Proc. Indian Acad. Sci. (Math. Sci.), Vol. 105, No. 2, May 1995, pp. 207–218.

Solution of a system of nonstrictly hyperbolic conservation laws

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MS received 30 November 1993; revised 17 June 1994

Abstract. In this paper we study a special case of the initial value problem for a 2×2 system of nonstrictly hyperbolic conservation laws studied by Lefloch, whose solution does not belong to the class of L^∞ functions always but may contain δ-measures as well. Lefloch's theory leaves open the possibility of nonuniqueness for some initial data. We give here a uniqueness criteria to select the entropy solution for the Riemann problem. We write the system in a matrix form and use a finite difference scheme of Lax to the initial value problem and obtain an explicit formula for the approximate solution. Then the solution of initial value problem is obtained as the limit of this approximate solution.

Keywords. System of conservation laws; delta waves; explicit formula

1. Introduction

The standard theory of hyperbolic systems of conservation laws assumes usually the systems to be strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields, see Lax [6] and Glimm [1]. But many of the hyperbolic systems which come in applications do not satisfy these assumptions and such cases were studied by many authors [3, 5, 8]. In all these papers solutions are found in the sense of distributions, say in the class of L^∞ functions. In a very interesting paper, Lefloch [7] considered a system of conservation laws, namely

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(a(u)v) = 0,$$

with initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \tag{1.2}$$

where $a(u) = f'(u)$ and $f\colon R \to R$ is a strictly convex function. For systems of this type generally there is neither existence nor uniqueness in the class of entropy weak solutions in the sense of distributions. He has shown that when $u_0 \in L^1(R) \cap BV(R)$ and $v_0 \in L^\infty(R) \cap L^1(R)$ (1.1) and (1.2) has at least one solution $(u, v) \in L^\infty(R_+, BV(R)) \times L^\infty(R_+, M(R))$ given by

$$u(x,t) = (f^*)'\left(\frac{x - y_0(x,t)}{t}\right),$$

$$v(x,t) = \frac{\partial}{\partial x}\int_{-\infty}^{y_0(x,t)} v_0(z)\,\mathrm{d}z,$$

where $y = y_0(x,t)$ minimizes

$$\min_{-\infty < y < \infty}\left[\int_{-\infty}^{y} u_0(z)\,\mathrm{d}z + tf^*\left(\frac{x-y}{t}\right)\right]$$

and f^* is the convex dual of $f(u)$ and $M(R)$ is the space of bounded borel measures on R. Further he proved that if u_0 satisfies

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} \leqslant K_0 \tag{1.3}$$

in the sense of distributions for some K_0, then the problem (1.1) and (1.2) has one and only one entropy solution. If we take

$$u_0(x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases}$$

then (1.3) is equivalent to saying

$$(u_R - u_L)\varphi(0) \leqslant K_0 \quad \text{for all } \varphi \in C_0^\infty(R), \varphi \geqslant 0$$

and this will be true for some K_0 and for all $\varphi \in C_0^\infty(R)$, $\varphi \geqslant 0$, iff $u_L \geqslant u_R$. In fact for the Riemann problem, i.e., when the initial data for (1.1) is of the form

$$(u(x,0), v(x,0)) = \begin{matrix} (u_L, v_L) & \text{if } x < 0 \\ (u_R, v_R) & \text{if } x > 0, \end{matrix} \tag{1.4}$$

Lefloch [7] has given an infinite number of solution for the case $u_L < u_R$.

In this paper we study a criteria to choose the correct entropy solution. Classically, vanishing viscosity method or proper numerical approximations are used to choose the correct entropy solution. Following Hopf [2], vanishing viscosity method was used by Joseph [4] to pick up the unique solution for the Riemann problem when $f(u) = u^2/2$ in (1.1). It was shown that in the case, $u_L < u_R$, which is the case of non-uniqueness, the v component of the vanishing viscosity solution is

$$v(x,t) = \begin{cases} v_L, & \text{if } x < u_L t \\ 0, & \text{if } u_L t < x < u_R t \\ v_R, & \text{if } x > u_R t. \end{cases}$$

In other words in the rarefaction fan region of u component, the v component is zero.

