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GODUNOV-TYPE METHODS FOR CONSERVATION LAWS WITH A FLUX FUNCTION DISCONTINUOUS IN SPACE*

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This paper is dedicated to the memory of Jacques-Louis Lions

Abstract. Scalar conservation laws with a flux function discontinuous in space are approximated using a Godunov-type method for which a convergence theorem is proved. The case where the flux functions at the interface intersect is emphasized. A very simple formula is given for the interface flux. A numerical comparison between the Godunov numerical flux and the upstream mobility flux is presented for two-phase flow in porous media. A consequence of the convergence theorem is an existence theorem for the solution of the scalar conservation laws under consideration. Furthermore, for regular solutions, uniqueness has been shown.

Key words. conservation laws, discontinuous coefficients, finite difference, finite volume, flow in porous media

AMS subject classifications. 35F25, 35L65, 65M06, 65M12, 76S05, 76M12, 76M20

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1. Introduction.

Let f and g be continuous functions on an interval $I \subset \mathbb{R}$, and define the flux function $F(x,u) = H(x)f(u) + (1-H(x))g(u)$, where $H(x)$ is the Heaviside function. Let $u_0 \in L^\infty(\mathbb{R}, I)$, and consider the following scalar conservation law:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}F(x,u) = 0 \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

This type of problem appears, for example, in modelling two-phase flow in a porous medium [8, 13], in sedimentation problems [7, 5], and in traffic flow [24].

It is well known that after a finite time (1.1) does not in general possess a continuous solution even if u_0 is sufficiently smooth. Hence by a solution of (1.1) we mean a solution in the weak sense. That is, $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ such that for all $\varphi \in C_0^\infty(\mathbb{R} \times \overline{\mathbb{R}}_+)$

$$\int_{-\infty}^{\infty}\int_0^\infty \left(u\frac{\partial \varphi}{\partial t} + F(x,u)\frac{\partial \varphi}{\partial x}\right) dtdx + \int_{-\infty}^{\infty} u_0(x)\varphi(x,0)dx = 0\,. \tag{1.2}$$

Denoting $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, then u satisfies (1.2) if and only if in the weak sense u satisfies

$$\begin{aligned} u_t + g(u)_x &= 0, \quad x < 0, \quad t > 0, \\ u_t + f(u)_x &= 0, \quad x > 0, \quad t > 0, \end{aligned} \tag{1.3}$$

and, at $x = 0$, u satisfies the Rankine–Hugoniot condition; namely, for almost all t,

$$f(u^+(t)) = g(u^-(t)), \tag{1.4}$$

where $u^+(t) = \lim_{x\to 0^+} u(x,t)$, $\quad u^-(t) = \lim_{x\to 0^-} u(x,t)\,.$

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Because of the discontinuity of the flux F at $x = 0$, the Kruzkov method [19] does not guarantee a weak solution of (1.1), and, even if the solution exists, it may not be unique.

When there is no discontinuity of F at $x = 0$, that is, when $f \equiv g$, the problem has been studied and well understood. In this case, existence of a weak solution was obtained by Kruzkov [19] in the class of functions satisfying the Lax–Oleinik entropy condition [21, 25]. The solution thus obtained can be represented by an L^1-contractive semigroup [18, 19]. The method adopted in this case is that of vanishing viscosity. Furthermore, finite difference schemes are constructed using a numerical flux based on exact or approximate Riemann solvers such as Lax–Friedrich, Godunov, Engquist–Osher, upstream mobility, etc.... Convergence of these schemes is based on the following properties: conservation, consistency, monotonicity, and Lipschitz continuity. Using these properties, one obtains that the finite difference schemes are TVD (total variation diminishing) and satisfy the maximum principle and a numerical entropy condition. This allows one to pass to a limit to obtain a unique weak solution satisfying the Lax–Oleinik entropy condition.

When $f \neq g$, this problem was considered from the theoretical or numerical point of view in several papers [3, 20, 8, 13, 6, 5, 15, 1, 28, 29]. In general, the solution to (1.2) is not unique. To choose a correct solution, in [8] it was suggested to choose a solution which has $|u^+(t) - u^-(t)|$ minimum, but the problem of uniqueness was left open in the case of a general Cauchy problem. Nevertheless, this led to the construction of a numerical flux which was actually the same as the one used in [3, 14]. It turns out that the solution to the Riemann problem and the flux function given in [8] and the numerical scheme given in [3, 14, 28, 29] are correct when assuming that the flux functions f and g are not intersecting, even though this was not stated explicitly. Actually, they may intersect but in such a way that no undercompressive waves are produced, which is not the case when $f' > 0$, $g' < 0$ at the intersection point. It should be noted that in [28, 29] at the intersection points derivatives of fluxes f and g have the same sign. At an intersection point, if the derivative of g is negative and that of f is positive, then the problem becomes more difficult. Later, in [6, 7, 5] the problem was studied in the general case with a source term, and it was suggested to choose a solution with a minimal variation in the x-direction. For this purpose a condition called the Γ-condition was introduced, an explicit formula was given for a solution to the Riemann problem, and uniqueness was proved. Diehl's construction allows undercompressive waves; hence it is not clear that the solution thus obtained can be represented by an L^1-contractive semigroup.

In [15] it was shown that the solution to the Riemann problem with the numerical flux built upon it in [3, 8, 13] was not correct when the flux functions f and g intersect in the undercompressive case, and a correct solution was given for this case. Independently, in [1], the authors asked themselves the following question: "What is an appropriate condition on $x = 0$ so that the solution can be represented by an L^1-contractive semigroup?" Assuming that f and g are strictly convex with superlinear growth, using the Hamilton–Jacobi theory, they constructed an explicit weak solution satisfying an explicit interface entropy condition at $x = 0$, different from the Lax–Oleinik entropy condition satisfied for $x \neq 0$. This interface entropy condition means that it does not allow the undercompressive waves. Furthermore, it was shown that this solution is unique by proving that the solution can be represented by an L^1-contractive semigroup. The solution to the Riemann problem thus obtained is actually the same as in [15], though written in a more compact form. This leads to a very simple way of calculating the interface flux and to derive its main properties: monotonicity and Lipschitz continuity. One should also note that the solution to the Riemann problem does not satisfy the maximum principle nor is it TVD.

Finally, we mention two papers which recently appeared and also investigate the problem of a nonlinear conservation law with a discontinuous flux function [17, 16].

In this paper, we consider the general case which includes the case where the flux functions are intersecting. Using the solution to the Riemann problem obtain in [15, 1] and the corresponding numerical flux, we study the resulting finite difference scheme and prove its convergence. In section 2 we present the continuous problem, defining in particular an interface entropy condition at $x = 0$, and we state an existence and uniqueness theorem for the solution to the continuous problem. In section 3 we present a Godunov-type method to calculate this solution, and in section 4, the core of this paper, we prove convergence of this numerical scheme. This scheme is conservative and monotone but not consistent in the usual sense. Due to the nonconsistency, it does not satisfy the maximum principle. In spite of this we show that the scheme is L^∞-bounded and L^1-stable. Furthermore, using the singular mapping technique introduced by Temple [27], we show that the scheme converges pointwise to weak solutions. These weak solutions satisfy the Lax–Oleinik entropy condition for $x \neq 0$ and the interface entropy condition at $x = 0$. This ensures the uniqueness of the limit solutions.

In section 5 we study the case of two-phase flow in porous media and introduce the alternative of the upstream mobility numerical flux [2]. One-dimensional numerical experiments are presented in section 6, and a comparison is made between these two numerical fluxes.

A consequence of the convergence theorem proved in section 4 is an existence theorem for the continuous problem for a larger class of functions f and g than the one studied in [1], where they were assumed to be convex. Uniqueness is shown in the appendix by proving that the solutions to the continuous problem form an L^1-contractive semigroup.

2. The continuous problem. Let $s < S$ denote the endpoints of the interval of the definition of f and g. In the following we will assume that f and g are smooth functions with the same endpoints and each one with one global minimum, reached at θ_f and θ_g, respectively, and with no other local minimum (see Figure 1).

Hypotheses. Assume that f, g are Lipschitz continuous functions on $[s, S]$ satisfying

(H_1) $f(s) = g(s), \quad f(S) = g(S),$

(H_2) f and g have one global minimum and no other local minimum in $[s, S]$.

Denote by Lip (f) and Lip (g) the Lipschitz constants of f and g. We will need also the constant

$$M = \max \{\text{Lip }(f), \text{Lip }(g)\}.$$

In order to state an existence and uniqueness theorem for the continuous problem we need to define regular solutions and entropy conditions. Since the flux function is not continuous, there are actually two different entropy conditions, one in the interior (which is the same as the usual Lax–Oleinik entropy condition) and the other at the interface which was introduced in [1].

