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On the complete integrability and linearization of nonlinear ordinary differential equations. II. Third-order equations

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We introduce a method for finding general solutions of third-order nonlinear differential equations by extending the modified Prelle–Singer method. We describe a procedure to deduce all the integrals of motion associated with the given equation, so that the general solution follows straightforwardly from these integrals. The method is illustrated with several examples. Further, we propose a powerful method of identifying linearizing transformations. The proposed method not only unifies all the known linearizing transformations systematically but also introduces a new and generalized linearizing transformation. In addition to the above, we provide an algorithm to invert the non-local linearizing transformation. Through this procedure the general solution for the original nonlinear equation can be obtained from the solution of the linear ordinary differential equation.

Keywords: integrability; integrating factor; linearization; equivalence problem

1. Introduction

In a previous paper (Chandrasekar *et al.* 2005) we have discussed the complete integrability aspects of a class of second-order nonlinear ordinary differential equations (ODEs) through a non-trivial extension of the so-called Prelle–Singer (PS) (Prelle & Singer 1983; Duarte *et al.* 2001) procedure. We have illustrated the procedure with several physically interesting nonlinear oscillator examples. We have also developed a straightforward algorithmic way to transform the given second-order nonlinear ODE to a linear free particle equation, if it is linearizable.

One of the questions raised at the final stage of our earlier work (Chandrasekar *et al.* 2005) was what are the implications of the novel features which we introduced in the extended PS procedure to obtain the second constant of motion (in the case of second-order ODEs) to third and higher-order ODEs. To have a closer look at the problem, let us recall our earlier work briefly here. We considered a second-order ODE of the form $\mathrm{d}^2x/\mathrm{d}t^2 = P(t, x, \dot{x})/Q(t, x, \dot{x})$, $P, Q \in \mathbb{C}[t, x, \dot{x}]$, and explored two pairs of independent functions, say, R_i and S_i, $i=1, 2$, associated with the underlying ODE. These functions are nothing but

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the integrating factors and null forms, respectively. Once these two pairs of functions are determined (by solving an overdetermined system of first-order partial differential equations (PDEs)), then each pair leads to an independent integral of motion, which can then be used to find the general solution for the given equation. Thus, instead of integrating the first integral and obtaining the general solution which is conventionally followed in the literature, we implemented some novel ideas in the PS method, such that one can construct the general solution for the given equation in a self-contained way and, in fact, our procedure works for a class of problems.

In the case of third-order ODEs, one should have three independent integrals of motion in order to establish the complete integrability. To deduce these three integrals, one should have three pairs of independent functions (R_i, U_i and S_i), $i=1, 2, 3$. When we extend the PS procedure to third-order ODEs, we find that the determining equations for the integrating factors and null forms straightforwardly provide either one or two integrals of motion only. Again the hidden form of the functions (R_3, U_3, S_3) should be explored in order to establish the complete integrability of the given equation within the framework of PS procedure. In this paper we describe a procedure to capture the required set of functions. *With the completion of this task we formulate a simple, straightforward and powerful method to solve a wide class of third-order ODEs of contemporary literature.*

We stress at this point that the application of PS procedure to third-order ODEs is not a straightforward extension of the second-order case. In fact, one has to overcome many faceted problems. The first and foremost one is how to solve the determining equations in such a way that one could obtain three sets of independent functions, namely, (R_i, U_i ,S_i), $i=1, 2, 3$, in a systematic way. In the present case we have six equations for three unknown functions (in the case of second-order equations we have three equations for two unknowns). We overcome this problem by adopting suitable methodologies, the details of which we present in §3. Another obstacle one could face in higher-order ODEs, at least in some cases, is that one may be able to get only one integral of motion and, in this situation, how one would be able to generate the remaining integrals of motion from the first integral is also tackled by us in this paper.

Our main goal, besides the above, is to bring out a novel and straightforward way to construct linearizing transformations for third-order ODEs. The latter can be used to transform the given third-order nonlinear ODEs to a linear equation. We note that unlike the second-order equations, the third-order ODEs can be linearized through different kinds of transformations, namely, invertible point transformation (Ibragimov & Meleshko 2005), contact transformation (Bocharov *et al.* 1993; Ibragimov & Meleshko 2005) and generalized Sundman transformation (Berkovich & Orlova 2000; Euler *et al.* 2003; Euler & Euler 2004).

In this paper we introduce a new kind of transformation, which can be effectively used to linearize a class of nonlinear third-order ODEs. In fact, one can linearize certain equations only through this transformation alone and not by the known ones in the literature. We call this transformation generalized linearizing transformation (GLT). We note that generalized Sundman transformation is a special case of this transformation. In the generalized Sundman transformation, the new independent variable is a non-local one, and so even though one is able to transform the given nonlinear third-order ODE to a linear one, due to the nature of the non-local independent variable, it is not easy to write down the general solution. In the case of GLT, both the new dependent and independent variables also contain derivative terms in addition to the independent variable being non-local. Even for this general case, in this paper, we succeed in presenting an efficient algorithm to deduce the general solution.

Another fundamental problem regarding linearization is how to deduce the linearizing transformations systematically. Generally, Lie symmetry analysis and direct methods are often used to deduce the point and contact transformations (Steeb 1993; Bocharov *et al.* 1993; Olver 1995; Bluman & Anco 2002; Ibragimov & Meleshko 2005). In this work we propose a simple and straightforward method to deduce linearizing transformations and we derive them from the first integral. Our method of deducing linearizing transformations has several salient features. Irrespective of the form of the linearizing transformation (point/contact/generalized Sundman transformation), it can be derived from the first integral itself. We also note that one can also linearize a third-order ODE to the second-order free particle equation through our method. An added advantage of our method is that suppose a given equation is linearizable through one or more kinds of transformations, then our procedure provides all these transformations in a straightforward way and as far as our knowledge goes *no such single method has been formulated in the literature*.

The plan of the paper is as follows. In §2, we extend the PS procedure to third-order ODEs and indicate new features in finding the three independent integrals of motion. In §3, we describe the methods of solving the determining equations and how one can obtain compatible solutions from them. We illustrate the procedure with several examples. In §4, we propose a powerful method of identifying linearizing transformations. This method not only brings out all the known transformations systematically, but also a new GLT for the third-order ODEs. We emphasize the validity of the method with several illustrative examples arising in different areas of mathematics and physics in §5. We present our conclusions in §6.

2. Prelle–Singer method for third-order ODEs

(*a*) *General theory*

Let us consider a class of third-order ODEs of the form

$$\dddot{x} = \frac{P}{Q}, \quad P, Q \in \mathbb{C}[t, x, \dot{x}, \ddot{x}] \left(\cdot = \frac{\mathrm{d}}{\mathrm{d}t} \right), \tag{2.1}$$

where the overdot denotes differentiation with respect to time and P and Q are polynomials in t, x, $\dot{x}$ and $\ddot{x}$ with coefficients in the field of complex numbers, $\mathbb{C}$. Let us assume that the third-order ODE (2.1) admits a first integral $I(t, x, \dot{x}, \ddot{x}) = C$, with C being constant on the solutions, so that the total differential of I gives

$$\mathrm{d}I = I_t \, \mathrm{d}t + I_x \, \mathrm{d}x + I_{\dot{x}} \, \mathrm{d}\dot{x} + I_{\ddot{x}} \, \mathrm{d}\ddot{x} = 0, \tag{2.2}$$

where each subscript denotes partial differentiation with respect to that variable. Equation (2.1) can be rewritten as $(P/Q)\mathrm{d}t - \mathrm{d}\ddot{x} = 0$. Now adding the null terms

$U(t,x,\dot{x},\ddot{x})\ddot{x}\,\mathrm{d}t - U(t,x,\dot{x},\ddot{x})\mathrm{d}\dot{x}$ and $S(t,x,\dot{x},\ddot{x})\dot{x}\,\mathrm{d}t - S(t,x,\dot{x},\ddot{x})\mathrm{d}x$ to this we obtain that, in the solutions, the one-form

$$\left(\frac{P}{Q} + S\dot{x} + U\ddot{x}\right)\mathrm{d}t - S\,\mathrm{d}x - U\,\mathrm{d}\dot{x} - \mathrm{d}\ddot{x} = 0. \tag{2.3}$$