In the present paper, we consider the special case $f(u) = \log[ae^u + be^{-u}]$, $a + b = 1$, $a > 0$, $b > 0$ are constants in (1.1). Then we have

$$u_t + (\log(ae^u + be^{-u}))_x = 0, \tag{1.5}$$

$$v_t + \left(\frac{ae^u - be^{-u}}{ae^u + be^{-u}}v\right)_x = 0,$$

and study the unique choice of solution. Here we use a numerical approximation of Lax [6], which he used to pick the correct entropy solution for a scalar conservation law. For the Riemann problem, we show that in the rarefaction fan region of u, the v component is zero, see Theorem 1. These examples suggest a uniqueness criteria at least for the Riemann problem.

Before stating our main results let us introduce the difference approximation. To do this first we note that (1.5) can be written in the matrix form

$$A_t + [\log(ae^{A} + be^{-A})]_x = 0, \tag{1.6}$$

where

$$A = \begin{pmatrix} u & 0 \\ v & u \end{pmatrix}. \tag{1.7}$$

Let Δx and Δt be spatial and time mesh sizes and let

$$A_k^n \simeq A(k\Delta x, n\Delta t), \quad k = 0, \pm 1, \pm 2, \ldots, \quad n = 0, 1, 2, \ldots \tag{1.8}$$

and following Lax [6], define the difference approximation

$$A_k^n = A_k^{n-1} + \frac{\Delta t}{\Delta x}[g(A_{k-1}^{n-1}, A_k^{n-1}) - g(A_k^{n-1}, A_{k+1}^{n-1})], \tag{1.9}$$

where the numerical flux $g(A, B)$ is given by

$$g(A, B) = \log[ae^{A} + be^{-B}]. \tag{1.10}$$

Here we can take $\Delta t = \Delta x = \Delta$, since the characteristic speed of the eigenvalues $\lambda_1 = \lambda_2 = \dfrac{ae^u - be^{-u}}{ae^u + be^{-u}}$ of (1.5) which are less than one in modulus. Then we note that (1.9) and (1.10) become

$$A_k^n = A_k^{n-1} + \log[ae^{A_{k-1}^{n-1}} + be^{-A_k^{n-1}}] - \log[ae^{A_k^{n-1}} + be^{-A_{k+1}^{n-1}}] \tag{1.11}$$

with initial condition

$$A_k^0 = \begin{pmatrix} u_k^0 & 0 \\ v_k^0 & u_k^0 \end{pmatrix}. \tag{1.12}$$

When

$$A = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},$$

(1.11) is nothing but the Lax scheme for the scalar equation $u_t + (\log[ae^u + be^{-u}])_x = 0$.

With the notations

$$s = (\log[ae^{u_R} + be^{-u_R}] - \log[ae^{u_L} + be^{-u_L}])/(u_R - u_L) \tag{1.13}$$

and

$$R(u_L, u_R, v_L, v_R) = s(v_R - v_L) - \frac{ae^{u_R} - be^{-u_R}}{ae^{u_R} + be^{-u_R}} v_R + \frac{ae^{u_L} - be^{-u_L}}{ae^{u_L} + be^{-u_L}} v_L, \tag{1.14}$$

we shall prove the following results.

Theorem 1. *Let $(u^{\Delta}(x,t), v^{\Delta}(x,t))$ be the approximate solution of* (1.1) *defined by* (1.11) *and* (1.12) *with Riemann initial data* (1.4), *then*

$$\lim_{\Delta\to 0}(u^{\Delta}(x,t), v^{\Delta}(x,t)) = (u(x,t), v(x,t))$$

exists in the sense of distributions and $(u(x,t), v(x,t))$ is given by the following explicit formula:

(i) *When $u_L > u_R$, then*

$$(u(x,t), v(x,t)) = \{u_L + (u_R - u_L)H(x - st), v_L + (v_R - v_L)H(x - st) + R(u_L, u_R, v_L, v_R)t\delta_{x=st}\},$$

where $H(x)$ is the Heaviside function.

