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POSITIVE VALUES OF INHOMOGENEOUS QUINARY QUADRATIC FORMS OF TYPE (4, 1)

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Abstract

Here it is proved that if $Q(x, y, z, t, u)$ is a real indefinite quinary quadratic form of type (4, 1) and determinant D, then given any real numbers x_0, y_0, z_0, t_0, u_0 there exist integers x, y, z, t, u such that

$$0 < Q(x + x_0, y + y_0, z + z_0, t + t_0, u + u_0) < (8|D|)^{1/5}.$$

All critical forms are also obtained.

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1. Introduction

Let $Q(x_1, x_2, \ldots, x_n)$ be a real indefinite quadratic form in n variables with signature $(r, n - r)$, $0 < r < n$ and determinant $D \neq 0$. It is known (see Blaney (1948)) that there exists a real number κ, depending upon n and r only, such that given any real numbers $c_1, c_2, \ldots, c_n$ the inequality

$$0 < Q(x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n) \leqslant (\kappa|D|)^{1/n}$$

has a solution in integers $x_1, x_2, \ldots, x_n$. Let $\Gamma_{r,n-r}$ denote the infimum of all such numbers κ. Davenport and Heilbronn (1947) proved that $\Gamma_{1,1} = 4$. $\Gamma_{2,1} = 4$ was proved by Barnes (1961) and $\Gamma_{1,2} = 8$ was obtained by Dumir (1967). Dumir (1968a, b) has also shown that $\Gamma_{3,1} = 16/3$ and $\Gamma_{2,2} = 16$. The authors (1980) proved that $\Gamma_{3,2} = 16$. In this paper we prove that $\Gamma_{4,1} = 8$. All the critical forms are also obtained. More precisely we prove:

THEOREM. *Let $Q(x, y, z, t, u)$ be a real indefinite quinary quadratic form of type* (4, 1) *and determinant D (< 0) then given any real numbers x_0, y_0, z_0, t_0, u_0, there exist integers x, y, z, t, u such that*

$$(1.1) \qquad 0 < Q(x + x_0, y + y_0, z + z_0, t + t_0, u + u_0) \leqslant (8|D|)^{1/5}.$$

The sign of equality in (1.1) *is necessary if and only if either*

$$(1.2) \qquad Q(x, y, z, t, u) \sim \rho Q_1 = \rho(xy + z^2 + t^2 + u^2 + zt + tu + uz)$$

or

$$(1.3) \qquad Q(x, y, z, t, u) \sim \rho Q_2 = \rho(x^2 + y^2 + z^2 + t^2 - 4u^2),$$

where $\rho > 0$.

For Q_1, the sign of equality in (1.1) *is necessary if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0) \pmod 1$ *while for Q_2 it is needed if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$.

2. Some lemmas

In the course of the proof we shall use the following lemmas:

LEMMA 1. *If Q is as in the theorem, there exist integers x_1, y_1, z_1, t_1, u_1 such that*

$$(2.1) \qquad 0 < Q(x_1, y_1, z_1, t_1, u_1) \leqslant (8|D|)^{1/5}.$$

The sign of equality in (2.1) *is necessary if and only if $Q \sim \rho Q_1$, $\rho > 0$.*

This follows from some results of Watson (1968), Jackson (1969) and Oppenheim (1953a). Also see Watson (1958).

Let $\varphi(y, z, t, u)$ be a real indefinite quaternary quadratic form of type (3, 1) and determinant D (< 0). We need the following results:

LEMMA 2. *Given any real numbers y_0, z_0, t_0, u_0, there exist* $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ *such that*

$$(2.2) \qquad |\varphi(y, z, t, u)| \leqslant (|D|/3)^{1/4}.$$

This is a theorem due to Dumir (1967).

LEMMA 3. *There exist integers y_2, z_2, t_2, u_2 such that*

$$(2.3) \qquad 0 < \varphi(y_2, z_2, t_2, u_2) < (4|D|)^{1/4}$$

except when $\varphi(y, z, t, u) \sim \rho\varphi_1 = \rho(y^2 + yz + z^2 + tu)$ and $\varphi(y, z, t, u) \sim \rho\varphi_2 = \rho(y^2 + z^2 + tu)$, $\rho > 0$.

This is Theorem 2 of Oppenheim (1953b).

