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On the complete integrability and linearization of nonlinear ordinary differential equations. V. Linearization of coupled second-order equations

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Linearization of coupled second-order nonlinear ordinary differential equations (SNODEs) is one of the open and challenging problems in the theory of differential equations. In this paper, we describe a simple and straightforward method to derive linearizing transformations for a class of two coupled SNODEs. Our procedure gives several new types of linearizing transformations of both invertible and non-invertible kinds. In both cases, we provide algorithms to derive the general solution of the given SNODE. We illustrate the theory with potentially important examples.

1. Introduction

Continuing our study on the integrability and linearization of coupled second-order nonlinear ordinary differential equations (SNODEs), in this paper, we focus our attention on the linearization of two coupled SNODEs. This study arises not only for the completeness of part IV (Chandrasekar *et al.* 2009), but also to show the importance of unfinished tasks that exist in the theory of linearization of two coupled SNODEs. As far as the first point is concerned, we show that one can also solve a class of coupled SNODEs by transforming them into two second-order free particle equations and, from the solutions of the latter, one can construct the solution of the former, even though this is a non-trivial problem in many situations (one can also transform coupled nonlinear ODEs into uncoupled nonlinear ones, which has already been pointed out by us in the previous paper, i.e. part IV). Regarding the second point, we wish to stress the fact that linearization of coupled nonlinear ODEs is a vast area of research that is still in its early stage. In this paper, we show that, in spite of the difficulties which exist in this topic, one can make useful progress on certain issues, namely: (i) developing a method to deduce all linearizing transformations wherein the new dependent variables are functions of only the old dependent and independent variables and not derivatives of the dependent variables and (ii) developing a method of constructing solutions of nonlinear ODEs from the linear ones in the case of non-point transformations.

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Even though the modern theory of linearization of nonlinear ODEs had originated and developed with the works of Lie, Tresse and Cartan (Mahomed & Leach 1989; Steeb 1993; Olver 1995; Ibragimov 1999; Chandrasekar *et al.* 2005), the entire subject was lying dormant for more than a century. Only recently, during the past two decades or so, has notable progress been made to linearize nonlinear ODEs through non-point (Duarte *et al.* 1994) or generalized transformations (Chandrasekar *et al.* 2006). For example, focussing our attention on single second-order ODEs, generalized Sundmann (Euler *et al.* 2003; Euler & Euler 2004) and generalized linearizing transformations (Chandrasekar *et al.* 2006) have been introduced to linearize a class of equations that cannot be linearized by invertible point transformations. As far as two coupled SNODEs are concerned, to our knowledge, most of the studies were focussed only on invertible point transformations, irrespective of whether it is an analytical approach or a geometrical formulation. For a survey on this topic, one may refer to the recent papers of Merker (2006) and Mahomed & Qadir (2007) and, for the earlier works in this direction, we cite Crampin *et al.* (1996), Fels (1995), Grossman (2000), Soh & Mahomed (2001) and Qadir (2007). More recently, Sookmee & Meleshko (2008) proposed a new algorithm to linearize the coupled second-order ODEs by sequentially reducing the order of the equation.

In this work, we aim to give a new dimension to the theoretical development of linearization of nonlinear dynamical systems having two degrees of freedom by proving that one can unearth a wide class of linearizing transformations besides invertible point transformations. Of course, the latter ones form a subclass of the new ones that we construct in this paper. In this study, we not only derive several new types of linearizing transformations, but also propose systematic procedures to derive the general solution in all these cases. We also wish to emphasize here that we derive all these transformations from the first two integrals alone, and thereby establish a potentially simple, straightforward and powerful approach in the theory of differential equations.

The plan of the paper is as follows. In §2*a*, we briefly describe the method of deriving linearizing transformations for a system of two coupled second-order ODEs. We show that one can have two classes of linearizing transformations, depending on the nature of the independent variables. If the new independent variables are the same ($z_1 = z_2$), we put them in class A category, and if they are different ($z_1 \neq z_2$), then we put them in class B category. In §2*b*(i), we consider class A category and identify three types of linearizing transformations. In §2*b*(ii), we consider class B category and identify six types of linearizing transformations. In §3, we consider one specific example for each of the nine types of linearizing transformations we have identified and obtain general solutions to each one of them to demonstrate our procedure. Finally, we present our conclusions in §4.

2. Linearizing transformations

(*a*) *Method of deriving linearizing transformations*

To begin with, let us consider a system of two coupled SNODEs, $R\{t, \mathbf{x}\}$ (eqn (2.1) in part IV (Chandrasekar *et al.* 2009)),

$$\ddot{x} = \phi_1(t, x, y, \dot{x}, \dot{y}) \quad \text{and} \quad \ddot{y} = \phi_2(t, x, y, \dot{x}, \dot{y}). \tag{2.1}$$

Any transformation of the form $T\{t, \mathbf{x}\}$, defined by

$$
\left.
\begin{aligned}
w_1 &= f_1(t, x, y), \quad z_1 = \int f_3(t, x, y, \dot{x}, \dot{y})\,\mathrm{d}t \\[2mm]
w_2 &= f_2(t, x, y), \quad z_2 = \int f_4(t, x, y, \dot{x}, \dot{y})\,\mathrm{d}t,
\end{aligned}
\right\}
\tag{2.2}
$$

and

which transforms the given set of nonlinear ODEs (2.1) to the free particle equations

$$
\frac{\mathrm{d}^2 w_1}{\mathrm{d}z_1^2} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2 w_2}{\mathrm{d}z_2^2} = 0
\tag{2.3}
$$

is called a linearizing transformation in the present work.

Let

$$
I_1 = \mathcal{F}(t, x, y, \dot{x}, \dot{y}) \quad \text{and} \quad I_2 = \mathcal{G}(t, x, y, \dot{x}, \dot{y})
\tag{2.4}
$$

be the first two integrals of motion of the coupled system (2.1) and that they can be explicitly found, if they exist, e.g. by using the generalized modified Prelle–Singer (PS) method formulated in part IV (Chandrasekar *et al.* 2009). Then, the following theorem ensures that the transformation can be deduced from I_i, $i = 1, 2$.

Theorem 2.1. *Suppose a given nonlinear system $R\{t, \mathbf{x}\}$ of ODEs (2.1) is linearizable to a system of two uncoupled free particle equations through the linearizing transformation $T\{t, \mathbf{x}\}$ of the form (2.2), then the latter can be deduced from the first integrals $I_i(t, x, y, \dot{x}, \dot{y})$, $i = 1, 2$.*

Proof. Let us re-express each of the functions $\mathcal{F}$ and $\mathcal{G}$ in equation (2.4) as a product of two new functions, i.e.

$$
\left.
\begin{aligned}
I_1 &= \frac{1}{f_3(t, x, y, \dot{x}, \dot{y})} \frac{\mathrm{d}}{\mathrm{d}t} f_1(t, x, y) \\[2mm]
I_2 &= \frac{1}{f_4(t, x, y, \dot{x}, \dot{y})} \frac{\mathrm{d}}{\mathrm{d}t} f_2(t, x, y).
\end{aligned}
\right\}
\tag{2.5}
$$

and

Again, rewriting f_3 and f_4 as total time derivatives of another set of functions, say z_1 and z_2, respectively, i.e. $\mathrm{d}z_1/\mathrm{d}t = f_3(t, x, y, \dot{x}, \dot{y})$ and $\mathrm{d}z_2/\mathrm{d}t = f_4(t, x, y, \dot{x}, \dot{y})$, equation (2.5) can be further recast as

$$
I_1 = \frac{1}{\mathrm{d}z_1/\mathrm{d}t} \frac{\mathrm{d}f_1}{\mathrm{d}t} = \frac{\mathrm{d}f_1}{\mathrm{d}z_1} \quad \text{and} \quad I_2 = \frac{1}{\mathrm{d}z_2/\mathrm{d}t} \frac{\mathrm{d}f_2}{\mathrm{d}t} = \frac{\mathrm{d}f_2}{\mathrm{d}z_2}.
\tag{2.6}
$$

Now identifying the functions $f_1(t, x, y) = w_1$ and $f_2(t, x, y) = w_2$ as the new dependent variables, equation (2.6) can be further recast in the form

$$
\frac{\mathrm{d}w_1}{\mathrm{d}z_1} = \hat{I}_1 \quad \text{and} \quad \frac{\mathrm{d}w_2}{\mathrm{d}z_2} = \hat{I}_2,
\tag{2.7}
$$

where $\hat{I}_1$ and $\hat{I}_2$ are the redefined constants. Obviously, equation (2.3) follows straightforwardly from equation (2.7). Consequently, the new variables, z_i and w_i, $i = 1, 2$, defined by equation (2.2) help us to transform the given set of coupled

SNODEs into two linear second-order ODEs that, in turn, lead to the solution by trivial integration. The variables w_i and z_i, $i = 1, 2$, then define the linearizing transformations for the given equation (2.1). ∎

It may be noted that, in general, the new dependent variables, w_1 and w_2, may also involve $\dot{x}$ and $\dot{y}$, i.e. $w_1 = G_1(t, x, y, \dot{x}, \dot{y})$ and $w_2 = G_3(t, x, y, \dot{x}, \dot{y})$, and this possibility may lead us to identify more generalized transformations such as point-contact and generalized-contact transformations. However, in this paper, we will confine ourselves only to the forms of w_1 and w_2 given by equation (2.2).