(ii) *When $u_L < u_R$, then*

$$(u(x,t), v(x,t)) = \begin{cases} (u_L, v_L) & \text{if } x < \left(\dfrac{ae^{u_L} - be^{-u_L}}{ae^{u_L} + be^{-u_L}}\right)t \\ \left(\dfrac{1}{2}\log\left(\dfrac{b}{a}\cdot\dfrac{t+x}{t-x}\right), 0\right) & \text{if } \dfrac{ae^{u_L} - be^{-u_L}}{ae^{u_L} + be^{-u_L}}t < x < \dfrac{ae^{u_R} - be^{-u_L}}{ae^{u_R} + be^{-u_L}}t \\ (u_R, v_R) & \text{if } x > \dfrac{ae^{u_R} - be^{-u_R}}{ae^{u_R} + be^{-u_R}}t. \end{cases}$$

(iii) *When $u_L = u_R = \bar{u}$, then*

$$(u(x,t), v(x,t)) = \begin{cases} (\bar{u}, v_L), & \text{if } x < a(\bar{u})t, \\ (\bar{u}, v_R), & \text{if } x \geqslant a(\bar{u})t. \end{cases}$$

Theorem 2. *Let the initial data $u^0(x)$* and *$v^0(x) \in L^{\infty}(R) \cap L^1(R)$. Then $(u^{\Delta}(x,t), v^{\Delta}(x,t))$ defined by* (1.11) *and* (1.12) *tends to $(u(x,t), v(x,t))$ in the sense of distributions and is given by*

$$u(x,t) = \frac{1}{2}\log\left[\frac{b}{a}\cdot\frac{t + x - y_0(x,t)}{t - x + y_0(x,t)}\right],$$

$$v(x,t) = -\frac{\partial}{\partial x}\int_{y_0(x,t)}^{\infty} v_0(z)\,dz,$$

where $y = y_0(x,t)$ maximizes

$$\max_{x-t \leqslant y \leqslant x+t}\left[\int_y^{\infty} u_0(z)\,dz - tf^*\left(\frac{x-y}{t}\right)\right].$$

Here $f^*(\lambda)$ is the convex dual of $f(u) = \log[ae^u + be^{-u}]$ and is given by

$$f^*(\lambda) = (1/2)\log(1+\lambda)^{1+\lambda}(1-\lambda)^{1-\lambda} - 1/2\log\{4a^{1+\lambda}b^{1-\lambda}\}.$$

2. Proof of Theorem 1

As a first step in the proof of Theorem 1, we obtain $(u^{\Delta}(x,t), v^{\Delta}(x,t))$ explicitly. In order to do this we recall from (1.11), (1.12) and (1.4),

$$A_k^n = A_k^{n-1} + \log[ae^{A_{k-1}^{n-1}} + be^{-A_k^{n-1}}] - \log[ae^{A_k^{n-1}} + be^{-A_{k+1}^{n-1}}] \tag{2.1}$$

for $n = 1, 2, 3, \ldots, k = 0, \pm 1, \pm 2, \ldots,$ with

$$A_k^0 = \begin{cases} A_R = \begin{pmatrix} u_R & 0 \\ v_R & u_R \end{pmatrix} & \text{if } k \geq 0 \\ A_L = \begin{pmatrix} u_L & 0 \\ v_L & u_L \end{pmatrix} & \text{if } k < 0. \end{cases} \tag{2.2}$$

Let us set

$$C_k^n = A_k^n - A_R \tag{2.3}$$

then, (2.1) becomes

$$\begin{aligned} C_k^n = C_k^{n-1} &+ \log[ae^{A_R + C_{k-1}^{n-1}} + be^{-A_R - C_k^{n-1}}] \\ &- \log[ae^{A_R + C_k^{n-1}} + be^{-A_R - C_{k+1}^{n-1}}]. \end{aligned} \tag{2.4}$$