LEMMA 4. *There exist* $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ *such that*

$$0 < -\varphi(y, z, t, u) < (22|D|)^{1/4}. \tag{2.4}$$

This follows from Theorem 1 of the authors (1980).

LEMMA 5. *Let* $\psi(z, t, u)$ *be a real indefinite ternary quadratic form of type* (2, 1) *and determinant* D (< 0). *Then given any real numbers* z_0, t_0, u_0 *there exist* $(z, t, u) = (z_0, t_0, u_0) \pmod 1$ *such that*

$$|\psi(z, t, u)| \leqslant (27|D|/100)^{1/3}. \tag{2.5}$$

This is a theorem due to Davenport (1948).

LEMMA 6. *Let* $\psi(z, t, u)$ *be as in Lemma* 5. *Let* $c = \frac{9}{8}, \frac{1}{2}$ *or* $\frac{1}{3}$. *Then given any real numbers* z_0, t_0, u_0 *there exist* $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ *such that*

$$-c(f(c)|D|)^{1/3} < \psi(z, t, u) \leqslant (f(c)|D|)^{1/3}, \tag{2.6}$$

where $f(\frac{9}{8}) = \frac{512}{2187}$, $f(\frac{1}{2}) = \frac{256}{429}$ *and* $f(\frac{1}{3}) = \frac{27}{32}$. *The sign of equality in* (2.6) *is necessary if and only if* $c = \frac{1}{3}$ *and* $\psi \sim \rho\psi_1$, $\rho > 0$ *where* $\psi_1 = z^2 + t^2 - 4u^2$. *For* ψ_1 *the equality is needed if and only if* $(z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$.

For $c = \frac{9}{8}$ and $\frac{1}{2}$, the result follows from a theorem of Dumir (1969). For $c = \frac{1}{3}$, it is due to the authors (1979).

LEMMA 7. *Let* α, β, *and* d *be real numbers with* $d \geqslant 1$. *Then given any real number* x_0, *there exists* $x \equiv x_0 \pmod 1$ *such that*

$$0 < (x + \alpha)^2 - \beta^2 \leqslant d \tag{2.7}$$

provided

$$\beta^2 \begin{cases} \leqslant (d-1)^2/4 & \text{if } d \text{ is an integer,} \\ < [d]^2/4 & \text{if } d \text{ is not an integer.} \end{cases} \tag{2.8}$$

Further strict inequality in (2.8) *implies strict inequality in* (2.7).

This is Lemma 6 of Dumir (1968a).

LEMMA 8. *Let* n *be an integer* $\geqslant 1$. *If* $f(d)$ *is an increasing function of* d *for* $d > n$ *and if*

$$f(d) < (d-1)^2/4 \quad \text{for } d \geqslant n+1, \tag{2.9}$$

then for $n < d < n + 1$,

(2.10) $$f(d) < [d]^2/4.$$

This obvious lemma is useful in many calculations.

3. Proof of the theorem

Let

(3.1) $$m = \inf_{\substack{x, y, z, t, u \in Z, \\ Q(x, y, z, t, u) > 0}} Q(x, y, z, t, u).$$

By Lemma 1,

$$0 \leqslant m \leqslant (8|D|)^{1/5}.$$

If $m = 0$, the result follows from a result of Watson (1960). So we can suppose that $m > 0$ in the rest of the paper. Let $0 < \varepsilon_0 < \frac{1}{16}$ be a sufficiently small number. Then we can find integers x_1, y_1, z_1, t_1, u_1 such that

$$Q(x_1, y_1, z_1, t_1, u_1) = \frac{m}{1 - \varepsilon} \leqslant (8|D|)^{1/5},$$

where $0 \leqslant \varepsilon < \varepsilon_0$ and g.c.d. $(x_1, y_1, z_1, t_1, u_1) = 1$. By a suitable unimodular transformation we can suppose that

$$Q(1, 0, 0, 0, 0) = m/(1 - \varepsilon)$$

and write

$$Q(x, y, z, t, u) = m(1 - \varepsilon)^{-1}\{(x + hy + gz + h't + g'u)^2 + \varphi(y, z, t, u)\},$$

where $|h| \leqslant \frac{1}{2}$, $|g| \leqslant \frac{1}{2}$, $|h'| \leqslant \frac{1}{2}$, $|g'| \leqslant \frac{1}{2}$ and $\varphi(y, z, t, u)$ is a real indefinite quadratic form of type (3, 1) with determinant

