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POSITIVE VALUES OF INHOMOGENEOUS 5-ARY QUADRATIC FORMS OF TYPE (3, 2)

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Abstract

Let $Q(x, y, z, t, u)$ be a real indefinite 5-ary quadratic form of type (3, 2) and determinant $D(>0)$. Then given any real numbers x_0, y_0, z_0, t_0, u_0 there exist integers x, y, z, t, u such that

$$0 < Q(x+x_0,\ y+y_0,\ z+z_0,\ t+t_0,\ u+u_0) \leqslant (16\,D)^{1/5}.$$

All the critical forms are also determined.

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1. Introduction

Let $Q(x_1, x_2, \ldots, x_n)$ be a real indefinite quadratic form in n variables with signature $(r, n-r), 0 < r < n$ and determinant $D \neq 0$. It is known (see Blaney (1948)) that there exists a real number κ, depending only on n and r, such that given any real numbers $c_1, c_2, \ldots, c_n$ the inequality

$$0 < Q(x_1+c_1, x_2+c_2, \ldots, x_n+c_n) \leqslant (\kappa\,|\,D\,|)^{1/n}$$

has a solution in integers $x_1, x_2, \ldots, x_n$. Let $\Gamma_{r,n-r}$ denote the infimum of all such constants κ. Davenport and Heilbronn (1947) proved that $\Gamma_{1,1} = 4$. $\Gamma_{2,1} = 4$ was proved by Barnes (1961) and $\Gamma_{1,2} = 8$ was obtained by Dumir (1967). Dumir (1968a, b) has also shown that $\Gamma_{3,1} = \frac{16}{3}$ and $\Gamma_{2,2} = 16$. In this paper we shall prove that $\Gamma_{3,2} = 16$. All the critical forms are also obtained. In a later paper we shall prove that $\Gamma_{4,1} = 8$. More precisely here we prove:

THEOREM. *Let $Q(x, y, z, t, u)$ be a real indefinite 5-ary quadratic form of type* (3, 2) *and determinant $D(>0)$. Then given any real numbers x_0, y_0, z_0, t_0, u_0 there exist integers x, y, z, t, u such that*

(1.1) $$0 < Q(x+x_0, y+y_0, z+z_0, t+t_0, u+u_0) \leqslant (16D)^{1/5}.$$

The sign of equality in (1.1) *is necessary if and only if either*

(1.2) $$Q(x,y,z,t,u) \sim \rho Q_1 = \rho(x^2+yz+tu)$$

or

(1.3) $$Q(x,y,z,t,u) \sim \rho Q_2 = \rho(x^2+y^2-2z^2-2tu),$$

where $\rho > 0$. *For* Q_1, *the sign of equality in* (1.1) *is necessary if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (0,0,0,0,0) \pmod 1$ *while for* Q_2 *it is so if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0) \pmod 1$.

2. Some lemmas

In the course of the proof we shall use the following lemmas :

LEMMA 1. *If Q is as in the theorem, there exist integers x_1, y_1, z_1, t_1, u_1 such that*

(2.1) $$0 < Q(x_1, y_1, z_1, t_1, u_1) \leqslant (16D)^{1/5}.$$

The sign of equality in (2.1) *is necessary if and only if* $Q \sim \rho Q_1$, $\rho > 0$.

This follows from some results of Watson (1958, 1968), Jackson (1969) and Oppenheim (1953a).

Let $\varphi(y,z,t,u)$ be a real indefinite quaternary quadratic form of type (2, 2) and determinant $D(>0)$. We shall need the following results :

LEMMA 2. *Given any real numbers y_0, z_0, t_0, u_0, there exist*

$$(y,z,t,u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$$

such that

(2.2) $$|\varphi(y,z,t,u)| \leqslant (\tfrac{1}{4}D)^{1/4}.$$

This is a theorem due to Birch (1958).

LEMMA 3. *There exist integers y_2, z_2, t_2, u_2 such that*

(2.3) $$0 < \varphi(y_2, z_2, t_2, u_2) \leqslant (\tfrac{81}{16}D)^{1/4}$$

except when $\varphi(y,z,t,u) \sim \rho(yz+tu)$, $\rho > 0$.

This is a theorem of Oppenheim (1953b).

LEMMA 4. *There exist* $(y,z,t,u) \equiv (y_0, z_0, t_0, u_0)$ (*mod* 1) *such that*

$$0 < \varphi(y,z,t,u) \leqslant (16D)^{1/4}. \tag{2.4}$$

This is a theorem of Dumir (1968b).

