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# Low-cost control problems on perforated and non-perforated domains

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**Abstract.** We study the homogenization of a class of optimal control problems whose state equations are given by second order elliptic boundary value problems with oscillating coefficients posed on perforated and non-perforated domains. We attempt to describe the limit problem when the cost of the control is also of the same order as that describing the oscillations of the coefficients. We study the situations where the control and the state are both defined over the entire domain or when both are defined on the boundary.

**Keywords.** Homogenization; $H$-convergence; optimal control.

## 1. Introduction

This paper discusses the asymptotic properties of some low-cost control problems with distributed or boundary control. We consider both the perforated and non-perforated cases of the problem. The low-cost control problems were studied by Kesavan and Saint Jean Paulin in [7] and this paper addresses some problems left open there. A thorough study of the low-cost problems is done in [11].

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Let $0 < a < b$ be given constants. We denote by $\mathcal{M}(a, b, \Omega)$ the set of all $n \times n$ matrices, $A = A(x)$, whose entries are in $L^\infty(\Omega)$ such that

$$a|\xi|^2 \leq A(x)\xi \cdot \xi \leq b|\xi|^2 \quad \text{a.e. } x \quad \forall \xi = (\xi_i) \in \mathbb{R}^n.$$

Let $A \in \mathcal{M}(a, b, \Omega)$, $U$ be a closed convex subset of $L^2(\Omega)$ and let $f \in L^2(\Omega)$ be a given function. Also, let $N > 0$ be a given constant. The basic optimal control problem that we consider is the following: Find $\theta^* \in U$ such that

$$J(\theta^*) = \min_{\theta \in U} J(\theta),$$

where the cost functional, $J(\theta)$, is defined by

$$J(\theta) = \frac{1}{2}\|u\|^2_{2,\Omega} + \frac{N}{2}\|\theta\|^2_{2,\Omega} \tag{1.1}$$

and the state $u = u(\theta)$ is the weak solution in $H^1_0(\Omega)$ of the boundary value problem

$$\begin{cases} -\operatorname{div}(A\nabla u) = f + \theta, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It can be shown by the direct method in the calculus of variations that there is a unique optimal control, $\theta^* \in U$, minimizing $J$ over $U$ (cf. Theorem 1.15 and Proposition 1.20 of [2]).

One is interested in the situation where the matrix $A$ and the cost of the control $N$ (in (1.1)) above depend on $\varepsilon$, a parameter which tends to zero. Such problems are called *low-cost control* problems and the terminology is due to the fact that the cost of the control $N$ is of the order of $\varepsilon$ that tends to zero. The notion of low-cost control was introduced by Lions in [8]. Lions had originally called it *cheap control* and the current terminology was used by Kesavan and Saint Jean Paulin in [7]. Kesavan and Saint Jean Paulin (cf. [7]) had considered the low-cost problems when the admissible set was either $L^2(\Omega)$ (unconstrained case) or the positive cone in $L^2(\Omega)$.

For the rest of the paper we consider, given a parameter $\varepsilon > 0$ which tends to zero, a family of matrix $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$.

The paper is organized as follows: In §2, we deal with an optimal control problem involving $L^2$-norm in the cost functional. In §2.1, we develop the preliminaries to prove our result and we study the case of non-perforated domains in §2.2. In §3, we take up the version of the low-cost problems in perforated domains and study situations when the control is either from domain or from boundary.

## 2. Low-cost controls on non-perforated domains

We are interested in the limiting behaviour of the following optimal control problem:

Given $\theta \in U$, the cost functional is defined as

$$J_\varepsilon(\theta) = \frac{1}{2}\|u_\varepsilon\|^2_{2,\Omega} + \frac{\varepsilon}{2}\|\theta\|^2_{2,\Omega}, \tag{2.1}$$

where the state $u_\varepsilon \in H^1_0(\Omega)$ is the weak solution of

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = f + \theta, & \text{in } \Omega \\ \qquad\qquad u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Thus, there exists a unique optimal control $\theta^*_\varepsilon \in U$ such that

$$J_\varepsilon(\theta^*_\varepsilon) = \min_{\theta \in U} J_\varepsilon(\theta).$$

Let $u^*_\varepsilon$ denote the state corresponding to $\theta^*_\varepsilon$. We are interested in identifying the limit problem of the above system. This problem was considered by Kesavan and Saint Jean Paulin in [7]. Though they were unable to identify the limit system, however, they proved the following result when the admissible control set $U \subset L^2(\Omega)$ is the positive cone $L^2(\Omega)$.

**Theorem 2.1 (Theorem 2.1 of [7]).** *If*

$$U = \{\theta \in L^2(\Omega) | \theta \geq 0 \ a.e. \ in \ \Omega\}$$

*is the admissible control set for the system* (2.1) *solving* (2.2), *then there exists* $u^*$ *and* $\theta^*$ *such that*

$$u^*_\varepsilon \rightharpoonup u^* \text{ weakly in } H^1_0(\Omega) \text{ and strongly in } L^2(\Omega), \tag{2.3}$$

$$\varepsilon^{\frac{1}{2}}\theta^*_\varepsilon \rightharpoonup 0 \text{ weakly in } H^1_0(\Omega) \text{ and strongly in } L^2(\Omega), \tag{2.4}$$

$$J_\varepsilon(\theta^*_\varepsilon) \to \frac{1}{2}\|u^*\|^2_{2,\Omega}, \tag{2.5}$$

$$\text{and for a subsequence, } \theta^*_\varepsilon \rightharpoonup \theta^* \text{ weakly in } H^{-1}(\Omega). \tag{2.6}$$

*Further, $u^*$ is the projection of* 0 *on to $\bar{K}$ in $L^2(\Omega)$, i.e., $u^* \in \bar{K}$ and*

$$\int_\Omega u^*(v - u^*)\,\mathrm{d}x \geq 0 \quad \forall v \in \bar{K}$$

*where*

$$K = \left\{ v \in H^1_0(\Omega) \,\middle|\, \begin{array}{c} \exists \text{ a sequence } v_\varepsilon \in H^1_0(\Omega) \text{ s.t. } v_\varepsilon \rightharpoonup v \text{ in } H^1_0(\Omega), \\ -\mathrm{div}(A_\varepsilon \nabla v_\varepsilon) \in L^2(\Omega) \text{ and is } \geq f \text{ a.e. in } \Omega \end{array} \right\}$$

*and $\bar{K}$ is the closure of $K$ in $L^2(\Omega)$.* □

In the above theorem though the limit optimal state, $u^*$, was shown to satisfy a kind of variational inequality, no relation was noted between $u^*$ and $\theta^*$ and the description of the set $K$ is somewhat complicated. Also, the limit control $\theta^*$ was not given as an optimal control of a homogenized problem.

One also observes that the possible limit cost functional

$$J(\theta) = \frac{1}{2}\|u(\theta)\|^2_{2,\Omega}, \tag{2.7}$$

may not be coercive in the weak topology of $L^2(\Omega)$ (cf. Example 2.1). Thus $J$ may not have a minimiser in $U$. This is quite different from the case of fixed cost of the control $N$ in the cost functional, since there the $J_\varepsilon$ were equi-coercive and the possible limit functional $J$ stayed coercive; thus admitting a minimiser. Refer [5] for the study of fixed cost case in non-perforated domains.

*Example* 2.1. The cost functional $J$ as defined in (2.7) is not coercive, in general, in the weak topology of $L^2(\Omega)$. We give a one-dimensional example to observe this fact. Let $\Omega = (-1, 1)$. Let $\rho_\varepsilon$ denote the sequence of mollifiers defined as

$$\rho_\varepsilon(x) = \begin{cases} k\varepsilon^{-1}\exp\left(\dfrac{-\varepsilon^2}{\varepsilon^2 - |x|^2}\right), & |x| < \varepsilon \\ 0, & |x| \geq \varepsilon \end{cases} \tag{2.8}$$

where $k^{-1} = \int_{|x|\leq 1}\exp\left(\frac{-1}{1-|x|^2}\right)\mathrm{d}x$, so that $\int_{-1}^{1}\rho_\varepsilon(x)\,\mathrm{d}x = 1$. We now observe that $\|\rho_\varepsilon\|^2_{2,(-1,1)} \to +\infty$ as $\varepsilon \to 0$.

$$\int_{-1}^{1}\rho^2_\varepsilon(x)\,\mathrm{d}x = \frac{k^2}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\exp\left(\frac{-2\varepsilon^2}{\varepsilon^2 - |x|^2}\right)\mathrm{d}x$$

$$= \frac{k^2}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\exp\left(\frac{-2}{1 - \frac{|x|^2}{\varepsilon^2}}\right)\mathrm{d}x.$$

Putting $y = \frac{x}{\varepsilon}$, we have

$$= \frac{k^2}{\varepsilon} \int_{-1}^{1} \exp\left(\frac{-2}{1-|y|^2}\right) \mathrm{d}y \to +\infty \text{ as } \varepsilon \to 0.$$

Using the mollifiers as controls we define $u_\varepsilon$ as the solution of

$$-\frac{\mathrm{d}^2 u_\varepsilon}{\mathrm{d}x^2} = \rho_\varepsilon \text{ in } \Omega = (-1, 1)$$

such that $u_\varepsilon(-1) = u_\varepsilon(1) = 0$. Hence

$$-u_\varepsilon'(x) = \int_{-1}^{x} \rho_\varepsilon(y)\, \mathrm{d}y - u_\varepsilon'(-1)$$

and $|u_\varepsilon'(-1)| \leq 1 + |u_\varepsilon'(x)|$. Integrating both sides over $(-1, 1)$, we have

$$2|u_\varepsilon'(-1)| \leq 2 + \int_{-1}^{1} |u_\varepsilon'(x)|\, \mathrm{d}x \leq 2 + \left(\int_{-1}^{1} |u_\varepsilon'|^2\right)^{1/2} \sqrt{2}.$$

By the variational formulation of the equation, we have

$$\int_{-1}^{1} |u_\varepsilon'(x)|^2\, \mathrm{d}x = \int_{-1}^{1} \rho_\varepsilon u_\varepsilon\, \mathrm{d}x \leq \|u_\varepsilon\|_{\infty,(-1,1)}$$

and hence

$$\begin{aligned} \|u_\varepsilon'\|_{\infty,(-1,1)} &\leq 1 + |u_\varepsilon'(-1)| \\ &\leq 1 + 1 + \frac{1}{\sqrt{2}} \left(\int_{-1}^{1} |u_\varepsilon'|^2\right)^{1/2} \\ &\leq 2 + \frac{1}{\sqrt{2}} \|u_\varepsilon\|_{\infty,(-1,1)}^{1/2}. \end{aligned}$$

Now, since $u_\varepsilon(x) = \int_{-1}^{x} u_\varepsilon'(y)\, \mathrm{d}y$, we have

$$|u_\varepsilon(x)| \leq \|u_\varepsilon'\|_{\infty,(-1,1)} |x+1| \leq 2\|u_\varepsilon'\|_{\infty,(-1,1)}.$$

Hence, $\|u_\varepsilon\|_{\infty,(-1,1)} \leq 4 + \sqrt{2}\|u_\varepsilon\|_{\infty,(-1,1)}^{\frac{1}{2}}$. The (positive) root of the quadratic equation $\alpha^2 - \sqrt{2}\alpha - 4 = 0$ is $2\sqrt{2}$ and so $\|u_\varepsilon\|_{\infty,(-1,1)} \leq 8$ and hence $\|u_\varepsilon\|_{2,(-1,1)} = \left(\int_{-1}^{1} u_\varepsilon^2\, \mathrm{d}x\right)^{1/2} \leq 8\sqrt{2}$. Thus, $\|u_\varepsilon\|_{2,(-1,1)}$ is bounded while $\|\rho_\varepsilon\|_{2,(-1,1)}^2 \to \infty$. Thus, $J$ as defined in (2.7) is not coercive in the weak topology of $L^2(-1, 1)$. $\square$

In this paper, we settle the low-cost problem for the positive cone case which was considered in [7] and in the following section we state the results crucial to settle our problem.