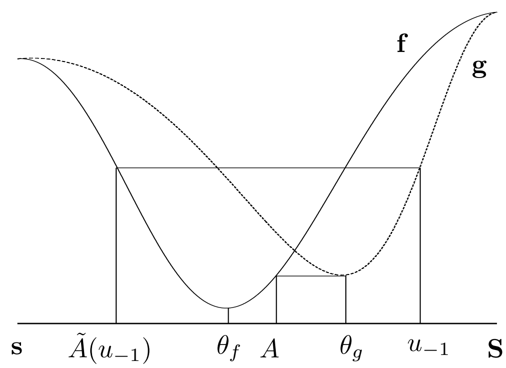

FIG. 1. *Flux functions f and g satisfying hypothesis* (H_2).

Entropy pairs. For $i = 1, 2, (\varphi_i, \psi_i)$ are said to be entropy pairs if φ_i is a convex function on $[s, S]$ and $(\psi_1'(\theta), \psi_2'(\theta)) = (\varphi_1'(\theta)f'(\theta), \varphi_2'(\theta)g'(\theta))$ for $\theta \in [s, S]$.

Let $u_0 \in L^\infty(\mathbb{R})$ be the initial data with $s \le u_0(x) \le S$ for all $x \in \mathbb{R}$, and let u be a weak solution of (1.2) with $s \le u(x,t) \le S$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Interior entropy condition. With u_0 and u as above, u is said to satisfy an interior entropy condition if for any entropy pairs $(\varphi_i, \psi_i), i = 1, 2,$ u satisfies in the sense of distributions

$$\begin{aligned} \frac{\partial \varphi_1(u)}{\partial t} + \frac{\partial \psi_1(u)}{\partial x} &\le 0 \text{ in } x > 0, \quad t > 0, \\ \frac{\partial \varphi_2(u)}{\partial t} + \frac{\partial \psi_2(u)}{\partial x} &\le 0 \text{ in } x < 0, \quad t > 0. \end{aligned} \tag{2.1}$$

Interface entropy condition. With u_0 and u as above, assume that $u^+(t) = \lim_{x\to 0^+} u(x,t)$ and $u^-(t) = \lim_{x \to 0^-} u(x,t)$ exist for almost all $t > 0$, and define

$$L = \left\{ t > 0;\ u^+(t) \in (\theta_f, S], u^-(t) \in [s, \theta_g) \right\},$$
$$U = \left\{ t \in L;\ u^+(t) = u^-(t) = S \right\} \cup \left\{ t \in L;\ u^-(t) = u^+(t) = s \right\}.$$

Then u is said to satisfy the interface entropy condition if

$$\text{meas } \{L \setminus U\} = 0. \tag{2.2}$$

This means that the characteristics must connect back to the x-axis on at least one side of the jump in F; i.e., undercompressive waves are not allowed.

Regular solution. u is said to be a regular solution of (1.2) if the discontinuities of u form a discrete set of Lipschitz curves.

We need also an estimator $N(f, g, u_0)$ of the total variation of the flux function evaluated at u_0. This estimator will be defined precisely below at (3.5).

We can now state our existence and uniqueness theorem for the continuous problem.

THEOREM 2.1. *Let $u_0 \in L^\infty(\mathbb{R})$ such that $s \le u_0(x) \le S$ for all $x \in \mathbb{R}$ and $N_h(f, g, u_0) < \infty$. Then there exists a weak solution $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ of* (1.2) *satisfying the following:*

(i) *For almost all $t > 0$ and $x \in \mathbb{R}$, $u(x+, t), u(x-, t)$ exist.*

(ii) *u satisfies the interior entropy condition* (2.1).

(iii) *If u is regular, then it satisfies also the interface entropy condition* (2.2) *and it is unique. Moreover, if $f = g$, then u is the unique entropy solution for the initial value problem studied in* [19].

An existence and uniqueness theorem was proved in [1] for convex functions f and g using arguments from the Hamilton–Jacobi theory. However, functions satisfying hypotheses (H_1) and (H_2) are not necessarily convex, as shown Figure 1, and a consequence of the convergence theorem, Theorem 3.2, proved below is that existence in Theorem 2.1 is valid for such functions. Uniqueness follows by showing that the solutions form an L^1-contractive family, which is done in the appendix.

We remark that a similar analysis to what is done in this paper for f and g satisfying hypothesis (H_2) can be done for the case where f and g satisfy hypothesis (H_3) instead:

(H_3) f and g have one global maximum and no other local maximum in $[s, S]$,

as shown in Figure 2. θ_f and θ_g would denote the points at which the maxima of f and g are reached. In the analysis below only the case where f and g satisfy (H_2) will be considered.

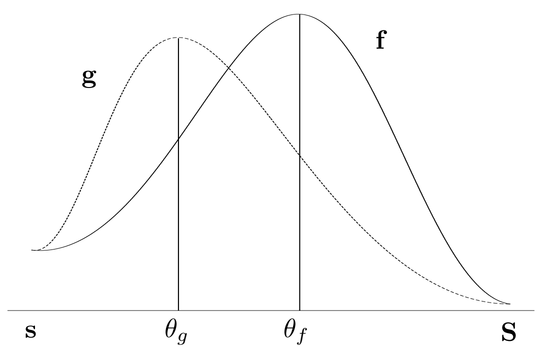

FIG. 2. *Flux functions f and g satisfying hypothesis* (H_3).

3. A Godunov-type finite volume method. Let F be the Godunov numerical flux with respect to f:

$$F(a,b) = \begin{cases} \min_{\theta\in[a,b]} f(\theta) & \text{if} \quad a < b, \\ \max_{\theta\in[b,a]} f(\theta) & \text{if} \quad a \geq b, \end{cases} \tag{3.1}$$

and similarly for the numerical flux G with respect to g.

Taking advantage of hypothesis (H_2), equivalent formulas can be used [1]:

$$\begin{aligned} F(a,b) &= \max\{F(a,S),\ F(s,b)\} = \max\{F(a,\theta_f),\ F(\theta_f,b)\} \\ &= \max\{f(\theta_f)(1-H(a_1)) + f(a)H(a_1), f(\theta_f)H(a_1) + f(b)(1-H(a_1))\} \\ &= \max\{f(\max\{a,\theta_f\}), f(\min\{\theta_f,b\})\}, \end{aligned}$$

where $a_1 = (a-\theta_f)$ and H is the Heaviside function. Note that the last two expressions are much simpler to use in calculations than formula (3.1).

In the case where f satisfies hypothesis (H_3) instead, the equivalent formulas are

$$\begin{aligned} F(a,b) &= \min\{F(a,s),\ F(S,b)\} = \min\{F(a,\theta_f),\ F(\theta_f,b)\} \\ &= \min\{f(\theta_f)H(a_1) + f(a)(1-H(a_1)), f(\theta_f)(1-H(a_1)) + f(b)H(a_1)\} \\ &= \min\{f(\min\{a,\theta_f\}), f(\max\{\theta_f,b\})\}. \end{aligned}$$

Interface flux $\overline{F}$. At the point $x = 0$ where the flux function changes we introduce the numerical flux $\overline{F}$ calculated by using the Riemann problem solution given in [1]:

$$\begin{aligned}\overline{F}(a,b) &= \max\{G(a,S),\ F(s,b)\} = \max\{G(a,\theta_g),\ F(\theta_f,b)\}\\ &= \max\{g(\theta_g)(1-H(a_1)) + g(a)H(a_1),\\ &\qquad\qquad f(\theta_f)H(b_1) + f(b)(1-H(b_1))\}\\ &= \max\{g(\max\{a,\theta_g\}), f(\min\{\theta_f,b\})\},\end{aligned} \tag{3.2}$$

where $a_1 = (a-\theta_g), b_1 = (b-\theta_f)$.

These four expressions of $\overline{F}$ are equivalent, but only the last two are useful for computational purposes. This flux $\overline{F}$ coincides with the one given in [15]. When f and g do not intersect this numerical flux reduces to the one given in [3, 9, 13, 6].

Remark 3.1. In the case where f and g satisfy hypothesis (H_3) the definition of the interface flux should be

$$\begin{aligned}\overline{F}(a,b) &= \min\{G(a,s),\ F(S,b)\} = \min\{G(a,\theta_g),\ F(\theta_f,b)\}\\ &= \min\{g(\theta_g)H(a_1) + g(a)(1-H(a_1)),\\ &\qquad\qquad f(\theta_f)(1-H(b_1)) + f(b)H(b_1)\}\\ &= \min\{g(\min\{a,\theta_g\}), f(\max\{\theta_f,b\})\},\end{aligned} \tag{3.3}$$

where θ_f and θ_g are now the maxima of f and g.