LEMMA 5. *Let* $\psi(z,t,u)$ *be an indefinite ternary quadratic form of type* (1, 2) *and determinant* $D(>0)$. *Then given any real numbers* z_0, t_0, u_0 *there exist* $(z,t,u) \equiv (z_0, t_0, u_0)$ (*mod* 1) *such that*

$$|\psi(z,t,u)| \leqslant (\tfrac{27}{100} D)^{1/3}. \tag{2.5}$$

This is a theorem of Davenport (1948).

LEMMA 6. *Let* $\psi(z,t,u)$ *be as in Lemma* 5. *Then given any real* z_0, t_0, u_0 *there exist* $(z,t,u) \equiv (z_0, t_0, u_0)$ (*mod* 1) *such that*

$$-(D/16)^{1/3} \leqslant \psi(z,t,u) < 3.(D/16)^{1/3}. \tag{2.6}$$

The sign of equality in (2.6) *is necessary if and only if* $\psi \sim \rho\psi_1$ *or* $\rho\psi_2$, $p > 0$; *where* $\psi_1 = -(z^2+tu)$, $\psi_2 = -2z^2 - t^2 + u^2$. *For* ψ_1, *the sign of equality in* (2.6) *is necessary if and only if* $(z_0, t_0, u_0) \equiv (\frac{1}{2}, 0, 0)$ (*mod* 1), *while for* ψ_2 *it is necessary if and only if* $(z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (*mod* 1).

This follows from the theorem of Dumir (1969).

LEMMA 7. *Let* α, β, d, *be a real numbers with* $d \geqslant 1$. *Then given any real number* x_0, *there exists* $x \equiv x_0$ (*mod* 1) *such that*

$$0 < (x+\alpha)^2 - \beta^2 \leqslant d \tag{2.7}$$

provided

$$\beta^2 \begin{cases} \leqslant \left(\dfrac{d-1}{2}\right)^2 & \text{if } d \text{ is an integer,} \\[2ex] < \left(\dfrac{[d]}{2}\right)^2 & \text{if } d \text{ is not an integer.} \end{cases} \tag{2.8}$$

Further strict inequality in (2.8) *implies strict inequality in* (2.7).

This is Lemma 6 of Dumir (1968a).

LEMMA 8. *Let α, β, d, be as above. Then given any real y_0, there exists $y \equiv y_0$ (mod 1) such that*

$$0 \leqslant (y+\alpha)^2 - \beta^2 < d \tag{2.9}$$

provided

$$\beta^2 \leqslant \begin{cases} \left(\dfrac{d-1}{2}\right)^2 & \text{if } d \text{ is an integer,} \\ \left(\dfrac{[d]}{2}\right)^2 & \text{if } d \text{ is not an integer.} \end{cases} \tag{2.10}$$

Further strict inequality in (2.10) *implies strict inequality in* (2.9).

This lemma is a simple modification of Lemma 7 stated above, so we omit the proof.

LEMMA 9. *Let n be an integer $\geqslant 1$. If for $d > n, f(d)$ is an increasing function of d and if*

$$f(d) < \left(\frac{d-1}{2}\right)^2 \quad \text{for } d \geqslant n+1, \tag{2.11}$$

then for $n < d < n+1$, we have

$$f(d) < \left(\frac{[d]}{2}\right)^2. \tag{2.12}$$

This obvious result is useful in many calculations.

3. Proof of the Theorem

Let

$$m = \inf_{\substack{x,y,z,t,u \text{ integers} \\ Q(x,y,z,t,u)>0}} Q(x,y,z,t,u) \tag{3.1}$$

By Lemma 1,

$$0 \leqslant m \leqslant (16D)^{1/5}.$$

If $m = 0$, the result follows from a result of Watson (1960). So we can suppose $m > 0$. Let $0 < \varepsilon_0 < \frac{1}{16}$ be a sufficiently small number. Then we can find integers x_1, y_1, z_1, t_1, u_1 such that

$$Q(x_1, y_1, z_1, t_1, u_1) = \frac{m}{1-\varepsilon} \leqslant (16D)^{1/5},$$

where $0 \leqslant \varepsilon < \varepsilon_0$. Since $\varepsilon < \frac{1}{16}$, we have g.c.d. $(x_1, y_1, z_1, t_1, u_1) = 1$. By a suitable unimodular transformation we can suppose that