## 2.1 *Data from the positive cone of $H^{-1}$*

In this section, we state some known results that extends the notion of $H$-convergence to weakly converging data from the positive cone of $H^{-1}(\Omega)$. To begin, we shall state the Meyers' regularity result, whose proof can be found in (page 38 of [1]) (or cf. [9]).

**Theorem 2.2.** *Let $A \in \mathcal{M}(a, b, \Omega)$ and $u \in H_0^1(\Omega)$ be the solution of*

$$\begin{cases} -\operatorname{div}(A\nabla u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

*where $f \in H^{-1}(\Omega)$. There exists a number $p > 2$ (which depends on $a, b, \Omega$ and on the dimension $n$) such that if $f \in W^{-1,p}(\Omega)$, then the solution $u$ belongs to $W_0^{1,p}(\Omega)$ and satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)} \leq C_0 \|f\|_{W^{-1,p}(\Omega)} \tag{2.10}$$

(*where $C_0$ depends on the same quantities as $p$ does*). □

The highlight of the above theorem, other than the regularity aspect, is that $p$ and $C_0$ will be independent of $\varepsilon$, if the equation involves oscillating coefficients, say $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$, and also that the $p$ is same for ${}^tA$ instead of $A$ in the state equation above.

We now state a result proved by Murat [10].

**Theorem 2.3 [10].** *Let $\Omega$ be an open subset of $\mathbb{R}^n$. Consider a sequence $\{g_\varepsilon\} \subset H^{-1}(\Omega)$ such that*

$$g_\varepsilon \rightharpoonup g \text{ weakly in } H^{-1}(\Omega)$$

*and $g_\varepsilon \geq 0$ for all $\varepsilon$. Then*

$$g_\varepsilon \to g \text{ strongly in } W_{\mathrm{loc}}^{-1,q}(\Omega), \quad \forall q < 2$$

*i.e.*,

$$\phi g_\varepsilon \to \phi g \text{ strongly in } W^{-1,q}(\Omega), \quad \forall q < 2 \text{ and } \forall \phi \in \mathcal{D}(\Omega).$$

□

The following is a $H$-convergence result for weak data from the positive cone of $H^{-1}(\Omega)$. We now prove the theorem in a particular case. The theorem in its full generality is stated and proved in Theorem 3.1 of [3].

**Theorem 2.4.** *Let $\{A_\varepsilon\}$ be a sequence of matrices in $\mathcal{M}(a, b, \Omega)$ which $H$-converges to a matrix $A_0$ and let $f \in H^{-1}(\Omega)$. If $u_\varepsilon \in H_0^1(\Omega)$ is the weak solution of*

$$\begin{cases} -\operatorname{div}(A_\varepsilon\nabla u_\varepsilon) = f + g_\varepsilon, & \text{in } \Omega \\ u_\varepsilon = 0, & \text{on } \partial\Omega \end{cases} \tag{2.11}$$

*with $g_\varepsilon \rightharpoonup g$ weakly in $H^{-1}(\Omega)$ and $g_\varepsilon$'s belong to the positive cone of $H^{-1}(\Omega)$. Then,*

$$\left.\begin{aligned} u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega),\\ A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 \text{ weakly in } (L^2(\Omega))^n, \end{aligned}\right\} \tag{2.12}$$

*where $u_0 \in H_0^1(\Omega)$ is the unique solution of*

$$\begin{cases} -\mathrm{div}(A_0 \nabla u_0) = f + g, & \text{in } \Omega \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.13}$$

*Remark* 2.1. We note that, in general, the energy functional does not converge for weakly converging data (from the positive cone) in $H^{-1}(\Omega)$, even if the coefficients are fixed, as the following example shows. Let $\Omega = (-1, 1) \subset \mathbb{R}$. Define $u_\varepsilon \colon \Omega \to \mathbb{R}$ as

$$u_\varepsilon(x) = \begin{cases} \dfrac{1+x}{\varepsilon}, & \text{if } x \in (-1, -1+\varepsilon^2), \\ \varepsilon, & \text{if } x \in [-1+\varepsilon^2, 1-\varepsilon^2], \\ \dfrac{1-x}{\varepsilon}, & \text{if } x \in (1-\varepsilon^2, 1), \end{cases}$$

then its first derivative $u_\varepsilon'$ is given as

$$u_\varepsilon'(x) = \begin{cases} \dfrac{1}{\varepsilon}, & \text{if } x \in (-1, -1+\varepsilon^2), \\ 0, & \text{if } x \in [-1+\varepsilon^2, 1-\varepsilon^2], \\ \dfrac{-1}{\varepsilon}, & \text{if } x \in (1-\varepsilon^2, 1). \end{cases}$$

Observe that the distribution $-u_\varepsilon'' = \frac{1}{\varepsilon}(\delta_{-1+\varepsilon^2} + \delta_{1-\varepsilon^2}) \geq 0$ is in the positive cone of $H^{-1}(\Omega)$ and converges weakly to 0. Also $u_\varepsilon \in H_0^1(\Omega)$ and $u_\varepsilon \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, while the associated energy functional $\|u_\varepsilon'\|_2^2 = 2$ is a constant independent of $\varepsilon$. $\square$

We now state some results which seem to be intuitively obvious but do not appear to have been proved anywhere in the available literature. For a proof of the following results we refer to [11].

A distribution is said to be *non-negative* if it takes non-negative values for all non-negative test functions. Now, if $f, g \in L^2(\Omega)$ are non-negative functions then, clearly, $\int_\Omega fg\,\mathrm{d}x \geq 0$. At this juncture one is interested to know if a similar statement is also valid in the dual of $H_0^1(\Omega)$, i.e., if $w \geq 0$ in $H^{-1}(\Omega)$ and $v \geq 0$ in $H_0^1(\Omega)$ then is it true that $\langle w, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0$? The answer is trivial to observe in the case when $\Omega = \mathbb{R}^n$ than in the case of a bounded open set in $\mathbb{R}^n$.

The basic idea for the $\Omega = \mathbb{R}^n$ case is that for any $v \in H_0^1(\mathbb{R}^n)$ such that $v \geq 0$ there exists a sequence of positive test functions converging strongly to $v$ in $H_0^1(\mathbb{R}^n)$. These positive test functions are obtained by the convolution of $v$ with the mollifiers (cf. (2.8)) and then using the cut-off function technique to make the support compact, i.e., define $v_k(x) = \zeta_k(x)(\rho_{\varepsilon_k} * v)(x)$ where the cut-off function $\zeta_k(x) = \zeta(x/k)$ for a function

$\zeta \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on $B(0,1)$ and $\mathrm{Supp}(\zeta) \subset B(0,2)$. This is a standard technique in the theory of Sobolev spaces to prove results on the entire space $\mathbb{R}^n$. But these techniques break down when $\Omega$ is a bounded open subset of $\mathbb{R}^n$. This difficulty is overcome by Proposition 2.1.

One knows that $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. In the following proposition we prove that for a given *positive* $H_0^1$ function we can extract a sequence of *positive* $H_0^1$ functions with compact support in $\Omega$ which converges to the given function in $H_0^1$.

PROPOSITION 2.1

*Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $v \in H_0^1(\Omega)$ and $v \geq 0$. Then there exists a sequence $\{\psi_n\} \subset H_0^1(\Omega)$ such that $\psi_n \to v$ in $H_0^1(\Omega)$, $\psi_n \geq 0$ for all n and $\psi_n$ has compact support in $\Omega$.*

*Remark* 2.2. In the result proved above if we choose $\psi_n = v - (v - \phi_n^+)^+$ then, in addition to the properties proved above, we also have that $0 \leq \psi_n \leq v$ for all $n$. Since, $v - \phi_n^+ \to 0$ strongly in $H_0^1(\Omega)$ we have $\|\nabla(v - \psi_n)\|_{2,\Omega}^2 \to 0$ because

$$\|\nabla(v - \phi_n^+)^+\|_{2,\Omega}^2 \leq \|\nabla(v - \phi_n^+)\|_{2,\Omega}^2 \to 0.$$

Hence $\psi_n \to v$ strongly in $H_0^1(\Omega)$ and by definition $0 \leq \psi_n \leq v$. □

PROPOSITION 2.2

*Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $g \in H^{-1}(\Omega)$ be such that $g \geq 0$ and let $u \in H_0^1(\Omega)$ be such that $u \geq 0$ a.e. in $\Omega$ then $\langle g, u\rangle \geq 0$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.*

We shall now prove a result which shows the equivalence of the above result to a statement on the closure of the positive cone.

PROPOSITION 2.3

*Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The following statements are true and are equivalent*:

(i) *The closure of the positive cone of $L^2(\Omega)$ in $H^{-1}(\Omega)$ is the positive cone of $H^{-1}(\Omega)$.*
(ii) *The closure of the positive cone of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$ is the positive cone of $H_0^1(\Omega)$.*
(iii) *If $g \in H^{-1}(\Omega)$ is such that $g \geq 0$ and $u \in H_0^1(\Omega)$ is such that $u \geq 0$ a.e. in $\Omega$ then $\langle g, u\rangle \geq 0$.*

## 2.2 *The limit problem*

We now completely settle the problem (2.1)–(2.2) for the positive cone case using the machinery developed in §2.1.

Let the admissible control set $U$ be the positive cone in $L^2(\Omega)$, i.e.,

$$U = \{\theta \in L^2(\Omega) | \theta \geq 0 \text{ a.e. in } \Omega\}.$$

We shall now introduce the adjoint problem and the optimality condition associated with the above described system.

The minimizer $\theta_\varepsilon^*$ is characterised by the optimality condition

$$\int_\Omega (u_\varepsilon^*(u_\varepsilon - u_\varepsilon^*) + \varepsilon\theta_\varepsilon^*(\theta - \theta_\varepsilon^*))\,\mathrm{d}x \geq 0, \quad \forall\theta \in U \tag{2.14}$$

where $u_\varepsilon$ is the state corresponding to $\theta$. We can rewrite the optimality condition as

$$\int_\Omega (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)(\theta - \theta_\varepsilon^*)\,\mathrm{d}x \geq 0 \quad \forall\theta \in U,$$

using the adjoint optimal state $p_\varepsilon^* \in H_0^1(\Omega)$ given as the weak solution of

$$\begin{cases} -\mathrm{div}({}^tA_\varepsilon\nabla p_\varepsilon^*) = u_\varepsilon^*, & \text{in } \Omega \\ p_\varepsilon^* = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Now,

$$\|u_\varepsilon^*\|_{2,\Omega}^2 \leq J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta), \quad \forall\theta \in U.$$

Therefore

$$\begin{aligned} \|u_\varepsilon^*\|_{2,\Omega}^2 &\leq \frac{1}{2}\|u_\varepsilon\|_{2,\Omega}^2 + \frac{\varepsilon}{2}\|\theta\|_{2,\Omega}^2 \\ &\leq \frac{1}{2}\|u_\varepsilon\|_{H_0^1(\Omega)}^2 + \frac{1}{2}\|\theta\|_{2,\Omega}^2 \\ &\leq \frac{1}{2a}\|f + \theta\|_{2,\Omega}^2 + \frac{1}{2}\|\theta\|_{2,\Omega}^2. \end{aligned}$$

Thus, since $\{u_\varepsilon^*\}$ is bounded in $L^2(\Omega)$, by $H$-convergence, there exists a matrix $A_0$ (called the $H$-limit of $\{A_\varepsilon\}$) such that

$$\begin{cases} -\mathrm{div}({}^tA_0\nabla p^*) = u^*, & \text{in } \Omega \\ p^* = 0, & \text{on } \partial\Omega \end{cases} \tag{2.16}$$

and $p_\varepsilon^* \rightharpoonup p^*$ weakly in $H_0^1(\Omega)$.

