TIFR/TH/09-18

Anomalies at finite density and chiral fermions

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Using perturbation theory in the Euclidean (imaginary time) formalism as well as the nonperturbative Fujikawa method, we verify that the chiral anomaly equation remains unaffected in the presence of nonzero chemical potential, μ. We extend our considerations to fermions with exact chiral symmetry on the lattice and discuss the consequences for the recent Bloch-Wettig proposal for the Dirac operator at finite chemical potential. We propose a new simpler method of incorporating μ and compare it with the Bloch-Wettig idea.

PACS numbers: 12.38.Gc, 11.30.Rd, 11.15.Ha

I. INTRODUCTION

As we know from Noether's theorem, invariance of a Lagrangian of a classical field theory under a continuous symmetry leads to conserved currents. Inclusion of quantum loop corrections can, however, make some currents anomalous, and thus lead to the breaking of the corresponding symmetry. Chiral anomalies are a well-known example of this phenomenon. Chiral anomalies arise in a theory of massless fermions interacting with the gauge fields . The flavorless axial current of the fermions is classically conserved but is violated at one-loop level, as was shown in the famous calculation of the Adler-Bell-Jackiw(ABJ) triangle diagram for the $U(1)$ case [1, 2]. The anomalous contribution is a universal feature of the theory and is independent of the ultraviolet regulator used for the quantum theory. Fujikawa provided a new insight on anomalies by showing that they arise due to the change of the fermion measure under the corresponding transformation of the fermion fields[3] in the path integral method. Chiral anomalies have a deeper physical significance, as they relate the exact zero modes of the Dirac operator to the nontrivial topological sectors of the gauge fields. Consequently, the chiral anomaly in Quantum Chromodynamics(QCD) is thought to give rise to η' mass [4]. For the physically interesting case of two massless flavor QCD ($N_f = 2$), the order of the chiral phase transition depends [5] on the size of the coefficient of the chiral anomaly term. It is of second order, with critical exponents of the $O(4)$ spin model, if the anomaly contribution is sizeable at finite temperature. One could expect a QCD-critical point in the $T - \mu$ plane for light quarks in that case. In view of this, it is important to ascertain what change occurs in the anomaly in the presence of finite temperature and densities.

In this paper we address both the perturbative and nonperturbative aspects of the chiral anomaly at finite temperature/density. In Sec. I, we compute the triangle anomaly in the imaginary time formalism of thermal field theory. This method has the advantage that it can be linked to the weak coupling lattice calculations. Lattice QCD deals with the imaginary time Euclidean propagators, and hence anomaly calculation in the Euclidean space-time would be directly relevant for numerical studies. In Sec. II, we extend Fujikawa's analysis to finite density in the continuum. We show that the anomaly equation arising due to the change in the measure of the functional integrals under chiral transformation of the fermion fields remains the same at nonzero densities as well. We extend these considerations in Sec. III to the case of fermions with exact chiral invariance on the lattice. We propose a lattice Dirac operator with a term linear in the chemical potential μ, i.e., similar to the continuum and also suggest a way to get rid of the spurious divergences in the thermodynamic quantities. Its potential to handle higher order terms in the Taylor expansion in chemical potential μ in full QCD is commented upon.

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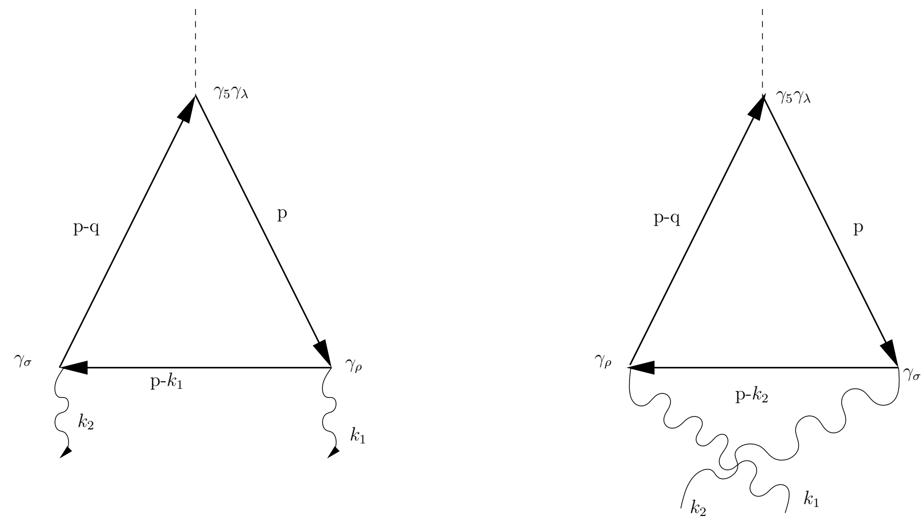

FIG. 1: The ABJ triangle diagram(left panel) and its crossed counter part(right panel).

II. ANOMALY AT $T = 0$ AND $\mu \neq 0$ IN CONTINUUM

A. Perturbative calculation

In this section we calculate the expectation value of the gradient of flavor singlet axial vector current of QCD perturbatively in the presence of finite fermion density to check how the anomaly equation is affected in the presence of a nonzero chemical potential. The lowest order diagrams are the ABJ triangle diagrams shown in Fig. 1. It is well-known that the higher order diagrams do not contribute to the anomaly equation at zero density, neither do other diagrams like the square and pentagon diagrams. We therefore compute only the triangle diagrams at finite density. Our starting point is the QCD Lagrangian in the Euclidean space with the finite number density term as defined in [6]. In order to maintain consistency with the lattice literature, we have however chosen the Dirac gamma matrices to be Hermitian:

$$\mathcal{L} = -\bar{\psi}(\not{D} + m)\psi - \frac{1}{2}\mathrm{Tr} \;\; F_{\alpha\beta}F_{\alpha\beta} + \mu\bar{\psi}\gamma_4\psi \; , \tag{1}$$

where $\not{D} = \gamma_\nu(\partial_\nu - igA^a_\nu T_a)$ with T_a being the generators of the SU(3) gauge group. The ghost terms are not important in such a calculation as these do not directly couple to the fermions. The $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ is also Hermitian in our case. The inverse free fermion propagator is seen to acquire a μ dependence and become $[i\not{p} - m + \mu\gamma_4]$. In order to find out whether the chiral current $j_{\mu 5} = \bar{\psi}\gamma_\mu\gamma_5\psi$ for massless quarks is conserved at finite density in one-loop perturbation theory, we compute the quantum mechanical expectation value of the derivative of the chiral current i.e. ,

$$\langle \partial_\mu j_{\mu,5}\rangle = -\frac{1}{2}\int d^4x_1 d^4x_2 \partial_\lambda \langle T\{j_{5,\lambda}(x) j_\rho(x_1) j_\sigma(x_2)\}\rangle A^\rho(x_1) A^\sigma(x_2) \; . \tag{2}$$

where the expectation value of the time ordered product of the three currents at one-loop level is the axialvector-vector-vector (AVV) triangle diagram shown in Fig. 1. Any deviation of this quantity from its classical value would give us the anomaly. Using the Euclidean space Feynman rules, the amplitude of the AVV triangle diagram can be computed. The crossed diagram is the one with the gluon legs exchanged among the two vector (VV) vertices, and it corresponds to the process which is quantum mechanically equally favored.