$$Q(1,0,0,0,0) = \frac{m}{1-\varepsilon}$$

and write

$$Q(x,y,z,t,u) = \frac{m}{1-\varepsilon}\{(x+hy+gz+h't+g'u)^2 + \varphi(y,z,t,u)\},$$

where

$$|h| \leqslant \tfrac{1}{2}, \quad |g| \leqslant \tfrac{1}{2}, \quad |h'| \leqslant \tfrac{1}{2}, \quad |g'| \leqslant \tfrac{1}{2}$$

and where $\varphi(y,z,t,u)$ is a real indefinite quadratic form of type $(2,2)$ with determinant

(3.2) $$D\bigg/\left(\frac{m}{1-\varepsilon}\right)^5 \geqslant \frac{1}{16}.$$

Equality in (3.2) occurs if and only if $Q \sim \rho Q_1$(by Lemma 1). Also by the definition of m, for any integers x, y, z, t, u, we must have either $Q(x,y,z,t,u) \leqslant 0$ or $Q(x,y,z,t,u) \geqslant m$.

Because of homogeneity it suffices to prove

Theorem A. *Let* $Q(x,y,z,t,u) = (x+hy+gz+h't+g'u)^2 + \varphi(y,z,t,u)$ *where* $\varphi(y,z,t,u)$ *is a real indefinite quaternary quadratic form of type* $(2,2)$ *and determinant* D *such that*

(3.3) $$D \geqslant \tfrac{1}{16}, \quad (D = \tfrac{1}{16} \textit{ if and only if } Q \sim Q_1)$$

and

(3.4) $$|h| \leqslant \tfrac{1}{2}, \quad |g| \leqslant \tfrac{1}{2}, \quad |h'| \leqslant \tfrac{1}{2}, \quad |g'| \leqslant \tfrac{1}{2}.$$

Suppose further that for integers x, y, z, t, u *we have either*

(3.5) $$Q(x,y,z,t,u) \leqslant 0 \quad \textit{or} \quad Q(x,y,z,t,u) \geqslant 1-\varepsilon$$

where $0 \leqslant \varepsilon < \frac{1}{16}$ *is sufficiently small.*

Let

(3.6) $$d = (16D)^{1/5}.$$

Then given any real x_0, y_0, z_0, t_0, u_0, *we can find*

$$(x, y, z, t, u) \equiv (x_0, y_0, z_0, t_0, u_0) \quad (mod\ 1)$$

satisfying

$$0 < Q(x, y, z, t, u) \leqslant d. \tag{3.7}$$

The sign of equality in (3.7) *is necessary if and only if* $Q \sim Q_1$ *or* Q_2.

3.1. Proof of Theorem A.

LEMMA 10. *If* $Q(x, y, z, t, u)$ *is as defined in Theorem* A, *then for integers* y, z, t, u *we have either*

$$\varphi(y, z, t, u) \leqslant 0 \quad or \quad \varphi(y, z, t, u) \geqslant \tfrac{3}{4} - \varepsilon. \tag{3.8}$$

This result is similar to Lemma 4.1 of Dumir (1969), so we omit the proof.

LEMMA 11. *If* $Q = Q_1$, *then* (3.7) *holds with strict inequality unless* $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0)$ (*mod* 1).

PROOF. Here $D = \frac{1}{16}$, so that $d = 1$.

Case (i) $(y_0, z_0, t_0, u_0) \not\equiv (0, 0, 0, 0)$ (mod 1).
Without loss of generality we can suppose that $t_0 \not\equiv 0$ (mod 1). Choose $(x, y, z) \equiv (x_0, y_0, z_0)$ (mod 1) arbitrarily, $t \equiv t_0$ (mod 1) such that $0 < |t| \leqslant \frac{1}{2}$ and then choose $u \equiv u_0$ (mod 1) to satisfy

$$0 < x^2 + yz + tu \leqslant |t| \leqslant \tfrac{1}{2} < d.$$

Case (ii) $(y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0)$ (mod 1).
Take $y = z = t = u = 0$ and choose $x \equiv x_0$ (mod 1) such that $0 < x \leqslant 1$, so that

$$0 < x^2 + yz + tu = x^2 \leqslant 1 = d.$$

Strict inequality holds if $x_0 \not\equiv 0$ (mod 1). If $x_0 \equiv 0$ (mod 1), then the sign of equality is necessary because $x^2 + yz + tu$ takes only integral values.