**Theorem 2.5.** *If $U = \{\theta \in L^2(\Omega) | \theta \geq 0$ a.e. in $\Omega\}$ is the admissible control set for the system* (2.1) *solving* (2.2), *then there exist $u^*$ and $\theta^*$ such that*

(a)

$$u_\varepsilon^* \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{2.17}$$

$$\varepsilon^{\frac{1}{2}}\theta_\varepsilon^* \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{2.18}$$

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\|u^*\|_{2,\Omega}^2, \tag{2.19}$$

(b) $\theta_\varepsilon^* \rightharpoonup \theta^*$ *weakly in $H^{-1}(\Omega)$ for the entire sequence.*

(c) $u^*$ *solves*

$$\begin{cases} -\mathrm{div}(A_0\nabla u^*) = f + \theta^*, & \text{in } \Omega \\ u^* = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

*where, now,* $\theta^* \in H^{-1}(\Omega)$.

(d) $\theta^*$ *is the unique minimizer of* $J(\theta) = \frac{1}{2}\|u(\theta)\|^2_{2,\Omega}$ *over* $V$, *the positive cone of* $H^{-1}(\Omega)$.

(e) $u^*$ *is the projection of* 0 *on to* $\bar{K}'$ *in* $L^2(\Omega)$, *i.e.,* $u^* \in \bar{K}'$ *and*

$$\int_\Omega u^*(v - u^*)\,\mathrm{d}x \geq 0 \quad \forall v \in \bar{K}'$$

*where*

$$K' = \{v \in H_0^1(\Omega) \mid -\mathrm{div}(A_0\nabla v) - f \in V\}.$$

*Proof.* (a) follows from Theorem 2.1. Also, (b) holds for a subsequence (cf. (2.6)) and by Theorem 2.4 we have that $u^*$ is the solution of (2.20), thus proving (c).

It follows from Proposition 2.3 that $V$ is the strong closure of $U$ in $H^{-1}(\Omega)$. Observe that $V$ is a closed convex subset of $H^{-1}(\Omega)$. Thus, $V$ is also the weak closure of $U$ in $H^{-1}(\Omega)$ and hence $\theta^* \in V$. We know that

$$J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta), \quad \forall \theta \in U. \tag{2.21}$$

Therefore, passing to the limit as $\varepsilon$ goes to 0 we have

$$J(\theta^*) \leq J(\theta), \quad \forall \theta \in U$$

and hence

$$J(\theta^*) \leq J(\theta), \quad \forall \theta \in V. \tag{2.22}$$

By the strict convexity of $J$, $\theta^*$ is the unique minimizer of $J$ over $V$, thus proving (d). The uniqueness of $\theta^*$ implies (b).

Let $\bar{K}'$ denote the closure of $K'$ in $L^2(\Omega)$. This is then a closed convex subset of $L^2(\Omega)$. Observe that $u^* \in K' \subset \bar{K}'$, since $\theta^* \in V$. Let $\theta \in U$ and $v(\theta)$ be the solution of

$$\begin{cases} -\mathrm{div}(A_0\nabla v) = f + \theta, & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.23}$$

Then passing to the limit in the optimality condition (2.14) and noting that $u_\varepsilon \rightharpoonup v(\theta)$ in $H_0^1(\Omega)$, we have

$$\int_\Omega u^*(v(\theta) - u^*)\,\mathrm{d}x \geq 0 \quad \forall \theta \in U.$$

Let $v \in K'$ and let $\theta = -\mathrm{div}(A_0\nabla v) - f$. Then there exists a sequence $\{\theta_n\} \subset U$ such that $\theta_n \to \theta$ strongly in $H^{-1}(\Omega)$. Let $v_n \in K'$ be the states corresponding to $\theta_n$ for which the above inequality holds. Thus,

$$\int_\Omega u^*(v - u^*)\,\mathrm{d}x \geq 0 \quad \forall v \in K'$$

and a simple density argument proves (e). □

*Remark* 2.3. Since $\theta^*$ is a unique minimizer of $J$ over $V$, it is characterised by the condition

$$\langle \theta - \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0 \quad \forall \theta \in V.$$

Now, by choosing $\theta = 0$ and $\theta = 2\theta^*$, we deduce $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$. Also, by choosing $\theta = \theta^* + \eta$, for arbitrary $\eta \in V$, we get $\langle \eta, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0$ implying that $p^* \geq 0$ a.e. in $\Omega$. □

*Remark* 2.4. We now observe that the $K'$ we defined in the above theorem is same as the $K$ defined in Theorem 2.1, i.e., $K' = K$. Let $v \in K$. Then there exists a sequence $\{v_\varepsilon\} \subset H_0^1(\Omega)$ such that $v_\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and $\theta_\varepsilon = -\mathrm{div}(A_\varepsilon \nabla v_\varepsilon) - f \in U$. Then, by Theorem 2.4, it follows that $v \in K'$ for some $\theta \in V$ which comes as the weak limit of $\theta_\varepsilon$ in $H^{-1}(\Omega)$. Thus, $K \subset K'$. Now, let $v \in K'$ and $\theta \in V$. Then there exists a sequence $\{\theta_\varepsilon\} \subset U$ such that $\theta_\varepsilon \to \theta$ strongly in $H^{-1}(\Omega)$. Set $v_\varepsilon$ to be the solution of

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla v_\varepsilon) = f + \theta_\varepsilon, & \text{in } \Omega \\ v_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.24}$$

and thus $v_\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$. Hence, we have shown $v \in K$ and therefore $K' \subset K$. □

*Remark* 2.5. The highlight of Theorem 2.5 is the result (d). We conclude that the optimal controls $\theta_\varepsilon^*$ converge weakly in $H^{-1}(\Omega)$ to $\theta^*$ which is a unique optimal control for the problem of minimising

$$J(\theta) = \frac{1}{2} \|u_0(\theta)\|_{2,\Omega}^2$$

over the set $V$, the positive cone of $H^{-1}(\Omega)$, where $u_0 \in H_0^1(\Omega)$ solves

$$\begin{cases} -\mathrm{div}(A_0 \nabla u_0) = f + \theta, & \text{in } \Omega \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.25}$$

Further, $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$. This was a problem open in [7] (cf. Theorem 2.1). They were also unable to estabilsh the relation between $u^*$ and $\theta^*$. Also, the description of the set $K'$ was quite complicated. □

## 3. Low-cost controls on perforated domains

In this section, we study the asymptotic behaviour of low-cost control problems on perforated domains.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $S_\varepsilon \subset \Omega$ be a family of closed subsets (called the 'holes'). Let $\Omega_\varepsilon = \Omega \backslash S_\varepsilon$ represent the perforated domain.

Let $U_\varepsilon \subset L^2(\Omega_\varepsilon)$, the set of admissible controls, be a closed convex set and let $f \in L^2(\Omega)$ be given.

We consider the system with the cost functional similar to the one in the previous section and see if this can be homogenized as has been done for the non-perforated case. We shall consider the cases when both the control and state are given in the domain (cf. §3.1) and when they are prescribed on the boundary (cf. §3.2).

3.1 *Control and state on the domain*

In this section, we consider the analogue of the system (2.1)–(2.2). Before we describe the problem, we introduce some notations required to proceed further. Let $\chi_\varepsilon$ denote the characteristic function of the set $\Omega_\varepsilon$ in $\Omega$,

$$\chi_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in \Omega_\varepsilon \\ 0, & \text{if } x \in S_\varepsilon \end{cases}$$

and let $\chi_0$ be a weak* limit of $\chi_\varepsilon$ in $L^\infty(\Omega)$. Henceforth, we fix a (sub)sequence such that $\chi_\varepsilon \rightharpoonup \chi_0$ weak* in $L^\infty(\Omega)$. The extension of a function on $\Omega_\varepsilon$ by zero on the holes of $\Omega$ is denoted with a $\sim$ in the superscript. We shall now prove a result which will be useful in the sequel.

It is easy to observe that when a sequence $f_\varepsilon \to f$ strongly in $L^2(\Omega)$ then we have $\int_\Omega \chi_\varepsilon f_\varepsilon \,\mathrm{d}x \to \int_\Omega \chi_0 f \,\mathrm{d}x$. We shall now prove a lemma that discusses about the $L^2$-norm convergence of $\chi_\varepsilon f_\varepsilon$.

*Lemma* 3.1. *If $f_\varepsilon \to f$ strongly in $L^2(\Omega)$ then $\|\chi_\varepsilon f_\varepsilon\|^2_{2,\Omega} \to \int_\Omega \chi_0 f^2 \,\mathrm{d}x$.*

*Proof.* Since $f_\varepsilon \to f$ in $L^2(\Omega)$, we have $\|f_\varepsilon\|_{2,\Omega} \to \|f\|_{2,\Omega}$. Equivalently, we have $\|f_\varepsilon^2\|_{1,\Omega} \to \|f^2\|_{1,\Omega}$. Further, for a subsequence, $f_\varepsilon(x) \to f(x)$ pointwise a.e.. Now, it can be shown as a consequence of Egoroff's theorem and Fatou's lemma (cf. Exercise 17(b), page 73 of [12]) that $f_\varepsilon^2 \to f^2$ strongly in $L^1(\Omega)$. Thus, we have (recall that $\chi_\varepsilon^2 = \chi_\varepsilon$),

$$\|\chi_\varepsilon f_\varepsilon\|^2_{2,\Omega} = \int_\Omega \chi_\varepsilon f_\varepsilon^2 \,\mathrm{d}x \to \int_\Omega \chi_0 f^2 \,\mathrm{d}x$$

using the $L^\infty(\Omega)$ weak* convergence of $\{\chi_\varepsilon\}$. Since the limit obtained above is independent of the subsequence, the convergence occurs for the entire sequence. □

We begin by assuming the following two hypotheses:

**H1.** There exists, for each $\varepsilon > 0$, an extension operator

$$P_\varepsilon \colon V_\varepsilon \to H_0^1(\Omega)$$

where $V_\varepsilon = \{u \in H^1(\Omega_\varepsilon) | u = 0 \text{ on } \partial\Omega\}$, such that, for every $u \in V_\varepsilon$,

$$P_\varepsilon u|_{\Omega_\varepsilon} = u \quad \text{and} \quad \|\nabla P_\varepsilon u\|_{2,\Omega} \le C_0 \|\nabla u\|_{2,\Omega_\varepsilon},$$

where the constant $C_0$ is independent of $\varepsilon$.

