A (2+1) dimensional integrable spin model: Geometrical and gauge equivalent counterpart, solitons and localized coherent structures

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Abstract

A non-isospectral (2+1) dimensional integrable spin equation is investigated. It is shown that its geometrical and gauge equivalent counterparts is the (2+1) dimensional nonlinear Schrödinger equation introduced by Zakharov and studied recently by Strachan. Using a Hirota bilinearised form, line and curved soliton solutions are obtained. Using certain freedom (arbitrariness) in the solutions of the bilinearised equation, exponentially localized dromion-like solutions for the potential is found. Also, breaking soliton solutions (for the spin variables) of the shock wave type and algebraically localized nature are constructed.

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The nature of (1+1) dimensional integrable systems is now well understood [1]. On the other hand, examples of (2+1) dimensional integrable equations solvable by Inverse Scattering Transform method (IST) are fewer in number and such systems are being actively investigated from different points of view at present [1,2]. An interesting subclass of integrable systems, useful both from the mathematical and physical points of view, is the set of integrable spin systems. Since the identification of the first integrable spin model twenty years ago, namely the continuum isotropic Heisenberg spin systems [3,4], several other integrable spin systems in (1+1) dimensions have been identified and investigated (see for example, refs.[5,6]) through geometrical and gauge equivalence concepts and IST method.

Again in (2+1) dimensions, only a small number of integrable spin systems are known, among which the Ishimori equation (IE) is the most prominent one admitting different kinds of spin excitations such as solitons, vortices, dromions and so on[2]. Since the spin equations can also be considered as the nonrelativistic version of O(3) sigma models, which in turn have wide physical ramifications in (2+1) dimensions in problems such as high T_c superconductors and quark confinement, construction of any integrable spin model in higher dimensions assumes considerable significance. Moreover, since the spin systems have a natural connection with nonlinear Schrödinger family of equations in general, it is relevent from the soliton theory point of view to establish such connections in (2+1) dimensional cases also (as in the case of IE and Davey -Stewartson equation).

Recently a new family of integrable and nonintegrable (but admitting exact solitary wave solutions) (N+1) dimensional (N = 1,2) classical spin models was proposed in refs[7-9]. One interesting nonlocal integrable spin model is (the so called M-I equation)[7]

$$\vec{S}_t = \{\vec{S} \wedge \vec{S}_y + u\vec{S}\}_x, \tag{1a}$$

$$u_x = -\vec{S} \cdot \left(\vec{S}_x \wedge \vec{S}_y\right). \tag{1b}$$

Here, subscripts stand for partial derivatives, $\vec{S} = (S_1, S_2, S_3)$ and $\vec{S}^2 = S_3^2 + r^2(S_1^2 + S_2^2) = 1$, $r^2 = \pm 1$. As in the case of IE, the quantity

$$Q = \frac{1}{4\pi}\int dxdy\vec{S}\cdot\left(\vec{S}_x\wedge\vec{S}_y\right) \tag{2}$$

may be called the topological charge. Both eq.(1) as well as IE in the (1+1) dimensional case reduce to one and the same equation - the well known (1+1) dimensional isotropic classical continuous Heisenberg ferromagnet model[3]. The Lax representation of eq.(1) was given in ref.[7] and some of its properties were studied in refs.[8-11]. The aim of this letter is to find the equivalent (geometrical and gauge) counterpart namely the (2+1) dimensional nonlinear Schrödinger equation(NLSE) and to obtain physically interesting solutions such as line and curved solitons, exponentially localized and breaking solitons of eq.(1).