Denoting by $\Delta^{\lambda\rho\sigma}(k_1, k_2)$ the total amplitude and contracting it with q_λ, Eq. (2) can be written in the momentum space for massless quarks as

$$\begin{aligned} q_\lambda \Delta^{\lambda\rho\sigma} \;&=\; (-i)g^2 \mathrm{tr}[T^a T^b] \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \;\; \Bigg[\gamma^5 \frac{1}{\not{p} - \not{q} - i\mu\gamma^4}\gamma^\sigma \frac{1}{\not{p} - \not{k}_1 - i\mu\gamma^4}\gamma^\rho - \gamma^5 \frac{1}{\not{p} - i\mu\gamma^4}\gamma^\sigma \frac{1}{\not{p} - \not{k}_1 - i\mu\gamma^4}\gamma^\rho \\ &+\; \gamma^5 \frac{1}{\not{p} - \not{q} - i\mu\gamma^4}\gamma^\rho \frac{1}{\not{p} - \not{k}_2 - i\mu\gamma^4}\gamma^\sigma - \gamma^5 \frac{1}{\not{p} - i\mu\gamma^4}\gamma^\rho \frac{1}{\not{p} - \not{k}_2 - i\mu\gamma^4}\gamma^\sigma \Bigg] \; , \end{aligned} \tag{3}$$

with the tr (Tr) denoting trace over color (spin) indices. Combining further the first (second) term of the AVV diagram and the second (first) term of the corresponding crossed diagram respectively, we rewrite the contracted amplitude in terms of functions $f_1(p,k_1)$ and $f_2(p,k_2)$,

$$q_\lambda \Delta^{\lambda\rho\sigma} = (-i)\ \mathrm{tr}[T^a T^b] g^2 \int \frac{d^4p}{(2\pi)^4} \left[f_2(p-k_1,k_2) - f_2(p,k_2) + f_1(p-k_2,k_1) - f_1(p,k_1) \right] \ , \tag{4}$$

where the function $f_1(p,k_1)$ is defined as,

$$\begin{aligned} f_1(p,k_1) &= \mathrm{Tr}\ \left[\gamma^5 \frac{\not{p} - i\mu\gamma_4}{(p_4 - i\mu)^2 + \vec{p}^2} \gamma^\sigma \frac{\not{p} - \not{k}_1 - i\mu\gamma_4}{(p_4 - k_{14} - i\mu)^2 + (\vec{p} - \vec{k}_1)^2} \gamma^\rho \right] \\ &= - \left[\frac{4\epsilon^{\alpha\sigma\beta\rho} p_\alpha k_{1\beta} - 4i\mu \epsilon^{4\sigma\beta\rho} k_{1\beta}}{((p_4 - i\mu)^2 + \vec{p}^2)((p_4 - k_{14} - i\mu)^2 + (\vec{p} - \vec{k}_1)^2)} \right] \ , \text{ since Tr } \ [\gamma^5 \not{p} \gamma^\sigma \not{p} \gamma^\rho] = 0. \end{aligned} \tag{5}$$

f_2 can be obtained by substituting k_2 for k_1 and interchanging the indices ρ and σ in Eq. (5). We will use below a common notation f for denoting either in order to sketch the proof further. Although the numerator of Eq. (5) has terms up to quadratic order in μ, it should be noted that the μ^2 terms are $\sim \mu^2 \mathrm{Tr}\ [\gamma^5 \gamma^4 \gamma^\sigma \gamma^4 \gamma^\rho] \sim \epsilon^{4\sigma 4\rho}$ and therefore vanish. In order to further evaluate the right-hand side of Eq. (4), we note that the integrals are linearly divergent and hence must be regulated by introducing a cut-off scale, Λ. This procedure must be carried out in a gauge invariant manner such that the vector currents are conserved. In momentum space this amounts to

$$k_{1\rho} \Delta^{\lambda\rho\sigma}(k_1,k_2) = k_{2\sigma} \Delta^{\lambda\rho\sigma}(k_1,k_2) = 0 \ . \tag{6}$$

We follow the usual text book [7] method to impose these conditions above and compute the anomaly. In order to highlight the differences due to the $\mu \neq 0$ terms, we sketch below the evaluation of just the relevant part of Eq. (4). Expanding the first term and combining it with the second, we rewrite the first two integrals as,

$$\int \frac{d^4p}{(2\pi)^4} \left[f(p-k_1,k_2) - f(p,k_2) \right] = \mathcal{L}t_{\Lambda\to\infty} \int_0^\Lambda \frac{d^4p}{(2\pi)^4} \left[-k_{1\mu}\partial_\mu f + \frac{1}{2} k_{1\mu} k_{1\nu} \partial_\mu \partial_\nu f + \mathcal{O}(k^3) \right] \ . \tag{7}$$

where the derivatives are in the momentum space. The first term of the above integrand can be written as a surface integral using Gauss law,

$$\begin{aligned} \mathcal{L}t_{\Lambda\to\infty} \int_0^\Lambda \frac{d^4p}{(2\pi)^4} k_{1\mu} \partial_\mu f(p,k_2) &= \mathcal{L}t_{\Lambda\to\infty} \frac{k_{1\mu}\Lambda_\mu}{\Lambda} \frac{f(\Lambda,k_2) 2\pi^2 \Lambda^3}{(2\pi)^4} \\ &\sim \mathcal{L}t_{\Lambda\to\infty} \left[\frac{4\epsilon^{\alpha\sigma\beta\rho} \frac{\Lambda_\alpha k_{1\mu} k_{2\beta}}{\Lambda} - \frac{4i\mu}{\Lambda} \epsilon^{4\sigma\beta\rho} k_{1\mu} k_{2\beta}}{((1 - i\frac{\mu}{\Lambda})^2 + 1)((1 - \frac{k_{24} + i\mu}{\Lambda})^2 + (\hat{\Lambda} - \frac{\vec{k}_2}{\Lambda})^2)} \right] \frac{\Lambda_\mu \Lambda^3}{8\pi^2 \Lambda^4} \\ &= - \ \frac{\epsilon^{\alpha\beta\sigma\rho} k_{1\alpha} k_{2\beta}}{8\pi^2} \end{aligned} \tag{8}$$

where we uses the isotropy condition, $\Lambda_\nu \Lambda_\alpha / \Lambda^2 = g_{\nu\alpha}/4$. It is clear that the second term of the integrand in Eq. (7) when similarly integrated leads to the gradient of $f(p,k_2)$ at the Fermi surface of radius Λ, and therefore vanishes as $\mathcal{O}(\frac{1}{\Lambda})$. Hence this term, and the higher derivative terms, do not contribute in the limit when the cut-off is taken to infinity. The other two terms of Eq. (4), as well as the vector current conservation condition Eq. (6), can be similarly shown to be μ independent, leading to the canonical result even for $\mu \neq 0$:

$$q_\lambda \Delta^{\lambda\rho\sigma} = -\mathrm{tr}[T^a T^b] \frac{ig^2}{2\pi^2} \epsilon^{\alpha\beta\sigma\rho} k_{1\alpha} k_{2\beta} \ . \tag{9}$$