**H2.** Every weak* limit point in $L^\infty(\Omega)$ of $\{\chi_\varepsilon\}$ is positive a.e. in $\Omega$.

We say that the family of holes $\{S_\varepsilon\}$ is an *admissible* family of holes in $\Omega$, if the conditions (H1) and (H2) are satisfied. Throughout this paper $S_\varepsilon$ will denote an admissible family of holes in $\Omega$.

We now state the problem we are interested in. For a given $\theta_\varepsilon \in U_\varepsilon$, the cost functional is given by

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\|u_\varepsilon\|^2_{2,\Omega_\varepsilon} + \frac{\varepsilon}{2}\|\theta_\varepsilon\|^2_{2,\Omega_\varepsilon}, \tag{3.1}$$

where the state $u_\varepsilon = u_\varepsilon(\theta_\varepsilon) \in V_\varepsilon$ is the weak solution of

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = f + \theta_\varepsilon, & \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = 0, & \text{on } \partial S_\varepsilon \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

For $u \in V_\varepsilon$, we define the norm on $V_\varepsilon$ as $\|u\|_{V_\varepsilon} = \|\nabla u\|_{2,\Omega_\varepsilon}$.

The problem (3.1) solving (3.2) admits a unique optimal solution, which minimizes $J_\varepsilon$ in $U_\varepsilon$ and is denoted by $\theta_\varepsilon^*$. The corresponding optimal states is denoted by $u_\varepsilon^*$. We now introduce the adjoint optimal state $p_\varepsilon^* \in V_\varepsilon$ as the weak solution of the problem

$$\begin{cases} -\mathrm{div}({}^t A_\varepsilon \nabla p_\varepsilon^*) = u_\varepsilon^*, & \text{in } \Omega_\varepsilon \\ {}^t A_\varepsilon \nabla p_\varepsilon^* \cdot n_\varepsilon = 0, & \text{on } \partial S_\varepsilon \\ p_\varepsilon^* = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Then the optimality condition

$$\int_{\Omega_\varepsilon} [u_\varepsilon^*(u_\varepsilon - u_\varepsilon^*) + \varepsilon\theta_\varepsilon^*(\theta_\varepsilon - \theta_\varepsilon^*)]\, \mathrm{d}x \geq 0 \quad \forall \theta_\varepsilon \in U_\varepsilon \tag{3.4}$$

can be rewritten as

$$\int_{\Omega_\varepsilon} (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)(\theta_\varepsilon - \theta_\varepsilon^*)\, \mathrm{d}x \geq 0 \quad \forall \theta_\varepsilon \in U_\varepsilon.$$

We observe that $\theta_\varepsilon^*$ is the projection in $L^2(\Omega_\varepsilon)$ of $\frac{-p_\varepsilon^*}{\varepsilon}$ onto $U_\varepsilon$.

Given the hypothesis (H1) and assuming there exists, for each $\varepsilon > 0$, $\theta_\varepsilon \in U_\varepsilon$ such that $\{\tilde{\theta_\varepsilon}\}$ is bounded in $L^2(\Omega)$, then we have both $\{\chi_\varepsilon P_\varepsilon u_\varepsilon^*\}$, $\{\varepsilon^{1/2}\tilde{\theta_\varepsilon^*}\}$ bounded in $L^2(\Omega)$, and both $\{P_\varepsilon u_\varepsilon\}$ and $\{P_\varepsilon p_\varepsilon^*\}$ are bounded in $H_0^1(\Omega)$. The proofs of these are easy to check and can be found in [11]. It then follows that, up to a subsequence,

$$\varepsilon^{1/2}\tilde{\theta_\varepsilon^*} \rightharpoonup \theta' \text{ weakly in } L^2(\Omega), \tag{3.5}$$

$$\chi_\varepsilon P_\varepsilon u_\varepsilon^* \rightharpoonup u' \text{ weakly in } L^2(\Omega), \tag{3.6}$$

$$P_\varepsilon p_\varepsilon^* \rightharpoonup p^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega). \tag{3.7}$$

We observe that the adjoint equation (3.3) can be rewritten in the following way:

$$\begin{cases} -\mathrm{div}({}^t A_\varepsilon \nabla p_\varepsilon^*) = \chi_\varepsilon P_\varepsilon u_\varepsilon^*, & \text{in } \Omega_\varepsilon \\ {}^t A_\varepsilon \nabla p_\varepsilon^* \cdot n_\varepsilon = 0, & \text{on } \partial S_\varepsilon \\ p_\varepsilon^* = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Thus, we can homogenize the adjoint equation (3.3) (cf. Proposition 2.1 of [6]). In other words, by the theory of $H_0$-convergence, there exists a matrix $A_0$ such that (up to a subsequence) $A_\varepsilon$ $H_0$-converges to $A_0$ and $p^*$ is the solution of

$$\begin{cases} -\mathrm{div}({}^t A_0 \nabla p^*) = u', & \text{in } \Omega \\ p^* = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Let us now extend the admissible set to the space $L^2(\Omega)$ in the following way:

$$\tilde{U}_\varepsilon = \{\tilde{\theta_\varepsilon} \in L^2(\Omega) | \theta_\varepsilon \in U_\varepsilon\} \subset L^2(\Omega).$$

**Theorem 3.1.** *Let $A_0$ be the $H_0$-limit of $\{A_\varepsilon\}$ and let the sequential $K$-limit of $\{\widetilde{U_\varepsilon}\}$ in the weak topology of $L^2(\Omega)$ exist, denoted by $U$. Also let the optimal controls $\tilde{\theta_\varepsilon^*}$ converge to $\theta^*$ weakly in $L^2(\Omega)$. Then $\theta^*$ is the unique minimizer of*

$$J(\theta) = \frac{1}{2}\int_\Omega \chi_0 |u|^2 \,\mathrm{d}x$$

*in $U$, where $u = u(\theta) \in H_0^1(\Omega)$ is the weak solution of*

$$\begin{cases} -\mathrm{div}(A_0 \nabla u) = \chi_0 f + \theta, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

*Further*

$$P_\varepsilon u_\varepsilon^* \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega),$$

$$J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*),$$

*$u' = \chi_0 u^*$ and $\theta' = 0$.*

*Proof.* The fact that $\theta' = 0$ follows from the weak convergence hypothesis of the optimal controls $\theta_\varepsilon^*$. Now, since $U$ is the sequential $K$-limit of $\{\widetilde{U_\varepsilon}\}$, we have $\theta^* \in U$. Also, for any given $\theta \in U$, there exists a $\delta > 0$ and a sequence $\{\theta_\varepsilon\}$ such that $\theta_\varepsilon \rightharpoonup \theta$ weakly in $L^2(\Omega)$ and $\theta_\varepsilon \in \widetilde{U_\varepsilon}$, $\forall \varepsilon < \delta$. Now, since $\theta_\varepsilon^*$ is the minimizer of $J_\varepsilon$ in $U_\varepsilon$, we have, for $\varepsilon < \delta$,

$$J_\varepsilon(\theta_\varepsilon^*) \le J_\varepsilon(\theta_\varepsilon)$$

(we denote the restriction of $\theta_\varepsilon$ to $\Omega_\varepsilon$ by $\theta_\varepsilon$ itself). Taking limit on both sides of the above inequality, we have

$$\lim_{\varepsilon\to 0} \frac{1}{2}[\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}^2 + \varepsilon\|\tilde{\theta_\varepsilon^*}\|_{2,\Omega}^2] \le \lim_{\varepsilon\to 0} \frac{1}{2}[\|\chi_\varepsilon P_\varepsilon u_\varepsilon\|_{2,\Omega}^2 + \varepsilon\|\theta_\varepsilon\|_{2,\Omega}^2].$$

It now follows from the theory of $H_0$-convergence (cf. Proposition 2.1 of [6]) that $P_\varepsilon u_\varepsilon^* \rightharpoonup u^*$ and $P_\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ where the $u^*$ and $u$ are the solutions of the homogenized problem (3.10) corresponding to $\theta^*$ and $\theta$, respectively. Thus, $u' = \chi_0 u^*$. Hence, it now follows from Lemma 3.1 that

$$\frac{1}{2}\int_\Omega \chi_0 |u^*|^2 \,\mathrm{d}x \le \frac{1}{2}\int_\Omega \chi_0 |u|^2 \,\mathrm{d}x,$$

i.e. $J(\theta^*) \le J(\theta)$. Since $\theta \in U$ was arbitrary, we have shown that $\theta^*$ is the minimiser of $J$ over $U$. The uniqueness of $\theta^*$ is proved by passing to the limit in (3.4). Observe that (3.4) can be rewritten in the following way:

$$\int_\Omega [\chi_\varepsilon P_\varepsilon u_\varepsilon^* (P_\varepsilon u_\varepsilon - P_\varepsilon u_\varepsilon^*) + \varepsilon \tilde{\theta_\varepsilon^*}(\theta_\varepsilon - \tilde{\theta_\varepsilon^*})] \,\mathrm{d}x \ge 0, \quad \forall \theta_\varepsilon \in U_\varepsilon$$

where $\theta_\varepsilon$ is as chosen above that converges to $\theta$ weakly in $L^2(\Omega)$. Now passing to the limit in the above inequality, we have

$$\int_\Omega \chi_0 u^*(u - u^*)\,\mathrm{d}x \geq 0, \quad \forall u \in G(U)$$

where $G$ is the map $\theta \mapsto u$, where $u$ is the solution of (3.10). Note that, since $U$ is closed and convex, $G(U)$ is a closed convex subset of $L^2(\Omega)$ and thus we have $u^*$ as a projection of 0 onto $G(U)$ in $L^2_\mu(\Omega)$ where $\mathrm{d}\mu = \chi_0 \mathrm{d}x$. Thus, from the uniqueness of $u^*$ follows the uniqueness of $\theta^*$. □

*Remark* 3.1. We observe that the optimality condition involving the adjoint state

$$\int_{\Omega_\varepsilon} (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)(\theta_\varepsilon - \theta_\varepsilon^*)\,\mathrm{d}x \geq 0, \quad \forall \theta_\varepsilon \in U_\varepsilon$$

can be rewritten in the following way:

$$\int_\Omega (P_\varepsilon p_\varepsilon^* + \varepsilon\tilde{\theta}_\varepsilon^*)(\theta_\varepsilon - \tilde{\theta}_\varepsilon^*)\,\mathrm{d}x \geq 0, \quad \forall \theta_\varepsilon \in U_\varepsilon$$

and by passing to the limit, we obtain the optimality condition for the limit system

$$\int_\Omega p^*(\theta - \theta^*)\,\mathrm{d}x \geq 0, \quad \forall \theta \in U$$

where $p^*$ is the solution of (3.9) with $u' = \chi_0 u^*$. □

We observe that one is, in general, unable to verify the weak convergence hypothesis of the optimal controls as in Theorem 3.1 for the system (3.1) solving (3.2). However, we shall observe some trivial cases of the above mentioned system. Observe that, under the hypothesis of Theorem 3.1, if $-\chi_0 f \in U$ then by uniqueness of $\theta^*$, we have $\theta^* = -\chi_0 f$ and $u^* = 0$.

COROLLARY 3.1

*Under the hypothesis of Theorem* 3.1, *if* $-\chi_0 f \notin U$ *then* $\theta^* \in \partial U$.