(b) *The nature of transformations*

An important question that we will focus upon in this paper is what are the possible forms of linearizing transformations one can unearth through the above procedure. We recall here that, in the case of scalar SNODEs, one has point generalized Sundman and generalized linearizing transformations (Chandrasekar *et al.* 2005, 2006). As far as the coupled SNODEs (2.1) are concerned, as there are two independent variables z_1 and z_2 as given in equation (2.2), one can choose them to be either the same, $z_1 = z_2$ (class A), or different, $z_1 \neq z_2$ (class B). In the case of class A transformations, one can construct three different types of linearizing transformations, whereas for class B, one can formulate six different types of linearizing transformations, as we point out below. However, we also note that even further types of local transformations involving the variables $\dot{x}$ and $\dot{y}$ are possible, but these are not included in the present study.

(i) *Class A linearizing transformations* ($z_1 = z_2 = z$)

In the case of class A transformations, we have $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = z_2 = z = \int f_3(t, x, y, \dot{x}, \dot{y}) \, \mathrm{d}t = \int f_4(t, x, y, \dot{x}, \dot{y}) \, \mathrm{d}t$. Now appropriately restricting the form of f_3 $(= f_4)$, one can identify three different types of linearizing transformations.

(i) Suppose $z_1 = z_2 = z$ is a perfect differential function and w_i, $i = 1, 2$, and z do not contain the variables $\dot{x}$ and $\dot{y}$, then we call the resultant transformation, namely $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$ and $z = f_3(t, x, y)$, a point transformation of type I.

(ii) On the other hand, if z is not a perfect differential function, and w_i, $i = 1, 2$, and z do not contain the variables $\dot{x}$ and $\dot{y}$, then we call the resultant transformation, namely $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$ and $z = \int f_3(t, x, y) \, \mathrm{d}t$, a generalized Sundman transformation of type I.

(iii) As a more general case, if we consider the independent variable z to contain the derivative terms also, i.e. $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$ and $z = \int f_3(t, x, y, \dot{x}, \dot{y}) \, \mathrm{d}t$, then we call the resultant transformation a generalized linearizing transformation of type I.

In our analysis, we do not consider the possibility $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$ and $z = f_3(t, x, y, \dot{x}, \dot{y})$ because the procedure to handle it is different from the presently discussed linearizing transformations. This possibility will be studied separately.

(ii) *Class B linearizing transformations* ($z_1 \neq z_2$)

In the class B type of linearizing transformations, we have $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = \int f_3(t, x, y, \dot{x}, \dot{y}) \, dt$ and $z_2 = \int f_4(t, x, y, \dot{x}, \dot{y}) \, dt$, $z_1 \neq z_2$. Now appropriately restricting the forms of f_3 and f_4, one can obtain six different types of linearizing transformations.

(i) If z_1 and z_2 are perfect differential functions and w_i and z_i, $i = 1, 2$, do not contain the variables $\dot{x}$ and $\dot{y}$, then we call the resultant transformation, namely $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = f_3(t, x, y)$ and $z_2 = f_4(t, x, y)$, a point transformation of type II.

(ii) Suppose z_1 is a perfect differential function and z_2 is not a perfect differential function or vice versa, and if z_1 and z_2 do not contain the variables $\dot{x}$ and $\dot{y}$, then we call the resultant transformation, namely $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = f_3(t, x, y)$ and $z_2 = \int f_4(t, x, y) \, dt$ or $z_1 = \int f_3(t, x, y) \, dt$ and $z_2 = f_4(t, x, y)$, a mixed point-generalized Sundman transformation.

(iii) On the other hand, if any one of the independent variables contains the variables $\dot{x}$ and $\dot{y}$, then we call the resultant transformation, namely $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = f_3(t, x, y)$ and $z_2 = \int f_4(t, x, y, \dot{x}, \dot{y}) \, dt$ or $z_1 = \int f_3(t, x, y, \dot{x}, \dot{y}) \, dt$ and $z_2 = f_4(t, x, y)$, a mixed point-generalized linearizing transformation.

(iv) Suppose the independent variables are not perfect differential functions and are also not functions of $\dot{x}$ and $\dot{y}$, i.e. $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = \int f_3(t, x, y) \, dt$ and $z_2 = \int f_4(t, x, y) \, dt$, then we call the resultant transformation a generalized Sundman transformation of type II (GST II).

(v) Further, if one of the independent variables, say z_1, does not contain the derivative terms, whereas the other independent variable z_2 does contain the derivative terms or vice versa, i.e. $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = \int f_3(t, x, y) \, dt$ and $z_2 = \int f_4(t, x, y, \dot{x}, \dot{y}) \, dt$ or $z_1 = \int f_3(t, x, y, \dot{x}, \dot{y}) \, dt$ and $z_2 = \int f_4(t, x, y) \, dt$, then we call the resultant transformation a mixed generalized Sundman-generalized linearizing transformation.

(vi) As a general case, if we allow both the independent variables z_1 and z_2 to be non-perfect differential functions and also to contain derivative terms, i.e. $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = \int f_3(t, x, y, \dot{x}, \dot{y}) \, dt$ and $z_2 = \int f_4(t, x, y, \dot{x}, \dot{y}) \, dt$, then the resultant transformation will be termed a generalized linearizing transformation of type II.

Finally, the possibility that $w_1 = f_1(t, x, y)$, $w_2 = f_2(t, x, y)$, $z_1 = f_3(t, x, y, \dot{x}, \dot{y})$ and $z_2 = f_4(t, x, y, \dot{x}, \dot{y})$ is not considered in this study and will be pursued separately.

3. Applications

In this section, we consider specific examples and illustrate each one of the linearizing transformations identified in the previous section so as to make clear the applicability of them under different situations.

(*a*) *Class-A linearizing transformations* ($z_1 = z_2$)

(i) *Example 1: point transformation of type I*

Let us consider the system of SNODEs

$$\ddot{x} + \frac{(\dot{x}y - \dot{y}x)^2}{2xy(x - y)} + \omega^2 x = 0 \quad \text{and} \quad \ddot{y} - \frac{(\dot{x}y - \dot{y}x)^2}{2xy(x - y)} + \omega^2 y = 0, \tag{3.1}$$

where ω is an arbitrary constant. The first two integrals associated with equation (3.1), which can be obtained using the formulation given in §2 of part IV (Chandrasekar *et al.* 2009), can be written as

$$\left. \begin{aligned} I_1 &= (\dot{x} + \dot{y})\sin(\omega t) - \omega(x + y)\cos(\omega t) \\[1mm] I_2 &= \frac{1}{2\sqrt{xy}}((\dot{x}y + x\dot{y})\sin(\omega t) - 2\omega xy\cos(\omega t)). \end{aligned} \right\} \tag{3.2}$$

and

Rewriting equation (3.2) in the form of equations (2.5) and (2.6), we obtain

$$\left. \begin{aligned} I_1 &= \sin^2(\omega t)\frac{\mathrm{d}}{\mathrm{d}t}((x + y)\operatorname{cosec}(\omega t)) = \frac{\mathrm{d}t}{\mathrm{d}z_1}\frac{\mathrm{d}w_1}{\mathrm{d}t} = \frac{\mathrm{d}w_1}{\mathrm{d}z_1} \\[2mm] I_2 &= \sin^2(\omega t)\frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{xy}\operatorname{cosec}(\omega t)) = \frac{\mathrm{d}t}{\mathrm{d}z_2}\frac{\mathrm{d}w_1}{\mathrm{d}t} = \frac{\mathrm{d}w_2}{\mathrm{d}z_2}, \end{aligned} \right\} \tag{3.3}$$

and

so that we can identify point transformation of type I as

$$w_1 = (x + y)\operatorname{cosec}(\omega t), \quad w_2 = \sqrt{xy}\operatorname{cosec}(\omega t) \quad \text{and} \quad z_1 = z_2 = z = -\frac{1}{\omega}\cot(\omega t). \tag{3.4}$$