We have thus shown explicitly that the anomaly equation has no corrections due to nonzero μ or, equivalently, at nonzero finite density. It is easy to generalize the same computation to nonzero temperatures. At finite temperature, the temporal part of the momentum gets quantized as the well-known Matsubara frequencies : $p_4 = \frac{(2n+1)\pi}{\beta}$. Correspondingly, $\int_{-\infty}^{\infty} \frac{dp_4}{2\pi}$ gets replaced by $\frac{1}{\beta}\sum_n$, where $n = \pm 1, \pm 2, ..., \pm\infty$. The sum over discrete energy eigenvalues, can as usual, be split as a zero temperature contribution along with the finite temperature contributions weighted by the Fermi-Dirac distribution functions for the particles and antiparticles. Note that the finite temperature contributions will fall off to zero in the ultraviolet limit because these are regulated by the distribution functions. Thus,

$$\int \frac{d^3\vec{p}}{(2\pi)^3}\left[k_1^i\partial_i\left[f(|\vec{p}|)\left(\frac{1}{e^{\beta(|\vec{p}|-\mu)}+1}+\frac{1}{e^{\beta(|\vec{p}|+\mu)}+1}\right)\right]+\{\rho,k_1\leftrightarrow\sigma,k_2\}\right]$$
$$=\mathcal{L}t_{|\vec{p}|\to\infty}\frac{4\pi|\vec{p}|}{(2\pi)^3}\left[(\vec{k}_1\cdot\vec{p})f(|\vec{p}|)\left(\frac{1}{e^{\beta(|\vec{p}|-\mu)}+1}+\frac{1}{e^{\beta(|\vec{p}|+\mu)}+1}\right)+\{\rho,k_1\leftrightarrow\sigma,k_2\}\right]\longrightarrow 0 \tag{10}$$

Such perturbative calculations of the ABJ anomaly were reported earlier in the real time formalism at finite temperature and at both zero [8] and nonzero [9, 10] fermion densities as well as for finite density in Minkowski space-time [11]. We have shown above that these calculations are possible using the imaginary time formalism as well. An imaginary time calculation is useful as this can be generalized to weak coupling calculations in lattice gauge theory.

B. Nonperturbative calculation

The chiral anomaly in the path integral formalism can also be looked upon as arising due to the change of the measure under chiral transformation of the fermion fields[3]. In this section, Fujikawa's method of anomaly calculation in the path integral formalism, at zero temperature and zero fermion density, is extended to the finite fermion density case. But before analyzing the finite density problem, the method for $\mu = 0$ is summarized to point out the differences that would arise in the finite density case. The partition function for massless fermions interacting with $SU(N)$ gauge theory can be written in Euclidean space as

$$\mathrm{Z}=\int\mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\nu]\mathrm{e}^{-\int \mathrm{d^4x}\ \bar{\psi}\not{D}\psi-\mathrm{S_{YM}}}=\int\mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}\mathrm{A}_\nu]\mathrm{e}^{-\mathrm{S}} \tag{11}$$

where $S_{YM} = 1/2\int d^4x\left[\mathrm{Tr}\ \ F_{\alpha\beta}(x)F_{\alpha\beta}(x)+1/\xi(f^aA^a_\mu)^2\right]$ is the free Yang-Mills action with appropriate gauge fixing $f^aA^a_\mu = 0$. The action for the ghost term is included within the gauge field measure and hence denoted within square brackets. This is justified since we are interested in the change of the fermion fields under chiral transformations and the ghost fields do not interact with the fermions. Under the infinitesimal local chiral transformation of the fermion fields, given by

$$\delta\psi(x)=i\alpha(x)\gamma_5\psi(x)\quad\text{and}\quad\delta\bar{\psi}(x)=i\alpha(x)\bar{\psi}(x)\gamma_5\ , \tag{12}$$

the action changes as $S \to S - i\int d^4x\ \alpha(x)\partial_\nu j_5^\nu$. The measure changes as a result of the transformation of the fermion fields. The change of measure is,

$$\mathcal{D}\bar{\psi}^{'}\mathcal{D}\psi^{'}=\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathrm{Det}|\frac{\partial(\bar{\psi}^{'},\psi^{'})}{\partial(\bar{\psi},\psi)}|=\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathrm{e}^{-2\mathrm{i}\int \mathrm{d^4x}\ \alpha(\mathrm{x})\mathrm{Tr}\gamma_5} \tag{13}$$

where Tr stands for the trace over the spin and the color space. This trace can be computed using the eigenvectors of the operator $\not{D}$, since it is an anti-Hermitian operator. It has purely imaginary eigenvalues and the corresponding eigenvectors form a complete orthonormal basis. Splitting the trace computation into two parts, the trace over the nonzero eigenvalues can be done easily as follows. Since $\{\gamma_5, \not{D}\} = 0$, for every eigenvector ϕ_m with nonzero eigenvalue $\lambda_m \neq 0$, there is a corresponding eigenvector $\gamma_5\phi_m$ with eigenvalue $-\lambda_m$. Thus for each finite λ_m , $\phi^\pm_m = \phi_m \pm \gamma_5\phi_m$ are eigenvectors of γ_5 with eigenvalues ±1. Since trace is independent of the basis vectors we can also compute the trace of γ_5 in the $\phi^\pm_m$ basis. One obtains zero as the result since there are equal number of $\phi^\pm_m$ respectively. For the zero eigenmodes, $\not{D}$ and γ_5 commute hence each zero mode has a definite chirality, leading to a +1 contribution for those with $\gamma_5\phi_n = \phi_n$ and a -1 for the opposite chirality. Hence the complete evaluation of the trace gets a nonzero contribution corresponding to the difference between number of the two chiralities:

$$\mathrm{Tr}\gamma_5=\sum_n\phi_n^\dagger\gamma_5\phi_n=n_+-n_-. \tag{14}$$

Chiral Jacobian in the presence of μ

The presence of finite chemical potential, μ, in the action can be described as an effective change of the Dirac operator from $\not{D}$ to $\not{D} - \mu\gamma_4 = \not{D}(\mu)$. Under the chiral transformation given in Eq. (12) the action still remains invariant as in the zero density case. This is due to the fact that the μ dependent term of the action anticommutes with γ_5: $\{\gamma_5, \mu\gamma_4\} = 0$. Under the transformations given in Eq. (12) the fermion measure changes again by the same Jacobian factor $\mathrm{Tr}\gamma_5$. The corresponding $\mathrm{Tr}\gamma_5$ is now evaluated in the space of eigenvectors of the new Dirac operator $\not{D}(\mu)$. This is because the measure is defined by the complete set of states of the Dirac operator which appears in the action. Although $\not{D}(\mu)$ has both an anti-Hermitian and a Hermitian term, it is still diagonalizable. Consider an eigenvector ϕ_m of $\not{D}(0)$ with an eigenvalue λ_m. Let us define two new vectors, ζ_m and υ_m as follows:

$$\zeta_m(\mathbf{x},\tau) = \mathrm{e}^{\mu\tau}\phi_\mathrm{m}(\mathbf{x},\tau) \quad , \quad \upsilon^\dagger_\mathrm{m}(\mathbf{x},\tau) = \phi^\dagger_\mathrm{m}(\mathbf{x},\tau)\mathrm{e}^{-\mu\tau} \ . \tag{15}$$