(3.2) $$D(m/(1 - \varepsilon))^{-5} \leqslant -\frac{1}{8}.$$

Equality in (3.2) occurs if and only if $Q \sim \rho Q_1$ (by Lemma 1). Also by definition of m, we have, for any integers x, y, z, t, u either $Q(x, y, z, t, u) \leqslant 0$ or $Q(x, y, z, t, u) \geqslant m$. Because of homogeneity it suffices to prove:

THEOREM A. *Let* $Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + \varphi(y, z, t, u)$, *where* $\varphi(y, z, t, u)$ *is a real indefinite quaternary quadratic form of type* (3, 1) *and determinant* D *such that*

(3.3) $$D \leqslant -\frac{1}{8} \quad \left(D = -\frac{1}{8} \text{ if and only if } Q \sim Q_1\right)$$

and

(3.4) $$|h| < \frac{1}{2}, \quad |g| < \frac{1}{2}, \quad |g'| < \frac{1}{2}, \quad |h'| < \frac{1}{2}.$$

Suppose further that for integers x, y, z, t, u we have

(3.5) $$\text{either} \quad Q(x, y, z, t, u) \leq 0 \quad \text{or} \quad Q(x, y, z, t, u) \geq 1 - \varepsilon,$$

where ε (> 0) is sufficiently small. Let

(3.6) $$d = (8|D|)^{1/5}.$$

Then given any real numbers x_0, y_0, z_0, t_0, u_0 there exist $(x, y, z, t, u) \equiv (x_0, y_0, z_0, t_0, u_0) \pmod 1$ satisfying

(3.7) $$0 < Q(x, y, z, t, u) \leq d.$$

The sign of equality in (3.7) *is necessary if and only if $Q \sim Q_1$ or Q_2. For Q_1, equality occurs if and only if $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0) \pmod 1$ while for Q_2 it occurs if and only if $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$.*

3.1. Proof of Theorem A.

LEMMA 9. *If $Q(x, y, z, t, u)$ is as defined in Theorem* A, *then for integers y, z, t, u we have*

(3.8) $$\text{either} \quad \varphi(y, z, t, u) \leq 0 \quad \text{or} \quad \varphi(y, z, t, u) \geq \frac{3}{4} - \varepsilon.$$

This result and its proof is similar to Lemma 4.1 of Dumir (1969).

LEMMA 10. *If $Q = Q_1$, then* (3.7) *is true with strict inequality unless $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0) \pmod 1$, in which case equality is necessary.*

PROOF. Here $|D| = \frac{1}{8}$, so that $d = 1$.

Case (i). $(x_0, y_0) \not\equiv (0, 0) \pmod 1$. Suppose without loss of generality that $x_0 \not\equiv 0 \pmod 1$. Choose $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ arbitrarily, $x \equiv x_0 \pmod 1$ such that $0 < |x| \leq \frac{1}{2}$ and then choose $y \equiv y_0 \pmod 1$ to satisfy

$$0 < xy + z^2 + t^2 + u^2 + zt + tu + uz \leq |x| \leq \frac{1}{2} < d = 1.$$

Case (ii). $(x_0, y_0) \equiv (0, 0) \pmod 1$. First we deal with the case when $(z_0, t_0, u_0) \not\equiv (0, 0, 0) \pmod 1$. Without loss of generality we can suppose that $z_0 \not\equiv 0 \pmod 1$. Choose $z \equiv z_0 \pmod 1$ such that $0 < |z| \leq \frac{1}{2}$. Now choose $t \equiv$

$t_0 \pmod 1$ such that $0 \leqslant |t + z/3| \leqslant \frac{1}{2}$ and $u \equiv u_0 \pmod 1$ such that $0 \leqslant |u + t/2 + z/2| \leqslant \frac{1}{2}$. Take $x = y = 0$. So that

$$\begin{aligned}0 &< xy + z^2 + t^2 + u^2 + zt + tu + uz \\ &= xy + (z/2 + t/2 + u)^2 + 3(t + z/3)^2/4 + 2z^2/3 \\ &\leqslant \frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \frac{2}{3}\cdot\frac{1}{4} = \frac{29}{48} < 1.\end{aligned}$$

Now let $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0) \pmod 1$. Then equality is needed in (3.7) because $xy + z^2 + t^2 + u^2 + zt + tu + uz$ takes integral values only.