Looking at equations (2.2) and (2.3) one can conclude that, in the solutions, these two forms are proportional and the form of equation (2.3) is equivalent to equation (2.2), except for an overall multiplication factor. Thus, multiplying equation (2.3) by the factor $R(t,x,\dot{x},\ddot{x})$ which acts as the integrating factor for (2.3), we have in the solutions that

$$\mathrm{d}I = R(\phi + S\dot{x} + U\ddot{x})\mathrm{d}t - RS\,\mathrm{d}x - RU\,\mathrm{d}\dot{x} - R\,\mathrm{d}\ddot{x} = 0, \tag{2.4}$$

where $\phi \equiv P/Q$. Comparing equation (2.2) with (2.4) we have the following relations in the solutions:

$$I_t = R(\phi + S\dot{x} + U\ddot{x}), \quad I_x = -RS, \quad I_{\dot{x}} = -RU, \quad I_{\ddot{x}} = -R. \tag{2.5}$$

Now imposing the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{t\ddot{x}} = I_{\ddot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, $I_{x\ddot{x}} = I_{\ddot{x}x}$, $I_{\dot{x}\ddot{x}} = I_{\ddot{x}\dot{x}}$, which exist between equations (2.5), we have the following equations which constitute three determining equations ((2.6)–(2.8)) for the functions S, U and R along with three constraints ((2.9)–(2.11)) that they need to satisfy:

$$\mathrm{D}[S] = -\phi_x + S\phi_{\ddot{x}} + US, \tag{2.6}$$

$$\mathrm{D}[U] = -\phi_{\dot{x}} + U\phi_{\ddot{x}} - S + U^2, \tag{2.7}$$

$$\mathrm{D}[R] = -R(U + \phi_{\ddot{x}}), \tag{2.8}$$

$$R_x = R_{\ddot{x}}S + RS_{\ddot{x}}, \tag{2.9}$$

$$R_{\dot{x}}S = -RS_{\dot{x}} + R_xU + RU_x, \tag{2.10}$$

$$R_{\dot{x}} = R_{\ddot{x}}U + RU_{\ddot{x}}, \tag{2.11}$$

where

$$\mathrm{D} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \phi\frac{\partial}{\partial \ddot{x}}. \tag{2.12}$$

The task of solving equations (2.6)–(2.11) can be accomplished in the following way. Substituting the given expression of ϕ into (2.6) and (2.7) and solving them one can obtain expressions for S and U. With the known U, equation (2.8) becomes the determining equation for the function R. Solving the latter one can get an explicit form for R. Compatible solutions to equations (2.6)–(2.8) can also be obtained in alternative ways, the details of which are given in §3.

Now the functions R, U and S have to satisfy an extra set of constraints, i.e. equations (2.9)–(2.11). Suppose a compatible solution satisfying all the equations has been found, then the functions R, U and S fix the differential invariant

$I(t,x,\dot{x},\ddot{x})$ by the relation

$$I(t,x,\dot{x},\ddot{x}) = r_1 - r_2 - \int \left\{ RU + \frac{\mathrm{d}}{\mathrm{d}\dot{x}}[r_1 - r_2] \right\} \mathrm{d}\dot{x} - \int \left\{ R + \frac{\mathrm{d}}{\mathrm{d}\ddot{x}} \left[r_1 - r_2 - \int \left\{ RU + \frac{\mathrm{d}}{\mathrm{d}\dot{x}}[r_1 - r_2] \right\} \mathrm{d}\dot{x} \right] \right\} \mathrm{d}\ddot{x}, \qquad (2.13)$$

where

$$r_1 = \int R(\phi + S\dot{x} + U\ddot{x})\mathrm{d}t, \quad r_2 = \int \left(RS + \frac{\mathrm{d}}{\mathrm{d}x} \int r_1 \right) \mathrm{d}x.$$

Equation (2.13) can be derived straightforwardly by integrating equations (2.5). Here it is to be noted that for every independent set (S, U, R), equation (2.13) defines an integral.

(*b*) *Exploring the complete form of R: theory*

From the above discussion, it is clear that equation (2.1) may be considered as completely integrable once we obtain three independent sets of the solutions (S_i, U_i, R_i), $i=1, 2, 3$, which provide three independent integrals of motion through the relation (2.13). Here we note that since we are solving equations (2.6)–(2.8) first and then checking the compatibility of this solution with equations (2.9)–(2.11), one often meets the situation that all the solutions which satisfy equations (2.6)–(2.8) need not satisfy the constraints (2.9)–(2.11), since equations (2.6)–(2.11) constitute an overdetermined system for the unknowns R, S and U. In fact, for a class of problems one often gets one or two sets of S, U, R, which satisfy all equations (2.6)–(2.11) and another(other) set(s) (S, U, R), which satisfies(satisfy) only the first three equations and not the other, namely, (2.9)–(2.11). In this situation we find an interesting fact that one can use the integral(s) derived from the set(s) which satisfies(satisfy) all the six equations (2.6)–(2.11) and deduce the other compatible solution(s) (S, U, $\hat{R}$) (definition of $\hat{R}$ follows). For example, let the set (S_3, U_3, R_3) be a solution of the determining equations (2.6)–(2.8) and not of the constraints (2.9)–(2.11). After analysing several examples we find that one can make the set (S_3, U_3, R_3) compatible by modifying the form of R_3 as

$$\hat{R}_3 = F(t,x,\dot{x},\ddot{x})R_3, \qquad (2.14)$$

where $\hat{R}_3$ satisfies equation (2.8), so that we have

$$(F_t + \dot{x}F_x + \ddot{x}F_{\dot{x}} + \phi F_{\ddot{x}})R_3 + F\,\mathrm{D}[R_3] = -FR_3(U_3 + \phi_{\dot{x}}). \qquad (2.15)$$

Further, if F is a constant of motion (or a function of it), then the first term on the left-hand side vanishes and one gets the same equation (2.8) for R_3, provided F is non-zero. That is, whenever F is a constant of motion or a function of it, then the solution to (2.8) may provide only a factor of the complete solution $\hat{R}_3$ without the factor F in equation (2.14). This general form of $\hat{R}_3$ along with S_3 and U_3 can now provide a complete solution to equations (2.6)–(2.11) as discussed below.

(c) Exploring the complete form of R: method

Now if the sets (S_i, U_i, R_i), $i=1, 2$ are found to satisfy equations (2.6)–(2.11) and the third set (S_3, U_3, R_3) does not satisfy equations (2.9)–(2.11), then F may be a function of the integrals I_i, $i=1, 2$, derived from the sets (S_i, U_i, R_i), $i=1, 2$. We need to find the explicit form of $F(I_1, I_2)$ in order to obtain the compatible solution (S_3, U_3, R_3). To do so let us find the derivatives of $\hat{R}_3$ with respect to x, $\dot{x}$ and $\ddot{x}$:

$$\left.\begin{aligned}\hat{R}_{3x} &= (F_1' I_{1x} + F_2' I_{2x})R_3 + FR_{3x}, \quad \hat{R}_{3\dot{x}} = (F_1' I_{1\dot{x}} + F_2' I_{2\dot{x}})R_3 + FR_{3\dot{x}},\\ \hat{R}_{3\ddot{x}} &= (F_1' I_{1\ddot{x}} + F_2' I_{2\ddot{x}})R_3 + FR_{3\ddot{x}},\end{aligned}\right\} \tag{2.16}$$

where $F_1' = \partial F/\partial I_1$ and $F_2' = \partial F/\partial I_2$. Substituting equation (2.16) into equations (2.9)–(2.11), we have

$$\frac{(f_1 F_1' + f_2 F_2')}{f_3} = \frac{F}{R_3}, \quad \frac{(f_4 F_1' + f_5 F_2')}{f_6} = \frac{F}{R_3}, \quad \frac{(f_7 F_1' + f_8 F_2')}{f_9} = \frac{F}{R_3}, \tag{2.17a}$$

where

$$\left.\begin{aligned}&f_1 = (I_{1x} - I_{1\ddot{x}} S_3), \quad f_2 = (I_{2x} - I_{2\ddot{x}} S_3), \qquad f_3 = (S_3 R_{3\ddot{x}} + R_3 S_{3\ddot{x}} - R_{3x}),\\ &f_4 = (I_{1\dot{x}} - I_{1\ddot{x}} U_3), \quad f_5 = (I_{2\dot{x}} - I_{2\ddot{x}} U_3), \qquad f_6 = (U_3 R_{3\ddot{x}} + R_3 U_{3\ddot{x}} - R_{3\dot{x}}),\\ &f_7 = (S_3 I_{1\dot{x}} - I_{1x} U_3), \quad f_8 = (S_3 I_{2\dot{x}} - I_{2x} U_3), \; f_9 = (R_3 U_{3x} + U_3 R_{3x} - R_3 S_{3\dot{x}} - S_3 R_{3\dot{x}}).\end{aligned}\right\} \tag{2.17b}$$

