalized $(2+1)$ dimensional KdV equation (§ 5.4), one can obtain more general exponentially localized solution of the form

$$u = \tfrac{3}{8} k_1^2 \operatorname{sech}^2 \tfrac{1}{2}[k_1 x + g(\xi)], \tag{86a}$$

$$v = \tfrac{3}{8} k_1 h(\xi) \operatorname{sech}^2 \tfrac{1}{2}[k_1 x + g(\xi)], \tag{86b}$$

where $g(\xi)$ and $h(\xi)$ are arbitrary functions in the variable $\xi = y - k_1^2 t$. Choosing $h(\xi)$ suitably, one can indeed obtain a very large class of localized structures for the potential v.

Thus in the case of the breaking soliton equation (77), it is not the physical field which is localized but it is rather the potential which admits localized solutions. Looking from another point of view, one may take (78) along with (77b) as the basic evolution equation, in which case we have now the new physical field which admits localized structures.

6.3 $(2+1)$ *dimensional NLS equation*

Considering the symmetric generalized $(2+1)$ dimensional nonlinear Schrödinger equation (29) introduced by Fokas [17], it can be rewritten as

$$iq_t + (\alpha + \beta) q_{\eta\eta} - (\alpha - \beta) q_{\xi\xi} - 2\lambda q[(\alpha + \beta)V - (\alpha - \beta)U] = 0, \tag{87a}$$

$$|q|^2_\eta = V_\xi, \tag{87b}$$

$$|q|^2_\xi = U_\eta. \tag{87c}$$

Equation (87) contains three important systems:

(i) $\alpha = \beta = \frac{1}{2}$: Simplest complex scalar equation in $(2+1)$ dimensions
(ii) $\alpha = 0$, $\beta = 1$: Davey–Stewartson equation I (DSI)
(iii) $\alpha = 1$, $\beta = 0$: Davey–Stewartson equation III (DSIII).

The Painlevé property of system (87) can be established in the standard way and it can be bilinearized [31, 36]. With the transformation,

$$q = \frac{g}{\phi}, \quad q^* = \frac{g^*}{\phi}, \tag{88a}$$

$$V = -\lambda \partial_{\eta\eta} \log \phi, \quad U = -\lambda \partial_{\xi\xi} \log \phi, \tag{88b}$$

one obtains the bilinearized form

$$[iD_t - (\alpha - \beta) D_\xi^2 + (\alpha + \beta) D_\eta^2] g \cdot \phi = 0, \tag{89a}$$

$$D_\xi D_\eta \phi \cdot \phi = -2\lambda g g^*. \tag{89b}$$

Expanding

$$g = \varepsilon g_1 + \varepsilon^3 g_3 + \cdots, \quad \phi = 1 + \varepsilon^2 \phi_2 + \varepsilon^4 \phi_4 + \cdots \tag{90}$$

and following the usual procedure, the basic line solitons are obtained as

$$q(\xi, \eta, t) = \tfrac{1}{2} \exp(-\psi) \operatorname{sech}(\chi_{1R} + \psi) \exp(i\chi_{1I}), \tag{91a}$$

$$V(\xi, \eta, t) = -\lambda s_{1R}^2 \operatorname{sech}^2(\chi_{1R} + \psi), \tag{91b}$$

$$U(\xi, \eta, t) = -\lambda p_{1R}^2 \operatorname{sech}^2(\chi_{1R} + \psi), \tag{91c}$$

$$\chi_1 = p_1 \xi + s_1 \eta - i[(\alpha - \beta) p_1^2 - (\alpha + \beta) s_1^2] t + \chi_1^{(0)}. \tag{91d}$$

One identifies two nonparallel ghost solitons, one at $\eta = -\infty$,

$$U = -\lambda p_{1R}^2 \operatorname{sech}^2[\bar{\chi}_{1R} + \psi], \quad \bar{\chi}_{1R} = p_{1R}[\xi + 2(\alpha - \beta) p_{1I} t], \tag{92a}$$

and the other at $\xi = -\infty$.

$$V = -\lambda s_{1R}^2 \operatorname{sech}^2[\hat{\chi}_{1R} + \psi'], \quad \hat{\chi}_{1R} = s_{1R}[\eta + 2(\alpha + \beta) s_{1I} t]. \tag{92b}$$

