

A study of disordered systems with gain: Stochastic Amplification

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A study of statistics of transmission and reflection from a random medium with stochastic amplification as opposed to coherent amplification is presented. It is found that the transmission coefficient t , for sample length L less than the critical length L_c grows exponentially with L . In the limit $L \rightarrow \infty$ transmission decays exponentially as $\langle \ln t \rangle = -L/\xi$ where ξ is the localization length. In this limit reflection coefficient r saturates to a fixed value which shows a monotonic increase as a function of strength of amplification α . The stationary distribution of super-reflection coefficient agrees well with the analytical results obtained within the random phase approximation (RPA). Our model also exhibits the well known duality between absorption and amplification. We emphasize the major differences between coherent amplification and stochastic amplification where-ever appropriate.

PACS Numbers: 42.25.Bs, 71.55.Jv, 72.15.Rn, 05.40.+j

With the development of scaling theory of localization our understanding of the Anderson localization of electrons in disordered systems has advanced considerably [1]. In close analogy with this it was felt that light waves in a medium with random dielectric constant may also exhibit localization [2]. In the interaction of light with matter apart from scattering, absorption plays a very important role. Due to absorption, light intensity gets strongly attenuated after being scattered a few times. Presence of gain in the medium not only compensates for this absorption but also opens up the possibility of observing interesting phenomenon arising due to interplay of disorder and amplification in strongly scattering media. This gave birth to the fascinating idea of the so-called “random lasers” which are essentially mirror-less lasers [4]. The obvious importance of this in the field of laser instrumentation together with viability of multiple scattering experiments using powdered laser crystals or micro-particle suspensions in laser dyes fueled an intense research activity in the field in recent years [3]. The different techniques employed in the theoretical investigations include the time-independent studies of reflection and transmission using invariant imbedding [4–7] equations and transfer matrix technique [8–12] for one-dimensional systems, Fokker-Planck equation for many channel case [13–15] and time dependent diffusion equation approach [16]. In these studies the gain or amplification(absorption) is modeled by introducing an imaginary part into the dielectric constant in case of classical (electro-magnetic) waves in random medium or into the site energy for the tight-binding models [10,11]. This amounts to making the potential complex leading to a non-Hermitian Hamiltonian [4–6,8–19]. The resulting amplification is referred to as coherent amplification.

The presence of imaginary potential gives rise to many a counter-intuitive features and some interesting results.

Due to the real to complex mismatch experienced by the incident waves in the vicinity of the amplifying medium there is an enhanced back reflection. Thus the medium acts as an amplifier as well as a reflector. This dual role implies that amplification without reflection is not possible [17,18]. In fact in the limit of the strength of imaginary part of the potential becoming infinite, reflection probability tends to unity, i.e., the amplifier acts as a perfect reflector [10,11]. The stationary distribution from such a disordered coherently amplifying medium, calculated under the assumption of random phase approximation using invariant imbedding method is given by [4]

$$P_s(r) = \frac{|D| \exp\left(-\frac{|D|}{r-1}\right)}{(r-1)^2} \quad \text{for } r \geq 1 \quad (1)$$

$$= 0 \quad \text{for } r < 1$$

Here D is proportional to V_i/W , V_i and W being the strength of imaginary part of the potential and disorder respectively. The distribution peaks at a single value of $r > 1$ which keeps shifting to higher values of r with increasing amplification. On the contrary, the exact distribution obtained numerically [11] exhibits a double peaked structure for strong amplification or disorder and in the limit of perfect reflector ($V_i \rightarrow \infty$) becomes a delta function at $r = 1$ in consistency with the physics of the problem. The transmission coefficient is also known to exhibit equally surprising features. Despite amplification, in the limit of sample length $L \rightarrow \infty$ the transmission falls off exponentially as a function of L with a well defined localization length ξ not only when there is disorder along with but also in case of a perfectly periodic amplifying medium [8–15]. Moreover, the rate of exponential decay is the same as that of an absorbing medium, i.e., $\xi(V_i) = \xi(-V_i)$. This symmetry, which has been re-