Using the transformation (3.4), one can transform equation (3.1) to a set of free particle equations, namely $\mathrm{d}^2 w_1/\mathrm{d}z^2 = 0$ and $\mathrm{d}^2 w_2/\mathrm{d}z^2 = 0$, so that $w_1 = I_1 z + I_3$ and $w_2 = I_2 z + I_4$, where I_3 and I_4 are the integration constants. Substituting the expressions for w_i, $i = 1, 2$, and z in the free particle solutions and rewriting the resultant expressions in terms of the old variables x and y, one obtains the general solution for equation (3.1) in the form

$$x(t) = -\frac{1}{2}(A \mp \sqrt{A^2 - 4B^2}) \quad \text{and} \quad y(t) = \frac{1}{2}(A \pm \sqrt{A^2 - 4B^2}), \tag{3.5}$$

where $A = I_1\cos(\omega t) + I_3\sin(\omega t)$ and $B = I_2\cos(\omega t) + I_4\sin(\omega t)$. Here, we point out that the nonlinear system (3.1) admits amplitude-independent frequency of oscillations.

In the above example, we have considered the new dependent variables w_1 and w_2 and independent variable z to be functions of only x, y and t. We will now consider examples that admit more general transformations.

(ii) *Example 2: generalized Sundman transformation of type I*

Let us focus our attention on the two-dimensional Mathews–Lakshmanan oscillator system of the form (Cariñena *et al.* 2004; Chandrasekar *et al.* 2009)

$$\ddot{x} = \frac{\lambda(\dot{x}^2 + \dot{y}^2 + \lambda(x\dot{y} - y\dot{x})^2)x - \alpha^2 x}{(1 + \lambda r^2)}$$

and

$$\ddot{y} = \frac{\lambda(\dot{x}^2 + \dot{y}^2 + \lambda(x\dot{y} - y\dot{x})^2)y - \alpha^2 y}{(1 + \lambda r^2)},$$

(3.6)

where $r = \sqrt{x^2 + y^2}$ and λ and α are arbitrary parameters. For $\alpha = 0$, equation (3.6) admits the following two integrals of motion:

$$\hat{I}_1 = \frac{(1 + \lambda y^2)\dot{x} - \lambda xy\dot{y}}{\sqrt{1 + \lambda r^2}} \quad \text{and} \quad \hat{I}_2 = \frac{(1 + \lambda x^2)\dot{y} - \lambda xy\dot{x}}{\sqrt{1 + \lambda r^2}}. \quad (3.7)$$

We note that the integrals I_2 and I_3 (eqns (5.35) and (5.36) in part IV; Chandrasekar *et al.* 2009) can be derived from equation (3.7) by using the relations $I_2 = -\lambda(\hat{I}_1^2 + \hat{I}_2^2 + \lambda I_1^2)$ and $I_3 = (\hat{I}_1 + \hat{I}_2)^2 - 2\lambda I_1^2$, where I_1 is given in eqn (5.35) of part IV. The general case $(\alpha \neq 0)$ can be linearized through mixed generalized Sundman-generalized linearized transformation (example 8). To demonstrate the linearization through generalized Sundman transformation of type I, we here consider equation (3.6) with $\alpha = 0$.

For the case $\alpha = 0$, one may note that, on making a substitution $y(t) = y(x(t))$ into equation (3.6), one can obtain a non-autonomous second-order ODE in $y(x)$. Although this equation satisfies the linearization condition for point transformation (Sookmee & Meleshko 2008), finding the linearizing transformation and the general solution for the transformed ODE turns out to be non-trivial. On the other hand, we provide a straightforward procedure of linearization.

The above two integrals (3.7) can be rewritten as

$$\hat{I}_1 = (1 + \lambda r^2)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{\sqrt{1 + \lambda r^2}}\right) = \frac{\mathrm{d}t}{\mathrm{d}z_1}\frac{\mathrm{d}w_1}{\mathrm{d}t} = \frac{\mathrm{d}w_1}{\mathrm{d}z_1} \quad (3.8)$$

and

$$\hat{I}_2 = (1 + \lambda r^2)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{\sqrt{1 + \lambda r^2}}\right) = \frac{\mathrm{d}t}{\mathrm{d}z_2}\frac{\mathrm{d}w_2}{\mathrm{d}t} = \frac{\mathrm{d}w_2}{\mathrm{d}z_2}. \quad (3.9)$$

From the above equations, we identify the new dependent and independent variables as

$$w_1 = \frac{x}{\sqrt{1 + \lambda r^2}}, \quad w_2 = \frac{y}{\sqrt{1 + \lambda r^2}} \quad \text{and} \quad z_1 = z_2 = z = \int \frac{\mathrm{d}t}{(1 + \lambda r^2)}. \quad (3.10)$$

One may observe that the independent variables z_1 and z_2 are not perfect differentials, even though they turn out to be identical. By using the above new variables, one can transform equation (3.6), with $\alpha = 0$, to the free particle equations, i.e. $\mathrm{d}^2 w_1/\mathrm{d}z^2 = 0$ and $\mathrm{d}^2 w_2/\mathrm{d}z^2 = 0$.

Unlike the earlier example, one cannot unambiguously integrate these two linear equations in terms of the original variables because of the non-local nature

of the independent variable. To overcome this problem, one should express $(1 + \lambda r^2)$ in terms of either z_1 or z_2 so that the resultant expression $dz_1 = dz_2 = dt/(1 + \lambda r^2)$ can be integrated. In the following, we describe a procedure to obtain an expression for the new independent variable.

Now integrating equation (3.8) and using the first relation in equation (3.10), we obtain

$$\frac{x}{\sqrt{1 + \lambda r^2}} = \hat{I}_1 z_1, \tag{3.11}$$

where we have fixed the integration constant to be zero (without loss of generality). On the other hand, from expressions (3.8) and (3.9), we obtain $dw_1/dw_2 = \hat{I}_1/\hat{I}_2$, from which one obtains

$$w_1 = \frac{\hat{I}_1}{\hat{I}_2} w_2 + \hat{I}_3 \Rightarrow \frac{x}{\sqrt{1 + \lambda r^2}} = \frac{\hat{I}_1}{\hat{I}_2} \frac{y}{\sqrt{1 + \lambda r^2}} + \hat{I}_3, \tag{3.12}$$

where $\hat{I}_3$ is the integration constant. Equation (3.12) provides us with an identity

$$\frac{y}{\sqrt{1 + \lambda r^2}} = \hat{I}_2 z_1 - \frac{\hat{I}_2 \hat{I}_3}{\hat{I}_1}. \tag{3.13}$$

Now squaring and adding equations (3.11) and (3.13), we obtain

$$\frac{\lambda r^2}{1 + \lambda r^2} = \lambda \left((\hat{I}_1^2 + \hat{I}_2^2) z_1^2 - 2 \frac{\hat{I}_2^2 \hat{I}_3}{\hat{I}_1} z_1 + \frac{\hat{I}_2^2 \hat{I}_3^2}{\hat{I}_1^2} \right). \tag{3.14}$$

From equation (3.14), one can express $(1 + \lambda r^2)$ in terms of z_1 as

$$1 + \lambda r^2 = \frac{1}{1 - \lambda \left((\hat{I}_1^2 + \hat{I}_2^2) z_1^2 - (2\hat{I}_2^2 \hat{I}_3/\hat{I}_1) z_1 + (\hat{I}_2^2 \hat{I}_3^2/\hat{I}_1^2) \right)}. \tag{3.15}$$