Consequently,the exponentially localized (1,1) dromion is obtained with the choice

$$\phi_{11D} = 1 + j \exp(\chi_1 + \chi_1^*) + k \exp(\chi_2 + \chi_2^*) + l \exp(\chi_1 + \chi_1^* + \chi_2 + \chi_2^*), \tag{93a}$$

with

$$\chi_1 = p_1 \xi - i(\alpha - \beta) p_1^2 t + c_1, \quad \chi_2 = s_1 \eta + i(\alpha + \beta) s_1^2 t + c_2, \tag{93b}$$

where p_1, s_1, c_1 and c_2 are constants. Choosing g appropriately, one then obtains

$$q_{11D} = \frac{\rho \exp(p_{1R}\hat{\xi} + s_{1R}\hat{\eta} + i\{p_{1I}\hat{\xi} + s_{1I}\hat{\eta} + [(\alpha + \beta)|s_1|^2 - (\alpha - \beta)|p_1|^2]\})}{1 + j \exp(2 p_{1R} \hat{\xi}) + k \exp(2 s_{1R} \hat{\eta}) + l \exp(2[p_{1R}\hat{\xi} + s_{1R}\hat{\eta}])}, \tag{94a}$$

where

$$\hat{\xi} = \xi + 2(\alpha - \beta) p_{1I} t, \quad \hat{\eta} = \eta - 2(\alpha + \beta) s_{1I} t. \tag{94b}$$

The multidromions can also be constructed by generalizing the above procedure. Similarly exponentially localized breather solutions also follow by putting $s_{1I} = p_{1I} = 0$ in (94a) to obtain

$$q_{11B} = \frac{\rho \exp(p_{1R}\xi + s_{1R}\eta + i\{(\alpha+\beta)s_{1R}^2 - (\alpha-\beta)p_{1R}^2\}t)}{1 + j\exp(2p_{1R}\xi) + k\exp(2s_{1R}\eta) + l\exp(2[p_{1R}\xi + s_{1R}\eta])}. \tag{95}$$

6.4 *The* $(2+1)$ *dimensional generalized NLS equation used by Strachan [18]*

The system of equations (30) is completely integrable and satisfies the Painlevé property [27]. It can be bilinearized through the transformation

$$q = \frac{g}{\phi}, \quad q^* = \frac{g^*}{\phi}, \quad v = 2\partial_{xy}\log\phi \tag{96}$$

as

$$iD_t g\cdot\phi = D_x D_y g\cdot\phi, \tag{97a}$$

$$D_x^2\phi\cdot\phi = 2gg^*. \tag{97b}$$

While admitting line solitons, it does not admit two nonparallel ghost solitons driving the boundaries in the usual sense. Thus the basic dromion solutions of the form considered in the previous examples do not seem to arise in this case.

However, one finds that a more general form of localized solutions corresponding to curved solitons can exist. To generate such localized solutions, we expand g and ϕ in the form of a power series similar to (90) to give rise to the following set of equations:

$$\varepsilon : ig_t^{(1)} = g_{xy}^{(1)} \tag{98a}$$

$$\varepsilon^2 : \phi_{xx}^{(2)} = g^{(1)}g^{(1)*} \tag{98b}$$

and so on. Solving (98a), we obtain

$$g^{(1)} = \sum_{j=1}^{N}\exp(\chi_j), \quad \chi_j = k_j x + m_j(y,t) + c_j, \tag{99a}$$

where $m_j(y,t)$ is an arbitrary function of (y,t) chosen such that

$$m_j(y,t) = m_j(\rho_j) = m_j(y - ik_j t) \tag{99b}$$

and k_j and c_j are complex constants. To construct one soliton solution, we take $N = 1$ and substitute $g^{(1)}$ in (98b) to give

$$\phi^{(1)} = \exp(\chi_1 + \chi_1^* + 2\psi), \quad \exp(2\psi) = \frac{1}{4k_R^2}. \tag{100}$$

Hence, the physical field q and the potential V assume the form

$$q = k_{1R}\,\mathrm{sech}(\chi_{1R} + \psi)\exp(i\chi_{1I}), \tag{101a}$$

$$V = 2k_{1R}(m_{1R})_{\rho_{1R}}\,\mathrm{sech}(\chi_{1R} + \psi), \quad \rho_{1R} = y + k_{1I}t, \quad k_{1I} = Im(k_1). \tag{101b}$$

It is evident from (101) that both the physical field q and the potential V remain finite on the curve

$$C = \chi_{1R} + \psi = k_{1R}x + m_{1R}(y,t) + c_{1R} = 0 \tag{102}$$

and decay exponentially everywhere (as $x, y \to \infty$) apart from the curve $C = 0$ given by (102). We call such line solitons as "curved solitons". As $(m_{1R})_{\rho_{1R}}$ is arbitrary, one can choose it conveniently as

$$(m_{1R})_{\rho_{1R}} = \mathrm{sech}^2(\rho_{1R}) \tag{103}$$

so that a one dromion solution for the potential V becomes

$$V = 2k_{1R}\mathrm{sech}^2(\rho_{1R})\mathrm{sech}^2(\chi_{1R} + \psi). \tag{104}$$

One calls such localized solutions as "induced dromions" as they are induced by virtue of the arbitrary function present in the system. One can indeed construct a large class of localized solutions for the potential by choosing the arbitrary function properly.