To begin with, let us find the geometrically equivalent counterpart of eq.(1), for $r^2 = 1$, that is, when $\vec{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1$. For this purpose, we will extend the geometrical method applicable to (1+1) dimensional systems suitably to the (2+1) dimensional case. We associate a moving space curve parametrised by the arclength x, and endowed with an additional coordinate y with the spin system[3,15,16]. Then the Serret-Frenet equation associated with the curve has the form

$$\vec{e}_{ix} = \vec{D}\wedge\vec{e}_i, \tag{3a}$$

where

$$\vec{D} = \tau\vec{e}_1 + \kappa\vec{e}_3 \tag{3b}$$

and $\vec{e}_i$'s, $i = 1, 2, 3$ form the orthogonal trihedral. Mapping the spin on the unit tangent vector

$$\vec{S}(x, y, t) = \vec{e}_1, \tag{4}$$

the curvature and the torsion are given by

$$\kappa(x, y, t) = (\vec{S}_x^2)^{\frac{1}{2}}, \tag{5a}$$

$$\tau(x, y, t) = \kappa^{-2}\vec{S}\cdot(\vec{S}_x\wedge\vec{S}_{xx}). \tag{5b}$$

Due to the orthonormality nature of the trihedral, $\vec{e}_{it}.\vec{e}_i = 0$, $\vec{e}_{iy}.\vec{e}_i = 0$, $i, j = 1, 2, 3$ and using the compatibility condition $\vec{e}_{ixy} = \vec{e}_{iyx}$, we find the equation for the y-part

$$\vec{e}_{iy} = \vec{\gamma} \wedge \vec{e}_i, \tag{6}$$

where $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ and

$$\gamma_1 = u + \partial_x^{-1}\tau_y, \tag{7a}$$

$$\gamma_2 = -\frac{u_x}{\kappa}, \tag{7b}$$

$$\gamma_3 = \partial_x^{-1}\left(\kappa_y - \frac{\tau u_x}{\kappa}\right). \tag{7c}$$

Now, from eq.(1) and using eqs.(3) and (6), we can easily find the time evolution of the trihedral

$$\vec{e}_{it} = \vec{\omega} \wedge \vec{e}_i, \tag{8}$$

with

$$\vec{\Omega} = (\omega_1, \omega_2, \omega_3) = \left(\frac{\kappa_{xy}}{\kappa} - \tau\partial_x^{-1}\tau_y, -\kappa_y, -\kappa\partial_x^{-1}\tau_y\right). \tag{9}$$

Ultimately, the compatibility condition $\vec{e}_{ixt} = \vec{e}_{itx}$, which is also consistent with the relation $\vec{e}_{iyt} = \vec{e}_{ity}$, $i = 1, 2, 3$ yields the following evolution equations for the curvature and torsion

$$\kappa_t = -(\kappa\tau)_y - \kappa_x\partial_x^{-1}\tau_y, \tag{10a}$$

$$\tau_t = \left[\frac{\kappa_{xy}}{\kappa} - \tau\partial_x^{-1}\tau_y\right]_x + \kappa\kappa_y. \tag{10b}$$

On making the complex transformation[3],

$$\psi(x, y, t) = \frac{\kappa(x, y, t)}{2}\exp\left[-i\int_{-\infty}^{x}\tau(x', y, t)dx'\right], \tag{11}$$

the set of equations (10) reduce to the following (2+1) dimensional nonlinear Schrödinger equation(NLSE)

$$i\psi_t = \psi_{xy} + r^2 V\psi, \tag{12a}$$

$$V_x = 2\partial_y \mid \psi \mid^2. \tag{12b}$$

Here, $r^2 = +1$, that is, we have the attractive type NLSE (The case $r^2 = -1$ corresponds to the repulsive case). Eq.(12) was originally introduced by Zakharov[12] and was recently rederived by Strachan (for $r^2 = +1$)[13]. Its Painlevé property and some exact solutions were also obtained [14]. N- soliton solutions of eq.(12) for both the cases ($r^2 = \pm 1$) can be found in ref.[17]. Here, we have proved that eq.(12) is equivalent to eq.(1) in the geometrical sense.