*Proof.* Suppose $\theta^* \notin \partial U$, then for some $r > 0$ there exists a ball $B(\theta^*, r) \subset U$. Thus,

$$\theta^* + t\eta \in U \quad \forall \eta \in B(0,1) \text{ and } t < r.$$

Using this in the optimality condition of the limit system,

$$\int_\Omega p^*(\theta - \theta^*)\,\mathrm{d}x \geq 0, \quad \forall \theta \in U$$

we have, $\forall \eta \in B(0,1)$,

$$t\int_\Omega p^*\eta \geq 0.$$

Hence, $p^* = 0$ which in turn implies $u^* = 0$ and thus $\theta^* = -\chi_0 f \in U$, a contradiction. Thus, $\theta^* \in \partial U$. □

PROPOSITION 3.1

*If there exists a* $\delta > 0$ *such that* $-f \in U_\varepsilon,\ \forall \varepsilon < \delta$, *then*

$$P_\varepsilon u_\varepsilon^* \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega),$$

$$\tilde{\theta_\varepsilon^*} \rightharpoonup \theta^* = -\chi_0 f \text{ weakly in } L^2(\Omega),$$

$$J_\varepsilon(\theta_\varepsilon^*) \to 0.$$

*Proof.* It follows from the hypothesis that $J_\varepsilon(\theta_\varepsilon^*) \le J_\varepsilon(-f),\ \forall \varepsilon < \delta$. Thus,

$$\frac{1}{2}\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}^2 + \frac{\varepsilon}{2}\|\tilde{\theta_\varepsilon^*}\|_{2,\Omega}^2 \le \frac{\varepsilon}{2}\|\chi_\varepsilon f\|_{2,\Omega}^2.$$

Hence, we deduce that $\chi_\varepsilon P_\varepsilon u_\varepsilon^* \to 0$ strongly in $L^2(\Omega)$ and $\tilde{\theta_\varepsilon^*} \rightharpoonup \theta^*$ weakly (for a subsequence) in $L^2(\Omega)$. Also, we have $J_\varepsilon(\theta_\varepsilon^*) \to 0$. It now follows from the theory of $H_0$-convergence that $P_\varepsilon u_\varepsilon^* \rightharpoonup u^*$ weakly in $H_0^1(\Omega)$ and hence we observe that $u^* = 0$ and $\theta^* = -\chi_0 f$, also the convergence of the optimal states holds for the entire sequence. □

As we observe from the results developed so far that one lacks information on the optimal controls when the admissible sets are arbitrary. We now consider the case of the positive cone as the admissible set and hope to establish stronger convergence results for $u_\varepsilon^*$ and $\theta_\varepsilon^*$ without any hypothesis on the optimal controls.

**Theorem 3.2.** *Let* $U_\varepsilon = \{\theta \in L^2(\Omega_\varepsilon) | \tilde{\theta} \ge 0\, a.e. \text{ in } \Omega\}$. *Then* $\{P_\varepsilon u_\varepsilon^*\}$ *is bounded in* $H_0^1(\Omega)$ *and hence we have (for a subsequence),*

$$P_\varepsilon u_\varepsilon^* \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{3.11}$$

$$\tilde{\theta_\varepsilon^*} \rightharpoonup \theta^* \text{ weakly in } H^{-1}(\Omega), \tag{3.12}$$

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\int_\Omega \chi_0 |u^*|^2 \,\mathrm{d}x. \tag{3.13}$$

*Further* $u' = \chi_0 u^*$, $\theta' = 0$ *and* $p^* \ge 0$.

*Proof.* Since $U_\varepsilon$ is the positive cone, we have $\varepsilon\theta_\varepsilon^* = (p_\varepsilon^*)^-$ in $\Omega_\varepsilon$. Observe that $\varepsilon\tilde{\theta_\varepsilon^*} = \chi_\varepsilon P_\varepsilon (p_\varepsilon^*)^- = \chi_\varepsilon (P_\varepsilon p_\varepsilon^*)^-$ in $\Omega$. Since $0 \in U_\varepsilon$ for all $\varepsilon$, the convergences in (3.5), (3.6) and (3.7) are valid.

Using $u_\varepsilon^*$ as a test function in the weak form of the state equation satisfied by $u_\varepsilon^*$, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* \,\mathrm{d}x &= \int_{\Omega_\varepsilon} (f + \theta_\varepsilon^*) u_\varepsilon^* \,\mathrm{d}x \\ &= \int_\Omega \chi_\varepsilon f P_\varepsilon u_\varepsilon^* \,\mathrm{d}x + \varepsilon^{-1}\int_{\Omega_\varepsilon} (p_\varepsilon^*)^- u_\varepsilon^* \,\mathrm{d}x.\end{aligned}$$

Now using $(p_\varepsilon^*)^-$ as a test function in the weak form of the adjoint equation (3.9), we have

$$\int_{\Omega_\varepsilon} (p_\varepsilon^*)^- u_\varepsilon^* \,\mathrm{d}x = \int_{\Omega_\varepsilon} A_\varepsilon \nabla (p_\varepsilon^*)^- . \nabla p_\varepsilon^* \,\mathrm{d}x = -\int_{\Omega_\varepsilon} A_\varepsilon \nabla (p_\varepsilon^*)^- . \nabla (p_\varepsilon^*)^- \,\mathrm{d}x$$

and hence we derive the equality,

$$\int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* \, \mathrm{d}x + \varepsilon^{-1} \int_{\Omega_\varepsilon} A_\varepsilon \nabla (p_\varepsilon^*)^- . \nabla (p_\varepsilon^*)^- \, \mathrm{d}x = \int_\Omega \chi_\varepsilon f P_\varepsilon u_\varepsilon^* \, \mathrm{d}x. \tag{3.14}$$

Since $\{\chi_\varepsilon P_\varepsilon u_\varepsilon^*\}$ is bounded in $L^2(\Omega)$, we deduce from (3.14) that $\{P_\varepsilon u_\varepsilon^*\}$ and $\{\varepsilon^{-1/2} P_\varepsilon (p_\varepsilon^*)^-\}$ are bounded in $H_0^1(\Omega)$. Therefore, for a subsequence, (3.11) holds and

$$\varepsilon^{-1/2} P_\varepsilon (p_\varepsilon^*)^- \rightharpoonup q \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega). \tag{3.15}$$

Hence

$$\chi_\varepsilon P_\varepsilon u_\varepsilon^* \rightharpoonup \chi_0 u^* \text{ weakly in } L^2(\Omega)$$

and by (3.6) it follows that $u' = \chi_0 u^*$. Also

$$\varepsilon^{-1/2} \chi_\varepsilon P_\varepsilon (p_\varepsilon^*)^- \rightharpoonup \chi_0 q \text{ weakly in } L^2(\Omega)$$

i.e.

$$\varepsilon^{1/2} \tilde{\theta}_\varepsilon^* \rightharpoonup \chi_0 q \text{ weakly in } L^2(\Omega).$$

Therefore, by (3.5), we have $\theta' = \chi_0 q$.

For $v \in H_0^1(\Omega)$, consider

$$\begin{aligned}
\left| \int_\Omega \tilde{\theta}_\varepsilon^* v \, \mathrm{d}x \right| &= \left| \int_{\Omega_\varepsilon} \theta_\varepsilon^* v \, \mathrm{d}x \right| \\
&= \left| \int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla v \, \mathrm{d}x - \int_\Omega \chi_\varepsilon f v \, \mathrm{d}x \right| \\
&\leq \left( b \|u_\varepsilon^*\|_{V_\varepsilon} + C_0 \|\chi_\varepsilon f\|_{2,\Omega} \right) \|v\|_{H_0^1(\Omega)}.
\end{aligned}$$

Hence, it follows that $\{\tilde{\theta}_\varepsilon^*\}$ is bounded in $H^{-1}(\Omega)$ and thus there exists a $\theta^* \in H^{-1}(\Omega)$ such that (3.12) holds. Consequently,

$$\varepsilon^{1/2} \tilde{\theta}_\varepsilon^* \to 0 \text{ strongly in } H^{-1}(\Omega)$$

and thus $\theta' = \chi_0 q = 0$. Now, since $\varepsilon \tilde{\theta}_\varepsilon^* = \chi_\varepsilon (P_\varepsilon p_\varepsilon^*)^-$ in $\Omega$ we have, using (3.7)

$$\varepsilon \tilde{\theta}_\varepsilon^* \rightharpoonup \chi_0 (p^*)^- \text{ weakly in } L^2(\Omega).$$

Therefore, $\chi_0 (p^*)^- = 0$ which implies $(p^*)^- = 0$ and hence $p^* \geq 0$.

It now follows from (3.11) and Lemma 3.1 that

$$\|u_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 = \|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}^2 \to \int_\Omega \chi_0 |u^*|^2 \, \mathrm{d}x$$

and from (3.15) and Lemma 3.1 that

$$\|\varepsilon^{1/2} \tilde{\theta}_\varepsilon^*\|_{2,\Omega}^2 = \|\varepsilon^{-1/2} \chi_\varepsilon P_\varepsilon (p_\varepsilon^*)^-\|_{2,\Omega}^2 \to \int_\Omega \chi_0 q^2 \, \mathrm{d}x = 0.$$

Since $J_\varepsilon(\theta_\varepsilon^*) = \frac{1}{2} \left( \|u_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 + \|\varepsilon^{1/2} \tilde{\theta}_\varepsilon^*\|_{2,\Omega}^2 \right)$, (3.13) holds. □

*Remark* 3.2. The penultimate line in the above proof shows that, in fact, $\varepsilon^{1/2}\tilde{\theta}_\varepsilon^* \to 0$ strongly in $L^2(\Omega)$. Also, since $\theta^*$ and $p^*$ are positive, we have $\langle\theta^*, p^*\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \geq 0$. On the other hand, observe that $\int_{\Omega_\varepsilon}(p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)\theta_\varepsilon^*\,\mathrm{d}x = 0$ and hence $\int_{\Omega_\varepsilon} p_\varepsilon^*\theta_\varepsilon^*\,\mathrm{d}x = -\varepsilon\|\theta_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 \leq 0$. Thus $\int_{\Omega_\varepsilon} p_\varepsilon^*\theta_\varepsilon^*\,\mathrm{d}x \leq 0$. But we are unable to conclude that $\langle\theta^*, p^*\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \leq 0$, owing to the weak convergences of $p_\varepsilon^*$ in $H_0^1(\Omega)$ and $\theta_\varepsilon^*$ in $H^{-1}(\Omega)$. □

*Remark* 3.3. Using $p_\varepsilon^*$ as a test function in the state equation (3.2) corresponding to $\theta_\varepsilon^*$ and $u_\varepsilon^*$ as a test function in the adjoint-state equation (3.3), for the case $U_\varepsilon$ as in Theorem 3.2, we have

$$\begin{aligned}\int_\Omega \chi_\varepsilon(P_\varepsilon u_\varepsilon^*)^2\,\mathrm{d}x = \int_{\Omega_\varepsilon}(u_\varepsilon^*)^2\,\mathrm{d}x &= \int_{\Omega_\varepsilon} A_\varepsilon\nabla u_\varepsilon^*\cdot\nabla p_\varepsilon^*\,\mathrm{d}x\\ &= \int_{\Omega_\varepsilon}(f+\theta_\varepsilon^*)p_\varepsilon^*\,\mathrm{d}x\\ &= \int_\Omega \chi_\varepsilon f P_\varepsilon p_\varepsilon^*\,\mathrm{d}x - \varepsilon\int_{\Omega_\varepsilon}(\theta_\varepsilon^*)^2\,\mathrm{d}x.\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, it follows that

$$\int_\Omega \chi_0|u^*|^2\,\mathrm{d}x = \int_\Omega \chi_0 f p^*\,\mathrm{d}x.$$

This result is crucial in the sense that it hints to the fact that one can have $\langle\theta^*, p^*\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0$, if one could homogenize the state equation (3.2) with the controls $\theta_\varepsilon^*$. □

The absence of the result equivalent to Theorem 2.4 for the Neumann boundary condition problem hinders one from writing down the limit control problem for (3.1) solving (3.2) as was done for the non-perforated case in §2.2, which keeps the problem still open.