Since from (3.3), $d = (8|D|)^{1/5} \geqslant 1$ and $d = 1$ if and only if $Q \sim Q_1$, we can suppose that $d > 1$ in the rest of the paper.

LEMMA 11. *Let $\nu_1 = d - \frac{1}{4}$ and $\nu_2 > 0$ be a real number satisfying*

$$\nu_2 \begin{cases} \leqslant (d-1)^2/4 & \text{if } d \text{ is an integer}, \\ < [d]^2/4 & \text{if } d \text{ is not an integer}. \end{cases} \tag{3.9}$$

Suppose that we can find $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ such that

$$-\nu_2 < \varphi(y, z, t, u) \leqslant \nu_1 \tag{3.10}$$

then for any x_0, there exists $x \equiv x_0 \pmod 1$ satisfying (3.7). Further strict inequality in (3.10) implies strict inequality in (3.7).

PROOF. If $0 < \varphi(y, z, t, u) \leqslant \nu_1$, choose $x \equiv x_0 \pmod 1$ such that

$$|x + hy + gz + h't + g'u| \leqslant \frac{1}{2},$$

so that

$$0 < Q(x, y, z, t, u) \leqslant \nu_1 + \frac{1}{4} = d.$$

Strict inequality holds if we have strict inequality in (3.10). If $-\nu_2 < \varphi(y, z, t, u) \leqslant 0$, then the result follows from Lemma 7 with $\alpha = hy + gz + h't + g'u$ and $\beta^2 = -\varphi(y, z, t, u)$.

LEMMA 12. *If $d > 11$, then (3.7) is true with strict inequality.*

PROOF. By Lemma 4, there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ such that

$$0 < -\varphi(y, z, t, u) < (22|D|)^{1/4}$$

that is

$$-(11d^5/4)^{1/4} < \varphi(y, z, t, u) < 0.$$

Then the result will follow from Lemma 11, if we have

$$(11d^5/4)^{1/4} < \begin{cases} (d-1)^2/4 & \text{if } d \geqslant 12, \\ [d]^2/4 & \text{if } 11 < d < 12. \end{cases}$$

$f(d) = (11d^5/4)^{1/4}$ is an increasing function of d for $d > 1$. By Lemma 8, it is enough to verify the above inequality for $d \geqslant 12$. This verification is easy and we omit the proof.

LEMMA 13. *If* $4 < d \leqslant 11$, *then again* (3.7) *is true with strict inequality.*

PROOF. By Lemma 2, there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0)$ with

$$|\varphi(y, z, t, u)| \leqslant (|D|/3)^{1/4} = (d^5/24)^{1/4}.$$

The result will follow from Lemma 11, if we have

$$(d^5/24)^{1/4} < d - \frac{1}{4} \tag{3.11}$$

and

$$(d^5/24)^{1/4} < \begin{cases} (d-1)^2/4 & \text{if } 5 \leqslant d \leqslant 11, \\ [d]^2/4 & \text{if } 4 < d \leqslant 5. \end{cases} \tag{3.12}$$

We observe that by Lemma 8, it is enough to verify (3.12) for $5 \leqslant d \leqslant 11$. Verification of these inequalities is easy and is left to the reader.

REMARK. For $1 < d \leqslant 4$, we shall repeat the procedure of reduction described in Section 3. We shall use Lemma 3 on the homogeneous minimum of positive values of quaternary forms of type (3, 1). So we first dispose of the exceptional forms.

LEMMA 14. *If* $\varphi(y, z, t, u) \sim \rho\varphi_1$ *or* $\rho\varphi_2$, $1 < d \leqslant 4$, $\rho > 0$, *then again* (3.7) *is true with strict inequality.*

PROOF. *Case* (i). $\varphi \sim \rho\varphi_1$. It is enough to consider

$$\varphi = \rho\varphi_1 = \rho(y^2 + yz + z^2 + tu).$$

So that

$$Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + \rho(y^2 + yz + z^2 + tu).$$

If $g' \neq 0$, then by (3.4) we get $0 < Q(0, 0, 0, 0, 1) = g'^2 \leqslant \frac{1}{4} < 1 - \varepsilon$. This contradicts (3.5). Therefore $g' = 0$. Similarly $h' = 0$. Consideration of the values of Q at the points $(0, 0, 1, -1, 1)$ and $(0, 1, 0, -1, 1)$ gives $g = h = 0$. Therefore

$Q(x, y, z, t, u) = x^2 + \rho(y^2 + yz + z^2 + tu)$ and $|D| = 3\rho^4/16$. Here $\rho = (16|D|/3)^{1/4} = (2d^5/3)^{1/4} < 2d$ for $d \leqslant 4$.