Substituting equation (3.15) in the last relation given in equation (3.10), we arrive at the following integral relationship between z_1 and t, namely

$$dz_1 = \left(1 - \lambda \left((\hat{I}_1^2 + \hat{I}_2^2) z_1^2 - \frac{2\hat{I}_2^2 \hat{I}_3}{\hat{I}_1} z_1 + \frac{\hat{I}_2^2 \hat{I}_3^2}{\hat{I}_1^2} \right) \right) dt. \tag{3.16}$$

As the variables are separated now, one can integrate this equation and obtain an expression that connects the new independent variable with the old independent variable through the relation

$$z_1 = \frac{\sqrt{\lambda} \hat{I}_2^2 \hat{I}_3 - \hat{I}_1 \omega \tan[\omega(t - t_0)]}{\sqrt{\lambda} \hat{I}_1 (\hat{I}_1^2 + \hat{I}_2^2)}, \quad (\hat{I}_2^2 (\lambda \hat{I}_3^2 - 1) - \hat{I}_1^2) > 0, \tag{3.17}$$

where $\omega = \sqrt{\lambda}\sqrt{\hat{I}_2^2(\lambda\hat{I}_3^2 - 1) - \hat{I}_1^2}$ and t_0 is the fourth integration constant. From equations (3.11)–(3.13) and (3.15), we obtain

$$
\left.
\begin{aligned}
x(t) &= \hat{I}_1 z_1 \left[1 - \lambda \left((\hat{I}_1^2 + \hat{I}_2^2) z_1^2 - \frac{2\hat{I}_2^2 \hat{I}_3}{\hat{I}_1} z_1 + \frac{\hat{I}_2^2 \hat{I}_3^2}{\hat{I}_1^2} \right) \right]^{-1/2} \\
\text{and} \\
y(t) &= \left(\hat{I}_2 z_1 - \frac{\hat{I}_2 \hat{I}_3}{\hat{I}_1} \right) \left[1 - \lambda \left((\hat{I}_1^2 + \hat{I}_2^2) z_1^2 - \frac{2\hat{I}_2^2 \hat{I}_3}{\hat{I}_1} z_1 + \frac{\hat{I}_2^2 \hat{I}_3^2}{\hat{I}_1^2} \right) \right]^{-1/2} .
\end{aligned}
\right\} \tag{3.18}
$$

Substituting expression (3.17) in equation (3.18) and simplifying the resultant expressions, we arrive at the following general solution for equation (3.6), with $\alpha = 0$, in the form:

$$
\left.
\begin{aligned}
x(t) &= A \left(\lambda \hat{I}_2^2 \hat{I}_3 \cos[\omega(t - t_0)] - \omega \hat{I}_1 \sin[\omega(t - t_0)] \right) \\
\text{and} \\
y(t) &= -\hat{I}_2 A \left(\lambda \hat{I}_1 \hat{I}_3 \cos[\omega(t - t_0)] + \omega \sin[\omega(t - t_0)] \right),
\end{aligned}
\right\} \tag{3.19}
$$

where $A = (1/\lambda(\hat{I}_1^2 + \hat{I}_2^2))\sqrt{(\hat{I}_1^2 + \hat{I}_2^2)/(\hat{I}_1^2 + \hat{I}_2^2(1 - \lambda\hat{I}_3^2))}$ and $\omega = \sqrt{\lambda}\sqrt{\hat{I}_2^2(\lambda\hat{I}_3^2 - 1) - \hat{I}_1^2}$.

(iii) *Example 3: generalized linearizing transformation of type I*

In the previous example, we restricted the new independent variable to be a non-local one and a function of only t, x and y. Now we relax the latter condition and also allow the independent variables z_1 and z_2 to contain derivative terms, namely $\dot{x}$ and $\dot{y}$. To illustrate this case, let us consider the two coupled second-order equations of the form

$$
\left.
\begin{aligned}
\ddot{x} &= \frac{2\dot{x}\dot{y}(x\dot{x} + x^3) - 2y\dot{x}^3}{x^3 y} \\
\text{and} \\
\ddot{y} &= \frac{2\dot{x}\dot{y}(x^2 y - y\dot{x} + x\dot{y}) + x(x^2\dot{y}^2 - y^2\dot{x}^2)}{x^3 y} .
\end{aligned}
\right\} \tag{3.20}
$$

One can easily identify two integrals for equation (3.20) of the form

$$
I_1 = y^2 \left(\frac{1}{x^2} + \frac{1}{\dot{x}} \right) \quad \text{and} \quad I_2 = y \left(\frac{y}{x} - \frac{\dot{y}}{\dot{x}} \right). \tag{3.21}
$$

Rewriting the above integrals as

$$
I_1 = \frac{y^2}{\dot{x}} \frac{\mathrm{d}}{\mathrm{d}t} \left(t - \frac{1}{x} \right) = \frac{\mathrm{d}w_1}{\mathrm{d}z_1} \quad \text{and} \quad I_2 = \frac{y^2}{\dot{x}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\log\left[\frac{x}{y} \right] \right) = \frac{\mathrm{d}w_2}{\mathrm{d}z_2}, \tag{3.22}
$$

we identify the following set of linearizing transformations for equation (3.20), i.e.

$$
w_1 = t - \frac{1}{x}, \quad w_2 = \log\left[\frac{x}{y} \right] \quad \text{and} \quad z_1 = z_2 = z = \int \frac{\dot{x}}{y^2}\, \mathrm{d}t. \tag{3.23}
$$

One may note that the independent variables are not only non-local, but also involve derivative terms. It is easy to check that equation (3.23) transforms equation (3.20) to the linearized form (2.3).

Again, as in the previous example, one cannot directly obtain the solution for equation (3.20) by direct integration of the linear ODEs because of the non-local nature of the independent variables. This can be overcome by expressing the term $\dot{x}/y^2$ in terms of either z_1 or z_2 so that the resultant equation can be integrated to obtain an explicit form for the new independent variable in terms of the old variables, as we discuss below.

From equation (3.21), we have

$$\frac{\dot{x}}{y^2} = \frac{1}{(I_1 - (y^2/x^2))}. \tag{3.24}$$

Since $\mathrm{d}w_1/\mathrm{d}z_1 = I_1$ (from equation (3.22)), we have $w_1 = I_1 z_1$, so that

$$t - \frac{1}{x} = I_1 z_1, \tag{3.25}$$

where, without loss of generality, we have fixed the integration constant to be zero. On the other hand, from equation (3.22), we have $\mathrm{d}w_1/\mathrm{d}w_2 = I_1/I_2$, which, in turn, gives

$$w_1 = \frac{I_1}{I_2} w_2 + I_3. \tag{3.26}$$

In terms of old variables, equation (3.26) can be rewritten as

$$t - \frac{1}{x} = \frac{I_1}{I_2} \log\left[\frac{x}{y}\right] + I_3. \tag{3.27}$$

From the identites (3.25) and (3.27), we can express y/x in terms of z_1 in the form

$$\frac{y}{x} = \exp\left[-\frac{I_2}{I_1}(I_1 z_1 - I_3)\right]. \tag{3.28}$$

Now substituting equation (3.28) into equation (3.24), we can express $(\dot{x}/y^2)$ in terms of z_1 and, plugging the latter relation into the third relation in equation (3.23), we arrive at

$$\mathrm{d}t = \left(I_1 - \exp\left[-2\frac{I_2}{I_1}(I_1 z_1 - I_3)\right]\right)\mathrm{d}z_1. \tag{3.29}$$

Integrating the above equation, we obtain

$$t + t_0 = I_1 z_1 + \frac{\mathrm{e}^{-(2I_2/I_1)(I_1 z_1 - I_3)}}{2I_2}, \tag{3.30}$$

where t_0 is the fourth integration constant. Substituting the expression $z_1 = (t - 1/x)/I_1$ (equation (3.25)) into equations (3.28) and (3.30), we obtain

the general solution of equation (3.20) in the implicit form

$$
\left.
\begin{aligned}
x(t) &= -\frac{1}{t_0} + \frac{x(t)\, e^{-(2I_2/I_1)(t-(1/x(t))-I_3)}}{2 I_2 t_0} \\[2ex]
y(t) &= x(t) \exp\left[-\frac{I_2}{I_1}\left(t - \frac{1}{x(t)} - I_3\right)\right].
\end{aligned}
\right\}
\tag{3.31}
$$

(b) Class B linearizing transformations ($z_1 \neq z_2$)

In the class A category, in all the three examples, we considered that the new independent variables z_1 and z_2 are identical. However, this need not always be the case in the theory of linearizing transformations, as discussed in §2. We now present specific examples to illustrate more general transformations.