It is easy to check that ζ_m is the eigenvector of $\not{D}(\mu)$ with the same (purely imaginary) eigenvalue λ_m,

$$\not{D}(\mu)\zeta_m = \lambda_m\zeta_m \ , \tag{16}$$

and $\upsilon^\dagger_m$ is the eigenvector of $\not{D}(\mu)^\dagger$ with the eigenvalue $\lambda^*_m = -\lambda_m$,

$$\upsilon^\dagger_m \not{D}^\dagger(\mu) = -\lambda_m \upsilon^\dagger_m. \tag{17}$$

Note that the sets of eigenvectors $\{\zeta\}$ and $\{\upsilon\}$ are in one-to-one correspondence with the complete set $\{\phi\}$. Using the completeness relation for the latter,

$$\sum_m \int \phi_m(\mathbf{x},\tau)\phi^\dagger_m(\mathbf{x},\tau)\ d^4x = \mathbf{I} \ , \tag{18}$$

where $\mathbf{I}$ denotes the identity matrix, we note that

$$\sum_m \int \zeta_m(\mathbf{x},\tau)\upsilon^\dagger_m(\mathbf{x},\tau)\ d^4x = \sum_m \int \phi_m(\mathbf{x},\tau)\mathrm{e}^{\mu\tau}\mathrm{e}^{-\mu\tau}\phi^\dagger_\mathrm{m}(\mathbf{x},\tau)\ \mathrm{d^4x} = \mathbf{I} \ . \tag{19}$$

Moreover, $\{\zeta\}$ and $\{\upsilon\}$ satisfy the following normality condition,

$$\int \upsilon^\dagger_m(\mathbf{x},\tau)\zeta_m(\mathbf{x},\tau)\ d^4x = \int \phi^\dagger_m \mathrm{e}^{-\mu\tau}\mathrm{e}^{\mu\tau}\phi_\mathrm{m}\ \mathrm{d^4x} = \int \phi^\dagger_\mathrm{m}(\mathbf{x},\tau)\phi_\mathrm{m}(\mathbf{x},\tau)\ \mathrm{d^4x} = 1 \ , \tag{20}$$

leading to

$$\upsilon^\dagger_m(\mathbf{x},\tau)\gamma_5\zeta_m(\mathbf{x},\tau) = \phi^\dagger_m\mathrm{e}^{-\mu\tau}\gamma_5\mathrm{e}^{\mu\tau}\phi_\mathrm{m} = \phi^\dagger_\mathrm{m}(\mathbf{x},\tau)\gamma_5\phi_\mathrm{m}(\mathbf{x},\tau) \ , \tag{21}$$

Using these eigenvector spaces of $\not{D}(\mu)$, the calculation of $\mathrm{Tr}\gamma_5$ goes through in the same way as for $\not{D}(0)$ above. Since the new operator still anticommutes with γ_5 i.e $\{\gamma_5, \not{D}(\mu)\} = 0$, for each eigenvector ζ_m with eigenvalue λ_m there is an eigenvector $\gamma_5\zeta_m$ with the eigenvalue $-\lambda_m$. Thus the trace of γ_5 is zero for all nonzero λ_m. In the basis of the zero modes of $\not{D}(\mu)$, given by ζ_n and $\upsilon^\dagger_n$, the change in the fermion measure is given as,

$$\mathrm{Tr}\gamma_5 = \sum_n \upsilon^\dagger_n\gamma_5\zeta_n = \sum_n \phi^\dagger_n\mathrm{e}^{-\mu\tau}\gamma_5\mathrm{e}^{\mu\tau}\phi_\mathrm{n} = \mathrm{n}_+ - \mathrm{n}_-. \tag{22}$$

Thus the change in the fermion measure due to the chiral transformations is the same as in the zero density case with no additional μ dependent terms. Hence the anomaly is unaffected in the presence of μ. Some remarks on the proof may be in order. The definition of the vectors ζ_m and υ_m in Eq. (15) assumes that neither μ nor τ is infinite. The same assumption is also utilized in various steps in Eqs. (19)-(22). Clearly at strictly zero temperature, this is not tenable. However, an infinitesimally small temperature suffices for the proof to go through. Moreover, since the result is finally μ-independent, we expect the result to be valid at zero temperature, although our proof is valid only in the limit of zero temperature. The scaling of the eigenvectors, including the chiral zero modes, by the $\exp(\pm\mu\tau)$ factors can be related to a nonunitary transformation of the fermion fields in the QCD action in the presence of μ, given by

$$\psi^{'}(\mathbf{x},\tau) = \mathrm{e}^{\mu\tau}\psi(\mathbf{x},\tau) \quad , \quad \bar{\psi}^{'}(\mathbf{x},\tau) = \bar{\psi}(\mathbf{x},\tau)\mathrm{e}^{-\mu\tau} \ , \tag{23}$$

which makes the action μ-independent:

$$S = \int d^4x\ \bar{\psi}^{'}[\not{D} - \mu\gamma_4]\psi^{'} = \int d^4x\ \bar{\psi}\mathrm{e}^{-\mu\tau}\ [\not{D} - \mu\gamma_4]\ \mathrm{e}^{\mu\tau}\psi = \int \mathrm{d^4x}\ \bar{\psi}\ \not{D}\ \psi \ . \tag{24}$$

Note that the fields ψ and $\bar\psi$ at the same space-time point scale differently in the transformation in Eq. (23) which is permissible [12] in the Euclidean field theory since they are mutually independent fields. Let us also emphasize that the transformation in Eq. (23) is not unitary and thus not physical. Indeed, it merely relates the actions in two different physical situations of zero and nonzero μ. One clearly cannot employ it in the evaluation of the partition function due to its nonunitary nature. We have shown above that the transformation suggests how to extend the cancellation argument for nonzero eigenvalues of the Dirac operator for $\mu = 0$ to the nonzero μ case as well and is thus useful. Furthermore, since the transformation commutes with both flavor singlet and nonsinglet chiral transformations, employing it as a prescription to introduce the chemical potential will necessarily lead to a μ dependent action which has the same chiral invariance as for $\mu = 0$. Whether this way to introduce the chemical potential in any theory is the only way to do so without affecting its chiral invariance would be interesting to explore; we conjecture that this is the case.

