UICHEP-TH/97-11; IOP-BBSR/97-38

Exact Solution of a Class of Three-Body Scattering Problems in One Dimension

Avinash Khare[++]

Institute of Physics,
Sachivalaya Marg, Bhubaneswar-751 005, India

Uday P. Sukhatme[**]

Department of Physics (m/c 273),
University of Illinois at Chicago, 845 W. Taylor Street, Room 2236 SES,
Chicago, Illinois 60607-7059, U.S.A.

Abstract

We present an exact solution of the three-body scattering problem for a one parameter family of one dimensional potentials containing the Calogero and Wolfes potentials as special limiting cases. The result is an interesting nontrivial relationship between the final momenta p'_i and the initial momenta p_i of the three particles. We also discuss another one parameter family of potentials for all of which $p'_i = -p_i$ $(i = 1, 2, 3)$.

++ khare@beta.iopb.stpbh.soft.net
** sukhatme@uic.edu

More than two decades ago, Marchioro [1] gave the complete solution of both the classical and quantum mechanical scattering problems of three equal mass distinguishable particles in one dimension interacting via the two body Calogero potential [2]

$$V(x_1, x_2, x_3) = \frac{g}{(x_1 - x_2)^2} + c.p. \quad , \tag{1}$$

where we have used the notation c.p. to denote the terms obtained by cyclic permutation of the particle coordinates x_1, x_2, x_3. It was shown that if p_i and p'_i (i=1,2,3) denote the momenta of the incoming and outgoing particles respectively then there is a surprisingly simple relation between the momenta:

$$p'_1 = p_3 \quad , \quad p'_2 = p_2 \quad , \quad p'_3 = p_1 \quad . \tag{2}$$

For the classical case, additional simple relations between the next-to-leading terms in the asymptotic expressions for the positions of the particles were also derived. Four years later, Calogero and Marchioro [3] gave the solution to the classical as well as quantum mechanical three-body scattering problem when the particles interact via the three-body Wolfes potential [4] given by

$$V(x_1, x_2, x_3) = \frac{3g}{(x_1 + x_2 - 2x_3)^2} + c.p. \quad . \tag{3}$$

For this interaction, they showed that

$$p'_1 = -p_2 \quad , \quad p'_2 = -p_1 \quad , \quad p'_3 = -p_3 \quad . \tag{4}$$

The purpose of this note is to raise and answer the following question: are there other potentials for which the three-body scattering problem is exactly solvable, and if so, what is the relationship between the incoming and outgoing momenta? We will show that both in the quantum and classical cases, if the three equal mass particles are interacting via the three-body potential

$$V(x_1, x_2, x_3) = \frac{g}{[(x_1 - x_2)\cos\delta + \frac{(x_1 + x_2 - 2x_3)}{\sqrt{3}}\sin\delta]^2} + c.p. \quad , \tag{5}$$

then p'_i and p_i are related by

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ -a & b & 0 \\ b & 0 & -a \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad , \tag{6}$$

where

$$a = \frac{2}{\sqrt{3}}\sin(2\delta) \quad , \quad b = \frac{2}{\sqrt{3}}\sin(\frac{\pi}{3} - 2\delta) \quad . \tag{7}$$

The coupling constant g is non-zero, and as usual, we assume $g > -\hbar^2/4m$ to prevent collapse. Here δ is an arbitrary phase angle such that $0 \leq \delta \leq \pi/3$. The potential (5) is a fairly complicated, translationally invariant three-body potential which interpolates smoothly between the two-body Calogero potential ($\delta = 0$) and the three-body Wolfes potential ($\delta = \pi/6$). Note that the potential of eq. (5) has a period $\pi/3$ in the angle δ. We will take δ in the range $0 \leq \delta \leq \pi/6$. For $\pi/6 \leq \delta \leq \pi/3$, then p_i and p'_i are still related by eqs. (6) and (7) but with δ replaced by $\delta' = \pi/3 - \delta$.