Let

$$D_k^n = \sum_{j=k}^{\infty} C_j^n. \tag{2.5}$$

Taking summation in (2.4) from k to ∞, we have

$$\begin{aligned} D_k^n = D_k^{n-1} &+ \log[ae^{A_R} e^{(D_{k-1}^{n-1} - D_k^{n-1})} + be^{-A_R} e^{-(D_k^{n-1} - D_{k+1}^{n-1})}] \\ &- \log(ae^{A_R} + be^{-A_R}). \end{aligned} \tag{2.6}$$

Following Lax [6], we use the nonlinear transformation,

$$D = \log E \tag{2.7}$$

in (2.6), and obtain

$$\begin{aligned} \log E_k^n = \log E_k^{n-1} &+ \log[ae^{A_R}(E_k^{n-1})^{-1} E_{k-1}^{n-1} + be^{-A_R}(E_k^{n-1})^{-1} E_{k+1}^{n-1}] \\ &- \log(ae^{A_R} + be^{-A_R}). \end{aligned}$$

Simplifying this we get

$$E_k^n = \alpha E_{k-1}^{n-1} + \beta E_{k+1}^{n-1}, \tag{2.8}$$

where

$$\alpha = ae^{A_R}(ae^{A_R} + be^{-A_R})^{-1} \quad \text{and} \quad \beta = be^{-A_R}(ae^{A_R} + be^{-A_R})^{-1}. \tag{2.9}$$

We note that $\alpha + \beta = I$. It can be easily seen that the solution E_k^n of (2.8) is given by

$$E_k^n = \sum_{q=0}^{n} \binom{n}{q} \alpha^q \beta^{n-q} E_{n+k-2q}^0. \tag{2.10}$$

From (2.2), (2.3), (2.5) and (2.7) we get

$$E_{n+k-2q}^0 = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } n+k-2q \geq 0 \\ e^{(n+k-2q)(u_L - u_R)} \begin{pmatrix} 1 & 0 \\ (n+k-2q)(v_L - v_R) & 1 \end{pmatrix}, & \text{if } n+k-2q < 0. \end{cases} \tag{2.11}$$

Using (2.11) in (2.10) we get

$$E_k^n = \begin{pmatrix} \theta_k^n & 0 \\ \eta_k^n & \theta_k^n \end{pmatrix}, \tag{2.12}$$

where

$$\theta_k^n = \frac{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} e^{(2q-n)u_R} e^{S(n,k,q)(u_L-u_R)}}{(ae^{u_R}+be^{-u_R})^n} \tag{2.13}$$

and

$$\eta_k^n = \frac{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} e^{(2q-n)u_R} e^{s(n,k,q)(u_L-u_R)} \cdot \left\{ \frac{2v_R(qbe^{-u_R}+(q-n)ae^{u_R})}{(ae^{u_R}+be^{-u_R})} + S(n,k,q)(v_L-v_R) \right\}}{(ae^{u_R}+be^{-u_R})^n}. \tag{2.14}$$

Here we used the notation $S(n,k,q) = \frac{1}{2}(n+k-2q-|n+k-2q|)$. Now

$$\log E_k^n = (\log\theta_k^n)I + \log\left[I + \begin{pmatrix} 0 & 0 \\ \eta_k^n/\theta_k^n & 0 \end{pmatrix} \right],$$

i.e.

$$\log E_k^n = \begin{pmatrix} \log\theta_k^n & 0 \\ \eta_k^n/\theta_k^n & \log\theta_k^n \end{pmatrix}. \tag{2.15}$$

By the transformations (2.3), (2.5) and (2.7) we get,

$$\begin{aligned} \Delta \log E_k^n &= \Delta D_k^n \\ &= \Delta \sum_k^{\infty} C_k^n \\ &= \Delta \sum_k^{\infty} (A_k^n - A_R). \end{aligned} \tag{2.16}$$