Let $h > 0$ and define the space grid points as follows:

$$x_{-1/2} = x_{1/2} = 0, \quad x_{j+1/2} = j\,h \quad \text{for } j \geq 0, \quad x_{j-1/2} = jh \quad \text{for } j \leq 0.$$

We will also use the midpoints of the intervals:

$$x_j = \left(\frac{2j-1}{2}\right)h \quad \text{for } j \geq 1, \quad x_j = \left(\frac{2j+1}{2}\right)h \quad \text{for } j \leq -1.$$

For time discretization the time step is $\Delta t > 0$, and let $t_n = n\Delta t,\ \lambda = \frac{\Delta t}{h}$.

For an initial data $u_0 \in L^\infty(\mathbb{R})$ we define

$$u^0_{j+1} = \frac{1}{h}\int_{x_{j+1/2}}^{x_{j+3/2}} u_0(x)dx \quad \text{if} \quad j \geq 0, \quad u^0_{j-1} = \frac{1}{h}\int_{x_{j-3/2}}^{x_{j-1/2}} u_0(x)dx \quad \text{if} \quad j \leq 0,$$

$$\begin{aligned}N_h(f,g,u_0) = &\sum_{i<-1} |G(u^0_i,u^0_{i+1}) - G(u^0_{i-1},u^0_i)| + \sum_{i>1} |F(u^0_i,u^0_{i+1}) - F(u^0_{i-1},u^0_i)|\\ &+|\overline{F}(u^0_{-1},u^0_1) - G(u^0_{-2},u^0_{-1})| + |F(u^0_1,u^0_2) - \overline{F}(u^0_{-1},u^0_1)|,\end{aligned} \tag{3.4}$$

$$N(f,g,u_0) = \sup_{h>0} N_h(f,g,u_0). \tag{3.5}$$

It is easy to see that if $u_0 \in BV(\mathbb{R})$, then $N(f,g,u_0) \leq C||u_0||_{BV}$, where C is a constant depending only on the Lipschitz constants of f and g.

Now we can define the explicit finite volume scheme $\{u^n_i\}$ inductively as follows:

$$\begin{aligned}u^{n+1}_1 &= u^n_1 - \lambda(F(u^n_1,u^n_2) - \overline{F}(u^n_{-1},u^n_1)),\\ u^{n+1}_i &= u^n_1 - \lambda(F(u^n_i,u^n_{i+1}) - F(u^n_{i-1},u^n_i)) \quad \text{if } i > 1,\\ u^{n+1}_{-1} &= u^n_{-1} - \lambda(\overline{F}(u^n_{-1},u^n_1) - G(u^n_{-2},u^n_{-1})),\\ u^{n+1}_i &= u^n_i - \lambda(G(u^n_i,u^n_{i+1}) - G(u^n_{i-1},u^n_i)) \quad \text{if} \quad i < -1.\end{aligned} \tag{3.6}$$

Observe that this is, a Godunov scheme for $i \neq \pm 1$, that is, away from $x = 0$, and that for $i = \pm 1$ the scheme is not consistent; that is, in general $\overline{F}(u, u)$ need not be equal to $f(u)$ or $g(u)$. Because of this, the maximum principle does not hold.

For $u_0 \in L^\infty(\mathbb{R})$ and grid length h and Δt with $\lambda = \frac{\Delta t}{h}$ fixed, define the function $u_h \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ associated with $\{u_i^n\}$ calculated by the scheme (3.6):

$$u_h(x,t) = u_i^n \quad \text{for } (x,t) \in [x_{i-1/2}, x_{i+1/2}) \times [n\Delta t,\ (n+1)\Delta t), \quad i \neq 0. \tag{3.7}$$

Now we can state the following convergence theorem.

THEOREM 3.2. *Assume that λ, M satisfies the CFL condition $\lambda M \leq 1$. Let $u_0 \in L^\infty(\mathbb{R})$ such that $s \leq u_0(x) \leq S$ for all $x \in \mathbb{R}$ and $N(f, g, u_0) < \infty$. For $h > 0$, let $\lambda = \frac{\Delta t}{h}$ and u_h be the corresponding calculated solution given by* (3.6), (3.7). *Then there exists a subsequence $h_k \to 0$ such that u_{h_k} converges a.e. to a weak solution u of* (1.2) *satisfying interior entropy condition* (2.1). *Suppose the discontinuities of every limit function u of $\{u_h\}$ is a discrete set of Lipschitz curves; then $u_h \to u$ in $L^\infty_{loc}(\mathbb{R}_+, L^1_{loc}(\mathbb{R}))$ as $h \to 0$, and u satisfies the interface entropy condition* (2.2).

The proof of this theorem is the object of the next section.

Remark 3.3. The CFL condition still reads $\lambda M \leq 1$ in the case of a discontinuous flux function.

4. Proof of the convergence theorem, Theorem 3.2.

4.1. Properties of the numerical flux. Before going into the details of the proof, we need to study the properties of the numerical flux F, G, and $\overline{F}$.

From definitions (3.1), (3.2), $F, G, \overline{F}$, are nondecreasing functions in the first variable and nonincreasing functions in the second variable. Furthermore, the functions F, G, and $\overline{F}$ satisfy for any $a, a_1, a_2, b, b_1, b_2 \in [s, S]$

$$\begin{array}{rcl} (|F(a_1,b) - F(a_2,b)|,\ |F(a,b_1) - F(a,b_2)|) & \leq & M(|a_1 - a_2|,\ |b_1 - b_2|), \\ (|G(a_1,b) - G(a_2,b)|,\ |G(a,b_1) - G(a,b_2)|) & \leq & M(|a_1 - a_2|,\ |b_1 - b_2|), \\ (|\overline{F}(a_1,b) - \overline{F}(a_2,b)|,\ |\overline{F}(a,b_1) - \overline{F}(a,b_2)|) & \leq & M(|a_1 - a_2|,\ |b_1 - b_2|). \end{array} \tag{4.1}$$

The following lemma is easy to prove.

LEMMA 4.1. *Let f and g satisfy* (H_1) *and* (H_2). *Then $\overline{F}$ satisfies*

$$\begin{array}{rcl} \overline{F}(s,s) & = & f(s) = g(s), \quad \overline{F}(S,S) = f(S) = g(S), \\ \overline{F}(a,b) & = & F(a,b) \quad \text{if} \quad f \equiv g\,. \end{array}$$

Now we define for $X, Y, Z \in [s, S]$

$$\begin{array}{rcl} H_{-2}(X,Y,Z) & = & Y - \lambda(\overline{F}(Y,Z) - G(X,Y)), \\ H_{-1}(X,Y,Z) & = & Y - \lambda(G(Y,Z) - G(X,Y)), \\ H_1(X,Y,Z) & = & Y - \lambda(F(Y,Z) - F(X,Y)), \\ H_2(X,Y,Z) & = & Y - \lambda(F(Y,Z) - \overline{F}(X,Y)). \end{array}$$

Then we have the following lemma.

LEMMA 4.2. *Let $\lambda M \leq 1$ and $a \in [s, S]$; then we have the following:*

(i) $H_{\pm_1}(a,a,a) = a$, *and* $H_{\pm_2}(s,s,s) = s, H_{\pm_2}(S,S,S) = S$.