We also discuss an even more general class of three-body potentials given by

$$V(x_1, x_2, x_3) = \frac{g}{[(x_1 - x_2)\cos\delta + \frac{(x_1+x_2-2x_3)}{\sqrt{3}}\sin\delta]^2} + \frac{f}{[(x_1 - x_2)\sin\delta + \frac{(x_1+x_2-2x_3)}{\sqrt{3}}\cos\delta]^2} + c.p. \quad , \tag{8}$$

where the coupling constants g, f are both non-zero and $0 \leq \delta \leq \pi/3$. We show that these potentials are strictly isospectral, in the sense that, for all of them, irrespective of the value of δ, the incoming and outgoing momenta satisfy

$$p'_1 = -p_1 \quad , \quad p'_2 = -p_2 \quad , \quad p'_3 = -p_3 \quad . \tag{9}$$

At first sight, it is difficult to reconcile the results of eqs. (9) and (6). One might wonder how the simple result of eq. (9) is obtained for the potential of eq. (8) for all finite values of the coupling constant f, whereas when f goes from being small to zero, the result for the outgoing momenta suddenly and discontinuously jumps to eq. (6) whereas the potential goes smoothly to eq. (5). A similar situation also occurs when $g = 0$. We would like to remark that this discontinuous change of results arises and was discussed in the paper of Calogero and Marchioro [3], which corresponds to our potentials (5) and (8) for the value $\delta = 0$. It is also nicely discussed by Marchioro in ref. [1] in the classical context. This paper contains a figure which shows the scattering process for two values of the coupling constant. For larger g, the interaction is stronger and the particles have a larger "distance of closest approach" before backscattering. For smaller g, the interaction is weaker, and the particles come much closer before backscattering. Indeed, backscattering occurs no matter how small g is, provided that it is non-zero. [This type of argument is also true for one-dimensional Rutherford scattering.] However, if $g = 0$, there is no repulsive force, and hence no backscattering, which gives a heuristic argument for the discontinuous change of results when a coupling constant is made to vanish.

The proof of all the above results for both the classical and the quantum cases follows by generalizing the procedure of refs. [1] and [3]. Hence, in this paper, we stress only those points which are different from the discussion of refs. [1] and [3].

Calogero [2] showed that the scattering problem is best solved in Jacobi coordinates defined by

$$R = \frac{(x_1 + x_2 + x_3)}{3} \ , \ x = \frac{(x_1 - x_2)}{\sqrt{2}} \ , \ y = \frac{(x_1 + x_2 - 2x_3)}{\sqrt{6}} \quad . \tag{10}$$

Throughout this paper, we work in the center of mass frame and following [1], for simplicity, take the origin of the x-axis to coincide with the position of the center of mass of the three-body system so that

$$x_1 + x_2 + x_3 = 0 \, . \tag{11}$$

We now define the polar coordinates (r, ϕ) by

$$x = r \sin\phi \ , \ y = r \cos\phi \ , \ r^2 = \frac{1}{3}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2] \ , \tag{12}$$

where the variables r and ϕ have the ranges $0 \leq r \leq \infty$ and $0 \leq \phi \leq 2\pi$. In terms of these coordinates, the relative Hamiltonian (after the elimination of the center of mass) corresponding to the potential (5) takes the simple form

$$H = -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}) + \frac{M}{r^2} \quad , \tag{13}$$

where

$$M = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \phi^2} + \frac{9g}{2\sin^2(3\phi + 3\delta)} \quad , \tag{14}$$

where $0 \leq \delta \leq \pi/6$. Note that M is unchanged under the transformation $\delta \longrightarrow \delta + q\pi/3$ where $q = 0, 1, ..., 5$, which just reflects the periodicity of $V(x_1, x_2, x_3)$.