ferred to as duality between absorption and amplification in the literature follows from the time-independent wave equation [14]. Recently, Soukoulis et. al. have claimed that this paradoxical result is a mere artifact of time-independent equations. Based on the analysis of the geometric series for the transmission amplitude they argue that depending on the gain parameter the series becomes divergent beyond a particular length indicating the absence of stable time-independent solutions [19]. However, it is to be noted that the exact calculation of the transmission coefficient (as a scattering problem) as a function of length for an ordered lasing medium does not show any divergence. As a function of length transmission coefficient exhibits a maximum both for ordered as well as disordered lasing systems.

The interesting features discussed above seem to be a fallout of the fact that the phase coherence of the waves in a medium thus modeled is maintained in spite of the presence of amplification justifying the term coherent amplification. In the present work we endeavor to study a different model for amplification in the framework of time-independent wave equations which gives qualitatively different results for statistics of super-reflection in the strong amplification regime. The model was used to study stochastic absorption also [20]. In this simple minded model a chain of uniformly spaced random strength delta function scatterers act as the disordered medium. A negative attenuation constant per unit length α characterizes the stochastic amplification. The free propagator acquires a factor $exp(\alpha a)$ for each trip while traversing the free region of length a in between the scatterers [21]. We find that this method of incorporating amplification avoids the additional reflections and resonances observed in case of coherent amplification as mentioned above. The continuum limit of our model yields the same Langevin equation for reflection amplitude $R(L)$ as obtained by Pradhan [22,23] for the case of stochastic absorption except that α is to be replaced by -2α . The stationary solution of the Fokker-Planck equation for the probability distribution of reflection coefficient $P_s(r)$ in case of lasing medium obtained within RPA, however, is the same as given by Eq. 1 with the D in the expression now proportional to α/W . The numerically obtained exact stationary distribution of reflection coefficient agrees well with the analytical result for small values of α but in larger parameter space as compared to the coherent amplification case [11,20]. With increasing α the stationary value $\langle lnr \rangle_s$ keeps increasing monotonically as against the non-monotonic behavior observed in case of coherent amplification. For coherent amplification $\langle lnr \rangle_s$ approaches zero in the $\alpha \rightarrow \infty$ limit as the amplifier then acts as a perfect reflector. One does not expect this from a realistic amplifying medium. Our phenomenological model rectifies this problem. Although the properties of reflection coefficient for stochastic amplification are qualitatively contrasting, the properties of transmission show some similarities with those observed in case of coherent amplification. In the present model

the transmission decays exponentially with the sample length L with a well defined localization length ξ . This localization length turns out to be equal to the one obtained for stochastic absorption, i.e., we find that this model also exhibits the dual symmetry. However, there are some minor differences in regard to the critical system length L_c beyond which the transmission starts decaying. In the following we begin by briefly describing the model and numerical procedure. We then present results and discussions and finally we give some conclusions.

We take recourse to the transfer matrix technique for our numerical calculations [24,25]. As mentioned earlier the chain of random strength delta function scatterers spaced at regular intervals of length a plays the role of a disordered medium. The transfer matrix for j^{th} delta function having strength q_j is

$$M_j = \begin{pmatrix} 1 - iq_j/2k & -iq_j/2k \\ iq_j/2k & 1 + iq_j/2k \end{pmatrix}$$

The q_j 's are chosen from a flat distribution between $-W/2$ and $W/2$, i.e., $P(q_j) = 1/W$ where W is the strength of the disorder. We choose to work with units such that \hbar and $2m$ are unity and therefore $E = k^2$ is the energy of the incident wave. W and α are scaled to make them dimensionless. Stochastic amplification is introduced in the free propagation medium through the transfer matrix [20]

$$X = \begin{pmatrix} e^{ik+\alpha} & 0 \\ 0 & e^{-ik-\alpha} \end{pmatrix}.$$

The total transfer matrix for the stochastically amplifying medium is then obtained by recursive multiplication of these matrices i.e.,