Next, it is always of interest to note that eqs. (1) and (12) are also gauge equivalent in the sense of Zakharov and Takhtajan[18]. To this end, we write the Lax representation of eq.(1)[7],

$$\phi_{1x} = U_1\phi_1, \tag{13a}$$

$$\phi_{1t} = V_1\phi_1 + \lambda\phi_{1y}, \tag{13b}$$

where

$$U_1 = \frac{i\lambda}{2}S, \quad S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \tag{14a}$$

$$V_1 = \frac{\lambda}{4}\left([S, S_y] + 2iuS\right), \quad S^{\pm} = S_1 \pm iS_2. \tag{14b}$$

Here, λ is the eigen value parameter which satisfies the following equations of Riemann wave type

$$\lambda_t = \lambda\lambda_y. \tag{15}$$

It means that for solving eq.(1), we must in general use the non-isospectral IST. To obtain gauge equivalent counterpart of eq.(1), in the usual way we consider the following gauge transformation

$$\phi_1 = g^{-1}\phi_2, \tag{16}$$

where $g(x,y,t)$ and $\phi_2(x,y,t,\lambda)$ are temporarily arbitrary matrix functions. Substituting eq.(16) into eq.(13), after some algebra we get the following system of linear equations for ϕ_2

$$\phi_{2x} = U_2\phi_2, \tag{17a}$$

$$\phi_{2t} = V_2\phi_2 + \lambda\phi_{2y} \tag{17b}$$

with

$$U_2 = \frac{i\lambda}{2}\sigma_3 + G, \quad G = \begin{pmatrix} 0 & \phi \\ -r^2\phi & 0 \end{pmatrix}, \tag{18a}$$

$$V_2 = i\sigma_3\left(\frac{VI}{2} - G_y\right), \quad I = diag(1,1), \tag{18b}$$

$$V = 2\partial_x^{-1}\partial_y\left(\mid \psi \mid^2\right) \tag{18c}$$

The compatibility condition of eq.(17) with (15) becomes (12), that is, eq.(1) and eq.(12) are gauge equivalent to each other. The above transformation is in fact reversible and we can similarly prove that eq.(12) is gauge equivalent to eq.(1).

The integrable eq.(1) allows an infinite number of integrals of motion. The first two conservation laws are

$$\left(\vec{S}_x^2\right)_t + \frac{1}{4}\partial_x\left[\vec{S}_x^2\partial_x^{-1}\left(\frac{\vec{S}\cdot\vec{S}_x\wedge\vec{S}_{xx}}{\vec{S}_x^2}\right)_y\right] + \frac{1}{4}\partial_y\left[\vec{S}\cdot\vec{S}_x\wedge\vec{S}_{xx}\right] = 0, \tag{19a}$$

$$\left[\vec{S}\cdot\left(\vec{S}_x\wedge\vec{S}_{xx}\right)\right]_t+\frac{1}{2}\partial_x\left[\frac{(\vec{S}_x^2)_x(\vec{S}_x^2)_y}{4}+\vec{S}\cdot\left(\vec{S}_x\wedge\vec{S}_{xx}\right)\partial_x^{-1}\left(\vec{S}\cdot\left(\vec{S}_x\wedge\vec{S}_{xx}\right)\right)_y\right]$$

$$+\partial_y\left\{(\vec{S}_x^2)_x^2+\frac{2\left(\vec{S}\cdot\left(\vec{S}_x\wedge\vec{S}_{xx}\right)\right)^2}{\vec{S}_x^2}-2\left(\vec{S}_x^2\right)_{xx}-4\vec{S}_x^4\right\}=0 \tag{19b}$$

and so on. Next, we present some important formulae which are just consequences of geometrical/gauge equivalence of eqs. (1) and (12). We have

$$tr(S_x^2)=8\mid\psi\mid^2=2\vec{S}_x^2. \tag{20a}$$

In a similar manner we find that

$$-2i\vec{S}\cdot(\vec{S}_x\wedge\vec{S}_{xx})=tr(S_xSS_{xx})=4(\psi\psi_x^*-\psi^*\psi_x). \tag{20b}$$

These relations are obviously equivalent to eq.(11). One may note that these are of the same form as in the case of (1+1) dimensional Heisenberg chain [3].