Equation (2.17a) represents an overdetermined system of equations for the unknown F. A simple way to solve this equation is to uncouple it for $F_1'(=\partial F/\partial I_1)$ and $F_2'(=\partial F/\partial I_2)$ and solve the resultant equations. For example, eliminating F_2' from equation (2.17a) we obtain equations for F_1' in the form

$$\frac{R_3 F_1'}{F} = \frac{(f_3 f_5 - f_2 f_6)}{(f_1 f_5 - f_2 f_4)} = \frac{(f_3 f_8 - f_2 f_9)}{(f_1 f_8 - f_2 f_7)} = \frac{(f_6 f_8 - f_5 f_9)}{(f_4 f_8 - f_5 f_7)}. \tag{2.18}$$

On the other hand, eliminating F_1' from equation (2.17a) we arrive at equations for F_2' in the form

$$\frac{R_3 F_2'}{F} = \frac{(f_3 f_4 - f_1 f_6)}{(f_2 f_4 - f_1 f_5)} = \frac{(f_3 f_7 - f_1 f_9)}{(f_2 f_7 - f_1 f_8)} = \frac{(f_6 f_7 - f_4 f_9)}{(f_5 f_7 - f_4 f_8)}. \tag{2.19}$$

It can be easily cheeked that the compatibility of the right three expressions in equations (2.18) or (2.19) gives rise to relations which are effectively nothing but the constraint equations (2.9)–(2.11) and so no new constraint is added now. Consequently, equations (2.18) and (2.19) can be written as

$$\frac{\partial F}{\partial I_1} = g(I_1, I_2)F \text{ and } \frac{\partial F}{\partial I_2} = h(I_1, I_2)F, \tag{2.20}$$

respectively, where $g(I_1,I_2) = 1/R_3((f_3 f_5 - f_2 f_6)/(f_1 f_6 - f_3 f_4))$ and $h(I_1,I_2) = 1/R_3 \times ((f_3 f_4 - f_1 f_6)/(f_2 f_4 - f_1 f_5))$. Now we can solve equations (2.20) and obtain the form of $F(I_1, I_2)$. This is demonstrated for several examples in the following sections explicitly. Once F is known we can obtain the complete solution $\hat{R}_3$ from which, along with S_3 and U_3, the third integral I_3 can be constructed. *Thus, with the explicit forms of the three integrals of motion, the complete integrability of equation (2.1) is guaranteed.*

Finally, if the set (S_1, U_1, R_1) alone is found to satisfy equations (2.6)–(2.11) and the second set (S_2, U_2, R_2) also does not satisfy equations (2.9)–(2.11), then F may be a function of the integral I_1 alone which was derived from the set (S_1, U_1, R_1). We need to find the explicit form of $F(I_1)$ in order to obtain the compatible solution (S_2, U_2, R_2). To do so let us find the derivatives of $\hat{R}_2$ with respect to x, $\dot{x}$ and $\ddot{x}$:

$$\hat{R}_{2x} = F_1' I_{1x} R_2 + F R_{2x}, \quad \hat{R}_{2\dot{x}} = F_1' I_{1\dot{x}} R_2 + F R_{2\dot{x}}, \quad \hat{R}_{2\ddot{x}} = F_1' I_{1\ddot{x}} R_2 + F R_{2\ddot{x}}, \tag{2.21}$$

where $F_1' = \partial F / \partial I_1$. Substituting equation (2.21) into equations (2.9)–(2.11), we have

$$\frac{R_2 F_1'}{F} = \frac{s_2}{s_1} = \frac{s_4}{s_3} = \frac{s_6}{s_5}, \tag{2.22a}$$

where

$$\left.\begin{aligned} s_1 &= (I_{1x} - I_{1\ddot{x}} S_2), & s_2 &= (S_2 R_{2\ddot{x}} + R_2 S_{2\ddot{x}} - R_{2x}), \\ s_3 &= (I_{1\dot{x}} - I_{1\ddot{x}} U_2), & s_4 &= (U_2 R_{2\ddot{x}} + R_2 U_{2\ddot{x}} - R_{2\dot{x}}), \\ s_5 &= (S_2 I_{1\dot{x}} - I_{1x} U_2), & s_6 &= (R_2 U_{2x} + U_2 R_{2x} - R_2 S_{2\dot{x}} - S_2 R_{2\dot{x}}). \end{aligned}\right\} \tag{2.22b}$$

One can again check that the compatibility of the right three expressions in equation (2.22a) leads to a condition which is deducible from (2.9)–(2.11), and so no new condition is introduced in reality. Then rewriting equation (2.22a) as

$$\frac{\partial F}{\partial I_1} = \left(\frac{1}{R_2}\frac{s_2}{s_1}\right) F = g(I_1) F, \tag{2.23}$$

and solving it, one can obtain an explicit expression for F. Once F is known we can construct the complete form of $\hat{R}_2$ from which, along with S_2 and U_2, the second integral of motion can be obtained. Once I_1 and I_2 are known, we can proceed to find the third compatible set (S_3, U_3, R_3) as before and obtain the third integral I_3 to establish complete integrability.

3. Methods of finding the explicit form of *R*

In §2 we outlined the method of solving the determining equations. However, in practice, it is difficult to solve equations (2.6)–(2.8) straightforwardly as they constitute a set of coupled first-order nonlinear PDEs. So one has to look for some intuitive ideas to solve these equations. We solve them and obtain the forms R, U and S in the following way. For this purpose, we observe the important fact that when we rewrite the coupled equations (2.6)–(2.8) into an equation for a single variable, namely, R, the resultant equation turns out to be a linear PDE. Then we solve this 'R equation' with a suitable ansatz (say polynomial or rational in $\ddot{x}$). Once R is found the remaining functions U and S can be easily deduced.

As noted, rewriting equations (2.6)–(2.8), we arrive at a third-order linear PDE for R in the form

$$\mathrm{D}^3[R]-\mathrm{D}^2[R\phi\ddot{x}]+\mathrm{D}[R\phi_{\dot{x}}]-\phi_x R=0,\quad \mathrm{D}=\frac{\partial}{\partial t}+\dot{x}\frac{\partial}{\partial x}+\ddot{x}\frac{\partial}{\partial \dot{x}}+\phi\frac{\partial}{\partial \ddot{x}}. \tag{3.1}$$

Solving equation (3.1) with a suitable ansatz in $\ddot{x}$ is relatively easier in many cases than solving equations (2.6)–(2.8). Once the explicit form of R is obtained, U can be deduced from equation (2.8) as

$$U=-\left(\frac{\mathrm{D}[R]}{R}+\phi_{\ddot{x}}\right), \tag{3.2}$$

from which S can be fixed by using equation (2.7). Now if this set (S, U, R) forms a compatible set for the remaining equations (2.9)–(2.11), then the corresponding integral I can be found using equation (2.13). To illustrate this idea, let us look into the following examples.