(i) Example 4: point transformation of type II

Let us consider a quasi-periodic oscillator governed by a set of two coupled SNODEs of the form

$$
\left.
\begin{aligned}
\ddot{x} + \frac{(\dot{x}y - \dot{y}x)^2 + 2x^2 y(\omega_1^2(x+y) - 2\omega_2^2 y)}{2xy(x-y)} &= 0 \\[2ex]
\ddot{y} - \frac{(\dot{x}y - \dot{y}x)^2 + 2xy^2(\omega_1^2(x+y) - 2\omega_2^2 x)}{2xy(x-y)} &= 0.
\end{aligned}
\right\}
\tag{3.32}
$$

To explore the linearizing transformation for equation (3.32), we consider the two associated integrals

$$
\left.
\begin{aligned}
I_1 &= (\dot{x} + \dot{y})\cos\omega_1 t + \omega_1(x+y)\sin\omega_1 t \\[2ex]
I_2 &= \frac{1}{2\sqrt{xy}}((\dot{x}y + \dot{y}x)\sin\omega_2 t - 2\omega_2 xy \cos\omega_2 t).
\end{aligned}
\right\}
\tag{3.33}
$$

Rewriting equation (3.33) in the form

$$
I_1 = \cos^2(\omega_1 t)\frac{d}{dt}((x+y)\sec(\omega_1 t)) = \frac{dw_1}{dz_1}
\tag{3.34}
$$

and

$$
I_2 = \sin^2(\omega_2 t)\frac{d}{dt}(\sqrt{xy}\,\mathrm{cosec}(\omega_2 t)) = \frac{dw_2}{dz_2},
\tag{3.35}
$$

one can identify the new dependent and independent variables as

$$
\left.
\begin{aligned}
w_1 &= (x+y)\sec(\omega_1 t), \quad z_1 = \frac{1}{\omega_1}\tan(\omega_1 t), \\[2ex]
w_2 &= \sqrt{xy}\,\mathrm{cosec}(\omega_2 t), \quad z_2 = -\frac{1}{\omega_1}\cot(\omega_2 t).
\end{aligned}
\right\}
\tag{3.36}
$$

One may note that now the independent variables z_1 and z_2 are not the same.

The new variables transform equation (3.32) to the free particle equations $\mathrm{d}^2 w_1 / \mathrm{d} z_1^2 = 0$ and $\mathrm{d}^2 w_2 / \mathrm{d} z_2^2 = 0$. From the general solutions $w_1 = I_1 z_1 + I_3$ and $w_2 = I_2 z_2 + I_4$, where I_i, $i = 1, 2, 3, 4$, are the integration constants, and using the expressions for w_i and z_i, $i = 1, 2$, given in equation (3.36), we arrive at the general solution for equation (3.32) in the form

$$x(t) = -\frac{1}{2}(A \mp \sqrt{A^2 - 4B^2}) \quad \text{and} \quad y(t) = \frac{1}{2}(A \pm \sqrt{A^2 - 4B^2}), \qquad (3.37)$$

where $A = I_1 \sin(\omega_1 t) + I_3 \cos(\omega_1 t)$ and $B = I_2 \cos(\omega_2 t) + I_4 \sin(\omega_2 t)$.

(ii) *Example 5: mixed point-generalized Sundman transformation*

Let us consider the two-dimensional force-free coupled Duffing–van der Pol (DVP) oscillator equation of the form

$$\left.\begin{aligned}
&\ddot{x} + 4(\alpha + \beta(k_1 x + k_2 y)^2)\dot{x} + \alpha(3\alpha + 4\beta(k_1 x + k_2 y)^2)x = 0 \\
\text{and} \quad &\ddot{y} + 4(\alpha + \beta(k_1 x + k_2 y)^2)\dot{y} + \alpha(3\alpha + 4\beta(k_1 x + k_2 y)^2)y = 0.
\end{aligned}\right\} \qquad (3.38)$$

One may note that the point transformations $X = k_1 x + k_2 y$ and $Y = k_1 x - k_2 y$ help one to rewrite equation (3.38) in a separable form

$$\ddot{X} + 4(\alpha + \beta X^2)\dot{X} + \alpha(3\alpha + 4\beta X^2)X = 0 \qquad (3.39a)$$

and

$$\ddot{Y} + 4(\alpha + \beta X^2)\dot{Y} + \alpha(3\alpha + 4\beta X^2)Y = 0. \qquad (3.39b)$$

The solution to equation (3.39a) can only be obtained in implicit form (Chandrasekar *et al.* 2005). Consequently, equation (3.39b) cannot be solved explicitly in this way. Further, the linearization of the scalar DVP oscillator (3.39a) itself has not yet been reported. In the following, we use our procedure to find the linearizing transformation and general solution to equation (3.38) straightforwardly.

The first two integrals for equation (3.38) can easily be identified using the procedure given in Chandrasekar *et al.* (2009) in the form

$$\left.\begin{aligned}
&I_1 = \left(\frac{x\dot{y} - y\dot{x}}{\dot{y} + \alpha y}\right) \mathrm{e}^{\alpha t} \\
\text{and} \quad &I_2 = (k_1 \dot{x} + k_2 \dot{y} + \alpha(k_1 x + k_2 y) + \frac{4\beta}{3}(k_1 x + k_2 y)^3) \mathrm{e}^{3\alpha t}.
\end{aligned}\right\} \qquad (3.40)$$

The above integrals can be rewritten as

$$\left.\begin{aligned}
&I_1 = -\frac{y^2 \mathrm{e}^{\alpha t}}{(\dot{y} + \alpha y)} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{y}\right) = \frac{\mathrm{d}w_1}{\mathrm{d}z_1} \\
\text{and} \quad &I_2 = -((k_1 x + k_2 y) \mathrm{e}^{5/3\alpha t})^3 \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{2}(k_1 x + k_2 y)^{-2} + \frac{2\beta}{3\alpha}\right) \mathrm{e}^{-2\alpha t}\right] = \frac{\mathrm{d}w_2}{\mathrm{d}z_2},
\end{aligned}\right\} \qquad (3.41)$$

from which we can obtain the following linearizing transformations:

$$w_1 = \frac{x}{y}, \quad w_2 = \left(\frac{1}{2}(k_1 x + k_2 y)^{-2} + \frac{2\beta}{3\alpha}\right) e^{-2\alpha t}$$

and

$$z_1 = \frac{e^{-\alpha t}}{y}, \quad z_2 = -\int ((k_1 x + k_2 y) e^{(5/3)\alpha t})^{-3} \, dt. \qquad (3.42)$$

One may note that, in the present problem, one of the new independent variables, i.e. z_2, is in a non-local form. In terms of the above new variables, equation (3.38) assumes the forms $d^2 w_1/dz_1^2 = 0$ and $d^2 w_2/dz_2^2 = 0$.

Now we seek the general solution of equation (3.38) from the linearized equations. Integrating $d^2 w_1/dz_1^2 = 0$, we obtain

$$w_1 = I_1 z_1 + I_3, \qquad (3.43)$$

where I_3 is the integration constant. Rewriting equation (3.43) in terms of the old variables, we obtain

$$x = I_1 \, e^{-\alpha t} + I_3 y. \qquad (3.44)$$

However, the second linear equation, $d^2 w_2/dz_2^2 = 0$, cannot be integrated straightforwardly (in terms of the original variables) because of the non-local nature of the second independent variable. To obtain an explicit form for z_2, we rewrite I_2 in the integral form (equation (3.41)) to obtain

$$\left(\frac{1}{2}(k_1 x + k_2 y)^{-2} + \frac{2\beta}{3\alpha}\right) e^{-2\alpha t} = -I_2 \int [(k_1 x + k_2 y) e^{(5/3)\alpha t}]^{-3} \, dt = I_2 z_2. \qquad (3.45)$$

Equation (3.45) provides

$$(k_1 x + k_2 y)^{-1} = \sqrt{2 I_2 z_2 \, e^{2\alpha t} - \frac{4\beta}{3\alpha}}. \qquad (3.46)$$