Now, we wish to find a class of exact solutions of eq.(1), such as line and curved solitons as well as exponentially localized solutions. Introducing the stereographic variable

$$S^+=S_1+iS_2=\frac{2\omega}{1+\mid\omega\mid^2},\quad S_3=\frac{1-\mid\omega\mid^2}{1+\mid\omega\mid^2}, \tag{21}$$

eq.(1) takes the form

$$i\left(\omega_t-u\omega_x\right)+\omega_{xy}-\frac{2\omega^*\omega_x\omega_y}{(1+\mid\omega\mid^2)}=0, \tag{22a}$$

$$u_x+\frac{2i(\omega_x\omega_y^*-\omega_x^*\omega_y)}{(1+\mid\omega\mid^2)^2}=0. \tag{22b}$$

On writing

$$\omega=\frac{g}{f}, \tag{23}$$

where g and f are complex valued functions, and after using the Hirota's D-operators, eq.(22) becomes

$$(iD_t - D_x D_y)(f^* \circ g) = 0, \tag{24a}$$

$$(iD_t - D_x D_y)\left(f^* \circ f - g^* \circ g\right) = 0, \tag{24b}$$

$$D_x\left(f^* \circ f + g^* \circ g\right) = 0, \tag{24c}$$

while the potential $u(x,y,t)$ is

$$u(x,y,t) = -i\frac{D_y\left(f^* \circ f + g^* \circ g\right)}{f^* \circ f + g^* \circ g}. \tag{24d}$$

In terms of g and f, the spin field $\vec{S}$ takes the form

$$S^+ = \frac{2f^* g}{\mid f \mid^2 + \mid g \mid^2}, \quad S_3 = \frac{\mid f \mid^2 - \mid g \mid^2}{\mid f \mid^2 + \mid g \mid^2} \tag{25}$$

and for $u(x,y,t)$ we get the following formula (using the properties of D-operators)

$$u_x(x,y,t) = -2i\partial^2_{xy} \ln\left(\mid f \mid^2 + \mid g \mid^2\right). \tag{26}$$

The construction of the solutions to the M-I equation (1) now becomes standard. One expands the functions f and g as a power series in the arbitrary parameter ϵ,

$$g = \sum_{n=0}^{\infty} \epsilon^{2n+1} g_{2n+1}, \quad f = 1 + \sum_{n=1}^{\infty} \epsilon^{2n} f_{2n}. \tag{27}$$

Substituting these expansions into (24 a,b,c) and equating the coefficients of ϵ^n yields

$$\left[i\partial_t + \partial_x \partial_y\right] g_{2n+1} = -\sum_{k+m=n} D'(f^*_{2x} \circ g_{2m+1}), \tag{28a}$$

$$\left[i\partial_t - \partial^2_{xy}\right](f^*_{2n} - f_{2n}) = D'\left(\sum_{n_1+n_2=n-1} g^*_{2n_1+1} \circ g_{2n_2+1} - \sum_{m_1+m_2=n} f^*_{2m_1} \circ f_{2m_2}\right), \tag{28b}$$

$$\partial_x(f^*_{2n} - f_{2n}) = D_x\left(\sum_{n_1+n_2=n-1} g^*_{2n_1+1} \circ g_{2n_2+1} - \sum_{n_1+n_2=n} f^*_{2n_1} \circ f_{2n_2}\right), \tag{28c}$$

with $D' = iD_t - D_x D_y$, $f_0 = 0$. In order to construct exact N-soliton (line and curved) solutions (N-SS) of eq.(1), we make the ansatz

$$g_1 = \sum_{j=1}^{N} \exp \chi_j, \quad \chi_j = \lambda_j x + m_j(y,t) + c_j. \tag{29}$$