(a) *Example 1*

Let us begin with a simple example, namely a linear third-order ODE,

$$\dddot{x}+\lambda x=0, \tag{3.3}$$

where λ is an arbitrary parameter. Substituting $\phi=-\lambda x$ into (3.1), we get the following equation for R:

$$\mathrm{D}^3[R]+\lambda R=0,\quad \mathrm{D}=\frac{\partial}{\partial t}+\dot{x}\frac{\partial}{\partial x}+\ddot{x}\frac{\partial}{\partial \dot{x}}-\lambda x\frac{\partial}{\partial \ddot{x}}. \tag{3.4}$$

We now assume an ansatz for R in the form

$$R=a(t,x,\dot{x})+b(t,x,\dot{x})\ddot{x}, \tag{3.5}$$

where a and b are arbitrary functions of t, x and $\dot{x}$. Substituting (3.5) into (3.4) and equating the coefficients of different powers of $\ddot{x}$ to zero, we get a set of linear PDEs for the variables a and b. Solving them one obtains three non-trivial solutions,

$$R_1=-\mathrm{e}^{kt},\quad R_2=-(2\ddot{x}+k\dot{x}-k^2x)\mathrm{e}^{-kt},\quad R_3=-\frac{\sqrt{3}k}{2}(\dot{x}+kx)\mathrm{e}^{-kt}, \tag{3.6}$$

where $k^3=\lambda$. Now substituting the form of R_i's, $i=1, 2, 3$, separately into equation (3.2), we get

$$U_1=-k,\quad U_2=\frac{(2k^2\dot{x}+k\ddot{x}+k^3x)}{(2\ddot{x}+k\dot{x}-k^2x)},\quad U_3=-\frac{(\ddot{x}-k^2x)}{(\dot{x}+kx)}. \tag{3.7}$$

Now substituting the forms of U_i's, $i=1, 2, 3$, into (2.7), one can fix the forms of S_i's, $i=1, 2, 3$, as

$$S_1=k^2,\quad S_2=\frac{2k^4x+k^3\dot{x}-k^2\ddot{x}}{2\ddot{x}+k\dot{x}-k^2x},\quad S_3=-k\frac{(\ddot{x}+k\dot{x})}{(\dot{x}+kx)}. \tag{3.8}$$

As a consequence now we have three sets of independent solutions for equations (2.6)–(2.8). Now we check the compatibility of these solutions with the remaining equations (2.9)–(2.11).

We find that the solutions (S_1, U_1, R_1) and (S_2, U_2, R_2) satisfy equations (2.9)–(2.11), and become compatible solutions. Substituting the forms (S_1, U_1, R_1) and (S_2, U_2, R_2) separately into equation (2.13) and evaluating the integrals we get

$$I_1 = \frac{1}{3k^2}(\ddot{x} - k\dot{x} + k^2 x)\mathrm{e}^{kt}, \tag{3.9}$$

$$I_2 = \frac{2}{3k^2}(\ddot{x}^2 + k^2\dot{x}^2 + k^4x^2 + k\dot{x}\ddot{x} - k^2x\ddot{x} + k^3x\dot{x})\mathrm{e}^{-kt}. \tag{3.10}$$

However, the set (S_3, U_3, R_3) does not satisfy the extra constraints (2.9)–(2.11), which means that the form of R_3 may not be the 'complete form' but might be a factor of the complete form. As mentioned in §2, in order to recover the complete form $\hat{R}_3$, one may assume that $\hat{R}_3 = F(I_1, I_2)R_3$, where $F(I_1, I_2)$ is a function of the integrals I_1 and I_2. Now we have to determine the form of $F(I_1, I_2)$ explicitly and for this purpose we proceed as follows. Substituting

$$\hat{R}_3 = F(I_1, I_2)R_3 = -\left[\frac{\sqrt{3}k}{2}(\dot{x} + kx)\mathrm{e}^{-kt}\right]F(I_1, I_2), \tag{3.11}$$

into equations (2.18) and (2.19), along with (2.17*b*), we obtain two equations for F as

$$F_1' = 0, \quad I_2F_2' + F = 0, \tag{3.12}$$

where F_1' and F_2' denote partial derivatives of F with respect to I_1 and I_2, respectively. Upon integrating (3.12), we get $F = 1/I_2$ (the integration constants are set to zero for simplicity), which fixes the form of $\hat{R}_3$ as

$$\hat{R}_3 = \frac{R_3}{I_2} = -\frac{\sqrt{3}k}{2}\frac{(\dot{x} + kx)}{(\ddot{x}^2 + k^2\dot{x}^2 + k^4x^2 + k\dot{x}\ddot{x} - k^2x\ddot{x} + k^3x\dot{x})}. \tag{3.13}$$

Now one can easily check that this set $(S_3, U_3, \hat{R}_3)$ is a compatible solution for equations (2.6)–(2.11), which, in turn, provides I_3 through the relation (2.13) as

$$I_3 = -\frac{\sqrt{3}k}{2}t + \tan^{-1}\left[\frac{\ddot{x} - k\dot{x} - 2k^2x}{\sqrt{3}(\ddot{x} + k\dot{x})}\right]. \tag{3.14}$$

Using the explicit forms of the integrals I_1, I_2 and I_3, the solution to equation (3.3) can be deduced directly as

$$x(t) = I_1\,\mathrm{e}^{-kt} + \sqrt{I_2}\mathrm{e}^{(k/2)t}\cos\left(\frac{\sqrt{3}k}{2}t + I_3\right). \tag{3.15}$$

The result exactly coincides with the solution presented in Polyanin & Zaitsev (1995).

(*b*) *Example 2*

The applicability of this method to nonlinear ODEs can be illustrated by considering an equation of the form

$$\dddot{x} = \frac{\ddot{x}^2}{\dot{x}} + \frac{\dot{x}\ddot{x}}{x}. \tag{3.16}$$

Equation (3.16) is a sub-case of the general form of a scalar third-order ODE, which is invariant under the generators of time translation and rescaling (Polyanin & Zaitsev 1995; Feix *et al.* 1997). A sub-case of equation (3.16), namely, $\dddot{x}-c(\ddot{x}^2/\dot{x})=0$, has been considered by both Bocharov *et al.* (1993) and Ibragimov & Meleshko (2005) to show that it can be linearized to a linear third-order ODE through a contact transformation. On the other hand, Euler & Euler (2004) have considered the equation $\dddot{x}-(\dot{x}\ddot{x}/x)=0$ and showed that it can be linearized through the Sundman transformation (see §5*c*). Here we consider the combined form (3.16) and derive integrating factors, integrals of motion and the general solution of this equation. Further, we show that equation (3.16) itself can be linearized by the GLT (see §5*d*).

As before, substituting $\phi=(\ddot{x}^2/\dot{x})+(\dot{x}\ddot{x}/x)$ into (3.1), we get the following linear PDE for R:

$$\mathrm{D}^3[R]-\mathrm{D}^2\left[\left(\frac{2\ddot{x}}{\dot{x}}+\frac{\dot{x}}{x}\right)R\right]-\mathrm{D}\left[\left(\frac{\ddot{x}^2}{\dot{x}^2}-\frac{\ddot{x}}{x}\right)R\right]+\frac{\dot{x}\ddot{x}}{x^2}R=0, \tag{3.17}$$

where 'D' is defined by equation (2.12). Now substituting the ansatz (3.5) into (3.17) and proceeding as before, we get

$$R_1=-\frac{1}{\dot{x}x},\quad R_2=\frac{x}{\dot{x}},\quad R_3=\frac{t\dot{x}^2-x(\dot{x}+t\ddot{x})}{2x\dot{x}^2}. \tag{3.18}$$

Following the ideas given in example 1, one can deduce the corresponding forms of S_i's and U_i's, $i=1, 2, 3$, as

$$S_1=-\frac{\ddot{x}}{x},\quad U_1=-\frac{\ddot{x}}{\dot{x}}, \tag{3.19}$$

$$S_2=\frac{\ddot{x}}{x},\quad U_2=\frac{-2\dot{x}}{x}-\frac{\ddot{x}}{\dot{x}}, \tag{3.20}$$

$$S_3=\frac{\dot{x}\ddot{x}(x+t\dot{x})}{x(-t\dot{x}^2+x(\dot{x}+t\ddot{x}))},\quad U_3=\frac{x\ddot{x}(2\dot{x}+t\ddot{x})}{\dot{x}(t\dot{x}^2-x(\dot{x}+t\ddot{x}))}. \tag{3.21}$$

The solutions (S_i, U_i, R_i), $i=1, 2$, satisfy the constraints (2.9)–(2.11), so that they lead to first and second integrals of the form

$$I_1=\frac{\ddot{x}}{\dot{x}x},\quad I_2=\frac{2\dot{x}^2-x\ddot{x}}{\dot{x}}. \tag{3.22}$$

Also in the present case the set (S_3, U_3, R_3) does not satisfy the extra constraints and so one has to explore the complete form of $\hat{R}_3$. To do so, we proceed as before and obtain the forms of F and $\hat{R}_3$ as $F=1/\sqrt{I_1 I_2}$ and

$$\hat{R}_3=\frac{t\dot{x}^2-x(\dot{x}+t\ddot{x})}{2(\sqrt{I_1 I_2})x\dot{x}^2}, \tag{3.23}$$

where the explicit forms of I_1 and I_2 are given in equation (3.22). Now one can check that the set $(S_3, U_3, \hat{R}_3)$ satisfies all the six equations (2.6)–(2.11) and furnishes the third integral I_3 through the relation (2.13) as

$$I_3 = -\frac{1}{2}\left(\sqrt{I_1 I_2}\right)t + \tan^{-1}\sqrt{\frac{I_1}{I_2}}x. \tag{3.24}$$

Using the explicit forms of the integrals I_1, I_2 and I_3, the solution to equation (3.16) can be deduced directly as

$$x(t) = \sqrt{\frac{I_2}{I_1}}\tan\left[\frac{1}{2}\left(\sqrt{I_1 I_2}t + 2I_3\right)\right]. \tag{3.25}$$

As can be seen from equation (3.23), the complete compatible solution $\hat{R}_3$ has $\ddot{x}$ term which appear inside the square root sign. This form of $\hat{R}_3$ can be explored only by making a suitable ansatz. Moreover, one may also face more difficulties in solving the determining equations (2.6)–(2.8). In such complicated situations, the complete solution $\hat{R}$ can be obtained by using our procedure.