6.5 (2 + 1) *dimensional sine-Gordon equation*

The system (31) under the transformations

$$q = -\frac{i\sigma}{2}\theta_\eta, \quad r = -\frac{i\sigma}{2}\theta_\xi \tag{105}$$

can be recast in the form

$$q_{\xi t} + \tfrac{1}{2}\rho_\xi q + \tfrac{1}{2}\rho_\eta r = 0, \tag{106a}$$

$$r_{\xi t} + \tfrac{1}{2}\rho_\xi q + \tfrac{1}{2}\rho_\eta r = 0, \tag{106b}$$

$$\sigma^2\rho_{\xi\eta} = -2(qr)_t. \tag{106c}$$

Again the P-property can be established algorithmically [30]. One then obtains the bilinearizing transformation

$$q = \frac{q}{\phi}, \quad r = \frac{h}{\phi}, \tag{107a}$$

$$\rho = 2\partial_t(\log\phi) + 2\int_{-\infty}^{\xi} m_1(\xi', t)\,\mathrm{d}\xi' + 2\int_{-\infty}^{\eta} m_1(\eta', t)\,\mathrm{d}\eta' \tag{107b}$$

so that the Hirota bilinear form of eq. (106) becomes

$$D_\xi D_t g\cdot\phi + m_2(\xi, t)g\cdot\phi + m_1(\eta, t)h\cdot\phi = 0, \tag{108a}$$

$$D_\eta D_t g\cdot\phi + m_2(\xi, t)g\cdot\phi + m_1(\eta, t)h\cdot\phi = 0, \tag{108b}$$

$$\sigma^2 D_\xi D_\eta \phi\cdot\phi = -2gh, \tag{108c}$$

$$hD_\eta D_t\phi\cdot\phi - gD_\xi D_t\phi\cdot\phi = 0, \tag{108d}$$

where $m_1(\eta, t)$ and $m_2(\xi, t)$ are certain arbitrary functions which can be identified with boundary flows at $\xi = -\infty$ and $\eta = -\infty$ respectively.

Considering the 2DSGI ($\sigma^2 = +1$) case, we note that the presence of nonzero m_1 and m_2 are essential for the formation of line kink solitons as well as localized solutions. Expanding g, h and ϕ in a suitable formal power series, one obtains the line kinks for the case in which m_1 and m_2 are non-zero constants as

$$q = \frac{2iq_1 \exp\chi_1}{1+\exp(2\chi_1+2\delta)}, \quad r = \frac{2ip_1 \exp\chi_1}{1+\exp(2\chi_1+2\delta)}, \tag{109}$$

where p_1, q_1 and δ are constants and

$$\chi_1 = 2p_1\xi + 2q_1\eta - \frac{m_1}{2q_1}t - \frac{m_2}{2p_1} + \chi_1^{(0)}, \quad \chi_1^{(0)}\text{: consts.} \tag{110}$$

Then we have

$$\theta(\xi,\eta,t) = 4\tan^{-1}[\exp(\chi_1+\delta)]. \tag{111}$$

Taking m_1 and m_2 in the form

$$m_1(\eta,t) = m_1^0 + m_1'(\eta'), \quad \eta' = \eta + v_1 t \tag{112a}$$

$$m_2(\xi,t) = m_2^0 + m_1'(\xi'), \quad \xi' = \xi + v_2 t \tag{112b}$$

one ultimately obtains the $(1,1)$ dromion solution [30, 37]

$$\theta = 4\tan^{-1}\left\{\frac{\zeta\exp[-(p_1\xi' + q_1\eta')]}{(1+\exp[-2(p_1\xi'-\delta_1)])(1+\exp[-2(q_1\eta'-\delta_2)])}\right\}. \tag{113}$$