III. ANOMALY ON THE LATTICE AT FINITE DENSITY

The above discussion of the anomaly in the continuum suggests a way to introduce the chemical potential on the lattice. By preserving the transformation (23) on the lattice, one may expect to maintain the anomaly to remain μ independent on the lattice as well. Let us consider the naıve massless fermion action on the lattice,

$$S = -\sum_{x,y} \bar\psi_x \left[U_4^\dagger(x-\hat4)\frac{\gamma_4}{2}\delta_{x,y+\hat4} - U_4(x)\frac{\gamma_4}{2}\delta_{x,y-\hat4} + \sum_{i=1}^{3}\left(U_i^\dagger(x-\hat\imath)\frac{\gamma_i}{2}\delta_{x,y+\hat\imath} - U_i(x)\frac{\gamma_i}{2}\delta_{x,y-\hat\imath}\right)\right]\psi_y \ . \tag{25}$$

Replacing the ψ and $\bar\psi$ fields in the above action by ψ' and $\bar\psi'$ respectively, using the lattice analogue of the transformation (23), we indeed obtain a fermionic action on the lattice at finite density,

$$S = -\sum_{x,y} \bar\psi_x' \left[\mathrm{e}^{-\mu a_4} \mathrm{U}_4^\dagger(\mathrm{x}-\hat4)\frac{\gamma_4}{2}\delta_{\mathrm{x,y}+\hat4} - \mathrm{e}^{\mu a_4}\mathrm{U}_4(\mathrm{x})\frac{\gamma_4}{2}\delta_{\mathrm{x,y}-\hat4} + \sum_{\mathrm{i}=1}^{3}\left(\mathrm{U}_\mathrm{i}^\dagger(\mathrm{x}-\hat\imath)\frac{\gamma_\mathrm{i}}{2}\delta_{\mathrm{x,y}+\hat\imath} - \mathrm{U}_\mathrm{i}(\mathrm{x})\frac{\gamma_\mathrm{i}}{2}\delta_{\mathrm{x,y}-\hat\imath}\right)\right]\psi_y' \ . \tag{26}$$

with a_4 being the lattice spacing in the temporal direction. Unfortunately, the infamous fermion doubling problem is related to the fact that the anomaly on the lattice is canceled exactly for such naıve fermions. The "no-go" theorem of Nielsen and Ninomiya [13] states that it is impossible to construct lattice Dirac operators which simultaneously satisfy Hermiticity, and locality and have chiral symmetry while being free of the "doublers". The commonly used fermions on the lattice, like the Wilson and the Kogut-Susskind fermions do not have $U_A(1)$ chiral symmetry, and so there is no anomaly to speak of. Nevertheless, we note that a similar transformation for such fermions does lead to the action popularly used for nonzero chemical potential[14, 15].

Recently, Neuberger [16] constructed a fermion operator D_{ov}, commonly known as the overlap operator, which has exact chiral symmetry and satisfies the Ginsparg and Wilson[17] relation,

$$\{\gamma_5, D_{ov}\} = D_{ov}\gamma_5 D_{ov} \quad \text{with} \quad D_{ov} = 1 + \gamma_5\epsilon(\gamma_5 D_W) \ . \tag{27}$$

Here ϵ is the sign function and D_W is the canonical Wilson-Dirac operator with a parameter M,

$$D_W(x,y) = (4-M)\,\delta_{x,y} - \sum_{i=1}^{4}\left(U_i^\dagger(x-\hat\imath)\frac{1+\gamma_i}{2}\delta_{x,y+\hat\imath} + U_i(x)\frac{1-\gamma_i}{2}\delta_{x,y-\hat\imath}\right) \ . \tag{28}$$

The value of the parameter M is constrained to lie between 0 and 2 for simulating a one flavor quark on the lattice. The overlap fermion action is invariant under the following chiral transformation, as shown by Luscher [18],

$$\delta\psi = \alpha\gamma_5(1-\frac{1}{2}D_{ov})\psi \quad \text{and} \quad \delta\bar\psi = \alpha\bar\psi(1-\frac{1}{2}D_{ov})\gamma_5 \ . \tag{29}$$

At zero temperature and density, the change in the measure computed on the lattice due to the Luscher transformations was shown to be related to the index of the fermion operator [18–20] ,

$$\mathrm{Tr}\ [2\gamma_5(1-\frac{1}{2}D_{ov})] = -\mathrm{Tr}\ (\gamma_5 D_{ov}) = n_+ - n_- = \ 2\ \mathrm{Index} D_{ov} \ , \tag{30}$$

where $n_\pm$ are right and left handed fermion zero modes respectively.

Bloch and Wettig [21] proposed a method to incorporate the chemical potential in the overlap operator. It consisted of i) multiplying U_4 $[U_4^\dagger]$ by exp(μa_4)[exp(-μa_4)] in the D_W in Eq. (28) and ii) extending the definition of the sign function for the resultant complex matrix. The $D_{ov}(\mu)$ also satisfied the Ginsparg-Wilson relation :

$$\{\gamma_5, D_{ov}(\mu)\} = D_{ov}(\mu)\gamma_5 D_{ov}(\mu). \text{ with } D_{ov}(\mu) = 1 + \gamma_5 \epsilon(\gamma_5 D_W(\mu)) \ . \tag{31}$$

It should be noted that the resultant action does not have the property of eliminating the μ-dependence by any transformation like Eq. (23) due to the nonlocal nature of D_{ov}.

As we pointed out[22] earlier though, the action $S = \sum_{x,y} \bar{\psi}_x [D_{ov}(\mu)]_{xy} \psi_y$ is not invariant under Luscher's chiral transformations of Eq. (29). Indeed, its variation is easily seen to be

$$\delta S = \frac{a\alpha}{2} \sum_{x,y} \bar{\psi}_x \left[2D_{ov}(\mu)\gamma_5 D_{ov}(\mu) - D_{ov}(0)\gamma_5 D_{ov}(\mu) - D_{ov}(\mu)\gamma_5 D_{ov}(0)\right]_{xy} \psi_y \neq 0 \ .$$

The chiral symmetry violation is of the $\mathcal{O}(a)$ and hence the symmetry is restored in the continuum limit. One may alternatively propose modified chiral transformations,

$$\delta\psi = \alpha\gamma_5(1 - \frac{1}{2}D_{ov}(\mu))\psi \quad \text{and} \quad \delta\bar{\psi} = \alpha\bar{\psi}(1 - \frac{1}{2}D_{ov}(\mu))\gamma_5 \ , \tag{32}$$

which will ensure $\delta S = 0$. In that case, the anomaly equation $-\text{Tr}\ (\gamma_5 D_{ov}(\mu)) = 2\ \text{Index} D_{ov}(\mu)$ is valid [21] on the lattice even in the presence of μ, since the fermion measure changes under these transformations by a Jacobian factor Tr $[2\gamma_5(1 - 1/2 D_{ov}(\mu))]$. Note, however, that the relevant zero modes are now those of the $D_{ov}(\mu)$, and thus μ dependent, in contrast to our continuum result of the previous section.