Now substituting relation (3.46) into the non-local variable z_2 (equation (3.42)), one obtains

$$\frac{dz_2}{dt} = -I_2 \left(2 I_2 z_2 - \frac{4\beta}{3\alpha} e^{-2\alpha t}\right)^{3/2} e^{-2\alpha t}. \qquad (3.47)$$

Solving the above equation, we obtain

$$t_0 - \frac{1}{2\alpha} e^{2\alpha t} = \frac{a}{3 I_2} \left[\sqrt{3} \tan^{-1}\left(\frac{\sqrt{3}}{2 a(2 I_2 z_2 - (4\beta/3\alpha) e^{-2\alpha t})^{1/2} - 1}\right)\right.$$
$$\left. + \frac{1}{2}\log\left(\frac{(1 + a(2 I_2 z_2 - (4\beta/3\alpha) e^{-2\alpha t})^{1/2})^2}{1 - a(2 I_2 z_2 - (4\beta/3\alpha) e^{-2\alpha t})^{1/2} + a^2(2 I_2 z_2 \, e^{2\alpha t} - (4\beta/3\alpha) e^{-2\alpha t})}\right)\right], \qquad (3.48)$$

where $a = \sqrt[3]{-3 I_2/4\beta}$ and t_0 is the fourth integration constant. From expression (3.48) and equations (3.44) and (3.46), one can deduce the general solution for equation (3.38) in implicit form. The resultant expression coincides exactly with eqn (6.18) given in Chandrasekar *et al.* (2009).

In the present example, we considered one of the independent variables to be in a non-local form. As we have two independent variables, one can also have the possibility of having both the independent variables to be of non-local form. Indeed, this is the case in our next example.

(iii) *Example 6: generalized Sundman transformation of type II*

To illustrate the GST II, we consider the equation of the form

$$\left.\begin{array}{l} \ddot{x} - \dfrac{2}{(x^2+y^2)}((\dot{x}^2-\dot{y}^2)x + 2y\dot{x}\dot{y}) + \dfrac{2}{t^2}x = 0 \\[12pt] \ddot{y} - \dfrac{2}{(x^2+y^2)}(2x\dot{x}\dot{y} - (\dot{x}^2-\dot{y}^2)y) + \dfrac{2}{t^2}y = 0. \end{array}\right\} \tag{3.49}$$

and

The first two integrals for equation (3.49) can be evaluated as

$$\left.\begin{array}{l} I_1 = \dfrac{(2(t\dot{y}-y)xy) + (x^2-y^2)(t\dot{x}-x)}{t^2(x^2+y^2)^2} \\[12pt] I_2 = \dfrac{(2xy(t\dot{x}-x) + (x^2-y^2)(t\dot{y}-y))}{t^2(x^2+y^2)^2}. \end{array}\right\} \tag{3.50}$$

and

Rewriting these two integrals as

$$\left.\begin{array}{l} I_1 = \dfrac{x^2}{(x^2+y^2)^2}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{x}{t} + \dfrac{y^2}{tx}\right) = \dfrac{\mathrm{d}w_1}{\mathrm{d}z_1} \\[14pt] I_2 = \dfrac{y^2}{(x^2+y^2)^2}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{y}{t} + \dfrac{x^2}{ty}\right) = \dfrac{\mathrm{d}w_2}{\mathrm{d}z_2}, \end{array}\right\} \tag{3.51}$$

and

we identify the linearizing transformations in a more general form

$$w_1 = \frac{x}{t} + \frac{y^2}{tx}, \quad w_2 = \frac{y}{t} + \frac{x^2}{ty}, \quad z_1 = \int \frac{(x^2+y^2)^2}{x^2}\,\mathrm{d}t \quad \text{and} \quad z_2 = \int \frac{(x^2+y^2)^2}{y^2}\,\mathrm{d}t. \tag{3.52}$$

The GST II equation (3.52) takes equation (3.38) to the free particle equations, $\mathrm{d}^2 w_1/\mathrm{d}z_1^2 = 0$ and $\mathrm{d}^2 w_2/\mathrm{d}z_2^2 = 0$. To obtain the solution in terms of the original variables, we have to replace both $\int((x^2+y^2)^2/x^2)\,\mathrm{d}t$ and $\int((x^2+y^2)^2/y^2)\,\mathrm{d}t$ by the variables z_1 and t, and z_2 and t, respectively, and integrate the resultant equations.

To do so, first we rewrite the first integrals I_1 and I_2 given by equation (3.51) in integral forms, which in turn lead us to $w_1 = I_1 z_1$ and $w_2 = I_2 z_2$. As w_1 and w_2 do not contain non-local variables, we can replace them by the old variables (equation (3.52)), i.e.

$$\frac{x}{t} + \frac{y^2}{tx} = I_1 z_1 \quad \text{and} \quad \frac{y}{t} + \frac{x^2}{ty} = I_2 z_2, \tag{3.53}$$

where we have fixed the integration constants to be zero (without loss of generality).

We observe that, to integrate the last two expressions in equation (3.52), one should further replace z_1 and z_2 in terms of t. So, we substitute the above expressions for x and y in terms of z_1 and z_2, respectively, in the last two relations in equation (3.52), and obtain

$$\mathrm{d}z_1 = I_1^2 z_1^2 t^2 \, \mathrm{d}t \quad \text{and} \quad \mathrm{d}z_2 = I_2^2 z_2^2 t^2 \, \mathrm{d}t. \tag{3.54}$$

Now integrating both the equations, we obtain

$$z_1 = \left(I_1^2 \left(I_3 - \frac{t^3}{3} \right) \right)^{-1} \quad \text{and} \quad z_2 = \left(I_2^2 \left(I_4 - \frac{t^3}{3} \right) \right)^{-1}, \tag{3.55}$$

where I_3 and I_4 are the third and forth integration constants, respectively. Plugging equation (3.55) into equation (3.53), we arrive at the following general solution for equation (3.49):

$$\left.\begin{aligned}
x(t) &= \frac{3t(I_1(3\hat{I}_3 - (I_1^2 - I_2^2)t^3) - I_2(3\hat{I}_4 + 2I_1 I_2 t^3)}{(3\hat{I}_3 - (I_1^2 - I_2^2)t^3)^2 + (3\hat{I}_4 + 2I_1 I_2 t^3)^2} \\[2ex]
y(t) &= \frac{3t(I_2(I_1^2 - I_2^2)t^3 - 3\hat{I}_3) - I_1(3\hat{I}_4 + 2I_1 I_2 t^3)}{(3\hat{I}_3 - (I_1^2 - I_2^2)t^3)^2 + (3\hat{I}_4 + 2I_1 I_2 t^3)^2},
\end{aligned}\right\} \tag{3.56}$$

where $\hat{I}_3 = (I_1 I_3 - I_2 I_4)/(I_1^3 + I_1 I_2^2)$ and $\hat{I}_4 = (I_2 I_3 + I_1 I_4)/(I_2^3 - I_1^2 I_2)$.

In the previous two examples, we focussed our attention on the case in which the new independent variable(s) is (are) non-local and does (do) not admit velocity-dependent terms. Now we relax this condition and allow either one or both the independent variables to admit velocity-dependent terms but in non-local form.

(iv) *Example 7: mixed point-generalized linearizing transformation*

To demonstrate this, we consider a variant of the two-dimensional Mathews and Lakshmanan equation (3.6) of the form

$$\ddot{x} = \frac{\lambda(\dot{x}^2 + \dot{y}^2 + 2\lambda(\dot{y} - \dot{x})^2) - \alpha^2}{(1 + 2\lambda(x + y))} \quad \text{and} \quad \ddot{y} = \frac{\lambda(\dot{x}^2 + \dot{y}^2 + 2\lambda(\dot{y} - \dot{x})^2) - \alpha^2}{(1 + 2\lambda(x + y))}. \tag{3.57}$$

Equation (3.57) admits the following two integrals of motion:

$$I_1 = \dot{x} - \dot{y} \quad \text{and} \quad I_2 = \frac{\alpha^2 - \lambda((1 + 2\lambda)(\dot{y} - \dot{x})^2 + 2\dot{x}\dot{y})}{(1 + 2\lambda(x + y))}. \tag{3.58}$$

Rewriting equation (3.58) in the form

$$I_1 = \frac{\mathrm{d}}{\mathrm{d}t}(x - y) \tag{3.59}$$

and

$$I_2 = \frac{\alpha^2 - \lambda((1 + 2\lambda)(\dot{y} - \dot{x})^2 + 2\dot{x}\dot{y})}{2\lambda(\dot{x} + \dot{y})} \frac{\mathrm{d}}{\mathrm{d}t}\left[\log[1 + 2\lambda(x + y)]\right], \tag{3.60}$$

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one can easily identify the linearizing transformations for equation (3.57) as

and

$$
\left.
\begin{array}{l}
w_1 = (x - y), \quad w_2 = \log[1 + 2\lambda(x + y)] \\[2mm]
z_1 = t, \quad z_2 = \displaystyle\int \frac{2\lambda(\dot{x} + \dot{y})}{\alpha^2 - \lambda((1 + 2\lambda)(\dot{y} - \dot{x})^2 + 2\dot{x}\dot{y})}\,\mathrm{d}t.
\end{array}
\right\}
\tag{3.61}
$$

In terms of the above new variables, equation (3.57) gets transformed to the free particle equations (2.3). One may note that one of the new independent variables is not only in non-local form, but also contains derivative terms that, in turn, complicate the situation to obtain the general solution.