$$M = M_L X \dots X M_2 X M_1.$$

One can then calculate the reflection and transmission coefficient from the matrix M by using the well-know formulae [25]

$$R = -\frac{M(2,1)}{M(2,2)}$$

and

$$T = -\frac{\det M}{M(2,2)}.$$

Needless to say that since we are working with non-Hermitian systems $r + t \neq 1$. In the results presented here we have considered at least 10,000 realizations for calculating the various distributions and averages and the incident energy is chosen to be $E = k^2 = 1$ unless specified otherwise. We consider chains of length about 10 times the localization length in order to evaluate the stationary distributions. The distributions did

not evolve on increasing length beyond 10ξ . To show the exponential scaling of transmission coefficient with length L of the sample we plot $\langle lnt \rangle$ versus L/ξ in Fig. 1 for $W=1.0$ and different values of amplification strength α ($\alpha=0.01,0.05,0.1,0.15$). From the figure it is evident that there is a critical length L_c of the system upto which the transmission grows and beyond that the exponential decay takes over. For $L < L_c$ the transmission increases with sample length as $\langle lnt \rangle = (1/\xi_a - 1/\xi_w)L$ as found in earlier studies [5,12]. Here $\xi_a \sim 1/\alpha$ is the gain length characterizing the exponential growth of the transmission in absence of disorder and presence of amplification only. $\xi_w = 96k^2/W^2$ is the localization length in the presence of disorder alone [26]. In the limit $W \rightarrow 0$, $\langle lnt \rangle \sim L/\xi_a$ and $L_c \rightarrow \infty$ i.e., in absence of disorder transmission keeps growing with sample length. In contrast to this, for coherent amplification $L_c \rightarrow L_c^0$, a fixed value for $W \rightarrow 0$. This is due to the fact that even in the ordered case coherent amplification leads to backscattering, i.e., reflection. In the present case of stochastic amplification for an ordered system transmission coefficient grows exponentially with length. The presence of disorder induces coherent backscattering leading to the exponential decay (localization) of the transmission in the asymptotic limit in spite of amplification. For lengths greater than L_c , i.e., $L \gg L_c$, the behavior of $\langle lnt \rangle$ versus L is given by $\langle lnt \rangle = -L/\xi$, where ξ is the localization length scaled with respect to inter-scatterer spacing a . The plot of $1/\xi$, calculated numerically by changing α for various values of disorder strength, versus $(1/\xi_a + 1/\xi_w)$ shown in Fig. 2 suggests the scaling relation $1/\xi = (1/\xi_a + 1/\xi_w)$ for the localization length in presence of both disorder and amplification. The value of ξ thus obtained was used for the plot of Fig. 1. We note that this relation is identical to one obtained for the localization length in case of a stochastically absorbing medium, i.e., $\xi(\alpha, W) = \xi(-\alpha, W)$ [20]. This gives the numerical evidence for the fact that our model also exhibits the much debated duality between absorption and amplification. Although the same duality is observed in case of coherent amplification and absorption it is important to realize that in the present case the duality relation holds only for non-zero disorder strength ($W \neq 0$) whereas in the former it was valid even for $W = 0$. To explicate the point of dual symmetry we have shown the plot of $\langle lnt \rangle$ versus L for $\alpha = \pm 0.1$ in Fig. 3. From the figure we see that the asymptotic slope of the transmission coefficient for absorption as well as amplification ($L \gg L_c$) are equal.