(*c*) *Example 3*

Let us consider a Chazy class of equation of the form (Halburd 1999; Mugan & Jrad 2002; Euler & Euler 2004; Euler *et al.* 2005/2006)

$$\dddot{x} + 4\alpha x\ddot{x} + 3\alpha\dot{x}^2 + 6\alpha^2 x^2\dot{x} + \alpha^3 x^4 = 0. \tag{3.26}$$

Substituting $\phi = -(4\alpha x\ddot{x} + 3\alpha\dot{x}^2 + 6\alpha^2 x^2\dot{x} + \alpha^3 x^4)$ into (3.1), we get

$$\mathrm{D}^3[R] + 4\alpha\,\mathrm{D}^2[xR] - 6\alpha\,\mathrm{D}[(\dot{x} + \alpha x^2)R] + 4\alpha(\ddot{x} + \alpha^2 x^3 + 3\alpha x\dot{x})R = 0. \tag{3.27}$$

To solve the above equation, we assume the same ansatz (3.5) for the variable R. However, substituting this ansatz into equation (3.27) and solving the resultant equation, we obtain only trivial solutions. So we seek a rational form of ansatz for R in the form

$$R = \frac{a(t,x,\dot{x}) + b(t,x,\dot{x})\ddot{x}}{(c(t,x,\dot{x}) + d(t,x,\dot{x})\ddot{x})^r}, \tag{3.28}$$

where r is an arbitrary number. Substituting equation (3.28) into equation (3.27) and solving the resultant PDEs we get

$$\left.\begin{aligned} R_1 &= \frac{\dot{x} + \alpha x^2}{(\alpha^2 x^3 + 3\alpha x\dot{x} + \ddot{x})^2}, \quad R_2 = \frac{t(-2x + \alpha t x^2 + t\dot{x})}{(\alpha^2 x^3 + 3\alpha x\dot{x} + \ddot{x})^2}, \\ R_3 &= \frac{t(3 - 3\alpha t x + \alpha^2 t^2 x^2 + \alpha t^2\dot{x})}{(\alpha^2 x^3 + 3\alpha x\dot{x} + \ddot{x})^2}. \end{aligned}\right\} \tag{3.29}$$

The corresponding forms of S_i's and U_i's, $i=1, 2, 3$, are

$$\left.\begin{aligned} U_1 &= \frac{2\alpha^2 x^3 - \ddot{x}}{\alpha x^2 + \dot{x}}, \quad S_1 = \frac{\alpha(\alpha^2 x^4 + 3\dot{x}^2 - 2x\ddot{x})}{\alpha x^2 + \dot{x}}, \\ U_2 &= \frac{2x - 6\alpha t x^2 + 2\alpha^2 t^2 x^3 - t^2 \ddot{x}}{t(-2x + \alpha t x^2 + t\dot{x})}, \\ S_2 &= \frac{2\alpha x^2 - 4\alpha^2 t x^3 + \alpha^3 t^2 x^4 - 2\dot{x} + 3\alpha t^2 \dot{x}^2 + 2t\ddot{x}(1 - \alpha t x)}{t(-2x + \alpha t x^2 + t\dot{x})}, \\ U_3 &= -\frac{3 - 12\alpha t x + 9\alpha^2 t^2 x^2 - 2\alpha^3 t^3 x^3 + \alpha t^3 \ddot{x}}{t(3 - 3\alpha t x + \alpha^2 t^2 x^2 + \alpha t^2 \dot{x})}, \\ S_3 &= \frac{(\alpha(12\alpha t x^2 - 6\alpha^2 t^2 x^3 + \alpha^3 t^3 x^4 + 3t(2\dot{x} + \alpha t^2 \dot{x}^2 + t\ddot{x}) - 2x(3 + \alpha t^3 \ddot{x})))}{t(3 - 3\alpha t x + \alpha^2 t^2 x^2 + \alpha t^2 \dot{x})}. \end{aligned}\right\} \tag{3.30}$$

Now we proceed to find the integrals of motion. First we note that the functions (S_1, U_1, R_1) satisfy the constraints (2.9)–(2.11) and hence they are compatible. Thus, substituting them into (2.13), the first integral I_1 is fixed easily as

$$I_1 = -t + \frac{\alpha x^2 + \dot{x}}{\alpha^2 x^3 + 3\alpha x\dot{x} + \ddot{x}}. \tag{3.31}$$

However, the set (S_2, U_2, R_2) (and so also (S_3, U_3, R_3)) does not satisfy the extra constraints (2.9)–(2.11), which means the form of R_2 may not be the 'complete form' but might be a factor of the complete form. As mentioned in §2, in order to recover the complete form $\hat{R}_2$, one may assume that $\hat{R}_2 = F(I_1)R_2$. Here $F(I_1)$ is a function of the integral I_1. Now we have to determine the form of $F(I_1)$ explicitly and for this purpose we proceed as follows. Substituting the expression

$$\hat{R}_2 = F(I_1)R_2 = \frac{t(-2x + \alpha t x^2 + t\dot{x})}{(\alpha^2 x^3 + 3\alpha x\dot{x} + \ddot{x})^2} F(I_1), \tag{3.32}$$

into equation (2.22*a*), we obtain $I_1 F_1' + 2F = 0$, where F_1' denotes differentiation with respect to I_1. Upon integrating the latter, we get $F = I_1^{-2}$ (the integration constant is set to zero), which fixes the form of R_2 as

$$\hat{R}_2 = \frac{t(-2x + \alpha t x^2 + t\dot{x})}{(\alpha x^2 - \alpha^2 t x^3 + \dot{x} - 3\alpha t x\dot{x} - t\ddot{x})^2}. \tag{3.33}$$

Now one can easily check that this set $(S_2, U_2, \hat{R}_2)$ is a compatible solution for equations (2.6)–(2.11), which, in turn, provides I_2 through the relation (2.13) as

$$I_2 = -\frac{-2\alpha t x^2 + \alpha^2 t^2 x^3 + x(2 + 3\alpha t^2 \dot{x}) + t(-2\dot{x} + t\ddot{x})}{(\alpha x^2 - \alpha^2 t x^3 + \dot{x} - 3\alpha t x\dot{x} - t\ddot{x})}. \tag{3.34}$$

As in the previous examples, the set (S_3, U_3, R_3) does not satisfy the constraints (2.9)–(2.11) and hence one should seek a complete form for R_3, which we denote as $\hat{R}_3$, in the form

$$\hat{R}_3 = F(I_1, I_2)R_3 = F(I_1, I_2)\frac{t(3-3\alpha tx + \alpha^2 t^2 x^2 + \alpha t^2 \dot{x})}{(\alpha^2 x^3 + 3\alpha x\dot{x} + \ddot{x})^2}. \tag{3.35}$$

Substituting (3.35) into equations (2.18) and (2.19), we obtain the following equations for F, i.e. $I_1 F_1' + 2F = 0$, $F_2' = 0$, where again $F_1' = \partial F/\partial I_1$ and $F_2' = \partial F/\partial I_2$. Upon integrating the equations, we get the explicit form of F as $F = 1/I_1^2$, which, in turn, fixes the form of $\hat{R}_3$ as

$$\hat{R}_3 = \frac{t(3-3\alpha tx + k^2 t^2 x^2 + \alpha t^2 \dot{x})}{3\alpha(\alpha x^2 - \alpha^2 t x^3 + \dot{x} - 3\alpha t x\dot{x} - t\ddot{x})^2}. \tag{3.36}$$

Now the set $(S_3, U_3, \hat{R}_3)$ satisfies all the six equations (2.6)–(2.11) and the relation (2.13) gives the form of third integral I_3 as

$$I_3 = \frac{(6 + 3\alpha^2 t^2 x^2 - \alpha^3 t^3 x^3 + 3\alpha t^2 \dot{x} - 3\alpha t x(2 + \alpha t^2 \dot{x}) - \alpha t^3 \ddot{x})}{6\alpha(\alpha x^2 - \alpha^2 t x^3 + \dot{x} - 3\alpha t x\dot{x} - t\ddot{x})}. \tag{3.37}$$

Thus, we have obtained the explicit forms of the integrals I_1, I_2 and I_3 and hence the solution to equation (3.3) is obtained directly as

$$x(t) = \frac{\frac{\alpha t^2}{2} + I_1 t + I_1 I_2}{\frac{\alpha^2 t^3}{6} + \alpha I_1 \frac{t^2}{2} + \alpha I_1 I_2 t + I_1 I_3}. \tag{3.38}$$

Recently, equation (3.26) has been shown to belong to the Riccati hierarchy of linearizable ODEs (Euler *et al.* 2005/2006).