Componentwise this becomes

$$\Delta \sum_{j=k}^{\infty} (u_j^n - u_R) = \Delta \log\theta_k^n \tag{2.17}$$

$$\Delta \sum_{j=k}^{\infty} (v_j^n - v_R) = \Delta \frac{\eta^n k}{\theta_k^n}. \tag{2.18}$$

By using Stirling's formula,

$$n! \approx \left(\frac{n}{e}\right)^n (2\pi n)^{1/2}, \quad \text{as } n \to \infty,$$

we get

$$\binom{n}{q} = \frac{n!}{q!(n-q)!} \approx \frac{n^n n^{1/2}}{q^q q^{1/2}(n-q)^{n-q}(2\pi)^{1/2}(n-q)^{1/2}}, \quad \text{as } n, q, n-q \to \infty.$$

Let $t = n\Delta$, $x = k\Delta$, $y = (n+k-2q)\Delta$ be fixed, then

$$q\Delta=\frac{t+x-y}{2},\quad (2q-n)\Delta=x-y. \tag{2.19}$$

We have,

$$\lim_{\Delta\to 0}\Delta\log\theta_k^n=\max_{0\leqslant(t+x-y)/2\leqslant t}\left[\Delta\log\binom{n}{q}+\Delta q\log a+\Delta(n-q)\log b\right.$$
$$\left.+\Delta(2q-n)u_R-\tfrac{1}{2}(y-|y|)(u_L-u_R)-t\log(ae^{u_R}+be^{-u_R})\right]. \tag{2.20}$$

Also as $\Delta\to 0$ in the above fashion, we have

$$\Delta\log\binom{n}{q}\approx\log\left[\frac{t^t}{\left(\frac{t+x-y}{2}\right)^{(t+x-y)/2}\cdot\left(\frac{t-x+y}{2}\right)^{(t-x+y)/2}}\right]. \tag{2.21}$$

and hence from (2.19)–(2.21), we get

$$\lim_{\Delta\to 0}\Delta\log\theta_k^n=\max_{x-t\leqslant y\leqslant x+t}\left[-1/2(y-|y|)(u_L-u_R)\right.$$
$$\left.+(x-y)u_R-t\log(ae^{u_R}+be^{-u_R})\right]$$
$$+\frac{(t+x-y)}{2}\log a+\frac{(t-x+y)}{2}\log b$$
$$+\log\left[\frac{t^t}{\left(\frac{t+x-y}{2}\right)^{(t+x-y)/2}\left(\frac{t-x+y}{2}\right)^{(t-x+y)/2}}\right]. \tag{2.22}$$

Let $y_0(x,t)$ be the value of y for which maximum is attained on the RHS of (2.22). An easy calculation shows that the following is true.

Lemma. Let $y_0(x,t)$ be a point where maximum is attained on the RHS of (2.22), *then $y_0(x,t)$ is given by the following:*

(i) *Let $u_L>u_R$, then*

$$y_0(x,t)=\begin{cases}x-a(u_L)t, & \text{if } x<st\\ x-a(u_R)t, & \text{if } x>st.\end{cases}$$

(ii) *Let $u_L<u_R$, then*

$$y_0(x,t)=\begin{cases}x-a(u_L)t, & \text{if } x<a(u_L)t\\ 0, & \text{if } a(u_L)t<x<a(u_R)t\\ x-a(u_R)t, & \text{if } x>a(u_R)t,\end{cases}$$

where

$$a(u)=f'(u)=(ae^u-be^{-u})/(ae^u+be^{-u})$$

and s is given by (1.13).