6.6 *The* $(2+1)$ *dimensional long dispersive wave (2DLW) equation*

Considering the 2DLDW equation (32), we find that again the P-property is true and that with the transformation

$$q = \frac{g}{\phi}, \quad r = \frac{h}{\phi}, \quad v = \frac{\partial^2}{\partial x^2}\log\phi \tag{114}$$

it can be bilinearized in the form

$$(\lambda D_t + D_x^2) g\cdot\phi = 0, \tag{115a}$$

$$(\lambda D_t - D_x^2) h\cdot\phi = 0, \tag{115b}$$

$$D_x D_\eta \phi\cdot\phi = -2gh. \tag{115c}$$

Analysis of (115) leads to the line soliton solutions of the form

$$q = \frac{\exp(-p_1 x + s_1\eta - (p_1^2/\lambda)t + c_1)}{1+\exp(-[p_1+p_1']x+[s_1+s_1']\eta+(1/\lambda)[p_1'^2-p_1^2]t+c+2\psi)}, \tag{116a}$$

$$r = \frac{\exp(-p_1' x + s_1'\eta + (p_1'^2/\lambda)t + c_1')}{1+\exp(-[p_1+p_1']x+[s_1+s_1']\eta+(1/\lambda)[p_1'^2-p_1^2]t+c+2\psi)}, \tag{116b}$$

while the potential v is described by the line soliton

$$v=\frac{-(p_1+p_1')^2}{4}\,\mathrm{sech}^2\tfrac{1}{2}(\chi_1+\chi_1'+2\psi), \tag{116c}$$

where

$$\chi_1=-p_1x+s_1\eta-\frac{p_1^2}{\lambda}t+c_1,\quad \chi_1'=-p_1'x+s_1'\eta+\frac{p_1'^2}{\lambda}t+c_1',$$

$$\exp(2\psi)=\frac{1}{(p_1+p_1')(s_1+s_1')}. \tag{116d}$$

We also note that the composite field is given by

$$Q=qr=\frac{(p_1+p_1')(s_1+s_1')}{4}\,\mathrm{sech}\tfrac{1}{2}(\chi_1+\chi_1'+2\psi). \tag{117}$$

Further in the limit $(s_1+s_1')\to 0$ (or equivalently at the boundary $\eta=-\infty$), one has the ghost soliton

$$v=\frac{(p_1+p_1')^2}{4}\,\mathrm{sech}^2\frac{1}{2}\left([p_1+p_1']x+\frac{1}{\lambda}(p_1'^2-p_1^2)t+c_1\right). \tag{118}$$

The above features indicate that it is not the physical fields q and r that will get localized, but it is rather the composite field $Q=$ "qr" which will get localized. Indeed with the choice

$$\phi=1+\exp\chi_1+\exp\chi_2+K\exp(\chi_1+\chi_2), \tag{119a}$$

$$\chi_1=p_1x-\frac{p_1^2t}{2}+c_1,\quad \chi_2=s_1\eta+c_2, \tag{119b}$$

one obtains the (1,1) dromion for the composite field as

$$Q=\text{“}qr\text{”}=\int_{-\infty}^{x}v_\eta\,\mathrm{d}x'=\frac{(1-K)p_1s_1\exp(\chi_1+\chi_2)}{[1+\exp(\chi_1)+\exp(\chi_2)+K\exp(\chi_1+\chi_2)]^2},\quad K>0. \tag{120}$$

More general $(1,N)$ dromions can be given by

$$(qr)_{1N}=\frac{(1-K)p_1\exp(\chi_1)\sum_{i=1}^{N}s_i\exp(\chi_{i+1})}{[1+\sum_{j=1}^{N+1}\exp(\chi_j)+K\exp(\chi_1\sum_{i=1}^{N}\exp(\chi_{i+1}))]^2}. \tag{121}$$

Again, it should be mentioned that one can indeed generate induced localized solutions for the composite field by utilising the arbitrary function $f(\eta)$ present in the system.

7. Conclusions

In this paper, we have investigated the type of basic nonlinear coherent structures that can arise in a class of nonlinear evolution equations in $(2+1)$ dimensions. In contrast to the ordinary solitons in $(1+1)$ dimensions, it was pointed out that much richer structures can arise in $(2+1)$ dimensions: line solitons, lump solitons, exponentially localized dromions and their generalizations and so on. In particular, we have discussed in detail certain algorithmic methods of obtaining the localized dromion-like solutions, either in the field variables or in the potential variables or even in the composite variables. In all these cases, the relevance of boundary contributions to form localized structures was stressed.

The necessity of boundary contributions to form exponentially localized solutions also justifies the type of NLEEs we have discussed here, namely all of them have nonlocal terms or equivalently extra fields associated with them. Simple minded $(2+1)$ dimensional (local) generalizations of $(1+1)$ dimensional soliton equations do not seem to admit exact analytical localized structures in general. More work needs to be carried out to understand the basic structures in such systems and the analysis of such systems will constitute an important area of development in nonlinear dynamics in future.

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