Interestingly, one can also derive the solution (3.38) through an alternate way. For example, instead of solving the 'R equation' with rational ansatz, one can look for equations in other variables, i.e. either in U or in S. For example, from equation (2.7) we get

$$S = -(\mathrm{D}[U] + \phi_{\dot{x}} - U\phi_{\ddot{x}} - U^2). \tag{3.39}$$

Substituting equation (3.39) into equation (2.6), we get a nonlinear PDE for U:

$$\mathrm{D}^2[U] - 3U\,\mathrm{D}[U] - U\,\mathrm{D}[\phi_{\ddot{x}}] + \mathrm{D}[\phi_{\dot{x}}] - \phi_x - \phi_{\dot{x}}\phi_{\ddot{x}} + \phi_{\ddot{x}}^2 U + 2\phi_{\ddot{x}} U^2 - \phi_{\dot{x}} U + U^3 = 0. \tag{3.40}$$

Now one can look for solutions of equation (3.40) with polynomial in $\ddot{x}$. Once U is known one can make use of equations (2.8) and (3.39) to get the forms of corresponding S and R, respectively. It turns out that for some cases, like the present example (3.26) (see the actual forms of U in equation (3.30)), solving equation (3.40) is easier than solving equation (3.1). However, this can be decided only by actual calculation.

4. Linearization

In §§2 and 3, we discussed the complete integrability of third-order ODEs by investigating sufficient number of integrals of motion. However, one can also establish the complete integrability of the given nonlinear ODE by transforming it into a linear free particle second-order ODE or into a third-order linear ODE of the form $d^3w/dz^3 = w''' = 0$. Unlike the second-order ODEs, the third-order nonlinear ODEs can be linearized through a wide class of transformations, namely, invertible point transformation (Steeb 1993; Ibragimov & Meleshko 2005), contact transformation (Bocharov *et al.* 1993; Duarte *et al.* 1994; Ibragimov & Meleshko 2005), generalized Sundman transformation (Berkovich & Orlova 2000; Euler *et al.* 2003; Euler & Euler 2004) and their generalizations. In the following, we describe a procedure to deduce these transformations from the first integral itself and illustrate our ideas with relevant examples.

(a) *Transformation from third-order nonlinear ODEs to second-order free particle equation*

Let us suppose that the ODE (2.1) admits a first integral,

$$I_1 = F(t, x, \dot{x}, \ddot{x}), \tag{4.1}$$

where F is a function of $t, x, \dot{x}$ and $\ddot{x}$ only. Now extending our earlier proposal for second-order ODEs (Chandrasekar *et al.* 2005) to the third-order equations (2.1), let us split the function F into a product of two functions, such that one involves a perfect differentiable function $(d/dt)G_1(t, x, \dot{x})$ and another function $G_2(t, x, \dot{x}, \ddot{x})$, i.e.

$$I_1 = F\left(\frac{1}{G_2(t, x, \dot{x}, \ddot{x})}\frac{d}{dt}G_1(t, x, \dot{x})\right). \tag{4.2}$$

Suppose G_2 is a total time derivative of another function, say z, i.e. $dz/dt = G_2(t, x, \dot{x}, \ddot{x})$, then (4.2) can be further rewritten in the form

$$I_1 = F\left(\frac{1}{\frac{dz}{dt}}\frac{dG_1}{dt}\right) = F\left(\frac{dG_1}{dz}\right). \tag{4.3}$$

Now identifying the function $G_1(t, x, \dot{x}) = w$ as the new dependent variable, equation (4.3) can be recast in the form $I_1 = F(dw/dz)$. In other words, we obtain

$$\hat{I}_1 = \frac{dw}{dz}, \tag{4.4}$$

where $\hat{I}_1$ is a constant, from which one can get $d^2w/dz^2 = 0$. Rewriting w and z in terms of old variables, namely,

$$w = G_1(t, x, \dot{x}), \quad z = \int_o^t G_2(t', x, \dot{x}, \ddot{x})dt', \tag{4.5}$$

we can get a linearizing transformation to transform the third-order nonlinear ODE into the second-order free particle equation. Then to deduce the general solution, one has to carry out one more integration.

(*b*) *Transformation from third-order nonlinear equation (2.1) to third-order linear ODE* $w'''=0$

Next, we aim to transform equation (2.1) into a third-order linear ODE and so we try to rewrite the first integral as a perfect second-order derivative. We note that this can be done when one is able to evaluate the following integral explicitly:

$$\hat{w} = \int_o^t G_1(t', x, \dot{x}) G_2(t', x, \dot{x}, \ddot{x}) \mathrm{d}t' = G_3(t, x, \dot{x}), \tag{4.6}$$

where G_1 and G_2 are defined as in equation (4.2). Note that in the function G_3, the $\ddot{x}$ dependence has been integrated out. The reason for making such a specific decomposition is that in this case equation (4.2) can be rewritten as a simple second-order ODE for the variable $\hat{w}$ (see equation (4.8)). We pursued a similar kind of approach in the integrable force-free Duffing–van der Pol oscillator equation (Chandrasekar *et al.* 2004), which has now been generalized in the present case. Here one can rewrite (4.1) as a perfect second-order derivative as follows. Differentiating (4.6) with respect to t, we obtain $\mathrm{d}\hat{w}/\mathrm{d}t = G_1 G_2$. Rewriting the left-hand side in the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = G_1 G_2, \tag{4.7}$$

and using the identities already used in equation (4.5), namely $\mathrm{d}z/\mathrm{d}t = G_2$ and $G_1 = w$, in equation (4.7), the latter becomes

$$\frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = w. \tag{4.8}$$

Differentiating (4.8) with respect to z and using the identity $\mathrm{d}w/\mathrm{d}z = \hat{I}_1$ (vide equation (4.4)), one gets $\mathrm{d}^2\hat{w}/\mathrm{d}z^2 = \hat{I}_1$. In other words, we have

$$\frac{\mathrm{d}^3\hat{w}}{\mathrm{d}z^3} = 0, \tag{4.9}$$

so that $\hat{w}$ and z are the required transformation variables.

(*c*) *The nature of transformations*

In §4*a*,*b*, we demonstrated how to construct linearizing transformations from the first integral and how they effectively change the given third-order nonlinear ODE to either second or third-order linear equation. Depending upon the explicit form of the transformations, we can call them point, contact, generalized Sundman or GLTs. To demonstrate how different kinds of transformation arise, let us consider the transformation, $\hat{w} = G_3$, $z = \int_o^t G_2 \, \mathrm{d}t'$ (vide equations (4.5) and (4.6)), which takes the given nonlinear ODE into a linear equation. Now, in the above transformation, suppose z is a perfect differentiable function and $\hat{w}$ and z do not contain the variable $\dot{x}$ and $\ddot{x}$, then we call the resultant transformation, namely, $\hat{w} = f_1(x, t)$ and $z = f_2(x, t)$, as a point transformation. Further, if the transformation admits the variable $\dot{x}$ also explicitly, then it becomes the contact transformation. In this case, we have $w = f_1(t, x, \dot{x})$ and $z = f_2(t, x, \dot{x})$. On the other hand, if $\hat{w} = G_3(t, x)$ and $z = \int_o^t G_2(t', x)\mathrm{d}t'$, then the transformation is said to be a generalized Sundman transformation. Note that in the latter case, the new independent variable is in an integral form. Besides the above, as pointed out earlier, we find that there exists another kind of transformation, namely, GLTs, in which the new dependent and independent variables take the form $\hat{w} = G_3(t, x, \dot{x})$ and $z = \int_o^t G_2(t', x, \dot{x}, \ddot{x})\mathrm{d}t'$, respectively.