The log-scale plot shown in Fig. 4 illustrates the dependence of the average transmission $\langle t \rangle$, the root mean squared (rms) variance of t , namely $t_v = \sqrt{\langle t^2 \rangle - \langle t \rangle^2}$ and the rms relative variance $t_{rv} = t_v/\langle t \rangle$ on the sample length L . For $L < \xi$, rms variance of t is less than the average value of t , i.e., $t_v < \langle t \rangle$ and consequently rms relative variance is less than unity ($t_{rv} < 1$). However, for $L > \xi$ the rms variance of t exceeds the average

transmission ($t_v > \langle t \rangle$) and rms relative variance crosses unity ($t_{rv} > 1$) indicating the onset of non-self-averaging (NSA) nature of transmission [28,29]. Therefore for samples of length $L > \xi$ the average transmission over an ensemble of macroscopically identical samples is dominated over by the sample-to-sample fluctuations in t . The sensitivity of the transmission coefficient to the exact spatial realization of the impurities gets highly enhanced. A look at the plot of the distribution of the transmission coefficient at different lengths shown in Fig. 5 helps to shed more light on this. The parameters for the figure are $\alpha=0.01$ and disorder strength $W=1.0$. Corresponding localization length is $\xi \sim 50.0$. For sample lengths (see $L=5$) much smaller than the localization length ($L \ll \xi$), though the resonant transmission dominates, the distribution peaks at a value of t close to unity because amplification strength is small. On increasing the length L further ($L=10,20$) the peak shifts to higher values of t . When the sample length becomes comparable to the localization length ($L \sim \xi \sim 50$), multiple reflections start dominating. Due to these multiple reflections time spent inside the medium increases thereby enhancing the amplification. This broadens the distribution and tails start appearing. This, however, does not continue indefinitely with increasing length. As seen earlier finally in the long length limit ($L \gg \xi$) transmission decays to zero exponentially with length. This reflects in the distributions of t also. The peak of the distribution starts shifting to values of t much less than one and simultaneously develops long tails. These tails are a clear signature of the sample-to-sample fluctuations which tender the non-self-averaging nature to the transmission coefficient. The origin of these tails can presumably be traced back to the so-called Azbel resonances observed in case of passive disordered media [27]. Since we have amplification in addition to disorder it would be naive to expect no change whatsoever in the structure of these resonances. Thus a study of resonances in our model is in order. For the purpose we assume that the ergodic hypothesis of relation between the ensemble fluctuations and the fluctuations for a given sample as a function of the chemical potential holds true in the present case. Plot of Fig. 6 shows the variation of transmission coefficient with the wave vector k for a single realization of the disorder, both in the absence and in the presence of amplification. The plot reveals that presence of amplification modifies the resonances observed in case of passive medium in two ways namely, the value of transmission at the resonances is either enhanced ($t > 1$) or reduced ($t < 1$) depending upon the sample length L and the widths of the resonance peaks is reduced in case of the amplifying medium. Unresolved or closely spaced resonance peaks in the case of passive disorder are resolved due to the narrowing effect of amplification. This also can be interpreted in the following manner: two nearby overlapping resonances may have characteristically different delay times [30] or dwell times [31]. Due to this enhancement of the transmittance by the amplification will be very different and hence the

resolution. This clearly indicates that amplification does not give rise to any new resonances and only modifies the existing resonances vis-a-vis their strength and resolution. We have verified the above observation for many different macroscopic realizations of the underlying disorder. This is in contrast to the case of coherent amplification where new resonances can appear due to the reflecting nature of imaginary potentials. Such complex potentials can give additional phase shifts leading to new resonances in addition to those seen in case of the same disordered realization but in the absence of imaginary potential.

We shall now discuss about the statistics of reflection coefficient which exhibits properties radically different from those observed for coherent amplification in the limit of strong amplification. Fig. 7 shows the variation of $\langle lnr \rangle$ with L for a fixed value of disorder strength $W = 1.0$ and various values of amplification strength α as indicated in the figure. As a function of length $\langle lnr \rangle$ increases monotonically. In the long length limit $L \gg \xi$, $\langle lnr \rangle$ saturates to a certain positive value $\langle lnr \rangle_s$ depending upon the value of disorder strength W and amplification strength α . At any length we notice that reflection (super-reflection) increases with increasing α .