(ii) *H_i is nondecreasing in each of its variables.*

(iii) *Let* $\{T_i\}_{i\in z\setminus\{0\}}$ *be a sequence in* $[s,S]$*, and define* $P_i = (T_{i-1}, T_i, T_{i+1})$ *if* $|i| \geq 2$*,* $P_1 = (T_{-1}, T_1, T_2)$*, and* $P_{-1} = (T_{-2}, T_{-1}, T_1)$*. Then for* $|i| \geq 3$*,*

$$\begin{array}{lllllll}
\frac{\partial H_1}{\partial X}(P_{i+1}) & + & \frac{\partial H_1}{\partial Y}(P_i) & + & \frac{\partial H_1}{\partial Z}(P_{i-1}) & = & 1,\\
\frac{\partial H_{-1}}{\partial X}(P_{i+1}) & + & \frac{\partial H_{-1}}{\partial Y}(P_i) & + & \frac{\partial H_{-1}}{\partial Z}(P_{i-1}) & = & 1,\\
\frac{\partial H_1}{\partial X}(P_3) & + & \frac{\partial H_1}{\partial Y}(P_2) & + & \frac{\partial H_2}{\partial Z}(P_1) & = & 1,\\
\frac{\partial H_{-2}}{\partial X}(P_{-1}) & + & \frac{\partial H_{-1}}{\partial Y}(P_{-2}) & + & \frac{\partial H_{-1}}{\partial Z}(P_{-3}) & = & 1,\\
\frac{\partial H_2}{\partial X}(P_1) & + & \frac{\partial H_{-2}}{\partial Y}(P_{-1}) & + & \frac{\partial H_{-1}}{\partial Z}(P_{-2}) & = & 1,\\
\frac{\partial H_1}{\partial X}(P_2) & + & \frac{\partial H_2}{\partial Y}(P_1) & + & \frac{\partial H_{-2}}{\partial Z}(P_{-1}) & = & 1.
\end{array}$$

Proof. From Lemma 4.1, $\overline{F}(s,s) = f(s) = g(s)$, $\overline{F}(S,S) = f(S) = g(S)$, and for all $a \in [s,S]$, $F(a,a) = f(a), G(a,a) = g(a)$. Hence $H_{\pm1}(a,a,a) = a, H_{\pm2}(s,s,s) = s, H_{\pm2}(S,S,S) = S$. This proves (i). By symmetry it is enough to prove (ii) for H_2. Let (X,Y,Z), $X_1 \leq X_2, Y_1 \leq Y_2, Z_1 \leq Z_2$, be given. Then

$$H_2(X_1,Y,Z) - H_2(X_2,Y,Z) = \lambda(\overline{F}(X_1,Y) - \overline{F}(X_2,Y)) \leq 0.$$

Without loss of generality we can assume that $g(\theta_g) = \min g \geq \min f = f(\theta_f)$. For $X \geq \theta_g, Z \leq \theta_f$, define $\tilde{A}(X) \leq \theta_f$ and $\tilde{B}(Z) \geq \theta_f$ by $f(\tilde{A}(X)) = g(X)$ and $f(Z) = f(\tilde{B}(Z))$. Then we have by direct calculations

$$I_1 = F(Y_1,Z) - F(Y_2,Z) = \begin{cases} f(\tilde{B}(Z)) - f(Y_2) & \text{if } Z \leq \theta_f, \quad Y_1 \leq \tilde{B}(Z) \leq Y_2,\\ f(Y_1) - f(Y_2) & \text{if } Z \leq \theta_f, \quad Y_1 \geq \tilde{B}(Z)\\ & \text{or } Z \geq \theta_f, \quad Y_1 \geq \theta_f,\\ f(\theta_f) - f(Y_2) & \text{if } Z \geq \theta_f, \quad Y_1 \leq \theta_f \leq Y_2,\\ 0 & \text{otherwise,} \end{cases}$$

$$I_2 = \overline{F}(X,Y_1) - \overline{F}(X,Y_2) = \begin{cases} f(Y_1) - f(\min(Y_2,\tilde{A}(X))) & \text{if } X \geq \theta_g, \quad Y_1 \leq \tilde{A}(X),\\ f(Y_1) - f(\min(Y_2,\tilde{A}(\theta_g))) & \text{if } X \leq \theta_g, \quad Y_1 \leq \tilde{A}(\theta_g),\\ 0 & \text{otherwise.} \end{cases}$$

Let $I = -\lambda(I_1 - I_2)$. Then from the above calculation, $I = -\lambda I_1$ if $Y_1 \geq \theta_f$ and $I = \lambda I_2$ if $Y_2 \leq \theta_f$. In either case we have $|I| \leq \lambda M|Y_1 - Y_2|$. Now suppose that $Y_1 \leq \theta_f \leq Y_2$; then we have

$$|I| \leq (|I_1| + |I_2|) \leq \lambda M(|Y_2 - \theta_f| + |Y_1 - \tilde{A}(\theta_g)|) = \lambda M|Y_1 - Y_2|.$$

Hence, since F and $\overline{F}$ are nondecreasing in the first variable and nonincreasing in the second variable, we obtain

$$\begin{array}{rcl}
H_2(X,Y_1,Z) - H_2(X,Y_2,Z) & = & Y_1 - Y_2 - \lambda(F(Y_1,Z) - F(Y_2,Z))\\
& & +\lambda(\overline{F}(X,Y_1) - \overline{F}(X,Y_2))\\
& = & Y_1 - Y_2 - \lambda(I_1 - I_2) \leq Y_1 - Y_2 + \lambda M|Y_1 - Y_2|\\
& \leq & (1-\lambda M)(Y_1 - Y_2) \leq 0,\\
H_2(X,Y,Z_1) - H_2(X,Y,Z_2) & = & -\lambda(F(Y,Z_1) - F(Y,Z_2)) \ \leq \ 0.
\end{array}$$

This proves (ii).

Let $i \geq 3$; then

$$\frac{\partial H_1}{\partial X}(P_{i+1}) + \frac{\partial H_1}{\partial Y}(P_i) + \frac{\partial H_1}{\partial Z}(P_{i-1})$$
$$= \lambda\frac{\partial F}{\partial a}(T_i, T_{i+1}) + 1 - \lambda\left(\frac{\partial F}{\partial a}(T_i, T_{i+1}) - \frac{\partial F}{\partial b}(T_{i-1}, T_i)\right) - \lambda\frac{\partial F}{\partial b}(T_{i-1}, T_i) = 1.$$

This proves the first equality in (iii). The proof is similar for the second equality in (iii). For the third, fourth, fifth, and sixth equalities we have

$$\begin{aligned}
\frac{\partial H_1}{\partial X}(P_3) + \frac{\partial H_1}{\partial Y}(P_2) + \frac{\partial H_2}{\partial Z}(P_1) &= \lambda\frac{\partial F}{\partial a}(T_2, T_3) + 1 \\
&\quad -\lambda\left(\frac{\partial F}{\partial a}(T_2, T_3) - \frac{\partial F}{\partial b}(T_1, T_2)\right) - \lambda\frac{\partial F}{\partial b}(T_1, T_2) \;=\; 1, \\
\frac{\partial H_{-2}}{\partial X}(P_{-1}) + \frac{\partial H_{-1}}{\partial Y}(P_{-2}) + \frac{\partial H_{-1}}{\partial Z}(P_3) &= \lambda\frac{\partial G}{\partial a}(T_{-2}, T_{-1}) + 1 \\
&\quad -\lambda\left(\frac{\partial G}{\partial a}(T_{-2}, T_{-1}) - \frac{\partial G}{\partial b}(T_{-3}, T_{-2})\right) - \lambda\frac{\partial G}{\partial b}(T_{-3}, T_{-2}) \;=\; 1, \\
\frac{\partial H_2}{\partial X}(P_1) + \frac{\partial H_{-2}}{\partial Y}(P_{-1}) + \frac{\partial H_{-1}}{\partial Z}(P_{-2}) &= \lambda\frac{\partial \overline{F}}{\partial a}(T_{-1}, T_1) + 1 \\
&\quad -\lambda\left(\frac{\partial \overline{F}}{\partial a}(T_{-1}, T_1) - \frac{\partial G}{\partial b}(T_{-2}, T_{-1})\right) - \lambda\frac{\partial G}{\partial b}(T_{-2}, T_{-1}) \;=\; 1, \\
\frac{\partial H_1}{\partial X}(P_2) + \frac{\partial H_2}{\partial Y}(P_1) + \frac{\partial H_{-2}}{\partial Z}(P_{-1}) &= \lambda\frac{\partial F}{\partial a}(T_1, T_2) + 1 \\
&\quad -\lambda\left(\frac{\partial F}{\partial a}(T_1, T_2) - \frac{\partial \overline{F}}{\partial b}(T_{-1}, T_1)\right) - \lambda\frac{\partial \overline{F}}{\partial b}(T_{-1}, T_1) \;=\; 1.
\end{aligned}$$

This completes the proof of Lemma 4.2.

4.2. L^∞ and TV bounds. The next lemmas show that the scheme (3.6) is L^1-contractive and the idea of the proof is taken from [11].

LEMMA 4.3. *Let $u_0 \in L^\infty(\mathbb{R}, [s, S])$ be the initial data, and let $\{u_i^n\}$ be the corresponding solution calculated by the finite volume scheme* (3.6). *When $\lambda M \leq 1$, then*

$$s \leq u_i^n \leq S \quad \forall i, n. \tag{4.2}$$

Proof. Since $s \leq u_0 \leq S$, hence for all i, $s \leq u_i^0 \leq S$. By induction, assume that (4.2) holds for n. Then from (i) and (ii) of Lemma 4.2 we have

$$\begin{aligned}
s &= H_{-1}(s, s, s) \leq H_{-1}(u_{i-1}^n, u_i^n, u_{i+1}^n) = u_1^{n+1} \leq H_{-1}(S, S, S) = S \text{ if } i \leq -2, \\
s &= H_1(s, s, s) \leq H_1(u_{i-1}^n, u_i^n, u_{i+1}^n) = u_i^{n+1} \leq H_1(S, S, S) = S \quad \text{if } i \geq 2, \\
s &= H_{-2}(s, s, s) \leq H_{-2}(u_{-2}^n, u_{-1}^n, u_1^n) = u_{-1}^{n+1} \leq H_{-2}(S, S, S) = S, \\
s &= H_2(s, s, s) \leq H_2(u_{-1}^n, u_1^n, u_2^n) = u_1^{n+1} \leq H_2(S, S, S) = S.
\end{aligned}$$

This proves (4.2).