Subcase (i). $(t_0, u_0) \not\equiv (0, 0) \pmod 1$. Without loss of generality we can suppose that $t_0 \not\equiv 0 \pmod 1$. Choose $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ arbitrarily, $t \equiv t_0 \pmod 1$ such that $0 < |t| \leqslant \frac{1}{2}$ and then choose $u \equiv u_0 \pmod 1$ satisfying

$$0 < x^2 + \rho(y^2 + yz + z^2 + tu) \leqslant \rho|t| \leqslant \rho/2 < d.$$

Subcase (ii). $(t_0, u_0) \equiv (0, 0) \pmod 1$. Take $t = u = 0$. Choose $x \equiv x_0 \pmod 1$ such that $|x| \leqslant \frac{1}{2}$, $z \equiv z_0 \pmod 1$ such that $|z| \leqslant \frac{1}{2}$ and $y \equiv y_0 \pmod 1$ such that $|y + z/2| \leqslant \frac{1}{2}$. So that

$$\begin{aligned} 0 &\leqslant x^2 + \rho(y^2 + yz + z^2 + tu) \\ &= x^2 + \rho(y + z/2)^2 + 3 \cdot \rho z^2/4 + \rho tu \leqslant 7\rho/16 + \frac{1}{4} < d. \end{aligned}$$

(It can be easily checked that $7\rho/4 < 4d - 1$, for $d \leqslant 4$.) Therefore (3.7) is satisfied with strict inequality unless $x = 0$, $y + z/2 = 0$, $z = 0$. In this case change x to 1, then (3.7) is satisfied with strict inequality.

Case (ii). $\varphi \sim \rho\varphi_2$, $\rho > 0$ is similar and is left to the reader.

3.2. Proof of Theorem A continued

From now on we can suppose that $1 < d \leqslant 4$ and $\varphi(y, z, t, u) \nsim \rho\varphi_1$ or $\rho\varphi_2$, $\rho > 0$. By Lemma 3, there exist integers y_2, z_2, t_2, u_2 with g.c.d. $(y_2, z_2, t_2, u_2) = 1$ such that

$$0 < a = \varphi(y_2, z_2, t_2, u_2) < (4|D|)^{1/4} = (d^5/2)^{1/4}. \tag{3.13}$$

Also from (3.8) we have $a \geqslant \frac{3}{4} - \varepsilon$. By a suitable unimodular transformation we can suppose that $\varphi(1, 0, 0, 0) = a$. So we can write

$$\varphi(y, z, t, u) = a\{(y + fz + f't + f''u)^2 + \psi(z, t, u)\},$$

where

$$\frac{3}{4} - \varepsilon \leqslant a < (d^5/2)^{1/4} \tag{3.14}$$

and $\psi(z, t, u)$ is a real indefinite ternary quadratic form of type (2, 1) and determinant D/a^4.

In view of Lemma 11, it is enough to prove that there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ such that

$$-\nu/a < (y + fz + f't + f''u)^2 + \psi(z, t, u) \leqslant (4d - 1)/4a, \tag{3.15}$$

where

$$(3.16)\qquad \nu = \begin{cases} \dfrac{9}{4} & \text{if } 3 < d \leqslant 4, \\ 1 & \text{if } 2 < d \leqslant 3, \\ \dfrac{1}{4} & \text{if } 1 < d \leqslant 2. \end{cases}$$

Let $\mu_1 = (4d - 1 - a)/4a$ and $\lambda = (4d - 1 + 4\nu)/4a$. Using (3.14) one can easily verify that $\mu_1 > 0$ and $\lambda > 1$.