So we can now suppose that $Q \nsim Q_1$. By (3.3) $d \geqslant 1$, and $d = 1$ if and only if $Q \sim Q_1$. Thus we have $d > 1$ in the rest of the paper.

LEMMA 12. *Let* $v_1 = d - \frac{1}{4}$ *and* $v_2 > 0$ *be a real number satisfying*

$$(3.9) \qquad v_2 \begin{cases} \leqslant \left(\dfrac{d-1}{2}\right)^2 & \textit{if } d \textit{ is an integer,} \\ < \left(\dfrac{[d]}{2}\right)^2 & \textit{if } d \textit{ is not an integer.} \end{cases}$$

Suppose that we can find $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0)$ *(mod* 1) *such that*

$$(3.10) \qquad -v_2 \leqslant \varphi(y, z, t, u) < v_1$$

then for any x_0, *there exists* $x \equiv x_0$ *(mod* 1) *satisfying* (3.7). *Further strict inequality in* (3.10) *implies strict inequality in* (3.7).

PROOF. If $0 < \varphi(y, z, t, u) < v_1$, choose $x \equiv x_0 \pmod 1$ such that

$$|x + hy + gz + h't + g'u| \leqslant \tfrac{1}{2},$$

so that

$$0 < Q(x, y, z, t, u) < \tfrac{1}{4} + v_1 = d.$$

If $-v_2 \leqslant \varphi(y, z, t, u) \leqslant 0$, then the result follows from Lemma 7 with $\alpha = hy + gz + h't + g'u$ and $\beta^2 = -\varphi(y, z, t, u)$.

LEMMA 13. *If* $d > 8$, *then* (3.7) *is true with strict inequality.*

PROOF. By Lemma 4 applied to $-\varphi(y, z, t, u)$, there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ such that

$$0 < -\varphi(y, z, t, u) \leqslant (16D)^{1/4}$$

i.e. $-d^{5/4} = -(16D)^{1/4} \leqslant \varphi(y, z, t, u) < 0$. The result will follow from Lemma 12 if we have

$$d^{5/4} < \begin{cases} \left(\dfrac{d-1}{2}\right)^2 & \text{if } d \geqslant 9 \\ \left(\dfrac{[d]}{2}\right)^2 & \text{if } 8 < d < 9. \end{cases}$$

$f(d) = d^{5/4}$ is an increasing function for $d > 1$. By Lemma 9 it is enough to verify the inequality for $d \geqslant 9$, which can be easily done.

LEMMA 14. *If* $3 < d \leqslant 8$, *then again* (3.7) *is true with strict inequality.*

PROOF. By Lemma 2, there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ such that

$$|\varphi(y,z,t,u)| \leqslant (\tfrac{1}{4}D)^{1/4} = \left(\frac{d^5}{64}\right)^{1/4}.$$

The result will follow from Lemma 12, if we have

$$\left(\frac{d^5}{64}\right)^{1/4} < d-\tfrac{1}{4} \tag{3.11}$$

and

$$\left(\frac{d^5}{64}\right)^{1/4} < \begin{cases} \left(\dfrac{d-1}{2}\right)^2 & \text{if } 4 \leqslant d \leqslant 8, \\ \left(\dfrac{[d]}{2}\right)^2 & \text{if } 3 < d < 4. \end{cases} \tag{3.12}$$

We observe that by Lemma 9, it is enough to verify (3.12) for $4 \leqslant d \leqslant 8$. Verifications of these inequalities are easy and are left to the reader.

REMARK. For $1 < d \leqslant 3$, we shall repeat the procedure of reduction described in Section 3. We use Lemma 3 on the homogeneous minimum of positive values of quaternary forms of type (2.2). So we first dispose of the exceptional forms.

LEMMA 15. *If* $\varphi(y,z,t,u) \sim \rho(yz+tu)$, $\rho > 0$ *and* $1 < d \leqslant 3$, *then* (3.7) *is true with strict inequality.*

PROOF. Without loss of generality we can suppose that $\varphi(y,z,t,u) = \rho(yz+tu)$. So

$$Q(x,y,z,t,u) = (x+hy+gz+h't+g'u)^2+\rho(yz+tu).$$

By (3.4), $0 \leqslant Q(0,1,0,0,0) = h^2 \leqslant \frac{1}{4} < 1-\varepsilon$. Therefore (3.5) implies $h = 0$. Similarly $g = g' = h' = 0$. Therefore $Q(x,y,z,t,u) = x^2+\rho(yz+tu)$ and $D = \rho^4/16$. Here $\rho/2 = \frac{1}{2}(16D)^{1/4} = \frac{1}{2}d^{5/4} < d$, for $d \leqslant 3$. Now one can easily verify that (3.7) is satisfied with strict inequality (proof is similar to that of Lemma 11).