In Fig. 8 we show the stationary distribution of reflection coefficient $P_s(r)$ for different values of amplification strength α . For small values of α ($\alpha = 0.001$) the distribution peaks at small values of r close to unity. On increasing α the peak shifts to larger values of r . The single parameter fit obtained using the analytical expression given in Eq. 1 is shown by a thick line. On increasing α further ($\alpha > 0.1$) the peak of the distribution continues to shift to higher values of r . Although the fit is no more valid, the qualitative features of the distribution, i.e., single prominent peak and long tail, do not change. We have observed in our numerical calculations that in this parameter range the distribution of phase of reflected wave shows two distinct peaks indicating breakdown of RPA. This is in striking contrast to the perfect reflector behavior observed for the case of coherent amplification [11] where on increasing amplification beyond a certain strength the distribution starts moving to lower values of r and in the limit $V_i \rightarrow \infty$, $P_s(r) \rightarrow \delta(r - 1)$. In the intermediate regime of V_i , the $P_s(r)$ in the coherent amplification exhibits a double peak structure reminiscent of the additional reflections introduced by the imaginary potential (see Ref. [11] for details). We do not have any such additional reflections arising due to amplification in the case of stochastic amplification. This is also reflected from the plot of $\langle lnr \rangle_s$ versus amplification strength α shown in Fig. 9. Recall that the $\langle lnr \rangle_s$ was a non-monotonic function of α in case of coherent amplification. In contrast here we find that $\langle lnr \rangle_s$ is a monotonically increasing function of α in accordance with the physical expectations from this model.

In summary, we have explored the effect of disorder on the stochastic amplification by numerically studying the statistics of transmission and reflection. It is found

that localization occurs solely due to the presence of disorder. The critical length L_c goes to infinity as disorder strength is reduced to zero. The transmittance for absorbing as well as amplifying medium falls off exponentially with length in the asymptotic limit with same localization length depending on the values of $|\alpha|$ and W . Thus the model exhibits duality between absorption and amplification. The transmittance is found to be NSA for $L > \xi$. Stochastic amplification does not introduce any new resonances in the behavior of transmittance as a function of energy. The average value of logarithm of reflection coefficient shows monotonic increase with increasing amplification strength α in contrast to the behavior observed in case of coherent amplification. The numerically calculated stationary distribution agrees well with the analytical result in a larger parameter space. For strong amplification regime the distribution still retains the same qualitative features with peak shifting to higher values of reflection. Thus there is no additional reflection due to presence of amplification as against the observation in case of coherent amplification.

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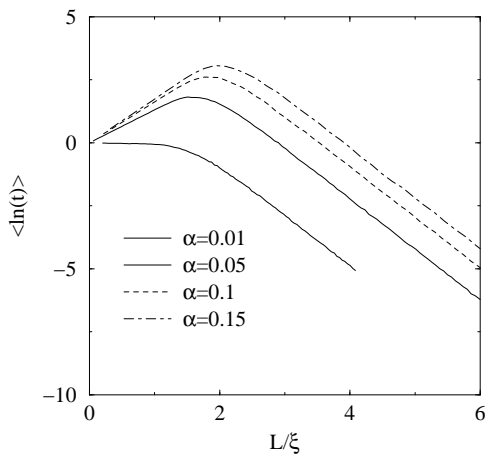


FIG. 1. Average of logarithm of transmission coefficient t versus L/ξ for different values of amplification strength α as indicated in the figure and disorder strength $W = 1.0$.

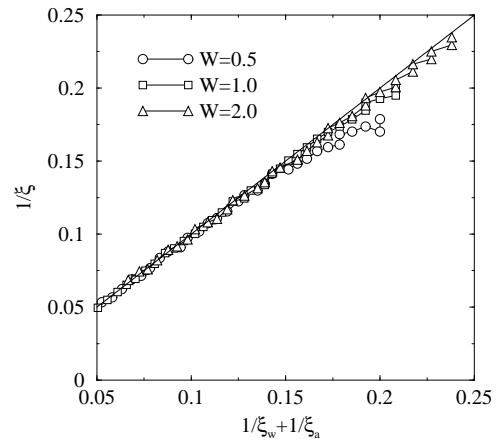


FIG. 2. $1/\xi$ versus $1/\xi_w + 1/\xi_a$, where ξ_w is the localization length for passive (non-amplifying) disordered system, ξ_a is the gain length for ordered ($W = 0$) amplifying system and ξ is the localization length for amplifying disordered system ($W = 1.0$).