We note here the important fact that $m_j(y,t)$ is an arbitrary complex function of (y,t) of the form (see eq.(28a))

$$m_j(y,t) = m_j(\rho), \quad \rho = y + i\lambda_j t, \tag{30}$$

where λ_j is an arbitrary complex parameter. As an example, we write the forms of g and f for N=1 as

$$g_1 = \exp \chi_1, \quad f_2 = \exp 2(\chi_{1R} + \psi), \tag{31}$$

where

$$\chi_1 = \chi_{1R} + i\chi_{1I}, \lambda_1 = \eta + i\xi, m_1 = m_{1R}(\rho) + im_{1I}(\rho), \chi_{1R} = \eta x + m_{1R}(\rho) + c_{1R},$$
$$\chi_{1I} = \xi x + m_{1I}(\rho) + c_{1I},\ c = ln(2\eta/\lambda_1^*),\ \exp 2\psi = \frac{-\lambda_1^2}{(\lambda_1 + \lambda_1^*)^2},$$
$$m_{1R}(\rho) = Rem_1(\rho), m_{1I}(\rho) = Imm_1(\rho). \tag{32}$$

The corresponding 1-SS of eq.(1) takes the form

$$S_3(x,y,t) = 1 - \frac{2\eta^2}{\eta^2 + \xi^2} sech^2 \chi_{1R}, \tag{33a}$$

$$S^+(x,y,t) = \frac{2\eta}{\eta^2 + \xi^2} \left[i\xi - \eta tanh \chi_{1R} \right] sech \chi_{1R}, \tag{33b}$$

while for the potential

$$u(x,y,t) = \frac{2\eta}{\eta^2 + \xi^2} \left(\xi m'_{1R} - \eta m'_{1I} \right) sech^2 \chi_{1R} \tag{33c}$$

in which the prime has been used to denote the differentiation with respect to the real part of the arguement. We note that the 1-SS(33) depends on two arbitrary functions $m_{1R}(\rho)$ and $m_{1I}(\rho)$ as in the case of some other (2+1) dimensional integrable equations.

Now for the particular choice,

$$m_1 = k_1(y + i\lambda_1 t),$$

where k_1 is a complex constant, in eq.(33), we get the usual line 1-SS of eq.(1). For fixed (y,t), it follows from (33) that $\vec{S} \to (0,0,1)$ as $x \to \pm\infty$ and the wavefront itself is defined by the equation $\chi_{1R} = \eta x + m_{1R}(\rho) + c_{1R} = \eta x + k_1 y - \xi t + c_{1R} = 0$. For other choices of m_1, we can obtain more general solutions. Particularly, we present the dromion type localized solutions of eq.(1), the so-called induced localized structures/or induced dromions[14] for the potential $u(x,y,t)$. This is possible by utilising the freedom in the choice of the arbitrary functions m_{1R} and m_{1I}. For example, if we make the ansatz

$$m_{1I}(\rho) = \kappa m_{1R}(\rho) = tanh(\rho_R), \tag{34}$$

$$u = 2\eta(\xi - \eta\kappa) sech^2 \rho_R sech[\eta x + tanh \rho_R - \eta x_0], \tag{35}$$

where $\rho_R = y - \xi t$ and k is a real constant. Similarly, the expressions for the spin can be obtained from eqs.(33). The solution (35) for $u(x,y,t)$ decays exponentially in all the directions, eventhough the spin $\vec{S}$ itself is not fully localized though bounded. Analogously we can construct another type of "induced dromion" solution with the choice

$$m_{1I} = \kappa m_{1R} = \int \frac{d\rho_R}{(\rho_R + \rho_0)^2 + 1} + m_0, \tag{36}$$

where ρ_0 and m_0 are constants, so that

$$u(x,y,t) = \frac{2\eta(\xi - \kappa y)}{(\rho_R + \rho_0)^2 + 1} sech^2 \left[\eta x + \int \frac{d\rho_R}{(\rho_R + \rho_0)^2 + 1} - \eta x_0\right]. \tag{37}$$

Generalizations of these solutions are also possible, which will be considered elsewhere.