3.2. Proof of Theorem A continued

From now on we can suppose that $1 < d \leqslant 3$ and $\varphi(y,z,t,u) \not\sim \rho(yz+tu)$, $\rho > 0$. By Lemmas 3 and 10, there exist integers y_2, z_2, t_2, u_2 with g.c.d.$(y_2,z_2,t_2,u_2) = 1$ such that

$$\tfrac{3}{4}-\varepsilon \leqslant a = \varphi(y_2,z_2,t_2,u_2) \leqslant (\tfrac{81}{16}D)^{1/4} = \tfrac{3}{4}d^{5/4}.$$

By a suitable unimodular transformation we can suppose

$$\varphi(1,0,0,0) = a.$$

So we can write

$$\varphi(y,z,t,u) = a\{(y+fz+f't+f''u)^2+\psi(z,t,u)\}$$

where

$$\frac{3}{4}-\varepsilon \leqslant a \leqslant \frac{3}{4}d^{5/4} \tag{3.13}$$

and $\psi(z,t,u)$ is a real indefinite ternary quadratic form of type (1,2) and determinant D/a^4. In view of Lemma 12, it is enough to prove that there exist $(y,z,t,u) \equiv (y_0,z_0,t_0,u_0) \pmod 1$ such that

$$-\frac{v}{a} \leqslant (y+fz+f't+f''u)^2+\psi(z,t,u) < \frac{4d-1}{4a} \tag{3.14}$$

and that strict inequality holds if d is not an integer; where

$$v = \begin{cases} 1 & \text{if } 2 < d \leqslant 3, \\ \frac{1}{4} & \text{if } 1 < d \leqslant 2. \end{cases} \tag{3.15}$$

Let

$$\mu_1 = \frac{4d-1-a}{4a} \quad \text{and} \quad \lambda = \frac{4d-1}{4a}+\frac{v}{a}.$$

Using (3.13) one can easily verify that $\mu_1 > 0$ and $\lambda > 1$. The proof of the following lemma is similar to that of Lemma 12 and is omitted. (Here we use Lemma 8 instead of Lemma 7.)

LEMMA 16. *Let*

$$0 < \mu_2 \leqslant \begin{cases} \left(\dfrac{\lambda-1}{2}\right)^2+\dfrac{v}{a} & \textit{if } \lambda \textit{ is an integer} \\[2ex] \left(\dfrac{[\lambda]}{2}\right)^2+\dfrac{v}{a} & \textit{if } \lambda \textit{ is not an integer.} \end{cases} \tag{3.16}$$

Suppose that we can find $(z,t,u) \equiv (z_0,t_0,u_0)$ *(mod* 1) *such that*

$$-\mu_2 \leqslant \psi(z,t,u) < \mu_1 \tag{3.17}$$

and strict inequality holds in (3.17) *if d is not an integer. Then there exists* $y \equiv y_0$ *(mod* 1) *such that* (3.14) *holds. Further strict inequality in* (3.17) *implies strict inequality in* (3.14).

LEMMA 17. *If* $2 < d \leqslant 3$, *then* (3.17) *and hence* (3.14) *is true with strict inequality.*

PROOF. In this case $\nu = 1$, so that $\lambda = (3+4d)/4a$. By Lemma 5, we can find $(z,t,u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$|\psi(z,t,u)| \leqslant \left(\frac{27}{100}\frac{D}{a^4}\right)^{1/3} = \left(\frac{27d^5}{1600a^4}\right)^{1/3}.$$

Then (3.17) will hold with strict inequality if we have

$$\left(\frac{27d^5}{1600a^4}\right)^{1/3} < \frac{4d-1-a}{4a} \tag{3.18}$$

and

$$\left(\frac{27d^5}{1600a^4}\right) < \begin{cases} \left(\dfrac{\lambda-1}{2}\right)^2 + \dfrac{1}{a} & \text{if } \lambda \text{ is an integer} \\ \left(\dfrac{[\lambda]}{2}\right)^2 + \dfrac{1}{a} & \text{if } \lambda \text{ is not an integer.} \end{cases} \tag{3.19}$$