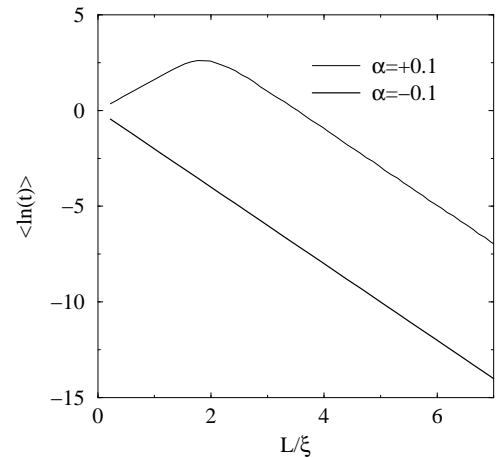


FIG. 3. Variation of average of logarithm of transmission coefficient with sample length L for $\alpha = 0.1$ (amplification) and $\alpha = -0.1$ (absorption). For both the cases the disorder strength $W = 1.0$.

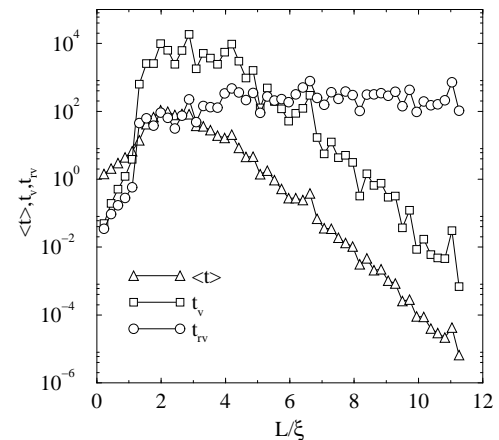


FIG. 4. Average transmission coefficient $\langle t \rangle$, rms variance of t and rms relative variance of t as a function of L/ξ for fixed disorder $W = 1.0$ and amplification $\alpha = 0.01$.

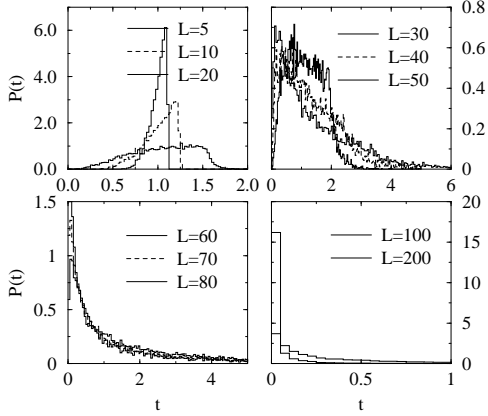


FIG. 5. Distribution of transmission coefficient t from a disordered amplifying system with $W = 1.0$ and $\alpha = 0.01$ at different lengths L . Localization length for these parameters is $\xi \sim 50$.