(d) *Connection between the functions G_1, G_2 and G_3*

Finally, we explore the connection between the functions G_1, G_2 and G_3. As we have seen above, for the given equation to be linearizable, it should be transformable to the form (4.8). Rewriting the latter in terms of the variables, G_1, G_2 and G_3, we get

$$G_{3t} + \dot{x}G_{3x} + \ddot{x}G_{3\dot{x}} = G_1(t, x, \dot{x})G_2(t, x, \dot{x}, \ddot{x}). \tag{4.10}$$

Note that the left-hand side of equation (4.10) contains the variable $\ddot{x}$ linearly. So the right-hand side should also be linear in $\ddot{x}$. Consequently, we can write

$$G_2(t, x, \dot{x}, \ddot{x}) = G_{21}(t, x, \dot{x})\ddot{x} + G_{22}(t, x, \dot{x}). \tag{4.11}$$

Using (4.11) we can rewrite equation (4.10) in the form

$$\frac{(G_{3t} + \dot{x}G_{3x})\left(1 + \frac{\ddot{x}G_{3\dot{x}}}{G_{3t}+\dot{x}G_{3x}}\right)}{G_{22}\left(1 + \frac{\ddot{x}G_{21}}{G_{22}}\right)} = G_1. \tag{4.12}$$

Since the right-hand side is independent of $\ddot{x}$, we have from (4.12) that

$$G_{21}(G_{3t} + \dot{x}G_{3x}) = G_{22}G_{3\dot{x}} \quad \text{and} \quad G_1 = \frac{G_{3\dot{x}}}{G_{21}}. \tag{4.13}$$

It may be noted that a similar condition that has been derived by Bocharov *et al.* (1993) and Ibragimov & Meleshko (2005) for the case G_2 is a perfect differential function. In other words, our procedure indicates that more generalized transformations are possible in the case of third-order ODEs. By imposing the condition (4.11) it becomes clear that whatever the type of linearizing transformation, the new independent variable should be at the maximum linear in $\ddot{x}$.

(e) *Transformation to fourth and higher-order linear ODEs*

In §4*a–d*, we have concentrated only on transforming a nonlinear third-order ODE either to a second-order or third-order linear equation only. However, one could also linearize certain third-order nonlinear ODEs to fourth-order linear ODEs. For example, equation (3.26) is linearizable to a fourth-order ODE of the form $\mathrm{d}^4w/\mathrm{d}z^4 = 0$, under the non-local transformation $x = \dot{w}/\alpha w$. This is not an isolated example and one can linearize a class of equations through this procedure. Besides the above one, we can also consider linearizing transformations, in which the new dependent variable, $\hat{w}$, is a non-local one. This choice leads us to classify another large class of equations, which we leave for future work.

5. Application

In this section, we consider specific examples to demonstrate the method given in §4.

(*a*) *Example 1: point transformation*

Let us consider a non-trivial example which was discussed by Steeb (1993) in the context of invertible point transformations, namely,

$$\dddot{x}+\frac{3\dot{x}\ddot{x}}{x}-3\ddot{x}-\frac{3\dot{x}^2}{x}+2\dot{x}=0. \tag{5.1}$$

The first integral, which can be obtained using the formulation in §2, can be written as

$$I_1=(\dot{x}^2+x\ddot{x}-x\dot{x})\mathrm{e}^{-2t}. \tag{5.2}$$

Rewriting (5.2) in the form (4.2), we get $I_1=\mathrm{e}^{-t}(\mathrm{d}/\mathrm{d}t)(x\dot{x}\,\mathrm{e}^{-t})$, so that

$$w=x\dot{x}\,\mathrm{e}^{-t},\quad z=\mathrm{e}^{t}. \tag{5.3}$$

As we noted earlier, one could transform (5.1) to the second-order free particle equation, $\mathrm{d}^2w/\mathrm{d}z^2=0$, by utilizing the transformation (5.3). Integrating the equation $\mathrm{d}^2w/\mathrm{d}z^2=0$, we get $w=I_1z+I_2$. Using (5.3) into this expression, the general solution of equation (5.1) can be obtained (after an integration) as

$$x(t)=(I_1\,\mathrm{e}^{2t}+I_2\,\mathrm{e}^{t}+I_3)^{1/2}, \tag{5.4}$$

where I_i, $i=1, 2, 3$ are the integration constants.

Further, using equation (4.6), we get $\hat{w}=\int x\dot{x}\,\mathrm{d}t=x^2/2$. Then we can directly check that the point transformation,

$$\hat{w}=\frac{x^2}{2},\quad z=\mathrm{e}^{t}, \tag{5.5}$$

transforms equation (5.1) to the form $\mathrm{d}^3\hat{w}/\mathrm{d}z^3=0$. As we mentioned earlier, since $\hat{w}$ and z involve only x and t, they are just point transformations. Integrating the linear equation $\mathrm{d}^3\hat{w}/\mathrm{d}z^3=0$, we get

$$\hat{w}=\frac{I_1}{2}z^2+I_2z+I_3. \tag{5.6}$$

Rewriting $\hat{w}$ and z in equation (5.6) in terms of the original variables x and t by using the transformation (5.5), we get the same solution as equation (5.4).

(*b*) *Example 2: contact transformation*

Let us consider an equation of the form

$$\dddot{x}=\frac{x\ddot{x}^3}{\dot{x}^3}. \tag{5.7}$$

Bocharov *et al.* (1993) have shown that equation (5.7) can be linearized through contact transformation. However, the explicit linearizing transformation is yet to be reported, which is also a difficult problem. Here we derive the explicit form of the linearizing transformation through our procedure.

The first integral can be easily deduced using the results of §2 as

$$I_1 = \frac{\dot{x}^2 - x\ddot{x}}{\dot{x}\ddot{x}}. \tag{5.8}$$

Rewriting (5.8) in the form (4.2), we get

$$I_1 = \frac{\dot{x}}{\ddot{x}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{\dot{x}}\right),$$

so that we have $w = x/\dot{x}$ and $z = \log \dot{x}$. The latter transforms equation (5.7) to the second-order free particle equation $\mathrm{d}^2w/\mathrm{d}z^2 = 0$, so that $w = I_1 z + I_2$, where I_1 and I_2 are integration constants. Rewriting w and z in terms of the old variables, we get $x = (I_1 \log(\dot{x}) + I_2)\dot{x}$. Unlike the earlier example, it is difficult to integrate this equation further and obtain the general solution. Therefore, one can look for variables which transform the third-order nonlinear ODE (5.7) to a third-order linear ODE, so that the non-trivial integration can be avoided. Now using equation (4.6), we get $\hat{w} = \int (x\ddot{x}/\dot{x}^2)\mathrm{d}t = t - (x/\dot{x})$. The new variables,

$$\hat{w} = \frac{t\dot{x} - x}{\dot{x}}, \quad z = \log \dot{x}, \tag{5.9}$$

transform equation (5.7) to the form (4.9). Unlike the earlier example, $\hat{w}$ and z admit the variable $\dot{x}$ explicitly and so they become contact transformation for the given equation.

Integrating the linear third-order equation (4.9), we get $\hat{w} = (I_1/2)z^2 + I_2 z + I_3$, where I_i, $i = 1, 2, 3$ are integration constants. Now replacing $\hat{w}$ and z in terms of the old variables and using the previous result $x = (I_1 \log(\dot{x}) + I_2)\dot{x}$, one can obtain the general solution for equation (5.7) in the form

$$x(t) = \left(-I_1 \pm \sqrt{I_1^2 + I_2^2 - 2I_1(I_3 - t)}\right)\exp\left(-\frac{I_1 + I_2 \mp \sqrt{I_1^2 + I_2^2 - 2I_1(I_3 - t)}}{I_1}\right). \tag{5.10}$$

(c) *Example 3: generalized Sundman transformation*

Next we consider the hydrodynamic type equation of the form (Berkovich & Orlova 2000; Euler & Euler 2004)

$$\dddot{x} = \frac{\ddot{x}\dot{x}}{x}, \tag{5.11}$$

which admits a first integral in the form $I_1 = \ddot{x}/x$ and the latter can be rewritten as

$$I_1 = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}t}\dot{x} = \frac{\mathrm{d}w}{\mathrm{d}z},$$

from which we identify $w = \dot{x}$ and $\mathrm{d}z = x\,\mathrm{d}t$. By utilizing the new variables, one can transform (5.11) to the second-order free particle equation, $\mathrm{d}^2w/\mathrm{d}z^2 = 0$. However, from equation (4.6), we get $\hat{w} = \int x\dot{x}\,\mathrm{d}t = x^2$. Then the Sundman transformation,

$$\hat{w} = x^2, \quad \mathrm{d}z = x\,\mathrm{d}t, \tag{5.12}$$

transforms equation (5.11) to the form (4.9), namely $\mathrm{d}^3\hat{w}/\mathrm{d}z^3 = 0$.