Furthermore, altering the symmetry transformations as above has undesirable physical consequences, as discussed in detail in [23]. Let us briefly outline here the main points. Non-Hermiticity of $\gamma_5 D_{ov}(\mu)$ makes the transformations nonunitary. The symmetry transformations should not depend on the intensive thermodynamic quantity μ, which is a tunable parameter of the physical system. The symmetry group itself changes with μ, leaving no physical order parameter which will characterize the chiral phase transition as a function of μ. In contrast, the chiral symmetry group remains the same at nonzero temperature (and zero density), allowing us to infer that vanishing of the chiral condensate would correspond to restoration of the symmetry for the vacuum.

A. A simple proposal

It is well-known that the overlap fermion operator can be obtained [24, 25] from the five dimensional domain wall fermions in the limit of infinite extent of the fifth dimension. The Bloch-Wettig proposal above was also shown to arise [26] in this way. It turns out that the chemical potential, μ enters in their action then as the Lagrange multiplier for the number of fermions on *each* slice of the fifth dimension. This means that all the unphysical "bulk" modes are considered with the same weightage in the partition function as the zero modes which eventually correspond to the massless quarks in four dimensions. The subsequent cancellation of the bulk contributions using Pauli-Villars fields to single out the contribution of a single chiral fermion thus becomes μ dependent on the lattice. Motivated by this physical fact, we propose to introduce the chemical potential only to count the fermion confined to the domain wall. Integrating out the fermions in the fifth dimension, one is led to the following action, which one would have written down naively to add a number density term :

$$D_{ov}(\hat{\mu})_{xy} = (D_{ov})_{xy} - \frac{a\hat{\mu}}{2a_4\ M} \left[(\gamma_4 + 1)U_4^\dagger(y)\delta_{x,y+\hat{4}} - (1 - \gamma_4)U_4(x)\delta_{x,y-\hat{4}}\right] \ . \tag{33}$$

Here D_{ov} is the same Neuberger-Dirac operator of Eq. (27), and $\hat{\mu} = \mu a_4$ is the chemical potential in lattice units. As usual, the volume of the system is $V = N^3 a^3$ and the temperature is $T = 1/(N_T a_4)$ on a $N^3 \times N_T$ lattice with lattice spacings a and a_4 in spatial and temporal directions respectively. The term containing the chemical potential is not unique. Improved density operators could be used for faster approach to the continuum limit, e.g., addition of three-link terms. We could have chosen $\hat{\mu}/s$ instead of $\hat{\mu}/M$ as the multiplying factor for the conserved number part. All such choices of actions are constrained by the fact that these have the correct continuum limit. However the finite lattice spacing errors in each of these operators would be different and we comment below on how they may affect the numerical simulations.

Note that our proposal, too, will break exact chiral invariance at the same $\mathcal{O}(a)$ as the Bloch-Wettig proposal. As a result, the anomaly equation on the lattice will get μ -dependent corrections anyway. A significant difference may be the fact that the change in the measure is μ independent for our proposal, as in the case of the continuum. We persist with it in the following, nevertheless, as it is simpler and easier to implement. Principally, this is due to the fact that one does not have to compute the sign function of a non-Hermitian matrix, with its inherent ambiguities, as in the Bloch-Wettig way of incorporating the chemical potential. The non-Hermitian sign function is numerically also more expensive to simulate for the full interacting case, whenever that becomes more practical.

For noninteracting fermions the $U_\mu = 1$ and the above Neuberger-Dirac operator with the chemical potential term can be diagonalized in momentum space in terms of the functions,

$$\begin{aligned} h_j &= -\sin ap_j \ , \ h_4 = -\frac{a}{a_4}\sin(a_4p_4) \ , \\ h_5 &= M - \sum_{j=1}^{3}(1-\cos ap_j) - \frac{a}{a_4}(1-\cos(a_4p_4)) \ , s = \sqrt{\sum_{j=1}^{3} h_j^2 + h_4^2 + h_5^2} \end{aligned} \tag{34}$$

such that $D_{ov}(\hat\mu)$ can be written as,

$$D_{ov}(\vec p, p_4, \hat\mu) = 1 - \sum_{i=1}^{4} i\gamma_i \frac{h_i}{s} - \frac{h_5}{s} - \frac{a\hat\mu}{a_4 M}\left[\gamma_4\cos(a_4p_4) - i\sin(a_4p_4)\right] \ . \tag{35}$$

To study thermodynamics of fermions one has to necessarily take antiperiodic boundary conditions along the temporal direction. Assuming periodic boundary conditions along the spatial directions we obtain

$$\begin{aligned} ap_j &= \frac{2n_j\pi}{N} \ , \ n_j = 0,..,(N-1), \ j = 1, \ 2, \ 3 \text{ and} \\ ap_4 &= \omega_n = \frac{(2n+1)\pi}{N_T} \ , \ n = 0,..,(N_T-1) \ , \end{aligned} \tag{36}$$

where ω_n are the Matsubara frequencies. The operator given by Eq. (35) can be shown to have correct continuum limit. The number density can be calculated at zero temperature by the contour integral method as was discussed for the Bloch-Wettig version of the overlap fermions at finite μ in [22]. The major difference one finds is the expected μ/a^2-divergence (μ^2/a^2-divergence) in the number (energy) density in the continuum limit of $a \to 0$. What is perhaps not widely appreciated from such calculations is that the leading term, corresponding to the Stefan-Boltzmann limit, also changes by a *finite* computable part. In the next section, we show through numerical evaluations of the sums, how one can deal with these problems.

B. Numerical Results

We compute two thermodynamic quantities of relevance to the above discussion as well as to the heavy-ion collision experiments: the change in the energy density due to the chemical potential, $\Delta\varepsilon(\mu, T) = \varepsilon(\mu, T) - \varepsilon(0, T)$ and the quark number susceptibility at zero chemical potential, $\chi(0)$. These thermodynamic quantities are computed by taking appropriate derivatives of the partition function $Z = \det D_{ov}$,

$$\chi(0) = \frac{1}{N^3a^2N_T}\left(\frac{\partial^2 \ln\det D_{ov}}{\partial\hat\mu^2}\right)_{a_4,\hat\mu\to 0, a_4=a} \ , \ \varepsilon(\hat\mu) = -\frac{1}{N^3a^3N_T}\left(\frac{\partial \ln\det D_{ov}}{\partial a_4}\right)_{\hat\mu N_T, \ a_4=a} \tag{37}$$

The quantities computed on the lattice are expected to have a $\Lambda^2 \sim 1/a^2$ dependence on the lattice. In order to eliminate these spurious Λ^2 terms, we follow the same prescription which was used for the energy density computation at zero temperature (which diverges as Λ^4). We compute these thermodynamic quantities at zero temperature and subtract them from the corresponding values computed on the lattice at nonzero temperatures. The zero temperature values were computed numerically on a lattice with a very large temporal extent N_T and fixed a_4 such that $T = 1/(N_Ta_4) \to 0$. The Matsubara frequencies then become continuous and hence could be integrated upon numerically.