LEMMA 4.4. *Let $u_0, v_0 \in L^\infty(\mathbb{R},\ [s, S])$ be initial datas, and let $\{u_i^n\}$ and $\{v_i^n\}$ be the corresponding solutions calculated by the finite volume scheme* (3.6). *Let $\lambda M \leq 1$*

and $i_0 \le j_0$*; then*

$$\sum_{\substack{i_0 \le i \le j_0 \\ i \ne 0}} |u_i^{n+1} - v_i^{n+1}| \le \sum_{\substack{i_0 - 1 \le i \le j_0 + 1 \\ i \ne 0}} |u_i^n - v_i^n|,$$
$$\sum_{i \ne 0} |u_i^{n+1} - u_i^n| \le \sum_{i \ne 0} |u_i^n - u_i^{n-1}|.$$

Lemma 4.4 is a localized version of the Crandall–Tartar lemma [4], which we will prove along the lines of [11].

Proof. The first inequality in Lemma 4.4 will be proved for $i_0 \le -1$ and $j_0 \ge 1$. The other cases follow in the same manner. For $\theta \in [0,1]$, let $p_i^n(\theta) = \theta u_i^n + (1-\theta) v_i^n$ and

$$P_i^n(\theta) = \begin{cases} (p_{i-1}^n(\theta),\ p_i^n(\theta), p_{i+1}^n(\theta)) & \text{if} \quad |i| \ge 2, \\ (p_{-2}^n(\theta),\ p_{-1}^n(\theta), p_1^n(\theta)) & \text{if} \quad i = -1, \\ (p_{-1}^n(\theta),\ p_1^n(\theta),\ p_2^n(\theta)) & \text{if} \quad i = 1. \end{cases}$$

From Lemma 4.3 we have $p_i^n(\theta) \in [s, S]$ for all i, n, and θ. From their definitions, the H_i's are uniformly continuous functions, and from (ii) in Lemma 4.2 a.e. (X, Y, Z), $\frac{\partial H_i}{\partial X} \ge 0$, $\frac{\partial H_i}{\partial Y} \ge 0$, $\frac{\partial H_i}{\partial Z} \ge 0$. Hence from the mean value theorem

$$\begin{aligned}
\sum_{i_0}^{-2} |u_i^{n+1} - v_i^{n+1}| &= \sum_{i_0}^{-2} |H_{-1}(u_{i-1}^n, u_i^n, u_{i+1}^n) - H_{-1}(v_{i-1}^n, v_i^n, v_{i+1}^n)| \\
&\le \sum_{i_0}^{-2} |u_{i-1}^n - v_{i-1}^n| \int_0^1 \frac{\partial H_{-1}}{\partial X}(P_i^n(\theta)) d\theta \\
&+ \sum_{i_0}^{-2} |u_i^n - v_i^n| \int_0^1 \frac{\partial H_{-1}}{\partial Y}(P_i^n(\theta)) d\theta + |u_{i+1}^n - v_{i+1}^n| \int_0^1 \frac{\partial H_{-1}}{\partial Z}(P_i^n(\theta)) d\theta \\
&= |u_{i_0-1}^n - v_{i_0-1}^n| \int_0^1 \frac{\partial H_{-1}}{\partial X}(P_{i_0}^n(\theta)) d\theta \\
&+ \sum_{i_0}^{-3} |u_i^n - v_i^n| \int_0^1 \left(\frac{\partial H_{-1}}{\partial X}(P_{i+1}^n(\theta)) + \frac{\partial H_{-1}}{\partial Y}(P_i^n(\theta)) + \frac{\partial H_{-1}}{\partial Z}(P_{i-1}^n(\theta)) \right) d\theta \\
&+ |u_{-2}^n - v_{-2}^n| \int_0^1 \left(\frac{\partial H_{-1}}{\partial Y}(P_{-2}^n(\theta)) + \frac{\partial H_{-1}}{\partial Z}(P_{-3}^n(\theta)) \right) d\theta \\
&+ |u_{-1}^n - v_{-1}^n| \int_0^1 \frac{\partial H_{-1}}{\partial Z}(P_{-2}^n(\theta)) d\theta.
\end{aligned}$$

Now $\frac{\partial H_{-1}}{\partial X}(X, Y, Z) = \lambda \frac{\partial G}{\partial a}(X, Y) \le \lambda M \le 1$, and from the second equality of (iii) in Lemma 4.2 we obtain

$$\begin{aligned}
\sum_{i_0}^{-2} |u_i^{n+1} - v_i^{n+1}| \le \sum_{i_0 - 1}^{-3} |u_i^n - v_i^n| &+ |u_{-1}^n - v_{-1}^n| \int_0^1 \frac{\partial H_{-1}}{\partial Z}(P_{-2}^n(\theta)) d\theta \\
&+ |u_{-2}^n - v_{-2}^n| \int_0^1 \left(\frac{\partial H_{-1}}{\partial Y}(P_{-2}^n(\theta)) + \frac{\partial H_{-1}}{\partial Z}(P_{-3}^n(\theta)) \right) d\theta.
\end{aligned}$$

Since $\frac{\partial H_1}{\partial Z} = -\lambda \frac{\partial F}{\partial b} \le \lambda M \le 1$, the following inequalities result from the first equality of (iii) in Lemma 4.2:

$$\sum_{2}^{j_0}|u_i^{n+1}-v_i^{n+1}| \le \sum_{3}^{j_0+1}|u_i^n-v_i^n|+|u_1^n-v_1^n|\int_0^1 \frac{\partial H_1}{\partial X}(P_2^n(\theta))d\theta + |u_2^n-v_2^n|\int_0^1\left(\frac{\partial H_1}{\partial Y}(P_2^n(\theta))+\frac{\partial H_1}{\partial X}(P_3^n(\theta))\right)d\theta.$$

Moreover,

$$\begin{aligned}
&|u_{-1}^{n+1}-v_{-1}^{n+1}|+|u_1^{n+1}-v_1^{n+1}|\\
&=|H_{-2}(u_{-2}^n,u_{-1}^n,u_1^n)-H_{-2}(v_{-2}^n,v_{-1}^n,v_1^n)|+|H_2(u_{-1}^n,u_1^n,u_2^n)-H_2(v_{-1}^n,v_1^n,v_2^n)|\\
&\le |u_{-2}^n-v_{-2}^n|\int_0^1 \frac{\partial H_{-2}}{\partial X}(P_{-1}^n(\theta))d\theta+|u_{-1}^n-v_{-1}^n|\int_0^1\left(\frac{\partial H_2}{\partial X}(P_1^n(\theta))+\frac{\partial H_{-2}}{\partial Y}(P_{-1}^n(\theta))\right)d\theta\\
&\quad+|u_1^n-v_1^n|\int_0^1\left(\frac{\partial H_2}{\partial Y}(P_1^n(\theta))+\frac{\partial H_{-2}}{\partial Z}(P_{-1}^n(\theta))\right)d\theta+|u_2^n-v_2^n|\int_0^1\frac{\partial H_2}{\partial Z}(P_1^n(\theta))d\theta.
\end{aligned}$$