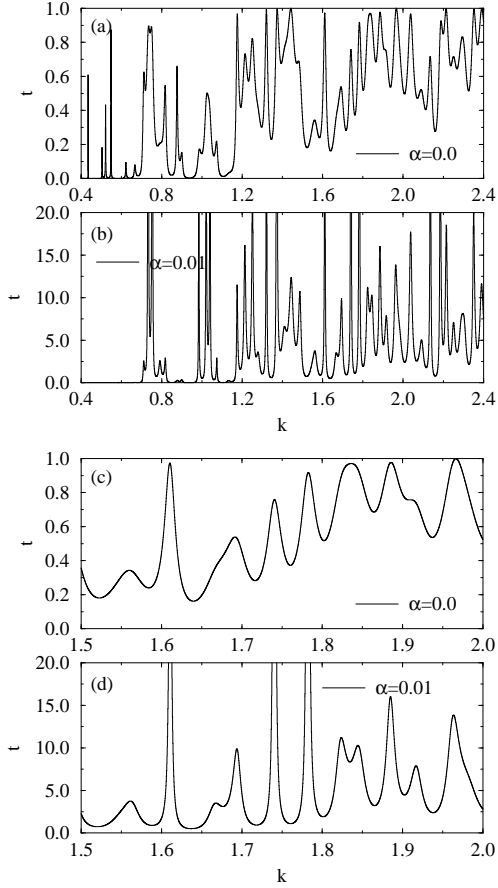


FIG. 6. Transmission coefficient as a function of incident wavenumber k for (a) disordered non-amplifying sample ($W = 1.0, \alpha = 0.0$) of length $L = 100$ and (b) disordered amplifying sample ($W = 1.0, \alpha = 0.01$) of length $L = 100$. (c) and (d) show a magnified view of the region between $k = 1.5$ to $k = 2.0$ of (a) and (b) respectively.

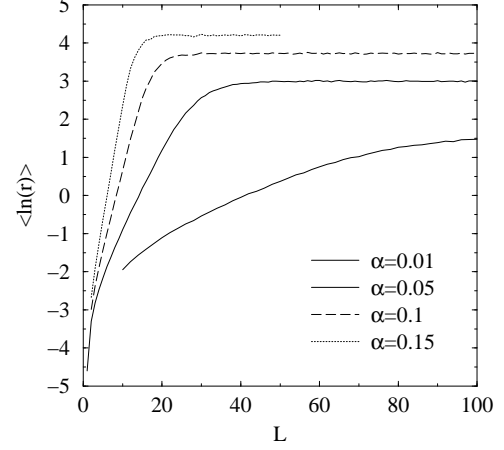


FIG. 7. Average of logarithm of reflection coefficient versus sample length for a fixed value of disorder $W = 1.0$ and different values of amplification α indicated in the figure.

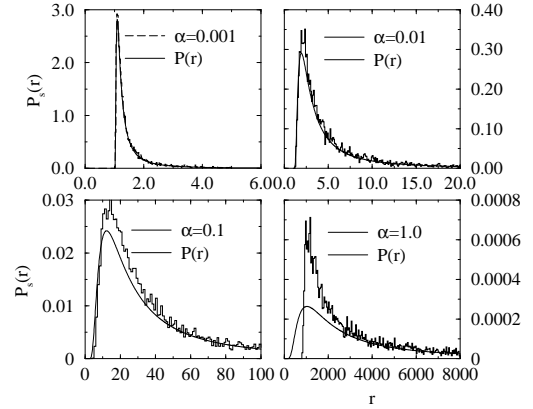


FIG. 8. Stationary distribution of reflection coefficient for a fixed value of disorder strength $W = 1.0$ and different values of amplification strength α . The thick line shows the single parameter fit of analytical expression Eqn.1 with (a) $D = 0.197$ for $\alpha = 0.001$, (b) $D = 1.92$ for $\alpha = 0.01$ and (c) $D = 20.53$ for $\alpha = 0.1$.

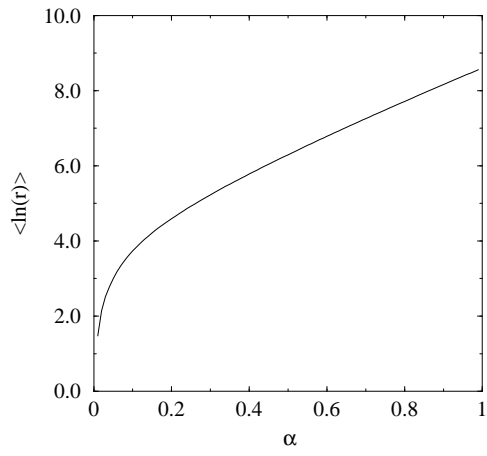


FIG. 9. Average value of logarithm of reflection coefficient ($\langle \ln r \rangle$) versus amplification strength α for $W = 1.0$ and $L/\xi = 10$.