Due to the nature of the problem we do not have the uniqueness characterization of $\theta^*$, in general. We compensate this lack by proving a uniqueness characterization of $u^*$.

Let us define the set

$$E = \left\{ v \in H_0^1(\Omega) \,\middle|\, \begin{array}{r}\exists v_\varepsilon \in V_\varepsilon \text{ s.t. } P_\varepsilon v_\varepsilon \rightharpoonup v \text{ in } H_0^1(\Omega),\\ -\mathrm{div}(A_\varepsilon\nabla v_\varepsilon) \in L^2(\Omega_\varepsilon) \text{ and is } \geq f \text{ a.e. in } \Omega_\varepsilon\end{array}\right\}$$

and let $\bar{E}$, a closed convex set in $L^2(\Omega)$, denote the norm-closure of $E$ in $L^2(\Omega)$. It follows from (3.11) that $u^* \in E \subset \bar{E}$ and hence $E$ is non-empty. Let $G_\varepsilon\colon L^2(\Omega_\varepsilon) \to V_\varepsilon$ be the map $\theta_\varepsilon \mapsto u_\varepsilon$ where $u_\varepsilon$ is the solution of (3.2).

PROPOSITION 3.2

*Let $U_\varepsilon$ be as given in Theorem* 3.2. *Then $E$ is the $K$-limit of the sets $E_\varepsilon = P_\varepsilon G_\varepsilon(U_\varepsilon)$ in the weak topology of $H_0^1(\Omega)$.*

*Proof.*

(a) Let $v \in E$. We need to find a $\eta > 0$ and a sequence $v_\varepsilon \rightharpoonup v$ in $H_0^1(\Omega)$ such that $v_\varepsilon \in E_\varepsilon$, $\forall \varepsilon \leq \eta$.
Given $v \in E$, by definition of $E$, there exists $w_\varepsilon \in V_\varepsilon$ s.t. $P_\varepsilon w_\varepsilon \rightharpoonup v$ in $H_0^1(\Omega)$. Set $\theta_\varepsilon = -\mathrm{div}(A_\varepsilon \nabla w_\varepsilon) - f$. Hence, by definition of $E$, $\theta_\varepsilon \in U_\varepsilon$, $\forall \varepsilon$. Therefore $w_\varepsilon = G_\varepsilon(\theta_\varepsilon)$. Now, choose $v_\varepsilon = P_\varepsilon w_\varepsilon$, $\forall \varepsilon$. Hence our claim.

(b) Suppose $v_\varepsilon \in E_\varepsilon$ and $v_\varepsilon \rightharpoonup v$ in $H_0^1(\Omega)$, then we need to show that $v \in E$.
Let $v_\varepsilon = P_\varepsilon w_\varepsilon$ where $w_\varepsilon \in G_\varepsilon(U_\varepsilon) \subset V_\varepsilon$. Note that, in fact, $w_\varepsilon$ is $v_\varepsilon$ restricted to $\Omega_\varepsilon$. Also, $\theta_\varepsilon = -\mathrm{div}(A_\varepsilon \nabla w_\varepsilon) - f$ is in $U_\varepsilon$ and hence $-\mathrm{div}(A_\varepsilon \nabla w_\varepsilon) \in L^2(\Omega_\varepsilon)$. Hence our claim.

Thus, we have shown that $E_\varepsilon \overset{K}{\rightharpoonup} E$ in the weak topology of $H_0^1(\Omega)$. □

*Remark* 3.4. In the non-perforated case the above proposition reduces to saying that $G_\varepsilon(U) \overset{K}{\rightharpoonup} E$ in the weak topology of $H_0^1(\Omega)$ where

$$U = \{\theta \in L^2(\Omega) | \theta \geq 0 \text{ a.e. in } \Omega\},$$

$$E = \left\{ v \in H_0^1(\Omega) | \quad \begin{array}{c} \exists v_\varepsilon \in H_0^1(\Omega) \text{ s.t. } v_\varepsilon \rightharpoonup v \text{ in } H_0^1(\Omega), \\ -\mathrm{div}(A_\varepsilon \nabla v_\varepsilon) \in L^2(\Omega) \text{ and is } \geq f \text{ a.e. in } \Omega \end{array} \right\}$$

and $G_\varepsilon$: $L^2(\Omega) \to H_0^1(\Omega)$ is the map $\theta_\varepsilon \mapsto u_\varepsilon$ where $u_\varepsilon$ is the solution of the counterpart of (3.2) in the non-perforated case. □

**Theorem 3.3.** *If $U_\varepsilon$ is as in Theorem* 3.2, *then $u^*$ is the projection of* 0 *onto $\bar{E}$ in $L^2_\mu(\Omega)$ where* $\mathrm{d}\mu = \chi_0 \mathrm{d}x$. *In other words,*

$$\int_\Omega \chi_0 u^*(v - u^*)\,\mathrm{d}x \geq 0, \quad \forall v \in \bar{E}.$$

*Proof.* Let $v \in E$ and set $\theta_\varepsilon = -\mathrm{div}(A_\varepsilon \nabla v_\varepsilon) - f$. Then we have $\theta_\varepsilon \in U_\varepsilon$ and arguing as in Theorem 3.2 we prove $\tilde{\theta_\varepsilon}$ is bounded in $H^{-1}(\Omega)$. Using this $\theta_\varepsilon$ in (3.4) we have

$$\int_{\Omega_\varepsilon} [u_\varepsilon^*(v_\varepsilon - u_\varepsilon^*) + \varepsilon \theta_\varepsilon^*(\theta_\varepsilon - \theta_\varepsilon^*)]\,\mathrm{d}x \geq 0$$

$$\text{i.e. } \int_{\Omega_\varepsilon} u_\varepsilon^* v_\varepsilon\,\mathrm{d}x + \varepsilon \int_{\Omega_\varepsilon} \theta_\varepsilon^* \theta_\varepsilon\,\mathrm{d}x \geq \int_{\Omega_\varepsilon} (u_\varepsilon^*)^2\,\mathrm{d}x + \varepsilon \int_{\Omega_\varepsilon} (\theta_\varepsilon^*)^2\,\mathrm{d}x,$$

$$\text{i.e. } \int_\Omega \chi_\varepsilon P_\varepsilon u_\varepsilon^* P_\varepsilon v_\varepsilon\,\mathrm{d}x + \varepsilon \int_\Omega \tilde{\theta_\varepsilon^*} \tilde{\theta_\varepsilon}\,\mathrm{d}x \geq \int_\Omega \chi_\varepsilon (P_\varepsilon u_\varepsilon^*)^2\,\mathrm{d}x + \varepsilon \int_\Omega (\tilde{\theta_\varepsilon^*})^2\,\mathrm{d}x$$

whence, on passing to the limit

$$\int_\Omega \chi_0 u^* v\,\mathrm{d}x \geq \int_\Omega \chi_0 (u^*)^2\,\mathrm{d}x.$$

Since $v \in E$ was arbitrary we have

$$\int_\Omega \chi_0 u^*(v - u^*)\,\mathrm{d}x \geq 0, \quad \forall v \in E$$

and by simple density argument we have the inequality for all $v \in \bar{E}$. □

*Remark* 3.5. By the uniqueness of $u^*$, the convergence in (3.5) and (3.11) holds for the entire sequence and not just for a subsequence. □

Let us now consider the cases where $f$ has a sign. If $f \leq 0$ a.e. in $\Omega$. Then $-f \in U_\varepsilon$ (as defined in Theorem 3.2) and hence the result of Proposition 3.1 holds. Moreover, from (3.14), we have $P_\varepsilon u_\varepsilon^* \to 0$ strongly in $H_0^1(\Omega)$.

Observe that the weak maximum principle remains valid for the state equation (3.2) due to the homogeneous Dirichlet boundary condition on $\partial\Omega$ and the homogeneous Neumann boundary condition on the holes. If $f \geq 0$ a.e. in $\Omega$ and since $\theta_\varepsilon^* \geq 0$ a.e. in $\Omega_\varepsilon$, it follows from the weak maximum principle that $u_\varepsilon^* \geq 0$ a.e. in $\Omega_\varepsilon$. Thus by using the weak maximum principle for the adjoint equation (3.3), we have $p_\varepsilon^* \geq 0$ a.e. in $\Omega_\varepsilon$ and hence $\theta_\varepsilon^* = 0$ in $\Omega_\varepsilon$. Thus, $\theta^* = 0$ and the state equation becomes

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon^*) = f, & \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon^* \cdot n_\varepsilon = 0, & \text{on } \partial S_\varepsilon \\ u_\varepsilon^* = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Then, by $H_0$ convergence, it follows that $u^*$ is the solution of the homogenized problem

$$\begin{cases} -\mathrm{div}(A_0 \nabla u^*) = \chi_0 f, & \text{in } \Omega \\ u^* = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

**Theorem 3.4.** *Let $U_\varepsilon = L^2(\Omega_\varepsilon)$ then we have, $u' = \theta' = p^* = 0$ and*

$P_\varepsilon u_\varepsilon^* \to 0$ *strongly in* $H_0^1(\Omega)$,

$P_\varepsilon p_\varepsilon^* \to 0$ *strongly in* $H_0^1(\Omega)$,

$\varepsilon^{1/2}\theta_\varepsilon^* \to 0$ *strongly in* $L^2(\Omega)$,

$\theta_\varepsilon^* \rightharpoonup \theta^*$ *weakly in* $L^2(\Omega)$ *and* $\theta^* = -\chi_0 f$,

$J_\varepsilon(\theta_\varepsilon^*) \to 0.$

*Proof.* Since $-f$ restricted to $\Omega_\varepsilon$ is in $U_\varepsilon = L^2(\Omega_\varepsilon)$, the results of Proposition 3.1 stays valid. Also, the convergences (3.5), (3.6) and (3.7) remain valid. It follows from the strong convergence of $P_\varepsilon u_\varepsilon^*$ that $u' = 0$ and hence $p^* = 0$. Now, since $\{\theta_\varepsilon^*\}$ is bounded in $L^2(\Omega)$, we have $\varepsilon^{1/2}\theta_\varepsilon^* \to 0$ strongly in $L^2(\Omega)$ and thus $\theta' = 0$.