Summing up all the above three inequalities and from the last four equalities of (iii) in Lemma 4.2 we obtain

$$\begin{aligned}
\sum_{\substack{i_0\le i\le j_0\\ i\ne 0}}|u_i^{n+1}-v_i^{n+1}| &\le \sum_{i_0-1}^{-3}|u_i^n-v_i^n|+\sum_{3}^{j_0+1}|u_i^n-v_i^n|\\
&\quad+|u_{-2}^n-v_{-2}^n|\int_0^1\left(\frac{\partial H_{-2}}{\partial X}(P_{-1}^n(\theta))+\frac{\partial H_{-1}}{\partial Y}(P_{-2}^n(\theta))+\frac{\partial H_{-1}}{\partial Z}(P_{-3}^n(\theta))\right)d\theta\\
&\quad+|u_{-1}^n-v_{-1}^n|\int_0^1\left(\frac{\partial H_2}{\partial X}(P_1^n(\theta))+\frac{\partial H_{-2}}{\partial Y}(P_{-1}^n(\theta))+\frac{\partial H_{-1}}{\partial Z}(P_{-2}^n(\theta))\right)d\theta\\
&\quad+|u_1^n-v_1^n|\int_0^1\left(\frac{\partial H_1}{\partial X}(P_2^n(\theta))+\frac{\partial H_2}{\partial Y}(P_1^n(\theta))+\frac{\partial H_{-2}}{\partial Z}(P_{-2}^n(\theta))\right)d\theta\\
&\quad+|u_2^n-v_2^n|\int_0^1\left(\frac{\partial H_1}{\partial X}(P_3^n(\theta))+\frac{\partial H_1}{\partial Y}(P_2^n(\theta))+\frac{\partial H_2}{\partial Z}(P_1^n(\theta))\right)d\theta\\
&=\sum_{\substack{i_0-1\le i\le j_0+1\\ i\ne 0}}|u_i^n-v_i^n|\,.
\end{aligned}$$

Take the special choice of v_0 by $v_0(x)=u_i^1$ in $[x_{i-1/2},x_{i+1/2})$. Then it follows easily that $v_i^n=u_i^{n+1}$. Now substituting this in the first inequality of the lemma and taking $i_0=-\infty$, $j_0=\infty$ we obtain the second inequality. This completes the proof of Lemma 4.4.

Next we use the singular mapping technique introduced in [27, 23, 22, 28] to obtain TV bounds for the transformed scheme, and this allows us to pass to the limit as $h\to 0$.

Let $k:[s,S]\to\mathbb{R}$ be a Lipschitz continuous function satisfying (H_2), and let K be the corresponding numerical flux as in (3.1). Let θ_k denote the unique minima of k. For $A\in[s,S], a,b\in\mathbb{R}$, $\{u_{j-1},u_j,u_{j+1},u_{j+2}\}\subset[s,S]$, define

$$\begin{aligned}
\psi_{k,A}(u) &= \int_A^u |k'(\theta)|d\theta, & \chi_-(k'(u)) &= \begin{cases}0 & \text{if } u\in(\theta_k,S],\\ 1 & \text{if } u\in[s,\theta_k],\end{cases}\\
\chi(a,b) &= \begin{cases}1 & \text{if } a\le b,\\ 0 & \text{if } a>b,\end{cases} & \chi_+(k'(u)) &= \begin{cases}1 & \text{if } u\in[\theta_k,S],\\ 0 & \text{if } u\in[s,\theta_k),\end{cases}
\end{aligned}$$

and $H_{i+1/2}=K(u_i,u_{i+1})$ for $j-1\le i\le j+1$.

Apart from χ_-, χ_+ we will use the standard notation

$$a_+ = \max(a,0), \quad a_- = \min(a,0), \quad a = a_+ + a_-, \quad |a| = a_+ - a_-.$$

LEMMA 4.5. *With the above notation we have the following inequalities:*

$$-\chi(u_j,u_{j+1})\int_{u_j}^{u_{j+1}} k'_-(\theta)d\theta \;\le\; \chi_-(k'(u_j))|H_{j+1/2}-H_{j-1/2}|, \tag{4.3}$$

$$\chi(u_j,u_{j+1})\int_{u_j}^{u_{j+1}} k'_+(\theta)d\theta \le \chi_+(k'(u_{j+1}))|H_{j+3/2}-H_{j+1/2}| \tag{4.4}$$

$$-(\psi_{k,A}(u_j)-\psi_{k,A}(u_{j+1}))_- = \chi(u_j,u_{j+1})\left\{\int_{u_j}^{u_{j+1}} k'_+(\theta)d\theta - \int_{u_j}^{u_{j+1}} k'_-(\theta)d\theta\right\}$$

$$\le \chi_-(k'(u_j))|H_{j+1/2}-H_{j-1/2}| + \chi_+(k'(u_{j+1}))|H_{j+3/2}-H_{j+1/2}|. \tag{4.5}$$

The proof of this lemma can be found in [28, Lemma 3.3], just replacing the requirement of a single maximum by a single minimum.

Singular mappings. Let f, g satisfy the hypotheses (H_1) and (H_2). Let θ_f, θ_g be the respective minima for f and g. Define the singular mappings ψ_1, ψ_2 associated with f and g as follows.

Case 1. $f(\theta_f) \le g(\theta_g)$. Choose $A \ge \theta_f$ such that $f(A) = g(\theta_g)$ and for $u \in [s,S]$

$$\psi_1(u) = \psi_{g,\theta_g}(u) = \int_{\theta_g}^{u} |g'(\theta)|d\theta, \quad \psi_2(u) = \psi_{f,A}(u) = \int_A^u |f'(\theta)|d\theta.$$

Case 2. $f(\theta_f) \ge g(\theta_g)$. Choose $A \le \theta_g$ such that $f(\theta_f) = g(A)$ and for $u \in [s,S]$

$$\psi_1(u) = \psi_{g,A}(u) = \int_A^u |g'(\theta)|d\theta, \quad \psi_2(u) = \psi_{f,\theta_f}(u) = \int_{\theta_f}^u |f'(\theta)|d\theta.$$

In order to obtain TV bounds for the transformed sequence under the singular mappings, we have to estimate the error term E defined as below. This error estimate will be carried out in the next two lemmas.

For $\{u_{-2}, u_{-1}, u_1, u_2\} \subset [s,S]$, define $z_1 = \psi_2(u_1)$, $z_{-1} = \psi_1(u_{-1})$, $H_{3/2} = F(u_1,u_2)$, $H_{1/2} = H_{-1/2} = \overline{F}(u_{-1},u_1)$, $H_{-3/2} = G(u_{-2},u_{-1})$, and

$$E = -(z_{-1}-z_1)_- - \chi(u_1,u_2)\int_{u_1}^{u_2} f'_-(\theta)d\theta + \chi(u_{-2},u_{-1})\int_{u_{-2}}^{u_{-1}} g'_+(\theta)d\theta$$
$$-|H_{-1/2}-H_{-3/2}| - |H_{3/2}-H_{1/2}|.$$

LEMMA 4.6. *With the above notation, for any sequence* $\{u_{-2}, u_{-1}, u_1, u_2\} \subset [s,S]$, *we have* $E \le 0$.

Proof. Without loss of generality we can assume that $\theta_f \le \theta_g$ and $f(\theta_f) \le g(\theta_g)$ (see Figure 1). Now $\psi_2(S) = f(S) - f(A) = g(S) - g(\theta_g) = \psi_1(S)$ and $\psi_2(s) = -(f(s) - f(\theta_f)) - (f(A) - f(\theta_f)) \le -(f(A) - f(s)) = (g(\theta_g) - g(s)) = \psi_1(s)$. Hence the range of ψ_1 is contained in the range of ψ_2. Therefore for each $u \in [s,S]$ there exists a unique $\rho(u) \in [s,S]$ such that $\psi_1(u) = \psi_2(\rho(u))$ and $u \mapsto \rho(u)$ is an increasing function since ψ_1, ψ_2 are increasing functions.

For $u_{-1} \ge \theta_g$, define $\tilde{A}(u_{-1}) \le \theta_f$ by $f(\tilde{A}(u_{-1})) = g(u_{-1})$ (see Figure 1).

Step 1. Let $u_{-1} \geq \theta_g$, $u_1 \geq \tilde{A}(u_{-1})$.

In this case it is easy to see that $H_{1/2} = H_{-1/2} = \overline{F}(u_{-1}, u_1) = g(u_{-1})$ and

$$\chi(u_{-2}, u_{-1})\int_{u_{-2}}^{u_{-1}} g'_+(\theta)d\theta = \chi(u_{-2}, u_{-1})(g(u_{-1}) - g(\max(\theta_g, u_{-2}))),$$

$$\begin{aligned}|H_{-1/2} - H_{-3/2}| &= |\overline{F}(u_{-1}, u_1) - G(u_{-2}, u_{-1})| = |g(u_{-1}) - g(\max(\theta_g, u_{-2}))| \\ &\geq \chi(u_{-2}, u_{-1})\int_{u_{-2}}^{u_{-1}} g'_+(\theta)d\theta.\end{aligned}$$

Hence

$$\chi(u_{-2}, u_{-1}) \int_{u_{-2}}^{u_{-1}} g'_+(\theta)d\theta - |H_{-1/2} - H_{-3/2}| \leq 0. \tag{4.6}$$

Since $u_{-1} \geq \theta_g$ this implies that $0 \leq g(u_{-1}) - g(\theta_g) = \psi_1(u_{-1}) = \psi_2(\rho(u_{-1}))$. Hence $\rho(u_{-1}) \geq A$ and $f(\rho(u_{-1})) = g(u_{-1})$.