From the above lemma and (2.22) we have if $u_L > u_R$, then $\lim_{\Delta\to 0}\Delta\log\theta_k^n = A_1(x,t)$, where

$$A_1(x,t)=\begin{cases} -(x-a(u_L)t)(u_L-u_R)+u_R a(u_L)t - t\log(ae^{u_R}+be^{-u_R}) \\ +\left(\dfrac{1+a(u_L)}{2}\right)t\log a+\left(\dfrac{1-a(u_L)}{2}\right)t\log b+t\log t \\ -\left(\dfrac{1+a(u_L)}{2}\right)t\log\left\{\left(\dfrac{1+a(u_L)}{2}\right)t\right\}-\left(\dfrac{1-a(u_L)}{2}\right)t\log\left(\dfrac{(1-a(u_L))}{2}t\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x<st \\ a(u_R)tu_R - t\log(ae^{u_R}+be^{-u_R})+\left(\dfrac{1+a(u_R)}{2}\right)t\log a \\ +\dfrac{1-a(u_R)}{2}t\log b+t\log t-\left(\dfrac{1+a(u_R)}{2}\right)t\log\left\{\left(\dfrac{1+a(u_R)}{2}\right)t\right\} \\ -\left(\dfrac{1-a(u_R)}{2}\right)t\log\left\{\left(\dfrac{1-a(u_R)}{2}\right)t\right\}, \quad \text{if } x>st. \end{cases}$$

If $u_L < u_R$, then $\lim_{\Delta\to 0}\Delta\log\theta_k^n = A_2(x,t)$, where

$$A_2(x,t)=\begin{cases} -(x-a(u_L)t)(u_L-u_R)+u_R a(u_L)t - t\log(ae^{u_R}+be^{u_R}) \\ +\dfrac{(1+a(u_L))}{2}t\log a+\dfrac{(1-a(u_L))}{2}t\log b+t\log t \\ -\dfrac{(1+a(u_L))}{2}t\log\left\{\left(\dfrac{1+a(u_L)}{2}\right)t\right\}-\left(\dfrac{1-a(u_L)}{2}\right)t\log\left\{\left(\dfrac{1-a(u_L)}{2}\right)t\right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x<a(u_L)t, \\ xu_R - t\log[ae^{u_R}+be^{-u_R})+\dfrac{t+x}{2}\log a+\dfrac{t-x}{2}\log b+t\log t \\ -\dfrac{(t+x)}{2}\log\dfrac{(t+x)}{2}-\dfrac{(t-x)}{2}\log\left(\dfrac{t-x}{2}\right), \quad \text{if } a(u_L)t<x<a(u_R)t \\ u_R a(u_R)t - t\log(ae^{u_R}+be^{-u_R})+\dfrac{1+a(u_R)}{2}t\log a \\ +\left(\dfrac{1-a(u_R)}{2}\right)t\log b+t\log t-\dfrac{(1+a(u_R))}{2}t\log\left\{\left(\dfrac{1+a(u_R)}{2}\right)t\right\} \\ -\left(\dfrac{1-a(u_R)}{2}\right)t\log\left\{\left(\dfrac{1-a(u_R)}{2}\right)t\right\} \quad \text{if } x>a(u_R)t. \end{cases}$$

Again from (2.13), (2.14) and (2.19), we get

$$\lim_{\Delta\to 0}\Delta\frac{\eta_k^n}{\theta_k^n}=\frac{v_R be^{-u_R}(t+x-y_0(x,t))+v_R ae^{u_R}(x-y_0(x,t)-t)}{ae^{u_R}+be^{-u_R}}.$$

$$-\tfrac{1}{2}(y_0(x,t)-|y_0(x,t)|)(v_L-v_R),$$

where $y_0(x,t)$ maximizes the RHS of (2.22). Using the lemma we have the following: If $u_L > u_R$, then

$$\lim_{\Delta\to 0} \Delta(\eta_k^n/\theta_k^n) = B_1(x,t),$$

where

$$B_1(x,t) = \begin{cases} -(x-a(u_L)t(v_L-v_R)) \\ \quad + \dfrac{v_R[ae^{u_R}(a(u_L)-1)t + be^{-u_R}(1+a(u_L))t]}{ae^{u_R}+be^{-u_R}}, & \text{if } x < st, \\ \dfrac{v_R[ae^{u_R}(a(u_R)-1)t + be^{-u_R}(1+a(u_R))t]}{ae^{u_R}+be^{-u_R}}, & \text{if } x > st. \end{cases}$$