Also, from the optimality condition, we have $\varepsilon\theta_\varepsilon^* = -p_\varepsilon^*$ in $\Omega_\varepsilon$ and hence $\varepsilon\widetilde{\theta_\varepsilon^*} = -\chi_\varepsilon P_\varepsilon p_\varepsilon^*$ in $\Omega$. An argument similar to the one in Theorem 3.2 gives the equality corresponding to (3.14), i.e.,

$$\int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* \,\mathrm{d}x + \varepsilon^{-1} \int_{\Omega_\varepsilon} A_\varepsilon \nabla p_\varepsilon^* . \nabla p_\varepsilon^* \,\mathrm{d}x = \int_\Omega \chi_\varepsilon f P_\varepsilon u_\varepsilon^* \,\mathrm{d}x.$$

We deduce from the above equality that $P_\varepsilon p_\varepsilon^* \to 0$ and $P_\varepsilon u_\varepsilon^* \to 0$ strongly in $H_0^1(\Omega)$. □

3.2 *Control and state on boundary*

In this section, we consider the case of perforated domain for the boundary control problem. To begin we need to reformulate the notion of *admissible* family of holes. For this section, the family of holes, $\{S_\varepsilon\}$, is said to be admissible in $\Omega$ if, along with (H2), the following is satisfied:

**H3.** There exists, for each $\varepsilon > 0$, an extension operator

$$Q_\varepsilon \colon H^1(\Omega_\varepsilon) \to H^1(\Omega)$$

such that, for every $u \in H^1(\Omega_\varepsilon)$,

$$Q_\varepsilon u|_{\Omega_\varepsilon} = u \quad \text{and} \quad \|Q_\varepsilon u\|_{H^1(\Omega)} \leq C_0 \|u\|_{H^1(\Omega_\varepsilon)}$$

where $C_0$ is independent of $\varepsilon$.

Such family of admissible holes has been considered by Hruslov in [4]. We note that the holes allowed by (H2) and (H3) is not very different from those allowed by (H2) and (H1). We can, in fact, construct $Q_\varepsilon$ from the extension operator $P_\varepsilon$ obtained in (H1), provided we have the following:

**H4.** There exists a positive constant $C_0$ independent of $\varepsilon$ such that for every $u \in H^1(\Omega_\varepsilon)$,

$$\|u\|_{H^{1/2}(\partial\Omega)} \leq C_0 \|u\|_{H^1(\Omega_\varepsilon)}.$$

(Recall that $H^{1/2}(\partial\Omega)$ is the range of the trace map $\gamma \colon H^1(\Omega) \to L^2(\partial\Omega)$.)

To see this, assume (H4). Let $u \in H^1(\Omega_\varepsilon)$. Since $u$ restricted to $\partial\Omega$ is in $H^{1/2}(\partial\Omega)$, there exists a $v \in H^1(\Omega)$ such that

$$\|v\|_{H^1(\Omega)} \leq C_0 \|u\|_{H^{1/2}(\partial\Omega)}. \tag{3.18}$$

Thus, $u - v \in V_\varepsilon$. Then, by (H1), $P_\varepsilon(u - v) \in H_0^1(\Omega)$. Define $Q_\varepsilon u = P_\varepsilon(u - v) + v$. Then $v$ restricted to $\partial\Omega$ is same as $Q_\varepsilon u$ restricted to $\partial\Omega$, which is $u$ restricted to $\partial\Omega$. Now, consider

$$\begin{aligned}
\|Q_\varepsilon u\|_{H^1(\Omega)} &= \|P_\varepsilon(u - v) + v\|_{H^1(\Omega)} \\
&\leq \|P_\varepsilon(u - v)\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \\
&= \|P_\varepsilon(u - v)\|_{H_0^1(\Omega)} + \|v\|_{H^1(\Omega)} \\
&\leq C_0 \|u - v\|_{V_\varepsilon} + \|v\|_{H^1(\Omega)} \\
&\leq C_0 \left(\|u\|_{V_\varepsilon} + \|v\|_{V_\varepsilon}\right) + \|v\|_{H^1(\Omega)} \\
&\leq C_0 \left(\|u\|_{V_\varepsilon} + \|\nabla v\|_{2,\Omega}\right) + \|v\|_{H^1(\Omega)} \\
&\leq C_0 \|u\|_{V_\varepsilon} + C_1 \|v\|_{H^1(\Omega)}.
\end{aligned}$$

Therefore, by (3.18), we have

$$\|Q_\varepsilon u\|_{H^1(\Omega)} \leq C_0\|u\|_{V_\varepsilon} + C_1\|u\|_{H^{1/2}(\partial\Omega)}$$

and then by, (H4),

$$\begin{aligned}\|Q_\varepsilon u\|_{H^1(\Omega)} &\leq C_0\|u\|_{V_\varepsilon} + C_1\|u\|_{H^1(\Omega_\varepsilon)}\\ &\leq C_2\|u\|_{H^1(\Omega_\varepsilon)}.\end{aligned}$$

Thus, we have constructed a $Q_\varepsilon$ such that (H3) is valid.

Conversely, if (H3) is valid then we always have (H4). To see this, note that for $u \in H^1(\Omega_\varepsilon)$, $u$ restricted to $\partial\Omega$ is same as $Q_\varepsilon u$ restricted to $\partial\Omega$. Now, it follows from trace theory that, for $Q_\varepsilon u \in H^1(\Omega)$,

$$\|u\|_{H^{1/2}(\partial\Omega)} \leq C_0\|Q_\varepsilon u\|_{H^1(\Omega)}$$

and from (H3), it follows that

$$\|u\|_{H^{1/2}(\partial\Omega)} \leq C_0\|u\|_{H^1(\Omega_\varepsilon)}.$$

In short, for state equations with Neumann (or more general) condition on the boundary $\partial\Omega$ in perforated domains, the discussion above suggests that the admissible family of holes are required to satisfy either (H2) and (H3) or, equivalently, (H1), (H2) and (H4). To maintain consistency throughout the section, we shall work with the hypotheses (H2) and (H3).

We now state the optimal control problem to be studied in this section. Let $U_\varepsilon \subset L^2(\partial\Omega)$ and $f \in L^2(\partial\Omega)$ be given. For $\theta_\varepsilon \in U_\varepsilon$, the cost functional is given by

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\|u_\varepsilon\|^2_{2,\partial\Omega} + \frac{\varepsilon}{2}\|\theta_\varepsilon\|^2_{2,\partial\Omega}, \tag{3.19}$$

where the state $u_\varepsilon = u_\varepsilon(\theta_\varepsilon)$ in $H^1(\Omega_\varepsilon)$ is the unique solution of

$$\begin{cases} -\operatorname{div}(A_\varepsilon\nabla u_\varepsilon) + u_\varepsilon = 0, & \text{in } \Omega_\varepsilon\\ A_\varepsilon\nabla u_\varepsilon\cdot n_\varepsilon = 0, & \text{on } \partial S_\varepsilon\\ A_\varepsilon\nabla u_\varepsilon\cdot \nu = f + \theta_\varepsilon, & \text{on } \partial\Omega \end{cases}. \tag{3.20}$$

$n_\varepsilon$ and $\nu$ are the unit outward normal on $\partial S_\varepsilon$ and $\partial\Omega$, respectively.

As usual, (3.19) and (3.20) admit a unique optimal solution, which minimizes $J_\varepsilon$ in $U_\varepsilon$ and is denoted by $\theta^*_\varepsilon$. The corresponding optimal states are denoted by $u^*_\varepsilon$. Also, the adjoint optimal state $p^*_\varepsilon \in H^1(\Omega_\varepsilon)$ is given as the weak solution of the problem

$$\begin{cases} -\operatorname{div}({}^tA_\varepsilon\nabla p^*_\varepsilon) + p^*_\varepsilon = 0, & \text{in } \Omega_\varepsilon\\ {}^tA_\varepsilon\nabla p^*_\varepsilon\cdot n_\varepsilon = 0, & \text{on } \partial S_\varepsilon\\ {}^tA_\varepsilon\nabla p^*_\varepsilon\cdot \nu = u^*_\varepsilon, & \text{on } \partial\Omega \end{cases}. \tag{3.21}$$

Then the optimality condition

$$\int_{\partial\Omega}[u^*_\varepsilon(u_\varepsilon - u^*_\varepsilon) + \varepsilon\theta^*_\varepsilon(\theta_\varepsilon - \theta^*_\varepsilon)]\,\mathrm{d}\sigma \geq 0 \quad \forall\theta_\varepsilon \in U_\varepsilon \tag{3.22}$$

can be rewritten as

$$\int_{\partial\Omega} (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)(\theta_\varepsilon - \theta_\varepsilon^*)\,\mathrm{d}\sigma \geq 0 \quad \forall\theta_\varepsilon \in U_\varepsilon$$

and hence $\varepsilon\theta_\varepsilon^*$ is the projection in $L^2(\partial\Omega)$ of $-p_\varepsilon^*$ onto $U_\varepsilon$.

Assume that there exists a sequence $\theta_\varepsilon \in U_\varepsilon$ such that $\{\theta_\varepsilon\}$ is bounded in $L^2(\partial\Omega)$. We can then show that (cf. [11]) $\{u_\varepsilon^*\}$ and $\{\varepsilon^{1/2}\theta_\varepsilon^*\}$ are bounded in $L^2(\partial\Omega)$ and, $\{Q_\varepsilon u_\varepsilon(\theta_\varepsilon)\}$ and $\{Q_\varepsilon p_\varepsilon^*\}$ are bounded in $H^1(\Omega)$. It then follows that, up to a subsequence,

$$\varepsilon^{1/2}\theta_\varepsilon^* \rightharpoonup \theta' \text{ weakly in } L^2(\partial\Omega), \tag{3.23}$$

$$u_\varepsilon^* \rightharpoonup u' \text{ weakly in } L^2(\partial\Omega), \tag{3.24}$$

$Q_\varepsilon p_\varepsilon^* \rightharpoonup p^*$ weakly in $H^1(\Omega)$ and hence we have

$$p_\varepsilon^* \rightharpoonup p^* \text{ weakly in } H^{1/2}(\partial\Omega) \text{ and strongly in } L^2(\partial\Omega). \tag{3.25}$$

We can homogenize the adjoint-state equation (3.21) (cf. Proposition 2.1 of [6]) and, by the theory of $H_0$ convergence, there exists a matrix $A_0$ such that $A_\varepsilon$ $H_0$-converges to $A_0$ and $p^*$ is the solution of

$$\begin{cases} -\mathrm{div}({}^tA_0\nabla p^*) + \chi_0 p^* = 0, & \text{in } \Omega \\ {}^tA_0\nabla p^* \cdot \nu = u', & \text{on } \partial\Omega \end{cases}. \tag{3.26}$$

A proof analogous to that of Theorem 3.1 will prove the following theorem.

**Theorem 3.5.** *Let $A_0$ be the $H_0$-limit of $\{A_\varepsilon\}$ and let the sequential $K$-limit of $\{U_\varepsilon\}$ in the weak topology of $L^2(\partial\Omega)$ exist, denoted by $U$. Also let the optimal controls $\theta_\varepsilon^*$ converge to $\theta^*$ weakly in $L^2(\partial\Omega)$. Then $\theta^*$ is the unique minimizer of*

$$J(\theta) = \frac{1}{2}\int_{\partial\Omega} u^2\,\mathrm{d}\sigma$$

*in $U$, where $u = u(\theta) \in H^1(\Omega)$ is the weak solution of*

$$\begin{cases} -\mathrm{div}(A_0\nabla u) + \chi_0 u = 0, & \text{in } \Omega \\ A_0\nabla u \cdot \nu = f + \theta, & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

*Further $u' = u^*$ and $\theta' = 0$.*

We now establish stronger convergence results for $u_\varepsilon^*$ and $\theta_\varepsilon^*$ and homogenize the system when the admissible control set is the positive cone of $L^2(\partial\Omega)$.