Now $(z_{-1} - z_1)_- \neq 0$ if and only if $\psi_2(\rho(u_{-1})) = \psi_1(u_{-1}) < \psi_2(u_1)$. This implies that $\rho(u_{-1}) < u_1$. Therefore for $A \leq \rho(u_{-1}) < u_1$ we have

$$-(z_{-1} - z_1)_- = \psi_2(u_1) - \psi_2(\rho(u_{-1})) = \int_{\rho(u_{-1})}^{u_1} |f'(\theta)|d\theta = f(u_1) - f(\rho(u_{-1})).$$

Hence

$$-(z_{-1} - z_1)_- = \begin{cases} f(u_1) - f(\rho(u_{-1})) & \text{if } A \leq \rho(u_{-1}) < u_1, \\ 0 & \text{otherwise.}\end{cases} \tag{4.7}$$

Now for $\tilde{A}(u_{-1}) \leq u_1$, $0 \leq \overline{F}(u_{-1}, u_1) - g(\theta_g) = g(u_{-1}) - g(\theta_g) = \psi_1(u_{-1}) = \psi_2(\rho(u_{-1})) = f(\rho(u_{-1})) - f(A)$, and therefore $\overline{F}(u_{-1}, u_1) = f(\rho(u_{-1}))$. Hence either $u_1 \leq u_2$ or $\rho(u_{-1}) < u_1$, and for all u_2 we have

$$\begin{aligned}|H_{3/2} - H_{1/2}| &= |F(u_1, u_2) - \overline{F}(u_{-1}, u_1)| = |F(u_1, u_2) - f(\rho(u_{-1}))| \\ &\geq \begin{cases} |f(u_1) - f(\rho(u_{-1}))| & \text{if } u_1 \geq \theta_f, \\ |f(\min(u_2, \theta_f)) - f(\rho(u_{-1}))| & \text{if } u_1 \leq \theta_f,\end{cases}\end{aligned} \tag{4.8}$$

$$-\chi(u_1, u_2) \int_{u_1}^{u_2} f'_-(\theta)d\theta = \begin{cases} 0 & \text{if } u_1 \geq \theta_f, \\ f(u_1) - f(\min(u_2, \theta_f)) & \text{if } u_1 \leq \theta_f.\end{cases} \tag{4.9}$$

Let $E_1 = -(z_{-1} - z_1)_- - \chi(u_1, u_2) \int_{u_1}^{u_2} f'_-(\theta)d\theta - |H_{1/2} - H_{3/2}|$. Suppose $\rho(u_{-1}) < u_1$; then $u_1 \geq \theta_f$, and hence from (4.7), (4.8), and (4.9) we have

$$E_1 \leq f(u_1) - f(\rho(u_{-1})) - |f(u_1) - f(\rho(u_{-1}))| \leq 0.$$

Suppose $u_1 \leq \rho(u_{-1})$; then $(z_{-1} - z_1)_- = 0$. If $\theta_f \leq u_1$, then $E_1 = -|H_{1/2} - H_{3/2}| \leq 0$. Let $\theta_f > u_1 \geq \tilde{A}(u_{-1})$; then by the definition of $\tilde{A}(u_{-1})$ we have $f(\tilde{A}(u_{-1})) = g(u_{-1}) = f(\rho(u_{-1}))$. If $u_2 \leq u_1$, then clearly $E_1 = -|H_{1/2} - H_{3/2}| \leq 0$. Let $u_1 \leq u_2$. Now $f(\tilde{A}(u_{-1})) = g(u_{-1}) = f(\rho(u_{-1}))$, and hence $f(u_1) \leq f(\tilde{A}(u_{-1})) = f(\rho(u_{-1}))$. Hence from (4.7), (4.8), and (4.9),

$$E_1 \leq f(u_1) - f(\min(u_2, \theta_f)) - |f(\rho(u_{-1})) - f(\min(u_2, \theta_f))| \leq 0.$$

This together with (4.6) implies that $E \leq 0$.

Step 2. $u_{-1} \geq \theta_g,\ u_1 < \tilde{A}(u_{-1})$.

In this case $\overline{F}(u_{-1}, u_1) = f(u_1)$ and $g(u_{-1}) \leq f(u_1)$.

$$\begin{aligned} |H_{-1/2} - H_{-3/2}| &= |\overline{F}(u_{-1}, u_1) - G(u_{-2}, u_{-1})| = |f(u_1) - g(\max(u_{-2}, \theta_g))| \\ &\geq \chi(u_{-2}, u_{-1})|g(u_{-1}) - g(\max(u_{-2}, \theta_g))| \\ &\geq \chi(u_{-2}, u_{-1}) \int_{u_{-2}}^{u_{-1}} g'_+(\theta) d\theta, \end{aligned} \tag{4.10}$$

$$\begin{aligned} |H_{3/2} - H_{1/2}| &= |F(u_1, u_2) - \overline{F}(u_{-1}, u_1)| = |f(\min(u_2, \theta_f)) - f(u_1)| \\ &\geq -\chi(u_1, u_2) \int_{u_1}^{u_2} f'_-(\theta) d\theta. \end{aligned} \tag{4.11}$$

From Step 1, $(z_{-1} - z_1)_- \neq 0$ if and only if $A < \rho(u_{-1}) < u_1$. Hence $(z_{-1} - z_1)_- = 0$. Combining this with (4.10) and (4.11) gives $E \leq 0$.

Step 3. $u_{-1} < \theta_g, u_1 \geq \tilde{A}(\theta_g)$ (see Figure 1).

In this case $\overline{F}(u_{-1}, u_1) = f(A) = g(\theta_g)$. Since $u_1 \geq \tilde{A}(\theta_g)$ this implies that $f(u_1) \leq f(A)$ if $u_1 \leq \theta_f$. Let $u_1 \leq u_2$; then

$$-\chi(u_1, u_2) \int_{u_1}^{u_2} f'_-(\theta) - |H_{3/2} - H_{1/2}| \leq \begin{cases} -|f(u_1) - f(A)| & \text{if } u_1 \geq \theta_f, \\ f(u_1) - f(\min(u_2, \theta_f)), & \\ -|f(u_1) - f(\min(u_2, \theta_f))| & \text{if } u_1 \leq \theta_f. \end{cases} \tag{4.12}$$

Since $u_{-1} \leq \theta_g$, hence $\chi(u_{-2}, u_{-1}) \int_{u_{-2}}^{u_{-1}} g'_+(\theta) d\theta = 0$. Let $(z_{-1} - z_1)_- = 0$; then from (4.12) we have $E \leq 0$. Suppose $(z_{-1} - z_1)_- \neq 0$; then $\rho(u_{-1}) \leq u_1$ and

$$\begin{aligned} -(z_{-1} - z_1)_- &= \psi_2(u_1) - \psi_2(\rho(u_{-1})) = \int_{\rho(u_{-1})}^{u_1} |f'(\theta)| d\theta, \\ |H_{-1/2} - H_{-3/2}| &= G(u_{-2}, u_{-1}) - \overline{F}(u_{-1}, u_1) \\ &\geq g(u_{-1}) - g(\theta_g) = -\psi_1(u_{-1}) = -\psi_2(\rho(u_{-1})). \end{aligned}$$

Hence from (4.12) we have

$$E \leq \begin{cases} -(z_{-1} - z_1)_- - |H_{-1/2} - H_{-3/2}| \leq \displaystyle\int_{\rho(u_{-1})}^{u_1} |f'(\theta)| d\theta - \int_{\rho(u_{-1})}^{A} |f'(\theta)| d\theta \leq 0 & \\ & \text{if } u_1 \leq A, \\ \displaystyle\int_{\rho(u_{-1})}^{u_1} |f'(\theta)| d\theta - \int_{\rho(u_{-1})}^{A} |f'(\theta)| d\theta - \int_{A}^{u_1} |f'(\theta)| d\theta = 0 & \text{if } u_1 \geq A. \end{cases}$$

Hence in all cases $E \leq 0$.