As both w_1 and z_1 are of point transformation types, one can integrate the first free particle equation, namely $\mathrm{d}^2 w_1/\mathrm{d}z_1^2 = 0$ and obtain $w_1 = I_1 z_1 + I_3$, where I_3 is an integration constant. Replacing the latter in terms of the old variables, one obtains the relation $(x - y) = I_1 t + I_3$. On the other hand, from the solution of the second linear equation $\mathrm{d}^2 w_2/\mathrm{d}z_2^2 = 0$, we can write $w_2 = I_2 z_2 \Rightarrow \log[1 + 2\lambda(x + y)] = I_2 z_2$ (again we assume the integration constant to be zero without loss of generality).

To evaluate z_2, let us first substitute equation (3.58) into equation (3.61) and rewrite the latter in the form

$$
\mathrm{d}z_2 = \frac{2\lambda(\dot{x} + \dot{y})}{I_2(1 + 2\lambda(x + y))}\,\mathrm{d}t = \frac{2\lambda(I_1 + 2\dot{y})}{I_2(1 + 2\lambda(x + y))}\,\mathrm{d}t.
\tag{3.62}
$$

Now substituting the form of $\dot{y}$ (equation (3.58)), i.e.

$$
\dot{y} = \frac{1}{2\lambda}\left(-\lambda I_1 \pm \sqrt{2\lambda\alpha^2 - \lambda^2(1 + 4\lambda)I_1^2 - 2\lambda I_2(1 + 2\lambda(x + y))}\right),
\tag{3.63}
$$

into equation (3.62) and using the relation $(1 + 2\lambda(x + y)) = \mathrm{e}^{I_2 z_2}$, we obtain

$$
\mathrm{d}z_2 = \frac{2\sqrt{\lambda}\sqrt{2\alpha^2 - \lambda(1 + 4\lambda)I_1^2 - 2I_2 \mathrm{e}^{I_2 z_2}}}{I_2 \mathrm{e}^{I_2 z_2}}\,\mathrm{d}t.
\tag{3.64}
$$

Integrating equation (3.64), we obtain

$$
z_2 = \frac{1}{I_2}\log\left(\frac{2\alpha^2 - \lambda(1 + 4\lambda)I_1^2 - 4I_2^2\lambda(t - t_0)^2}{2I_2}\right),
\tag{3.65}
$$

where t_0 is an integration constant. Substituting expression (3.65) into the relation $2\lambda(x + y) = \mathrm{e}^{I_2 z_2} - 1$, we obtain

$$
x + y = \frac{2\alpha^2 - \lambda(1 + 4\lambda)I_1^2 - 4I_2^2\lambda(t - t_0)^2 - 2I_2}{4\lambda I_2}.
\tag{3.66}
$$

From equation (3.66) and the relation $(x - y) = I_1 t + I_3$, we obtain the general solution for equation (3.57) in the form

$$\left.\begin{aligned} x(t) &= \frac{2\alpha^2 - \lambda(1 + 4\lambda)I_1^2 - 4I_2^2\lambda(t - t_0)^2 - I_2(2 - 4\lambda(I_1 t + I_3))}{8\lambda I_2} \\[2mm] y(t) &= \frac{2\alpha^2 - \lambda(1 + 4\lambda)I_1^2 - 4I_2^2\lambda(t - t_0)^2 - I_2(2 + 4\lambda(I_1 t + I_3))}{8\lambda I_2}. \end{aligned}\right\} \tag{3.67}$$

In this example, we considered the case in which only one of the independent variables is in non-local form. Now we consider the case in which both the independent variables are in non-local forms.

(v) *Example 8: mixed generalized Sundman–generalized linearizing transformation*

To illustrate this type of linearizing transformation, let us again consider equation (3.6), but now with $\alpha \neq 0$. Equation (3.6) admits the following two integrals of motion:

$$I_1 = (y\dot{x} - x\dot{y}) \quad \text{and} \quad I_2 = \frac{(\alpha^2 - \lambda(\dot{x}^2 + \dot{y}^2 + \lambda(y\dot{x} - x\dot{y})^2))}{1 + \lambda r^2}. \tag{3.68}$$

Rewriting these two integrals in the form

$$\hat{I}_1 = y^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{y}\right) = \frac{\mathrm{d}w_1}{\mathrm{d}z_1} \tag{3.69}$$

and

$$\hat{I}_2 = \frac{(\alpha^2 - \lambda(\dot{x}^2 + \dot{y}^2 + \lambda(y\dot{x} - x\dot{y})^2)}{2\lambda(x\dot{x} + y\dot{y})} \frac{\mathrm{d}}{\mathrm{d}t}\left(\log(1 + \lambda r^2)\right) = \frac{\mathrm{d}w_2}{\mathrm{d}z_2}, \tag{3.70}$$

and identifying the linearizing transformations, we obtain

$$\left.\begin{aligned} w_1 &= \frac{x}{y}, \quad w_2 = \log(1 + \lambda r^2) \\[2mm] z_1 &= \int \frac{\mathrm{d}t}{y^2}, \quad z_2 = \int \frac{2\lambda(x\dot{x} + y\dot{y})}{(\alpha^2 - \lambda(\dot{x}^2 + \dot{y}^2 + \lambda(y\dot{x} - x\dot{y})^2)} \, \mathrm{d}t. \end{aligned}\right\} \tag{3.71}$$

One can check that equation (3.71) transforms equation (3.6) to the form of equation (2.3).

Rewriting the first integrals I_1 and I_2 in the integral form and identifying them in terms of the new variables, we have $w_1 = I_1 z_1$ and $w_2 = I_2 z_2$ that, in turn, also give us a relationship between x and y with z_1 and z_2, respectively (after fixing

the integration constants to be zero without loss of generality), i.e.

$$x = I_1 z_1 y \quad \text{and} \quad 1 + \lambda r^2 = e^{I_2 z_2}. \tag{3.72}$$

Expressing $\dot{x}$ and $\dot{y}$ in terms of I_1, I_2, x and y (by using equation (3.68)) and substituting them in the expression for dz_2, we obtain

$$dz_2 = 2\frac{(\lambda\alpha^2 r^2 - \lambda(1 + \lambda r^2)(\lambda I_1^2 + I_2 r^2)^{1/2}}{I_2(1 + \lambda r^2)} dt. \tag{3.73}$$

Now, from the expression for $1 + \lambda r^2$ (equation (3.72)), we obtain

$$dz_2 = \frac{2}{I_2}\left[(\alpha^2 - \lambda^2 I_1^2 + I_2)e^{-I_2 z_2} - I_2 - \alpha^2 e^{-2I_2 z_2}\right]^{1/2} dt. \tag{3.74}$$

Integrating the above equation, we obtain

$$I_3 - t = \frac{1}{2\sqrt{I_2}} \tan^{-1}\left[\frac{-2I_2 + (\alpha^2 - \lambda^2 I_1^2 + I_2)e^{-I_2 z_2}}{2\sqrt{I_2}((\alpha^2 - \lambda^2 I_1^2 + I_2)e^{-I_2 z_2} - I_2 - \alpha^2 e^{-2I_2 z_2})^{1/2}}\right], \tag{3.75}$$

where I_3 is an integration constant that is nothing but the third integral of motion. In order to find the fourth integral, using $dz_1 = dt/y^2$ and equation (3.74), we eliminate dt to obtain

$$dz_1 = \frac{I_2}{2}\left[\frac{dz_2}{y^2((\alpha^2 - \lambda^2 I_1^2 + I_2)e^{-I_2 z_2} - I_2 - \alpha^2 e^{-2I_2 z_2})^{1/2}}\right]. \tag{3.76}$$