Verification of (3.18) is straightforward. So we proceed to verify (3.19). Let

$$n < \lambda = \frac{3+4d}{4a} \leqslant n+1, \quad n = 1, 2, 3, \ldots$$

Then (3.19) will be satisfied if we have

$$\left(\frac{27d^5}{1600a^4}\right)^{1/3} < \frac{n^2}{4} + \frac{1}{a} \quad \text{for all } n \geqslant 1.$$

That is,

$$\frac{27d^5}{1600a^4} < a^4\left(\frac{n^2}{4} + \frac{1}{a}\right)^3 = g(a) \quad \text{(say).} \tag{3.20}$$

$g(a)$ is an increasing function of a and $a \geqslant (3+4d)/4(n+1)$, therefore

$$g(a) \geqslant g\left(\frac{3+4d}{4(n+1)}\right) = \frac{(3+4d)\{n^2(3+4d)+16(n+1)\}^3}{4^7(n+1)^4}.$$

So we shall have (3.20) if

$$\frac{(3+4d)\{n^2(3+4d)+16(n+1)\}^3}{(n+1)^4 d^5} > 4^4 \cdot \frac{27}{25}. \tag{3.21}$$

As the left-hand side of (3.21) is clearly a decreasing function of d and $d \leqslant 3$, one can easily check that (3.21) is true for all $n \geqslant 1$. This proves (3.20) and hence (3.19).

LEMMA 18. *If* $1 < d \leqslant 2$, *then again* (3.17) *and hence* (3.14) *is true. Moreover* (3.14) *holds with strict inequality unless* $d = 2, a = 1$ *and* ψ, y_0, z_0, t_0 *are such that equality is necessary in* (2.6).

PROOF. In this case $\nu = \frac{1}{4}$, so that $\lambda = d/a$. By (3.13),

$$\lambda = \frac{d}{a} \leqslant \frac{2}{3/4-\varepsilon} < 3,$$

for sufficiently small ε. We distinguish two cases :

Case (i) $2 < \lambda < 3$.
In this case

$$\left(\frac{[\lambda]}{2}\right)^2 + \frac{\nu}{a} = 1 + \frac{1}{4a} = \frac{1+4a}{4a}.$$

So we have to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\frac{1+4a}{4a} < \psi(z, t, u) < \frac{4d-1-a}{4a}. \tag{3.22}$$

By Lemma 5, we can find $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$|\psi(z, t, u)| \leqslant \left(\frac{27d^5}{1600a^4}\right)^{1/3} \tag{3.23}$$

Therefore (3.22) follows from (3.23) if we have

$$\left(\frac{27d^5}{1600a^4}\right)^{1/3} < \min\left(\frac{1+4a}{4a}, \frac{4d-1-a}{4a}\right).$$

This inequality can be easily checked.

Case (ii) $1 < \lambda \leqslant 2$.
In this case we have to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\frac{1+a}{4a} \leqslant \psi(z, t, u) < \frac{4d-a-1}{4a} \tag{3.24}$$

and strict inequality holds if d is not an integer. By Lemma 6, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\left(\frac{D}{16a^4}\right)^{1/3} \leqslant \psi(z, t, u) < 3\left(\frac{D}{16a^4}\right)^{1/3}.$$

Therefore (3.24) will hold if we have

$$3\left(\frac{D}{16a^4}\right)^{1/3} = 3\left(\frac{d^5}{256a^4}\right)^{1/3} \leqslant \frac{4d-1-a}{4a} \tag{3.25}$$

and

$$\left(\frac{d^5}{256a^4}\right)^{1/3} \leqslant \frac{1+a}{4a}, \tag{3.26}$$

with strict inequality if $d \neq 2$.