LEMMA 15. *Let $\mu_2 > 0$ be a real number satisfying*

$$\mu_2 \leqslant \begin{cases} (\lambda - 1)^2/4 + \nu/a & \text{if } \lambda \text{ is an integer}, \\ [\lambda]^2/4 + \nu/a & \text{if } \lambda \text{ is not an integer}. \end{cases}$$

Suppose that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ *such that*

$$(3.17)\qquad -\mu_2 < \psi(z, t, u) \leqslant \mu_1.$$

Then we can find $y \equiv y_0 \pmod 1$ *satisfying* (3.15). *Further strict inequality in* (3.17) *implies strict inequality in* (3.15).

The proof is similar to that of Lemma 11, so we omit it.

LEMMA 16. *If $3 < d \leqslant 4$, then* (3.17) *and hence* (3.15) *holds with strict inequality.*

PROOF. In this case $\nu = \frac{9}{4}$, so that $\lambda = (d + 2)/a$.

By Lemma 6, we can find $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-(|D|/3a^4)^{1/3} < \psi(z, t, u) \leqslant \frac{8}{9}(|D|/3a^4)^{1/3} = \frac{8}{9}(d^5/24a^4)^{1/3}.$$

Then (3.17) will hold with strict inequality if we have

$$(3.18)\qquad \frac{8}{9}(d^5/24a^4)^{1/3} < (4d - 1 - a)/4a$$

and

$$(3.19)\qquad (d^5/24a^4)^{1/3} < \begin{cases} (\lambda - 1)^2/4 + \dfrac{9}{4}a & \text{if } \lambda \text{ is an integer}, \\ [\lambda]^2/4 + \dfrac{9}{4}a & \text{if } \lambda \text{ is not an integer}. \end{cases}$$

A simple calculation yields the inequality (3.18). So we proceed to verify (3.19). Let $n < \lambda = (d + 2)/a \leqslant n + 1$, $n = 1, 2, 3, \ldots$. Then (3.19) will be satisfied if we have

(3.20) $$d^5/24 < a\left(\frac{9}{4} + n^2a/4\right)^3 = g(a) \quad \text{(say)}.$$

Since $a \geqslant (d+2)/n + 1$, we have

$$g(a) \geqslant g((d+2)/(n+1)) = (d+2)(n+1)^{-4}4^{-3}\{9(n+1) + n^2(d+2)\}^3.$$

We shall have (3.20) if

$$h(d) = d^5 4^3 g((d+2)/(n+1)) > \frac{8}{3}.$$

For fixed n, $h(d)$ is a decreasing function of d and $d \leqslant 4$, therefore

$$h(d) \geqslant h(4) = 6 \cdot 27 \cdot 4^{-5}(n+1)^{-4}\{3(n+1) + 2n^2\}^3 \geqslant \frac{81}{16}$$

because $n \geqslant 1$. This proves (3.20) and hence (3.19).

LEMMA 17. *If* $2 < d \leqslant 3$, *then again* (3.17) *and hence* (3.15) *is satisfied with strict inequality.*

PROOF. In this case $\nu = 1$, so that $\lambda = (3+4d)/4a$. Let $n < (3+4d)/4a \leqslant n+1$, $n = 1, 2, \ldots$. In view of Lemma 15, it is enough to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

(3.21) $$-(n^2/4 + 1/a) < \psi(z, t, z) < (4d - 1 - a)/4a.$$

Case (I). $n \geqslant 2$. By Lemma 5, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$|\psi(z, t, u)| \leqslant (27|D|/100a^4)^{1/3} = (27d^5/800a^4)^{1/3}.$$

Then (3.21) will hold if we have

(3.22) $$(27d^5/800a^4)^{1/3} < (4d - 1 - a)/4a$$

and

(3.23) $$(27d^5/800a^4)^{1/3} < n^2/4 + 1/a.$$

We omit the straightforward verification of these inequalities.

Case (II). $n = 1$ that is $1 < (3+4d)/4a = \lambda \leqslant 2$. By Lemma 6, with $c = \frac{1}{2}$, we can find $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\frac{1}{2}(32d^5/429a^4)^{1/3} < \psi(z, t, u) \leqslant (32d^5/429a^4)^{1/3}.$$

Then (3.21) will hold if we have

(3.24) $$(32d^5/429a^4)^{1/3} < (4d - 1 - a)/4a$$

and

(3.25) $$(32d^5/429a^4)^{1/3} < 2(n^2/4 + 1/a) = (a + 4)/2a.$$

Since $(4d - 1 - a)/4a < (a + 4)/2a$ for $a \geq (3 + 4d)/8$ and $d \leq 4$, it is enough to verify (3.24), which can be easily done.

