UPPER MASS LIMIT FOR NEUTRON STAR STABILITY AGAINST BLACK HOLE FORMATION

(Letter to the Editor)

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Abstract. Starting from the exact general relativistic expression for the total energy of self-gravitating spherically distributed matter and using the minimum energy principle, we calculate the upper mass limit for a neutron star to be 3.1 solar masses.

The dynamical stability of a non-rotating neutron star against gravitational collapse into a black hole has been discussed by several authors (e.g., Chandrasekhar, 1964; Oppenheimer and Volkoff, 1939). These sutides have led to an upper critical mass limit M_c for the neutron star, beyond which the latter must collapse past the Schwarzschild singularity radius R_{Sch} and become a black hole. However, the calculated values of M_c reported in the literature show a wide scatter, ranging from $0.76\ M_\odot$ to about $5\ M_\odot$ (Lang, 1974). The point is that here one is interested in a phenomenon close to the Schwarzschild radius and hence the general relativistic (GR) effects must be taken fully into account. Since the value of M_c is of considerable astrophysical interest, we have attempted in this note a new derivation which is mathematically simple and physically transparent, and still incorporates the GR effects in an essentially exact manner. We obtain the critical baryon (neutron) number $N_c = 0.031\ [6\pi\hbar c/\sqrt{2}\ M_N^2 G]^{3/2}$, where M_N is the neutron mass and G is the Newtonian gravitational constant.

In the following we shall outline the calculations. We shall regard the neutron star as a degenerate gas of N neutrons distributed uniformly in a sphere of radius R. The mechanical energy–mass (E_0^N) content (i.e., in the absence of gravitational self-interaction) is then given by

$$E_0^N(R) = \left[\frac{R^3}{6\pi}\frac{m_N^4 c^5}{\hbar^3}\right]\left[\frac{\sinh 4x}{4x} - 1\right]x, \tag{1}$$

where

$$\sinh x = \frac{\lambda_N}{r_S}\left(\frac{9\pi}{2}\right)^{1/3}$$

with $\lambda_N = \hbar/m_N c$ = neutron Compton length and $r_S = R(2/N)^{1/3}$ = the mean inter-particle spacing for a given spin. Now, the exact expression for the gravitational mass content $M_G^N(R)$, taking into account the gravitational self-energy, is given by Arnowitt and Misner (1960) as

$$M_G^N(R) = \frac{c^2}{G}\left[-R + \sqrt{R^2 + \frac{2GRE_0^N(R)}{c^4}}\right]. \tag{2}$$

We should like to emphasize that expression (2) has the simple physical meaning within the Newtonian gravitational theory taken in conjunction with the universal energy–mass equivalence principle. This implies that the effective gravitational mass $M_G^N(R)$ can be obtained by subtracting from the mechanical energy $E_0^N(R)$ the gravitational binding energy $\frac{1}{2}G[M_G^N(R)]^2/R$ self-consistently and dividing by c^2 – i.e.,

$$M_G^N(R)c^2 = E_0^N - \tfrac{1}{2}G[M_G^N(R)]^2/R,$$

whose solution is given by Equation (2). This suggests that the GR effects are fully incorporated, as far as the total energy–mass constant is concerned, by supplementing Newtonian gravitational theory by the energy–mass equivalence principle and working as if in a flat space–time. We shall adopt this viewpoint. It is convenient at this point to introduce new variables

$$y = r_S/\lambda_N, \qquad \eta = N/N_0, \qquad m_u = \frac{\hbar c}{m_N G}, \qquad N_0 = \left(\frac{6\pi\hbar c}{2^{1/3} m_N^2 G}\right)^{3/2}$$

and

$$F(y) = x\left[\frac{\sinh 4x}{4x} - 1\right] \quad \text{with} \quad \sinh x = \frac{1}{y}\left(\frac{9\pi}{2}\right)^{1/3}$$

Then, Equations (1) and (2) can be rewritten as

$$M_0^N(y) = \frac{\eta}{12\pi}\left(\frac{6\pi}{2^{1/3}}\right)^{3/2}\left(\frac{m_u^3}{m_N}\right)^{1/2} y^3 F(y), \tag{3}$$

$$M_G^N(y) = m_u\left(\frac{N_0}{2}\right)^{1/3}\eta^{1/3} y[-1 + (1 + y^2\eta^{2/3}F(y))^{1/2}]. \tag{4}$$

The equilibrium radius $R_{\min}$ will be given by minimizing $M_G^N(y)$ with respect to y, giving $y_{\min}$. The corresponding Schwarzschild radius R_{Sch} will be given by $R_{\text{Sch}} = 2GM_0^N(y_{\min})/c^2$. Now, the condition of stability against collapse will be $R_{\text{Sch}} < R_{\min}$. It is seen from the examination of Equations (3) and (4) that as $\eta = N/N_0$ increases, the ratio $R_{\text{Sch}}/R_{\min}$ increases monotonically. Thus, the maximum value of the baryon number N is given by the condition $R_{\text{Sch}}(y_{\min}) = R(y_{\min})$. This gives a numerical solution $\eta_c = N_c/N_0 = 0.031$. Thus the maximum baryon number for a neutron star is

$$N_c = 0.031\left[\frac{6\pi\hbar c}{2^{1/3}M_N^2 G}\right]^{3/2} \simeq 4.079 \times 10^{57},$$

and the corresponding gravitational mass (the upper mass limit)

$$M_c \simeq 6.16 \times 10^{33}\,\mathrm{g} \simeq 3.1\,M_\odot.$$

We would like to conclude by stating that since the present calculation is based on total (global) energy considerations, it might be more relevant to the overall stability against collapse than the considerations based on the analysis of the local metric. This may be partly because a covariant gravitational energy–momentum tensor does not exist, but the total gravitational energy of a *closed* system can still be defined in a physically meaningful manner.

References

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