Now

$$\frac{4d-1-a}{12a} \leqslant \frac{1+a}{4a} \quad \text{for } a \geqslant \frac{d}{2} \quad \text{and} \quad d \leqslant 2.$$

Equality holds if and only if $d = 2$, $a = \frac{1}{2}d = 1$. Therefore it is enough to prove that (3.25) holds and equality is necessary only for $d = 2$. We shall have (3.25) if and only if

$$\tfrac{27}{256} d^5 \leqslant a\left(d - \frac{1-a}{4}\right)^3 = g(a) \quad \text{(say)} \tag{3.27}$$

$g(a)$ increases or decreases according as $a < d - \frac{1}{4}$ or $a > d - \frac{1}{4}$ and since $\frac{1}{2}d < d - \frac{1}{4}$, (3.27) will be true for $d/2 \leqslant a < \frac{3}{4}d^{5/4}$ if

$$\min\left(g\left(\frac{d}{2}\right), g(\tfrac{3}{4}d^{5/4})\right) \geqslant \tfrac{27}{256} d^2. \tag{3.28}$$

Now

$$g\left(\frac{d}{2}\right) = \frac{d(7d-2)^3}{2.8^3} \geqslant \tfrac{27}{256} d^5$$

if

$$f(d) = \frac{(7d-2)^3}{d^4} \geqslant 3^3 . 4.$$

$f(d)$ increases for $d \leqslant \frac{8}{7}$ and decreases for $d \geqslant \frac{8}{7}$, therefore for $1 < d \leqslant 2$,

$$f(d) \geqslant \min(f(1), f(2)) = f(2) = 3^3 . 4,$$

and strict inequality holds unless $d = 2$. The inequality $g(\frac{3}{4}d^{5/4}) > \frac{27}{256}d^5$ can be easily verified.

Therefore (3.27) is satisfied with strict inequality unless $d = 2$, $a = \frac{1}{2}d = 1$. Hence (3.24) is satisfied. Equality holds in (3.24) only if $d = 2$, $a = 1$ and ψ, z_0, t_0, u_0 are such that the sign of equality is necessary in (2.6).

This completes the proof of the lemma.

4. The case of equality

Lemma 19. *The sign of equality in* (3.7) *is necessary if and only if* $Q \sim Q_2$. *For* Q_2 *it is necessary if and only if*

$$(x_0, y_0, z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \quad (\textit{mod}\, 1).$$

Proof. Equality can be necessary in (3.7) only if it is necessary (3.14). This happens only if $d = 2$, $a = 1$ and ψ, z_0, t_0, u_0 are such that equality is necessary in (2.6) (see Lemma 18). Thus we must have $\psi \sim \rho\psi_1$ or $\rho\psi_2$, $\rho > 0$. For ψ_1 we must have $(z_0, t_0, u_0) \equiv (\frac{1}{2}, 0, 0)$, while for ψ_2 we have $(z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (mod 1).

Case (i). $\psi_1(z, t, u) = -\rho(z^2 + tu)$, $(z_0, t_0, u_0) \equiv (\frac{1}{2}, 0, 0)$ (mod 1). Then

$$\tfrac{1}{4}\rho^3 = \frac{D}{a^4} = \frac{d^5}{16a^4} = 2,$$

so that $\rho = 2$. Therefore

$$\varphi(y, z, t, u) = (y + fz + f't + f''u)^2 - 2z^2 - 2tu.$$

By a suitable unimodular transformation we can suppose that

$$|f| \leqslant \tfrac{1}{2}, |f'| \leqslant \tfrac{1}{2}, |f''| \leqslant \tfrac{1}{2}. \tag{4.1}$$

If $f'' \neq 0$, then

$$0 < \varphi(0, 0, 0, 1) = f''^2 \leqslant \tfrac{1}{4} < \tfrac{3}{4} - \varepsilon.$$

This contradicts (3.8) Therefore $f'' = 0$. Similarly, consideration of $\varphi(0, 0, 1, 0)$ and $\varphi(0, 1, -1, 1)$ gives $f' = f = 0$. Hence $\varphi(y, z, t, u) = y^2 - 2z^2 - 2tu$. For equality to occur in (3.14), the inequality

$$-\tfrac{1}{4} < (y + y_0)^2 - 2(z + \tfrac{1}{2})^2 - 2tu < d - \tfrac{1}{4} = \tfrac{7}{4} \tag{4.2}$$

should have no solution in integers y, z, t and u. Take $z = t = u = 0$ and choose the integer y such that $|y + y_0| \leqslant \frac{1}{2}$, then (4.2) is solvable unless $y_0 \equiv \frac{1}{2}$ (mod 1). Therefore,

$$Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + y^2 - 2z^2 - 2tu;$$

$$(y_0, z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \quad (\text{mod}\, 1).$$