To derive the solution, we proceed as follows. Rewriting the first integral I_1 in the integral form, we get

$$I_1 = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}t}\dot{x} \Rightarrow \dot{x} = I_1 \int x\,\mathrm{d}t.$$

Now using the identity (5.12) in the latter expression, we get $w = I_1 z$. From equation (4.6) (for the present case $G_1 = w$ and $G_2 = \mathrm{d}z/\mathrm{d}t$) we have

$$\hat{w} = \int w\,\mathrm{d}z = \int I_1 z\,\mathrm{d}z = \frac{I_1}{2}z^2 + I_2, \tag{5.13}$$

where I_2 is the integration constant. Using (5.12) in (5.13), we obtain $x^2 = (I_1/2)z^2 + I_2$, which, in turn, leads to a differential equation which connects the variables z and t in the form (using the relation $\mathrm{d}z = x\,\mathrm{d}t$)

$$\mathrm{d}z = \sqrt{\frac{I_1}{2}z^2 + I_2}\mathrm{d}t. \tag{5.14}$$

Integrating (5.14), we obtain $z = \sqrt{I_2/2}(\mathrm{e}^{\sqrt{I_1}(t+I_3)} - \mathrm{e}^{-\sqrt{I_1}(t+I_3)})$, where I_3 is the integration constant. Substituting the latter in the relation $x^2 = (I_1/2)z^2 + I_2$, we arrive at the general solution for (5.11) in the form

$$x(t) = \sqrt{I_2}\cosh\sqrt{I_1}(t + I_3). \tag{5.15}$$

We note that the solution for equation (5.11) has been already derived in an alternate way from the Sundman transformation (5.12) by Euler & Euler (2004). However, *the procedure which we described in the above is new and can also be used for more general linearizing transformations*, as we see below.

(*d*) *Example 4: generalized linearizing transformation*

As we noted earlier, *some nonlinear ODEs can be linearized only through more general non-local form of transformations*, which we designate here as GLTs. We illustrate the GLT with the same example discussed as example 2 in §3, which admits a first integral of the form $I_1 = \dot{x}x/\ddot{x}$ (vide equation (3.16)). Rewriting this first integral in the form (4.2), we get

$$I_1 = \frac{x}{\ddot{x}}\frac{\mathrm{d}}{\mathrm{d}t}(x),$$

so that

$$w = x, \quad \mathrm{d}z = \frac{\ddot{x}}{x}\mathrm{d}t, \tag{5.16}$$

which can be effectively used to transform the nonlinear ODE (3.16) to the equation $\mathrm{d}^2w/\mathrm{d}z^2 = 0$. Using equation (4.6), we get $\hat{w} = \int \ddot{x}\,\mathrm{d}t = \dot{x}$. Then the GLT becomes

$$\hat{w} = \dot{x}, \quad \mathrm{d}z = \frac{\ddot{x}}{x}\mathrm{d}t, \tag{5.17}$$

which can be used to transform equation (3.16) to the form $\hat{w}'''=0$. Note that in the present case, the new independent and dependent variables admit $\ddot{x}$ and $\dot{x}$ terms, respectively, and that the transformation is non-local. *Indeed, no such linearizing transformations have been reported in the literature at least to our knowledge. We also now establish a method of finding the general solution for this case.*

Integrating once the equation $\mathrm{d}^2w/\mathrm{d}z^2=0$, we get $w=I_1 z$ from which we obtain

$$x = I_1 z. \tag{5.18}$$

On the other hand, equation (4.6) provides us with a relation (after using (5.16) and (5.17))

$$\dot{x} = \frac{I_1}{2} z^2 + I_2. \tag{5.19}$$

Now using (5.18) in (5.19), we obtain

$$\left(\frac{2I_1}{I_1 z^2 + 2I_2}\right)\mathrm{d}z = \mathrm{d}t. \tag{5.20}$$

The variables are now separated out and one can integrate (5.20) and obtain

$$z = \sqrt{\frac{2I_2}{I_1}}\tan\sqrt{\frac{I_2}{I_1}}(t+I_3). \tag{5.21}$$

Now substituting (5.21) into (5.18), we get

$$x(t) = \sqrt{2I_1 I_2}\tan\sqrt{\frac{I_2}{I_1}}(t+I_3), \tag{5.22}$$

which is effectively the same as (3.25).

Finally, we note that the procedure given above can be profitably utilized for other examples which are also linearized by GLTs.

(*e*) *Example 5: an elementary non-trivial system of hydrodynamic type*

Finally, to show the importance of the GLT and how this transformation gives additional information about the linearization of nonlinear third-order ODEs, we consider the following specific example which was discussed in the literature (Berkovich 1996; Berkovich & Orlova 2000):

$$\dddot{x} = \frac{\ddot{x}\dot{x}}{x} - 4\alpha x^2\dot{x}, \quad \alpha : \text{parameter}. \tag{5.23}$$

Equation (5.23) is nothing but the dynamical equation of the Euler–Poinsot case of a rigid body written in terms of a single variable (Berkovich 1996; Berkovich & Orlova 2000). For a more general integrable version of this equation, see Euler & Leach (2003). As we have seen earlier, this equation is linearizable in the case $\alpha=0$ through generalized Sundman transformation. *However, we wish to show here that the general equation (5.23) itself is linearizable through the GLT.*

From the first integral $I_1=(\ddot{x}/x)+2\alpha x^2$ associated with equation (5.23), one can identify the GLT

$$\hat{w}=x^2,\quad dz=\frac{2\dot{x}x}{\sqrt{\dot{x}^2+\alpha x^4}}dt, \tag{5.24}$$

which transforms equation (5.23) to the form (4.9). Note that for the choice $\alpha=0$, the independent variable becomes $dz=2x\,dt$ and so it becomes the generalized Sundman transformation, equation (5.12), identified in the literature (Berkovich & Orlova 2000; Euler & Euler 2004). Now following the steps given in example 4, one can deduce the general solution for equation (5.23) in terms of Jacobian elliptic function as

$$x(t)=(I_1(c-(c-b)sn^2\sqrt{\alpha I_1(c-a)}(t-t_0),m)+I_2)^{1/2}, \tag{5.25}$$

where $a+b+c=(2I_1-3\alpha I_2)/(4\alpha I_1)$, $ab+ac+cb=(3\alpha I_2^2-2I_1I_2)/(4\alpha I_1^2)$, $abc=-I_2^3/4I_1^3$, $m^2=(b-c)/(a-c)$ and I_2 and t_0 are integration constants.

6. Conclusion

In this paper, we have discussed a method of finding the integrals of motion and general solution associated with third-order nonlinear ODEs through the modified PS method by a non-trivial extension of our earlier work on second-order ODEs (Chandrasekar *et al.* 2005). We illustrated the validity of the method with suitable examples. Further, we introduced a technique which can be utilized to derive linearizing transformations from the first integral. Interestingly, we showed that different types of transformations, namely, point, contact, Sundman and GLTs can be derived in a unique way from the first integral. We also indicated a procedure to derive general solution for the third-order ODEs when GLTs occur. We believe that the GLT introduced in this paper will be highly useful to tackle new systems, such as equation (5.23). Finally, the modified PS method can also be extended to higher-order ODEs and coupled systems of ODEs. As far as the linearization of higher-order ODEs is concerned, it is still an open and challenging problem. As we pointed out in §1, one can unearth a wide variety of linearizing transformations for the higher-order ODEs besides formulating the necessary and sufficient condition for linearizing these equations in each form of transformation. We hope to address some of these aspects shortly.

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