If $u_L < u_R$, then

$$\lim_{\Delta\to 0} \Delta(\eta_k^n/\theta_k^n) = B_2(x,t),$$

where

$$B_2(x,t) = \begin{cases} -(x-a(u_L)t)(v_L-v_R) \\ \quad + \dfrac{v_R[ae^{u_R}(a(u_L-1))t + be^{-u_R}(1+a(u_L))t]}{ae^{u_R}+be^{-u_R}}, & \text{if } x < a(u_L)t \\ \dfrac{v_R be^{-u_R}(t+x) + v_R ae^{u_R}(x-t)}{ae^{u_R}+be^{-u_R}}, & \text{if } a(u_L)t < x < (u_R)t \\ \dfrac{v_R[ae^{u_R}(a(u_R)-1)t + be^{-u_R}(1+a(u_R))t]}{ae^{u_R}+be^{-u_R}}, & \text{if } x > a(u_R)t. \end{cases}$$

Now it follows that, if $u_L > u_R$

$$\lim_{\Delta\to 0} \int_x^\infty (u^\Delta(y,t)-u_R)\,dy = A_1(x,t),$$

$$\lim_{\Delta\to 0} \int_x^\infty (v^\Delta(y,t)-v_R)\,dy = B_1(x,t)$$

and if $u_L < u_R$

$$\lim_{\Delta\to 0} \int_x^\infty (u^\Delta(y,t)-u_R)\,dy = A_2(x,t),$$

$$\lim_{\Delta\to 0} \int_x^\infty (v^\Delta(y,t)-v_R)\,dy = B_2(x,t).$$

Hence

$$\left.\begin{aligned} u^\Delta(x,t) &\to \frac{-\partial A_1}{\partial x} + u_R \\ v^\Delta(x,t) &\to -\frac{\partial B_1}{\partial x} + v_R \end{aligned}\right\} \text{ if } u_L > u_R$$

$$\left.\begin{aligned} u^{\Delta}(x,t) &\to \frac{-\partial A_2}{\partial x} + u_R \\ v^{\Delta}(x,t) &\to -\frac{\partial B_2}{\partial x} + v_R \end{aligned}\right\} \quad \text{if } u_L < u_R$$

in the sense of distribution as $\Delta \to 0$. An easy calculation shows that

$$u_R - \frac{\partial A_1}{\partial x} = \begin{cases} u_L, & \text{if } x < st, \\ u_R, & \text{if } x > st, \end{cases}$$

$$v_R - \frac{\partial B_1}{\partial x} = t[s(v_R - v_L) - a(u_R)v_R + a(u_L)v_L] \cdot \delta_{x=st} + v_R + (v_L - v_R)[1 - H(x - st)],$$

$$u_R - \frac{\partial A_2}{\partial x} = \begin{cases} u_L, & \text{if } x < a(u_L)t \\ \frac{1}{2}\log\left[\frac{b}{a}\frac{t+x}{t-x}\right], & \text{if } a(u_L)t < x < a(u_R)t \\ u_R, & \text{if } x > a(u_R)t, \end{cases}$$

$$v_R - \frac{\partial B_2}{\partial x} = \begin{cases} v_L, & \text{if } x < a(u_L)t \\ 0, & \text{if } a(u_L)t < x < a(u_R)t \\ v_R, & \text{if } x > a(u_R)t. \end{cases}$$

Proof of (iii) is similar. The proof of Theorem 1 is complete.