¿From eqs. (13) and (14) we find that the problem is effectively the Calogero problem with ϕ replaced by $\phi+\delta$. As in that case, the singular nature of the interaction disconnects the wave functions (apart from a symmetry requirement in the case of identical particles) in different sectors of the configuration space corresponding to different intervals of values of the angular variable ϕ. For simplicity, as in [1] and [3] we shall assume that the particles are distinguishable. Hence we consider the wave functions that differ from zero only in the sector

$$-\delta \leq \phi(t) \leq \pi/3 - \delta \, . \tag{15}$$

Of course, this may be replaced by any one of the other five sectors by permuting the particles. Now, in the center of mass frame, using eq. (11) it is easily seen that in terms of polar coordinates, the positions x_i are given by

$$x_i = -\sqrt{\frac{2}{3}} r \cos[\phi + i(\frac{2\pi}{3})] \ , \ i = 1, 2, 3 \, . \tag{16}$$

It is then not difficult to see that the sector defined by eq. (15) is characterized by the property

$$x_1 > x_3 < 0, x_2 > x_3, \mid x_1 - x_2 \mid < x_1 - x_3, \mid x_1 - x_2 \mid < x_2 - x_3 \,. \qquad (17)$$

The scattering problem is now easily discussed by following the steps given in [3] and replacing ϕ by $\phi + \delta$ at appropriate places. In particular, the eigenfunctions of the angular problem are given by eq. (2.17c) of [3] with ϕ replaced everywhere by $\phi + \delta$.

One major difference from [3] is that one now introduces a symmetry operation T such that

$$Tr = r, \; T\phi = \frac{\pi}{3} - \phi - 2\delta \quad . \qquad (18)$$

One can easily show that this T operator when applied to the angular eigenfunction Φ_l of the problem (as given by Eq. (2.17c) of [3] but with ϕ replaced by $\phi + \delta$) yields

$$T\Phi_l = (-1)^l \Phi_l \quad . \qquad (19)$$

Now following the steps of ref. [3], the relations (6) and (7) follow immediately, and furthermore the phase shift is as given by eq. (2.37c) of [3] which is independent of δ. It is worth noting here that while deriving these relations in the manner of ref. [3], one has to place additional restrictions on the initial momenta p_i. In particular, when the ordering of the particles is as given by eq. (17) then the initial momenta must satisfy

$$p_1 < p_3 > 0, p_2 < p_3, p_3 - p_1 < \mid p_1 - p_2 \mid, p_3 - p_2 < \mid p_1 - p_2 \mid \; . \qquad (20)$$

Let us now discuss the same scattering problem in the classical case. To exclude collapse, we must now assume that $g \geq 0$. We again work in the center of mass frame and use the polar coordinates r and ϕ. In terms of r and ϕ, the relative classical Hamiltonian corresponding to the potential (5) is given by

$$H = E = \frac{p_r^2}{2m} + \frac{B^2}{r^2} \qquad (21)$$

where

$$B^2 = \frac{p_\phi^2}{2m} + \frac{9g}{2\sin^2 3(\phi + \delta)} \quad , \qquad (22)$$

and E,B are two constants of motion. Using the fact that $p_r = m\frac{dr}{dt}, p_\phi = mr^2\frac{d\phi}{dt}$, it is easily seen that the solution to the classical eqs. (21) and (22) is

$$r(t) = \left[(\frac{2E}{m})(t - t_0)^2 + \frac{B^2}{E} \right]^{1/2} \quad , \qquad (23)$$

$$\cos 3(\phi(t)+\delta) = [1-\frac{9g}{2B^2}]^{1/2}\sin[\gamma - 3\tan^{-1}\frac{(t-t_0)}{\tau}] \quad , \tag{24}$$

where $\tau = \sqrt{\frac{m}{2}}(B/E)$ and

$$\sin\gamma = \frac{\cos 3(\phi(t_0)+\delta)}{\sqrt{1-\frac{9g}{2B^2}}} \quad , \quad \cos\gamma = \pm\frac{\cos 3(\phi(\pm\infty)+\delta)}{\sqrt{1-\frac{9g}{2B^2}}} \quad . \tag{25}$$