**Theorem 3.6.** *Let $U = \{\theta \in L^2(\partial\Omega) | \theta \geq 0 \; a.e. \text{ on } \partial\Omega\}$, for all $\varepsilon > 0$. Then $Q_\varepsilon u_\varepsilon^* \rightharpoonup u^*$ weakly in $H^1(\Omega)$ and hence,*

$$u_\varepsilon^* \rightharpoonup u^* = u' \text{ weakly in } H^{1/2}(\partial\Omega) \text{ and strongly in } L^2(\partial\Omega), \tag{3.28}$$

$$\theta_\varepsilon^* \rightharpoonup \theta^* \text{ weakly in } H^{-1/2}(\partial\Omega), \tag{3.29}$$

$$\varepsilon^{1/2}\theta_\varepsilon^* \rightharpoonup \theta' = 0 \text{ weakly in } H^{1/2}(\partial\Omega) \text{ strongly in } L^2(\partial\Omega), \tag{3.30}$$

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\int_{\partial\Omega} |u^*|^2 \,\mathrm{d}\sigma. \tag{3.31}$$

*Further*, $u^*$ *and* $\theta^*$ *satisfy the homogenized problem as in* (3.27).

*Proof.* Since $U$ is the positive cone, we have $\varepsilon\theta_\varepsilon^* = (p_\varepsilon^*)^-$ a.e. in $\partial\Omega$. Since $0 \in U$, the convergences in (3.23), (3.24) and (3.25) are valid.

Now computing, as done in the proof of Theorem 3.2, we derive the equality

$$a_\varepsilon(u_\varepsilon^*, u_\varepsilon^*) + \varepsilon^{-1} a_\varepsilon((p_\varepsilon^*)^-, (p_\varepsilon^*)^-) = \int_{\partial\Omega} f u_\varepsilon^* \,\mathrm{d}\sigma, \tag{3.32}$$

where

$$a_\varepsilon(v, w) = \int_{\Omega_\varepsilon} A_\varepsilon \nabla v \cdot \nabla w \,\mathrm{d}x + \int_{\Omega_\varepsilon} v w \,\mathrm{d}x$$

is the bilinear form on $H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$.

Since $\{u_\varepsilon^*\}$ is bounded in $L^2(\partial\Omega)$, we deduce from (3.32) that $\{Q_\varepsilon u_\varepsilon^*\}$ and $\{\varepsilon^{-1/2} Q_\varepsilon (p_\varepsilon^*)^-\}$ are bounded in $H^1(\Omega)$. Therefore, for a subsequence, (3.28) holds and (3.23) holds weakly in $H^{1/2}(\partial\Omega)$ and strongly in $L^2(\partial\Omega)$ . To show $\theta' = 0$, we shall show that $\theta_\varepsilon^*$ is bounded in $H^{-1/2}(\partial\Omega)$. We have for $v \in H^1(\Omega)$,

$$\int_{\partial\Omega} \theta_\varepsilon^* v \,\mathrm{d}\sigma = a_\varepsilon(u_\varepsilon^*, v) - \int_{\partial\Omega} f v \,\mathrm{d}\sigma.$$

Therefore, $(\theta_\varepsilon^*, \psi)_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$ is bounded uniformly with respect to $\varepsilon$ for each $\psi$, since any $\psi \in H^{1/2}(\partial\Omega)$ can be continuously lifted to a $v \in H^1(\Omega)$. Hence $\theta_\varepsilon^*$ is bounded in $H^{-1/2}(\partial\Omega)$. Thus, (3.29) holds for some $\theta^* \in H^{-1/2}(\partial\Omega)$ and also $\theta' = 0$. Thus we have shown (3.30), and (3.31) follows from (3.28) and (3.30). Moreover, since $\theta_\varepsilon^* \geq 0$, we have that $\theta^* \geq 0$ in the sense of $H^{-1/2}(\partial\Omega)$.

It follows from the $H_0$-convergence that

$$(\widetilde{A_\varepsilon \nabla u_\varepsilon^*}) \rightharpoonup A_0 \nabla u^* \text{ weakly in } (L^2(\Omega))^n.$$

Let $v \in H^1(\Omega)$. Then, by passing to the limit in

$$\begin{aligned}
\int_{\partial\Omega} \theta_\varepsilon^* v \,\mathrm{d}\sigma &= \int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla v \,\mathrm{d}x + \int_{\Omega_\varepsilon} u_\varepsilon^* v \,\mathrm{d}x - \int_{\partial\Omega} f v \,\mathrm{d}\sigma \\
&= \int_{\Omega} (\widetilde{A_\varepsilon \nabla u_\varepsilon^*}) \cdot \nabla v \,\mathrm{d}x + \int_{\Omega} Q_\varepsilon u_\varepsilon^* \chi_\varepsilon v \,\mathrm{d}x - \int_{\partial\Omega} f v \,\mathrm{d}\sigma,
\end{aligned}$$

we have

$$\begin{aligned}
\langle \theta^*, v \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} &= \int_{\Omega} A_0 \nabla u^* \cdot \nabla v \,\mathrm{d}x + \int_{\Omega} \chi_0 u^* v \,\mathrm{d}x - \int_{\partial\Omega} f v \,\mathrm{d}\sigma \\
&= \int_{\Omega} -\mathrm{div}(A_0 \nabla u^*) \cdot \nabla v \,\mathrm{d}x + \int_{\Omega} \chi_0 u^* v \,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} A_0 \nabla u^* \cdot \nu \, v \,\mathrm{d}\sigma - \int_{\partial\Omega} f v \,\mathrm{d}\sigma
\end{aligned}$$

and hence for all $v \in H^1(\Omega)$,

$$\int_{\partial\Omega} A_0 \nabla u^* \cdot \nu \, v \, \mathrm{d}\sigma = \langle \theta^*, v \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} + \int_{\partial\Omega} f v \, \mathrm{d}\sigma.$$

Thus, $\theta^*$ and $u^*$ satisfy the homogenized problem as in (3.27). □

*Remark* 3.6. Using $p_\varepsilon^*$ as a test function in the state equation for $u_\varepsilon^*$ and $u_\varepsilon^*$ as a test function in the adjoint-state equation, for $U$ as in Theorem 3.6, we have

$$\int_{\partial\Omega} (u_\varepsilon^*)^2 \, \mathrm{d}\sigma = a_\varepsilon(u_\varepsilon^*, p_\varepsilon^*) = \int_{\partial\Omega} (f + \theta_\varepsilon^*) p_\varepsilon^* \, \mathrm{d}\sigma$$

$$= \int_{\partial\Omega} f p_\varepsilon^* \, \mathrm{d}\sigma - \varepsilon \int_{\partial\Omega} (\theta_\varepsilon^*)^2 \, \mathrm{d}\sigma.$$

Passing to the limit as $\varepsilon \to 0$, it follows that

$$\int_{\partial\Omega} (u^*)^2 \, \mathrm{d}\sigma = \int_{\partial\Omega} f p^* \, \mathrm{d}x. \tag{3.33}$$

Since, we could homogenize the state equation, it follows that

$$\int_{\partial\Omega} f p^* \, \mathrm{d}\sigma + \langle \theta^*, p^* \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$$

$$= \int_{\Omega} A_0 \nabla u^* \cdot \nabla p^* \, \mathrm{d}x + \int_{\Omega} \chi_0 u^* p^* \, \mathrm{d}x$$

$$= \int_{\partial\Omega} {}^t A_0 \nabla p^* \cdot \nu \, u^* \, \mathrm{d}\sigma$$

$$= \int_{\partial\Omega} (u^*)^2 \, \mathrm{d}\sigma.$$

Hence, using (3.33), we deduce that $\langle \theta^*, p^* \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0$. □

We now study the unconstrained control set case.

**Theorem 3.7.** *Let $U = L^2(\partial\Omega)$. Then we have, $u' = \theta' = p^* = 0$ and*

$$Q_\varepsilon u_\varepsilon^* \to u^* = 0 \text{ strongly in } H^1(\Omega),$$

$$\theta^* = -f,$$

$$J_\varepsilon(\theta_\varepsilon^*) \to 0.$$

*Proof.* Since $0 \in U$, the convergences in (3.23), (3.24) and (3.25) are valid. Also, by the optimality condition, we have $\varepsilon \theta_\varepsilon^* = p_\varepsilon^*$ a.e. in $\partial\Omega$.

The analogous equality of (3.32) will be

$$a_\varepsilon(u_\varepsilon^*, u_\varepsilon^*) + \varepsilon^{-1} a_\varepsilon(p_\varepsilon^*, p_\varepsilon^*) = \int_{\partial\Omega} f u_\varepsilon^* \, \mathrm{d}\sigma.$$

It follows from the above equality that $p^* = 0$ and from the homogenized adjoint equation, it follows that $u' = 0$. Hence, by the above equality and (3.24), we have $Q_\varepsilon u_\varepsilon^* \to u^* = 0$ strongly in $H^1(\Omega)$. Also, we have $\varepsilon^{-1/2} Q_\varepsilon p_\varepsilon^* \to 0$ strongly in $H^1(\Omega)$ and hence, by (3.23), $\theta' = 0$. Thus $\theta^* = -f$ and $J_\varepsilon(\theta_\varepsilon^*) \to 0$. □

## References


[1] Bensoussan A, Lions J L and Papanicolaou G, Asymptotic Analysis for Periodic Structures (Amsterdam: North Holland) (1978)

[2] Dal Maso G, An Introduction to $\Gamma$-Convergence, vol. 8 of Progress in Nonlinear Differential Equations and Their Applications (Boston: Birkhäuser) (1993)

[3] Dal Maso G and Murat F, Asymptotic behaviour and correctors for linear Dirichlet problems with simultaneously varying operators and domains, *Ann. I. H. Poincaré-Analyse Non linéaire* **21** (2004) 445–486

[4] Hruslov E J, The asymptotic behavior of solutions of the second boundary value problem under fragmentation of the boundary of the domain, *Math USSR Sbornik* **35** (1979) 266–282

[5] Kesavan S and Saint Jean Paulin J, Homogenization of an optimal control problem, *SIAM J. Control Optim.* **35** (1997) 1557–1573

[6] Kesavan S and Saint Jean Paulin J, Optimal control on perforated domains, *J. Math. Anal. Appl.* **229** (1999) 563–586

[7] Kesavan S and Saint Jean Paulin J, Low cost control problems, Trends in Industrial and Applied Mathematics (2002) 251–274

[8] Lions J L, Remarks on 'Cheap Control', in: Topics in Numerical Analysis, J J H Miller (ed.) (Academic Press Inc.) (1973) pp. 203–209, ISBN 0-12-496950-X

[9] Meyers N G, An $L^p$-estimate for the gradient of solutions of second order elliptic divergence equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **17** (1963) 189–206

[10] Murat F, L'injection du cône positif de $H^{-1}$ dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl.* **60** (1981) 309–322

[11] Muthukumar T, Asymptotic Behaviour of some Optimal Control Problems, Ph.D. thesis, University of Madras, Chennai Institute of Mathematical Sciences, Chennai, India, May 2006

[12] Rudin W, Real and Complex Analysis, third ed. (McGraw-Hill Book Company) (1987) ISBN 0-07-100276-6