LEMMA 18. *If* $1 < d \leq 2$, *then again* (3.17) *and hence* (3.15) *is true.*

PROOF. In this case $\nu = \frac{1}{4}$, so that $\lambda = d/a$. Also from (3.13), $\lambda = d/a \leq 8/(3 - 4\varepsilon) < 3$, on taking ε sufficiently small. We distinguish two cases:

Case (i). $2 < \lambda < 3$. In this case $[\lambda]^2/4 + \nu/a = (1 + 4a)/4a$. So we have to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

(3.26) $$-(1 + 4a)/4a < \psi(z, t, u) \leq (4d - 1 - a)/4a.$$

By Lemma 6, with $c = \frac{1}{2}$, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\frac{1}{2}(32d^5/429a^4)^{1/3} < \psi(z, t, u) \leq (32d^5/429a^4)^{1/3}.$$

Then (3.26) will hold with strict inequality if

$$(32d^5/429a^4)^{1/3} < \min\left(\frac{4d - 1 - a}{4a}, \frac{1 + 4a}{2a}\right) = (4d - 1 - a)/4a.$$

This will be so if and only if

$$g(a) = a(d - (1 + a)/4)^3 > 32d^5/429.$$

$g(a)$ is an increasing function of a for $d/3 < a < d/2$, therefore

$$g(a) \geq g(d/3) = \frac{1}{3}d\{d - (1 + d/3)/4\}^3 > 32d^5/429$$

if $h(d) = (11d - 3)^3 d^{-4} > 12^3 \cdot 32/143$, which is true for $1 < d \leq 2$.

Case (ii). $1 < \lambda \leq 2$. In this case $[\lambda]^2/4 + \nu/a = (1 + a)/4a$. By Lemma 15, it is enough to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

(3.27) $$-(1 + a)/4a < \psi(z, t, u) \leq (4d - 1 - a)/4a.$$

By Lemma 6, with $c = \frac{1}{3}$, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\frac{1}{3}(27|D|/32a^4)^{1/3} < \psi(z, t, u) \leq (27|D|/32a^4)^{1/3}.$$

Then (3.27) will follow if we have

$$(27|D|/32a^4)^{1/3} = (27d^5/256a^4)^{1/3} \leq \min((4d - 1 - a)/4a, 3(1 + a)/4a).$$

Now $(4d - 1 - a)/4a \leq 3(1 + a)/4a$ if and only if $d \leq 1 + a$, which is true. (Strict inequality holds unless $d = 2$, $a = d/2 = 1$.) So it is enough to verify that

(3.28) $$(27d^5/256a^4)^{1/3} \leq (4d - 1 - a)/4a.$$

We shall have (3.28) if and only if

$$g(a) = a(d - (1 + a)/4)^3 > 27d^5/256. \tag{3.29}$$

$g(a)$ increases or decreases according as $a < d - \frac{1}{4}$ or $a > d - \frac{1}{4}$ and since $d/2 < d - \frac{1}{4}$, $d/2 \leqslant a < (\frac{1}{2}d^5)^{1/4}$, (3.29) will be true if

$$\min\left\{ g(d/2), g\left((d^5/2)^{1/4}\right)\right\} > 27d^5/256.$$

Now $g(d/2) = d(7d - 2)^3/1024 \geqslant 27d^5/256$ if $f(d) = (7d - 2)^3 d^{-4} \geqslant 108$. $f(d)$ increases or decreases according as $d < \frac{8}{7}$ or $d > \frac{8}{7}$. Therefore

$$f(d) \geqslant \min(f(1), f(2)) = f(2) = 108,$$

and strict inequality holds unless $d = 2$. The inequality $g((d^5/2)^{1/4}) > 27d^5/256$ can be easily verified.

Therefore (3.29) is satisfied with strict inequality unless $d = 2$, $a = d/2 = 1$. Hence (3.27) is true. Equality holds in (3.27) only if $d = 2$, $a = 1$, and ψ, z_0, t_0, u_0 are such that equality is needed in (2.6).