Finally, we note here that we have a non-isospectral problem, as the spectral parameter λ satisfies eq.(15). The above presented solutions correspond to the constant solution of eq.(15), that is $\lambda = \lambda_1 =$ constant. One may consider other interesting solutions of eq.(15). For example, one can have a special solution

$$\lambda = \lambda_1 = \eta(y,t) + i\xi(y,t) = \frac{y+k+i\eta}{b-t}, \tag{38}$$

where b, k and η are real constants. Corresponding to this case, we may call the solutions of eqs. (1) and (12) as breaking solitons[19]. Using the Hirota method, one can also construct the breaking 1-SS of eq.(1) associated with (38). For this purpose, we take g_1 in the form

$$g = g_1 = \exp\chi, \quad \chi = ax + m + c = \chi_R + i\chi_I, \tag{39}$$

where $a = a(y,t)$, $m = m(y,t)$ and $c = c(t)$ are functions to be determined. Substituting (39) into the first of eq.(28a), we get

$$ia_t + aa_y = 0, \quad im_t + am_y = 0, \quad iA_t + Aa_y = 0, \tag{40}$$

where $A = \exp(c)$. Particular solutions of eqs.(40) have the forms

$$a = -i\lambda = \frac{\eta - i(y+k)}{b-t}, \quad m = m\left(\frac{y+k+i\eta}{b-t}\right), \quad A = \frac{A_0}{b-t}, \tag{41}$$

where η, k, b and A_0 are some constants. From eqs. (28 b,c), we obtain

$$f_2 = B\exp 2\chi_R, \quad B = \frac{|A_0|^2 (y+k+i\eta)^2}{4\eta^2(b-t)^2}. \tag{42}$$

Now, we can write the breaking 1-SS of eq.(1) (using equations (25), (39)-(42))

$$S^+(x,y,t) = \frac{2\eta\exp i(\chi_I+\phi)(y+k-i\eta)}{[(y+k)^2+\eta^2]^{\frac{3}{2}}}\frac{[(y+k)cosh z - i\eta sinh z]}{cosh^2 z}, \tag{43a}$$

$$S_3(x,y,t) = 1 - \frac{2\eta^2}{[(y+k)^2+\eta^2]}sech^2 z, \tag{43b}$$

where $z = \frac{\eta}{(b-t)}x - \frac{1}{2}\ln[(y+k)^2 - \eta^2] + \psi$, $\psi = \ln \mid \frac{A_0(y+k+i\eta)}{2\eta(b-t)} \mid$, $\chi_I = -\frac{(y+k)}{(b-t)}x + m_I\left(\frac{y+k+i\eta}{b-t}\right)$. We see that the solution (43) corresponds to an algebraically decaying solution for large x, y.

Finally, we note that eq.(1) is a particular case of the following family of (2+1) dimensional equation

$$\vec{S}_t = \{\vec{S} \wedge \vec{S}_y + u\vec{S}\}_x + \vec{F}, \tag{44}$$

where u and $\vec{F}$ satisfy the eq.(1b) and $\vec{S} \cdot \vec{F} = 0$ respectively. Eq.(44) admits many integrable reductions, for example,

a) the isotropic M-I eq.(1), when $\vec{F} = 0$;

b) the anisotropic M-I equation, when $\vec{F} = \vec{S} \wedge A\vec{S}$, where $A = diag(a_1, a_2, a_3)$, and

c) the M-II equation, when $\vec{F} = mv\vec{S}_x + n\vec{S}_y$, where $v_x = k(\vec{S}_x^2)_y$, m, k, n are constants; and so on. All of these equations are integrable in the sense that the corresponding Lax representations exist[7] and their gauge equivalent counterparts can be constructed[10](see also [20]-[21]). So further studies of them will give more insight into the structure of nonlinear spin excitations in (2+1) dimensions.

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