¿From here one immediately obtains

$$\phi(\infty) = \frac{\pi}{3} - 2\delta - \phi(-\infty) \quad . \tag{26}$$

Let us now discuss the scattering process. Clearly the initial state of the system is completely specified by the asymptotic expression

$$x_i \stackrel{t\to-\infty}{\longrightarrow} \frac{p_i t}{m} + a_i, \; i = 1,2,3 \quad , \tag{27}$$

while the final state after the collision is specified by

$$x_i \stackrel{t\to+\infty}{\longrightarrow} \frac{p'_i t}{m} + a'_i \quad . \tag{28}$$

On following the steps given in [3] and using eqs. (23) to (25) it then follows that p'_i and p_i are related by eqs. (6) and (7) and further a'_i and a_i are also related by the same matrix.

Finally let us consider the class of potentials given by eq. (8). The scattering problem for the special case of $\delta = 0$ has already been discussed [3]. In terms of the polar coordinates (r,ϕ) defined by eqs. (10) and (12), the Hamiltonian is as given by eq. (13) but where

$$M = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\phi^2} + \frac{9}{2}\Big[\frac{g}{\sin^2(3\phi+3\delta)} + \frac{f}{\cos^2(3\phi+3\delta)}\Big] \quad . \tag{29}$$

This M is identical to that of [3] except that ϕ is replaced everywhere by $\phi+\delta$ (see their eq. (2.9)). One can now run through their arguments and show that irrespective of the value of δ, the outgoing and incoming momenta satisfy $p'_i = -p_i$ and further the phase shift in all the cases is as given by eq. (2.37a) of [3].

Similarly, one can show that the solution to the classical equations is as given by eq. (23) and

$$\sin 3(\phi(t)+\delta) = \alpha + \beta\sin[\gamma^{(1)} + 6\tan^{-1}\frac{(t-t_0)}{\tau}] \quad , \tag{30}$$

where $\alpha, \beta, \gamma^{(1)}$ and τ are given by eqs. (3.9) to (3.12) of [3]. It immediately follows from here that irrespective of the value of $\delta, \phi(+\infty) = \phi(-\infty)$, and hence following the steps of [3] one gets

$$p'_i = -p_i \quad , \quad a'_i = -a_i \quad , \quad i = 1,2,3 \quad . \tag{31}$$

Thus we have obtained a one continuous parameter family of strictly isospectral three-body potentials as given by eq. (8). Besides, we have also shown that the class of potentials as given by eq. (5) are strictly isospectral to the potentials with δ being replaced by $\delta + q\pi/3$ where $q = 0, 1, ..., 5$. Besides, they are also strictly isospectral to the potentials with δ being replaced by $q\pi/3 - \delta$. Thus for any δ between $0 < \delta < \pi/6$, there are twelve strictly isospectral potentials while for $\delta = 0$ and $\delta = \pi/6$ there are six strictly isospectral potentials. It is worth noting that if we add the two-body harmonic oscillator potential $\omega^2 \sum (x_i - x_j)^2$ [2] to eq. (8), these potentials still continue to be strictly isospectral in the sense that for all of them the spectrum is purely discrete, independent of δ and is as given by Wolfes [4]. Further, if we add the two-body harmonic potential to the class of potentials given by eq. (5), then they are also strictly isospectral having a purely discrete spectrum which is independent of δ. Note that this class includes both the Calogero two-body and the Wolfes three-body potentials.

Finally, it is remarkable that the result for three-body scattering is rather simple when the interaction is the two-body Calogero potential as given by eq. (1), or the three-body Wolfes potential as given by eq. (3), or even when the interaction is the sum of the Calogero two-body and the Wolfes three-body potentials! In this context it is worth recalling that a simple relationship also holds for a system of N-particles interacting via the two-body Calogero potential. It is then worth enquiring whether a similar simple relationship holds good for N particle scattering when they are interacting via the three-body Wolfes potential or the sum of Wolfes and Calogero potentials. If so, these will give us additional integrable N-body systems in one dimension.

References

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