Considering $Q(0, 0, 0, 0, 1)$, $Q(0, 0, 0, 1, 0)$ and $Q(0, 0, 1, -1, 1)$ and using (3.4), (3.5) we get $g = h' = g' = 0$. Therefore $Q(x, y, z, t, u) = (x + hy)^2 + y^2 - 2z^2 - 2tu$. If equality is to be necessary in (3.7), the inequality

(4.3) $$0 < F(x,y,z,t,u) = (x+x_0+h(y+\tfrac{1}{2}))^2+(y+\tfrac{1}{2})^2 - 2(z+\tfrac{1}{2})^2 - 2tu < d = 2$$

should have no solution in integers x, y, z, t, u. Now $0 < F(x,0,0,0,0) < 2$ is solvable for integer x unless $x_0+\frac{1}{2}h \equiv \frac{1}{2} \pmod 1$. Also $0 < F(x,-1,0,0,0) < 2$ is solvable in integer x unless $x_0-\frac{1}{2}h \equiv \frac{1}{2} \pmod 1$. Thus (4.3) is solvable unless $h \equiv 0 \pmod 1$. Since $|h| \leqslant \frac{1}{2}$ from (3.4), we must have $h = 0$ Then $x_0 \equiv \frac{1}{2} \pmod 1$. Hence

$$Q(x,y,z,t,u) = x^2+y^2-2z^2-2tu = Q_2$$

and

$$(x_0, y_0, z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \pmod 1.$$

Considering congruence module 8, one can see that the sign of equality is necessary in this case.

Case (ii). $\psi_2(z,t,u) = -\rho(2z^2+t^2-u^2)$, $(z_0,t_0,u_0) \equiv (\frac{1}{2},\frac{1}{2},\frac{1}{2}) \pmod 1$
Proceeding as above, one can see that equality is necessary in (3.7) if and only if

$$Q = x^2+y^2-2z^2-t^2+u^2$$

and

$$(x_0, y_0, z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \pmod 1.$$

Since

$$x^2+y^2-2z^2-t^2+u^2 = (x-t-u)^2+y^2-2z^2+2(x-t)(t+u),$$

$Q \sim x^2+y^2-2z^2+2tu \sim Q_2$. Therefore this case does not give us a new form.

The proof of Theorem A follows from Lemmas 10 to 19 and thus our theorem is proved.

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References

E. S. Barnes (1961), 'The positive values of inhomogeneous ternary quadratic forms', *J. Austral. Math. Soc.* **2**, 127–132.

B. J. Birch (1958), 'The inhomogeneous minimum of quadratic forms of signature zero'. *Acta Arith.* **4**, 85–98.

H. Blaney (1948), 'Indefinite quadratic forms in n variables', *J. London Math. Soc.* **23**, 153–160.

H. Davenport (1948), 'Non-homogeneous ternary quadratic forms', *Acta Math.* **80**, 65–95.

H. Davenport and H. Heilbronn (1947), 'Asymmetric inequalities for non-homogeneous linear forms,' *J. London Math. Soc.* **22**, 53–61.

V. C. Dumir (1967), 'Asymmetric inequalities for non-homogeneous ternary quadratic forms.' *Proc. Cambridge Philos. Soc.* **63**, 291–303.

V. C. Dumir (1968a), 'Positive values of inhomogeneous quaternary quadratic forms (I),' *J. Austral. Math. Soc.* **8**, 87–101.

V. C. Dumir (1968b), 'Positive values of inhomogeneous quaternary quadratic forms (II),' *J. Austral. Math. Soc.* **8**, 287–303.

V. C. Dumir (1969), 'Asymmetric inequality for non-homogeneous ternary quadratic forms', *J. Number Theory* **1**, 326–345.

H. Jackson (1969), 'Small positive values of indefinite quadratic forms.' *J. London Math. Soc.* (2) **1**, 643–659.

A. Oppenheim (1953a), 'Values of quadratic forms II', *Quart. J. Math. Oxford Ser.* **4**, 60–66.

A. Oppenheim (1953b), 'One sided inequalities for quadratic forms (II) quaternary forms', *Proc. London Math. Soc.* **3**, 417–429.

G. L. Watson (1958), 'One sided inequalities for integral quadratic forms,' *Quart J. Math. Oxford Ser.* **9**, 99–108.

G. L. Watson (1960), 'Indefinite quadratic polynomials', *Mathematika* **7**, 141–144.

G. L. Watson (1968), 'Asymmetric inequalities for indefinite quadratic forms,' *Proc. London Math. Soc.* (3) **18**, 95–113.

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