This completes the proof of Lemma 18.

4. The case of equality

LEMMA 19. *For $d > 1$, the sign of equality in* (3.7) *is necessary if and only if $Q \sim Q_2$. For Q_2, it is so if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$.

PROOF. Equality can hold in (3.7) only if it holds in (3.15). This happens only if $d = 2$, $a = 1$ and ψ, z_0, t_0, u_0 are such that equality is necessary in (2.6) (see Lemma 18). Thus we must have $\psi \sim \rho\psi_1 = \rho(z^2 + t^2 - 4u^2)$, $\rho > 0$ and $(z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$. Then $4\rho^3 = |D|/a^4 = d^5/8a^4 = 4$ so that $\rho = 1$. Therefore $\varphi(y, z, t, u) = (y + fz + f't + f''u)^2 + z^2 + t^2 - 4u^2$.

By a suitable unimodular transformation we can suppose that

$$|f| \leqslant \frac{1}{2}, \qquad |f'| \leqslant \frac{1}{2}, \qquad |f''| \leqslant \frac{1}{2}. \tag{4.1}$$

Again for equality to occur in (3.15), the following inequality

$$-\frac{1}{4} < F(y, z, t, u) = \left(y + y_0 + f\left(z + \frac{1}{2}\right) + f'\left(t + \frac{1}{2}\right) + f''\left(u + \frac{1}{2}\right)\right)^2 + \left(z + \frac{1}{2}\right)^2 + \left(t + \frac{1}{2}\right)^2 - 4\left(u + \frac{1}{2}\right)^2 < d - \frac{1}{4} = \frac{7}{4}$$

should not have any solution in integers y, z, t, and u. Now

$$-\frac{1}{4} < F(y, 0, 0, 0) = \left(y + y_0 + \frac{f}{2} + \frac{f'}{2} + \frac{f''}{2}\right)^2 + \frac{1}{4} + \frac{1}{4} - 1 < \frac{7}{4}$$

is solvable for integer y unless

$$y_0 + f/2 + f'/2 + f''/2 \equiv \frac{1}{2} \pmod 1. \tag{4.2}$$

Similarly considering $F(y, -1, 0, 0)$, $F(y, 0, -1, 0)$ and $F(y, 0, 0, -1)$ for equality to occur we must have

$$y_0 - f/2 + f'/2 + f''/2 \equiv \frac{1}{2} \pmod 1, \tag{4.3}$$

$$y_0 + f/2 - f'/2 + f''/2 \equiv \frac{1}{2} \pmod 1, \tag{4.4}$$

$$y_0 + f/2 + f'/2 - f''/2 \equiv \frac{1}{2} \pmod 1. \tag{4.5}$$

Subtracting (4.3), (4.4) and (4.5) from (4.2) we get

$$f \equiv f' \equiv f'' \equiv 0 \pmod 1.$$

Then from (4.1) we have

$$f = f' = f'' = 0, \qquad y_0 \equiv \frac{1}{2} \pmod 1.$$

Therefore $\varphi(y, z, t, u) = y^2 + z^2 + t^2 - 4u^2$, and $(y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (mod 1). Hence

$$Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + y^2 + z^2 + t^2 - 4u^2.$$

Again if equality is necessary in (3.7), the following inequality

$$0 < Q\left(x + x_0, y + \frac{1}{2}, z + \frac{1}{2}, t + \frac{1}{2}, u + \frac{1}{2}\right) < d = 2$$

should not have any solution in integers x, y, z, t, u. Proceeding as above, one can see that this is solvable unless

$$h \equiv g \equiv h' \equiv 0 \pmod 1.$$

Since $|h| < \frac{1}{2}$, $|g| < \frac{1}{2}$, $|h'| < \frac{1}{2}$, $|g'| < \frac{1}{2}$ from (3.4), we must have

$$h = g = h' = g' = 0 \quad \text{and} \quad x_0 \equiv \frac{1}{2} \pmod 1.$$

Hence $Q(x, y, z, t, u) = x^2 + y^2 + z^2 + t^2 - 4u^2$ and $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (mod 1). Considering congruences modulo 8, one can see that the sign of equality is necessary in this case.

The proof of Theorem A follows from Lemmas 10–19, and thus our theorem is proved.

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