From equation (3.72), we obtain $y^2 = (e^{I_2 z_2} - 1)/(\lambda(I_1^2 z_1^2 + 1))$, which on substitution into equation (3.76) leads to

$$\frac{dz_1}{(I_1^2 z_1^2 + 1)} = \frac{\lambda I_2}{2}\left[\frac{(e^{I_2 z_2} - 1)^{-1}dz_2}{y^2((\alpha^2 - \lambda^2 I_1^2 + I_2)e^{-I_2 z_2} - I_2 - \alpha^2 e^{-2I_2 z_2})^{1/2}}\right], \tag{3.77}$$

Now integrating equation (3.77), we obtain

$$I_4 = \tan^{-1}\left[I_1 z_1\right] - \frac{1}{2}\tan^{-1}\left[\frac{(I_2 - \alpha^2 - \lambda^2 I_1^2) + (\alpha^2 - \lambda^2 I_1^2 - I_2)e^{I_2 z_2}}{2\lambda I_1((\alpha^2 - \lambda^2 I_1^2 + I_2)e^{I_2 z_2} - I_2 e^{2I_2 z_2} - \alpha^2)^{1/2}}\right], \tag{3.78}$$

where I_4 is the fourth integration constant. Now making use of these four integrals of motion, namely equations (3.68), (3.75) and (3.78), the general solution to equation (3.6) can be straightforwardly constructed. The resultant solution also agrees with eqn (5.40) of Chandrasekar *et al.* (2009), obtained through the modified PS approach, after a redefinition of integration constants.

(vi) *Example 9: generalized linearizing transformation of type II*

To understand the generalized linearizing transformation, let us start with the following system of coupled second-order ODEs:

$$\ddot{x} + \frac{k(x\dot{x} - y\dot{y}) + k^2 x}{(x^2 - y^2)} + \lambda x = 0 \quad \text{and} \quad \ddot{y} + \frac{k(x\dot{y} - y\dot{x}) - k^2 y}{(x^2 - y^2)} + \lambda y = 0. \tag{3.79}$$

The associated first integrals are

$$
I_1 = \left(\frac{k + \sqrt{-\lambda}(x+y) + \dot{x} + \dot{y}}{k - \sqrt{-\lambda}(x+y) + \dot{x} + \dot{y}} \right) e^{-2\sqrt{-\lambda}\,t}
$$

and

$$
I_2 = \left(\frac{k + \sqrt{-\lambda}(x-y) + \dot{x} - \dot{y}}{k - \sqrt{-\lambda}(x-y) + \dot{x} - \dot{y}} \right) e^{-2\sqrt{-\lambda}\,t}.
$$

(3.80)

Rewriting equation (3.80) in the form

$$
I_1 = \frac{k\,e^{-3\sqrt{-\lambda}\,t}}{(k - \sqrt{-\lambda}(x+y) + \dot{x} + \dot{y})} \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{\sqrt{-\lambda}} + \frac{x+y}{k} \right) e^{\sqrt{-\lambda}\,t} \right]
$$

and

$$
I_2 = \frac{k\,e^{-3\sqrt{-\lambda}\,t}}{(k - \sqrt{-\lambda_4}(x-y) + \dot{x} - \dot{y})} \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{\sqrt{-\lambda}} + \frac{x-y}{k} \right) e^{\sqrt{-\lambda}\,t} \right],
$$

(3.81)

and identifying the new variables, we obtain the linearizing transformation

$$
w_1 = \left(\frac{1}{\sqrt{-\lambda}} + \frac{x+y}{k} \right) e^{\sqrt{-\lambda}\,t}, \quad z_1 = \int \frac{(1 - (\sqrt{-\lambda}/k)(x+y) + ((\dot{x}+\dot{y})/k))}{e^{3\sqrt{-\lambda}\,t}}\,\mathrm{d}t
$$

and

$$
w_2 = \left(\frac{1}{\sqrt{-\lambda}} + \frac{x-y}{k} \right) e^{\sqrt{-\lambda}\,t}, \quad z_2 = \int \frac{(1 - (\sqrt{-\lambda}/k)(x-y) + ((\dot{x}-\dot{y})/k))}{e^{3\sqrt{-\lambda}\,t}}\,\mathrm{d}t.
$$

(3.82)

From the first integrals, we have (after assuming the integration constants to be zero without loss of generality)

$$
w_1 = I_1 z_1 \quad \text{and} \quad w_2 = I_2 z_2.
$$

(3.83)

Using equation (3.82) in equation (3.83), we obtain

$$
x(t) = \frac{k}{2}(I_1 z_1 + I_2 z_2)e^{-\sqrt{-\lambda}\,t} - \frac{k}{\sqrt{-\lambda}} \quad \text{and} \quad y(t) = \frac{k}{2}(I_1 z_1 - I_2 z_2)e^{-\sqrt{-\lambda}\,t}.
$$

(3.84)

Substituting the expressions of x and y into equation (3.80) and solving the resultant equation for $\dot{x}$ and $\dot{y}$, we obtain

$$
\dot{x} = -k - \sqrt{-\lambda}\left[\frac{k}{2}(I_1 z_1 + I_2 z_2)e^{-\sqrt{-\lambda}\,t} - \frac{k}{\sqrt{-\lambda}} \right]\left(\frac{1 + e^{-\sqrt{-\lambda}\,t}}{1 - e^{-\sqrt{-\lambda}\,t}} \right)
$$

and

$$
\dot{y} = -\sqrt{-\lambda}\left[\frac{k}{2}(I_1 z_1 - I_2 z_2)e^{-\sqrt{-\lambda}\,t} \right]\left(\frac{1 + e^{-\sqrt{-\lambda}\,t}}{1 - e^{-\sqrt{-\lambda}\,t}} \right).
$$

(3.85)

Substituting equations (3.84) and (3.85) into the expressions (3.82) for dz_1 and dz_2 and integrating the resultant equation, we obtain

$$z_1 = \frac{1}{I_1}\left(\frac{e^{\sqrt{-\lambda}t}}{\sqrt{-\lambda}} + \left((e^{2\sqrt{-\lambda}t} - I_1)\left(I_3 + \tanh^{-1}\left[\frac{e^{\sqrt{-\lambda}t}}{\sqrt{I_1}}\right]\right)\right)\right)$$

and

$$z_2 = \frac{1}{I_2}\left(\frac{e^{\sqrt{-\lambda}t}}{\sqrt{-\lambda}} + \left((e^{2\sqrt{-\lambda}t} - I_2)\left(I_4 + \tanh^{-1}\left[\frac{e^{\sqrt{-\lambda}t}}{\sqrt{I_2}}\right]\right)\right)\right),$$

$$(3.86)$$

where I_3 and I_4 are the third and fourth integration constants, respectively. From equations (3.84) and (3.86), we can obtain the general solution for equation (3.79) straightforwardly.

4. Conclusions

In this paper, we have studied the linearization of two coupled SNODEs. In particular, we have introduced a new method of deriving linearizing transformations from the first integrals for the given equation. The procedure is simple and straightforward. From our analysis, we have demonstrated that one can have two wider classes of linearizing transformations, namely class A and class B, depending on the nature of the independent variables. In class A category, the independent variables are the same, and we identified three types of linearizing transformations in which two of them are new to the literature. On the other hand, in the class B category (the independent variables are different), we found six new types of linearizing transformations. We have explicitly demonstrated the method of deducing the linearizing transformations and the general solution for all of these cases with specific examples. However, in this paper, we have restricted our attention to two aspects: (i) dependent variables are functions of only (t, x, y) and (ii) independent variables are not of the local form $z_i = f_i(t, x, y, \dot{x}, \dot{y})$, where $i = 1, 2$. Linearization under these two types requires separate treatment and will be studied subsequently. The method proposed here can naturally be extended to any number of coupled second-order ODEs and indeed one can derive a very wide class of linearizing transformations in these cases.

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