3. Proof of Theorem 2

To prove Theorem 2, we first note that the approximate solutions are defined by

$$A_k^n = A_k^{n-1} + \log[ae^{A_{k-1}^{n-1}} + be^{-A_k^{n-1}}] - \log[ae^{A_k^{n-1}} + be^{-A_{k+1}^{n-1}}], \tag{3.1}$$

with

$$A_k^0 = \begin{pmatrix} u_k^0 & 0 \\ v_k^0 & u_k^0 \end{pmatrix} \tag{3.2}$$

where $u_k^0 = u^0(k\Delta)$, $v_k^0 = v^0(k\Delta)$. Following Lax [6], let us introduce

$$B_k^n = \sum_k^{\infty} A_k^n$$

and use the nonlinear transformation

$$B = \log E.$$

We get as before

$$E_k^{n+1} = aE_{k-1}^n + bE_{k+1}^n,$$

whose solution is

$$E_k^n = \sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \exp\left\{ \sum_{j=n+k-2q}^{\infty} u_j^0 \begin{pmatrix} 1 & 0 \\ \sum_{j=n+k-2q}^{\infty} v_j^0 & 1 \end{pmatrix} \right\}.$$

In terms of the original variable A_k^n, we have

$$A_k^n = \log\left[E_k^n (E_{k+1}^n)^{-1}\right].$$

Carrying out the explicit calculations as before, we get

$$u_k^n = \log\left[\frac{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \exp\left\{\sum_{j=n+k-2q}^{\infty} u_j^0\right\}}{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \exp\left\{\sum_{j=n+k+1-2q}^{\infty} u_j^0\right\}}\right],$$

$$\sum_{j=k}^{\infty} v_j^n = \frac{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \left(\sum_{j=n+k-2q}^{\infty} v_j^0\right) \exp\left\{\sum_{j=n+k-2q}^{\infty} u_j^0\right\}}{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \exp\left\{\sum_{j=n+k-2q}^{\infty} u_j^0\right\}}.$$

Now let $x = k\Delta$, $t = n\Delta$, $y = (n+k-2q)\Delta$ be fixed and let $\Delta \to 0$. Lax has shown that

$$u^{\Delta}(x,t) \to u(x,t) = \frac{1}{2}\log\left[\frac{b\,t + x - y_0(x,t)}{a\,t - x + y_0(x,t)}\right],$$

where $y = y_0(x,t)$ maximizes

$$\max_{x-t \leqslant y \leqslant x+t} \left[\int_y^{\infty} u_0(z)\,\mathrm{d}z - t f^*\left(\frac{x-y}{t}\right)\right], \tag{3.3}$$

where

$$f^*(\lambda) = \tfrac{1}{2}\log\left[(1+\lambda)^{1+\lambda}\cdot(1-\lambda)^{1-\lambda}\right] - \tfrac{1}{2}\log\left[4a^{1+\lambda}b^{1-\lambda}\right].$$

Again the same analysis of Lax [6] gives

$$\lim_{\Delta\to 0}\int_x^{\infty} v^{\Delta}(x,t)\,\mathrm{d}y = \lim_{\Delta\to 0} \Delta \sum_k^{\infty} v_j^n = \int_{y_0(x,t)}^{\infty} v_0(z)\,\mathrm{d}z.$$

Here again $y = y_0(x,t)$ maximizes (3.3). Since $\int_x^{\infty} v^{\Delta}(y,t)\,\mathrm{d}y$ is a sequence of bounded function converging to $\int_{y_0(x,t)}^{\infty} v_0\,\mathrm{d}x$ for a.e. (x,t), it follows that $v^{\Delta}(x,t)$ converges to

$$-\frac{\partial}{\partial x}\int_{y_0(x,t)}^{\infty} v_0(z)\,\mathrm{d}z$$

in distribution. The proof of Theorem 2 is complete.

Acknowledgements

One of the authors (KTJ) thanks the Indo-French centre for the Promotion of Advanced Research, New Delhi, for the financial support to visit France, where he had several fruitful discussions on this problem, and Prof. C. Bardos and Prof. O.

Pironneau for their hospitality. The authors are also thankful to the unknown referee for his constructive criticism which improved the presentation of the paper.

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