Fig. 2 displays the subtracted results for $\Delta\varepsilon(\mu, T)$ for $r = \mu/T = \hat\mu N_T = 0.5$ and $\chi(0)$. The former is displayed in units of T^4 and has the value 0.127 for $r = 0.5$ in the continuum limit, while the latter is normalized to the ideal gas value ($T^2/3$). The M values are as indicated along the symbol used. The subtraction constants had to be computed separately for energy density and susceptibility. From a comparison of the plots with the corresponding ones [22] for the Bloch-Wettig case, we find that

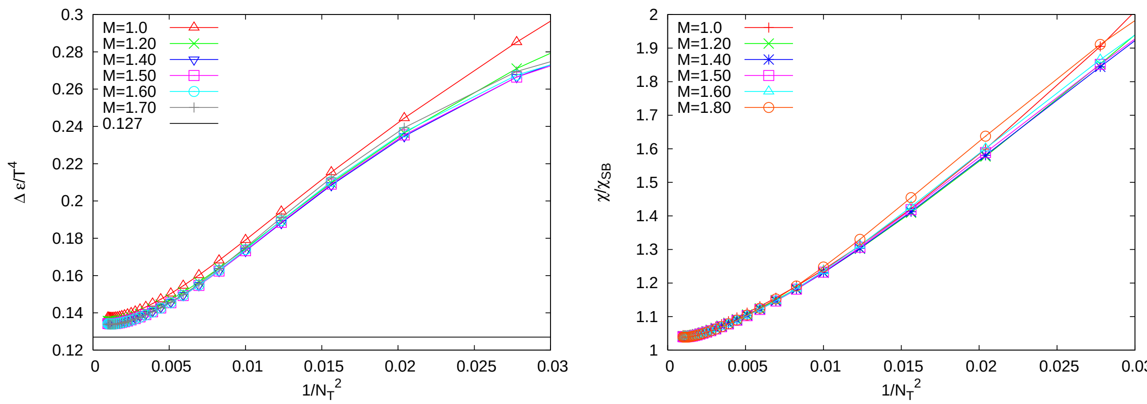

FIG. 2: The energy density(left panel) and quark number susceptibility (right panel)as a function of $1/N_T^2$ for M values as indicated for $\zeta = 4$.

- there are no leftover effects of divergences after the zero temperature subtraction,
- there are no oscillations for odd-even values of N_T,
- the M-dependence is much less pronounced, and
- the scaling towards the continuum value is linear with the possibility of an easier extrapolation.

We also computed the susceptibility using the Wilson fermions and compared the results with those above. We found that for $N_T = 6$ the cut-off effects of the Wilson operator are about 21% larger than the $M = 1.60$ overlap result shown in the right panel of Fig. (2). The difference reduces to about 3% at $N_T = 10$. Beyond $N_T = 10$, the approach to the continuum limit is almost identical for both the operators. The Wilson fermions have no chiral symmetry even for $\mu = 0$, which may make them less favored for the QCD critical point searches which are pivoted around the $\mu = 0$ transition.

We have also checked that there are no other divergent terms of the form $\mathcal{O}(a^{-n})$ with $n > 2$ in the number density, by calculating the fourth-order susceptibility since odd orders of susceptibilities vanish at $\mu = 0$. At zero chemical potential, the fourth-order susceptibility is given by,

$$\chi^{(4)}(0) = \frac{1}{N^3 N_T} \left(\frac{\partial^4 \ln \det D_{ov}}{\partial \hat{\mu}^4} \right)_{a_4, \hat{\mu} \to 0} \tag{38}$$

A term $\mathcal{O}(a^{-4})$ in the number density will show up as a divergence in this susceptibility, and will need a subtraction too. From Fig. (3), where we display our results for $\chi^{(4)}(0)$ for $M = 1.5$, we can conclude that there are indeed no divergences to be seen in the large N_T limit. The normalization in this case is also the expected continuum value. It is *not* identical to the Stefan-Boltzmann value of $2\pi^{-2}$. Using the contour integral method it can be easily shown to be $\chi_c^{(4)}(0) = 2/\pi^2(1 + 1/4)$, with the additional factor of 0.25 coming from the term usually cancelled in the usual prescriptions [14, 15, 27, 28]. We have found the convergence to the continuum value to be strongly M dependent and unfortunately very slow for all values of M, as seen in the plot B of Fig. (3). Introducing the chemical potential by choosing $\hat{\mu}/s$ as the coefficient of the number density term in Eq. (33), instead of the $\hat{\mu}/M$ we used, achieves a milder M dependence and a faster convergence towards the continuum. Perhaps improving the number density term can achieve a still faster convergence.

C. A new proposal for QCD critical point via Taylor expansion

Inspired by the above experience of dealing with the number density in the linear form, as in Eq. (33), we make a proposal valid for all fermions. Because of the infamous sign/phase problem for the fermion determinant with nonzero chemical potential, it has been proposed to look for the QCD critical point [29] by looking for the radius of convergence of the Taylor expansion [29, 30] in μ of the baryonic susceptibility. Computations have been done up to the eighth order so far [29, 31]. Extending these calculations to higher order is both necessary and desirable to

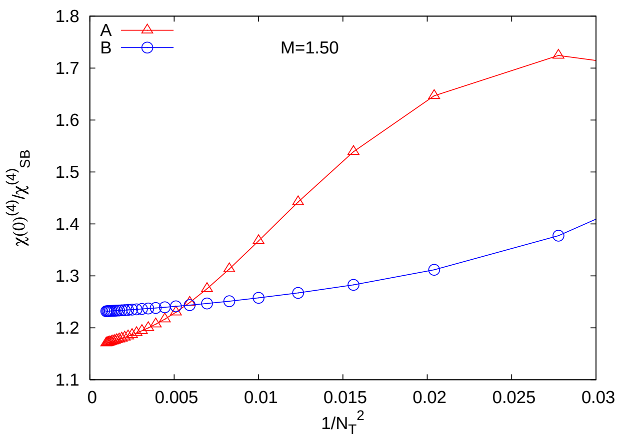

FIG. 3: The variation of the ratio of the fourth order susceptibility and the corresponding continuum value as a function of $1/N_T^2$ for $\zeta = 4,\ M = 1.5$ for the A) $\hat{\mu}/s$ and B)$\hat{\mu}/M$ ways of incorporating the chemical potential.

confirm the results already obtained. Our proposal can permit such an endeavor. We denote $M(\mu)$ to be any generic lattice fermionic operator with the chemical potential μ :

$$\begin{aligned} S_F &= \sum_{x,y} \bar{\Psi}(x) M(\mu; x, y) \Psi(y) \\ &= \sum_{x,y} \bar{\Psi}(x) D(x,y) \Psi(y) + \mu a \sum_{x,y} N(x,y) \end{aligned} \tag{39}$$

Here D can be the staggered, overlap, Wilson-Dirac or any other suitable fermion operator, and $N(x,y)$ is the corresponding point-split and gauge invariant number density. Eq. (33) provides a concrete example of the above for the overlap fermions. Note that any improvements in the fermion operator D or the number density N are generically included as long as the classical continuum limit is the same and μ appears linearly.

It is easy to see that only the first derivative of M with μ is nonzero. All others are zero. Thus denoting by M' the first derivative of M with respect to μ and adding more primes in the superscript for successively higher orders,

$$M' = \sum_{x,y} N(x,y), \qquad \text{and} \qquad M'' = M''' = M''''... = 0\ , \tag{40}$$

for our proposal to incorporate μ in contrast to the popular $\exp(\pm\mu)$ prescription where *all* derivatives are nonzero:

$$M' = M'''... = \sum_{x,y} N(x,y) \qquad \text{and} \qquad M'' = M'''' = M''''''... \neq 0\ . \tag{41}$$

As a consequence, the various nonlinear susceptibility expressions, or equivalently the expressions for Taylor series coefficients, are a lot simpler and have a lot fewer terms. For example, let us write down a fourth-order coefficient [by combining Eqs. (A.4), (A.7), and (A.8) of [29]] :

$$\chi^{(4)} = \frac{T}{V}\left[\left\langle \mathcal{O}_{1111} + 6\mathcal{O}_{112} + 4\mathcal{O}_{13} + 3\mathcal{O}_{22} + \mathcal{O}_4 \right\rangle - 3\left\langle \mathcal{O}_{11} + \mathcal{O}_2 \right\rangle^2\right]. \tag{42}$$

Here the notation $\mathcal{O}_{ij\cdots l}$ stands for the product, $\mathcal{O}_i\mathcal{O}_j \cdots O_l$. The expressions for $\mathcal{O}_n$, n=1,4 for our proposal above are

$$\begin{aligned} \mathcal{O}_1 &= \text{Tr}\ M^{-1}M', \\ \mathcal{O}_2 &= -\text{Tr}\ \ M^{-1}M'M^{-1}M', \\ \mathcal{O}_3 &= 2\ \text{Tr}\ \ (M^{-1}M')^3, \\ \mathcal{O}_4 &= -6\ \text{Tr}\ \ (M^{-1}M')^4, \end{aligned} \tag{43}$$

in contrast with those for the usual case given in [29] :

$$\begin{aligned}
\mathcal{O}_1 &= \text{Tr}\ \ M^{-1}M', \qquad (44)\\
\mathcal{O}_2 &= -\text{Tr}\ \ M^{-1}M'M^{-1}M' + \text{Tr}\ \ M^{-1}M'',\\
\mathcal{O}_3 &= 2\ \text{Tr}\ \ (M^{-1}M')^3 - 3\ \text{Tr}\ \ M^{-1}M'M^{-1}M'' + \text{Tr}\ \ M^{-1}M''',\\
\mathcal{O}_4 &= -6\ \text{Tr}\ \ (M^{-1}M')^4 + 12\ \text{Tr}\ \ (M^{-1}M')^2M^{-1}M'' - 3\ \text{Tr}\ \ (M^{-1}M'')^2\\
&- 4\ \text{Tr}\ \ M^{-1}M'M^{-1}M''' + \text{Tr}\ \ M^{-1}M''''.
\end{aligned}$$

The eighth-order term needs $\mathcal{O}_8$, which has 18 terms in the usual case whereas it will simply be $\mathcal{O}_8 = -5040$ Tr $(M^{-1}M')^8$ for our proposal.

The number of matrix inversions required to compute the higher order susceptibilities is also drastically reduced in this way of incorporating the chemical potential. This would save a considerable amount of computer time, as matrix inversions are the most time intensive operations. Following Fig. 3 of Ref. [29], one can see that all computations referred to on the leftmost branch of the algorithm tree need to be performed when M has a linear μ dependence. Thus for the eighth-order susceptibility computation we need to compute only eight matrix inversions as compared to the 20 required there, saving 60% of the computer time. For higher order susceptibilities, the number of matrix inversions is reduced by at least half, enabling us to compute even higher orders of the Taylor series of thermodynamic quantities and thus constrain the radius of convergence and the estimated location of the critical point better.

Of course, there is a price to pay, and we hope to demonstrate in the future from our ongoing work that it is not very big. All the coefficients that one evaluates this way will have the remnants of the terms which are otherwise eliminated by the usual prescriptions [14, 15, 27, 28]. Based on our computations in the previous section, we suggest that the zero temperature contribution to each of them be subtracted by evaluating them on a symmetric N^4 lattice at the same $\beta = 6/g^2$ as the finite temperature computation on the $N^3 \times N_T$ lattice. Since the second-order susceptibility $\chi^{(2)}$ has a divergence in the continuum limit, its computations may need higher precision to ensure the absence of the cut-off effects but the higher order coefficients have no such difficulties. One will also have to rescale the fourth-order susceptibility by a factor of 1.25 in order to use it in the ratio or the root method of estimating the radius of convergence. We hope that tenth- or even twelfth-order coefficient may thus be computable.

IV. CONCLUSIONS

We have shown perturbatively from the computation of the triangle diagram at zero temperature that the anomaly equation does not have any finite density correction terms. We have extended our calculations to the nonperturbative case where we have used Fujikawa's method to show that the anomaly relation is unaffected in the presence of a finite chemical potential. This has an important implication for the lattice field theory in designing the lattice Dirac operator for nonzero μ: It should lead to a μ-independent anomaly relation on the lattice. The recent Bloch-Wettig proposal for chiral fermion operators at finite density violates the chiral invariance on the lattice itself. While a μ-dependent modification of the chiral transformation can restore the chiral invariance, it leads to a μ-dependent anomaly relation unlike in the continuum theory. Such a modification has other physical consequences discussed in Ref. [23].

We have proposed a physically more justified way of introducing μ in the overlap Dirac operator. In this method the chiral symmetry is explicitly broken as well, but the contribution to the anomaly relation from the measure is likely to remain μ independent, with the lattice corrections to the anomaly relation falling off as a power law in the continuum limit. It has the expected μ^2/a^2-type divergences in the continuum limit. We showed how a simple subtraction scheme can take care of them in the free case. We proposed to use the simple linear in μ form for the Taylor series expansion technique of locating the QCD critical point. It has the advantage that the number of fermion matrix inversions goes down drastically when computing the higher order quark number susceptibilities. The higher order susceptibility computations are clearly important to accurately locate the critical point in the T-μ_B phase space for QCD. Our proposal would save much of the computational effort required for obtaining higher order susceptibilities, even for the staggered fermions.

Acknowledgments

R.V.G. would like to thank Rajamani Narayanan of FIU, Florida for his queries about QCD at finite temperature and density which inspired this work. S.S. would like to acknowledge the Council of Scientific and Industrial Research(CSIR) for financial support. We are indebted to Parthasarathi Mitra of SINP, Kolkata for carefully reading the earlier version of our manuscript and for his crucial remarks which led us to the proof in Sec. II.B.

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