Step 4. Let $u_{-1} \leq \theta_g,\ u_1 \leq \tilde{A}(\theta_g)$. In this case $\overline{F}(u_{-1}, u_1) = f(u_1)$ and $\chi(u_{-2}, u_{-1}) \int_{u_{-2}}^{u_{-1}} g'_+(\theta) d\theta = 0$. Let $u_1 \leq u_2, u_{-2} \leq u_{-1}$; then

$$\begin{aligned} |H_{3/2} - H_{1/2}| &= |F(u_1, u_2) - \overline{F}(u_{-1}, u_1)| = |f(\min(u_2, \theta_f)) - f(u_1)| \\ &= -\chi(u_1, u_2) \int_{u_1}^{u_2} f'_-(\theta) d\theta. \end{aligned} \tag{4.13}$$

If $u_1 \leq \rho(u_{-1})$, then $(z_{-1} - z_1)_- = 0$, and therefore from (4.13) $E \leq 0$. Hence assume that $\rho(u_{-1}) < u_1$; then $f(\rho(u_{-1})) > f(u_1)$. Since $\psi_2(\rho(u_{-1})) = \psi(u_{-1})$ this implies that

$$f(\rho(u_{-1})) - f(\theta_f) + f(A) - f(\theta_f) = \int_{\rho(u_{-1})}^{A} |f'(\theta)| d\theta = -\psi_2(\rho(u_{-1}))$$
$$= -\psi_1(u_{-1}) = \int_{u_{-1}}^{\theta_g} |g'(\theta)| d\theta = g(u_{-1}) - g(\theta_g).$$

Hence $f(\rho(u_{-1})) - g(u_{-1}) = 2(f(\theta_f) - f(A)) \le 0$, and therefore $f(u_1) \le f(\rho(u_{-1})) \le g(u_{-1})$. This implies that

$$|H_{-3/2} - H_{-1/2}| = |G(u_{-2}, u_{-1}) - \overline{F}(u_{-1}, u_1)| \ge g(u_{-1})_- f(u_1).$$

Since $f(A) = f(\tilde{A}(\theta_g))$ we have

$$\begin{aligned} E &\le -(z_{-1} - z_1)_- - |H_{-3/2} - H_{-1/2}| \le \int_{\rho(u_{-1})}^{u_1} |f'(\theta) d\theta - |g(u_{-1}) - f(u_1)| \\ &= f(\rho(u_{-1})) - g(u_{-1}) \le 0. \end{aligned}$$

This proves Lemma 4.6.

LEMMA 4.7. *Let $u_0 \in L^\infty(\mathbb{R})$ such that $s \le u_0(x) \le S$ for all $x \in \mathbb{R}$ and $N(f, g, u_0) < 0$. Let $\{u_i^n\}$ be the scheme defined as in* (3.6)*. Let ψ_1 and ψ_2 be as in Lemma* 4.6*. We introduce the constant*

$$L = \max\{Lip(\psi_1), Lip(\psi_2), \|\psi_1\|_\infty, \|\psi_2\|_\infty\}$$

and define

$$\text{(4.14)} \qquad \begin{aligned} z_i^n &= \begin{cases} \psi_2(u_i^n) & \text{if} \quad i \ge 1, \\ \psi_1(u_i^n) & \text{if} \quad i \le -1, \end{cases} \\ TV(z^n) &= \sum_{i \ne 0, -1} |z_i^n - z_{i+1}^n| + |z_{-1}^n - z_1^n|. \end{aligned}$$

Then

$$\text{(4.15)} \qquad TV(z^n) \le 2/\lambda \sum_{i \ne 0} |u_i^{n+1} - u_i^n| \le 2/\lambda \sum_{i \ne 0} |u_i^1 - u_i^0| = 2N_h(f, g, u_0),$$

$$\text{(4.16)} \qquad \sum_{i \ne 0} |z_i^n - z_i^m| \le \lambda L |n - m| N(f, g, P, u_0).$$

Proof.
Define $H_{1/2} = H_{-1/2} = \overline{F}(u_{-1}^n, u_1^n)$ and

$$H_{j+1/2} = \begin{cases} F(u_j^n, u_{j+1}^n) & \text{if } j \ge 1, \\ G(u_j^n, u_{j+1}^n) & \text{if } j \le -2. \end{cases}$$

Since $0 = \sum_{i \ne 0,-1} (z_i^n - z_{i+1}^n) + (z_{-1}^n - z_1^n)$,

$$\begin{aligned} \frac{1}{2} TV(z^n) &= \frac{1}{2} \left(\sum_{i \ne 0,-1} |z_i^n - z_{i+1}^n| + |z_{-1}^n - z_1^n| \right) \\ &= -\left(\sum_{i \ne 0,-1} (z_i^n - z_{i+1}^n)_- + (z_{-1}^n - z_1^n)_- \right) = I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = -\sum_{i\le -3}(z_i^n - z_{i+1}^n)_-, \quad I_2 = -\sum_{i\ge 2}(z_i^n - z_{i+1}^n)_-,$$
$$I_3 = -(z_{-2}^n - z_{-1}^n)_- - (z_{-1}^n - z_1^n)_- - (z_1^n - z_2^n)_-.$$

From (4.3) to (4.5) we have

$$\begin{aligned} I_1 &= -\sum_{i\le -3}(z_i^n - z_{i+1}^n)_- = -\sum_{i\le -3}(\psi_1(u_i^n) - \psi_1(u_{i+1}^n))_- \\ &\le \sum_{i\le -3}\chi_-(g'(u_i^n))|H_{i+1/2} - H_{i-1/2}| + \chi_+(g'(u_{i+1}^n))|H_{i+3/2} - H_{i+1/2}| \\ &\le \sum_{i\le -3}|H_{i+1/2} - H_{i-1/2}| + \chi_+(g'(u_{-2}^n))|H_{-3/2} - H_{-5/2}|, \\ I_2 &= -\sum_{i\ge 2}(z_i^n - z_{i+1}^n)_- = -\sum_{i\ge 2}(\psi_2(u_i^n) - \psi_2(u_{i+1}^n))_- \\ &\le \sum_{i\ge 2}\chi_-(f'(u_i^n))|H_{i+1/2} - H_{i-1/2}| + \chi_+(f'(u_{i+1}^n))|H_{i+3/2} - H_{i+1/2}| \\ &\le \sum_{i\ge 3}|H_{i+1/2} - H_{i-1/2}| + \chi_-(f'(u_2^n))|H_{5/2} - H_{3/2}|, \\ -(z_{-2}^n - z_{-1}^n)_- &= \chi(u_{-2}^n, u_{-1}^n)\left(\int_{u_{-2}^n}^{u_{-1}^n} g'_+(\theta)d\theta - \int_{u_{-2}^n}^{u_{-1}^n} g'_-(\theta)d\theta\right) \\ &\le \chi(u_{-2}^n, u_{-1}^n)\int_{u_{-2}^n}^{u_{-1}^n} g'_+(\theta)d\theta + \chi_-(g'(u_{-2}^n))|H_{-3/2} - H_{-5/2}|, \\ -(z_1^n - z_2^n)_- &= \chi(u_1, u_2)\left(\int_{u_1^n}^{u_2^n} f'_+(\theta)d\theta - \int_{u_1^n}^{u_2^n} f'_-(\theta)d\theta\right) \\ &\le \chi_+(f'(u_2^n))|H_{5/2} - H_{3/2}| - \chi(u_1^n, u_2^n)\int_{u_1^n}^{u_2^n} f'_-(\theta)d\theta. \end{aligned}$$

Combining all the above three inequalities we obtain

$$\begin{aligned} \frac{1}{2}TV(z^n) &\le \sum_{|i|\ge 2}|H_{i+1/2} - H_{i-1/2}| + \chi(u_{-2}^n, u_{-1}^n)\int_{u_{-2}^n}^{u_{-1}^n} g'_+(\theta)d\theta \\ &\quad -\chi(u_1^n, u_2^n)\int_{u_1^n}^{u_2^n} f'_-(\theta)d\theta - (z_{-1}^n - z_1^n)_- \\ &= \sum_{i=-\infty}^{\infty}|H_{i+1/2} - H_{i-1/2}| + E, \end{aligned}$$

where

$$E = -(z_{-1}^n - z_1^n)_- - \chi(u_1^n, u_2^n)\int_{u_1^n}^{u_2^n} f'_-(\theta)d\theta + \chi(u_{-2}^n, u_{-1}^n)\int_{u_{-2}^n}^{u_{-1}^n} g'_+(\theta)d\theta$$
$$- |H_{-1/2} - H_{-3/2}| - |H_{3/2} - H_{1/2}|.$$

From Lemma 4.6, $E \le 0$; hence from Lemma 4.4

$$\begin{aligned} TV(z^n) &= \sum_{i\ne 0,-1}|z_i^n - z_{i+1}^n| + |z_{-1}^n - z_1^n| \le 2\sum|H_{i+1/2} - H_{i-1/2}| \\ &= \frac{2}{\lambda}\sum_{i\ne 0}|u_i^{n+1} - u_i^n| \le \frac{2}{\lambda}\sum_{i\ne 0}|u_i^1 - u_i^0| = 2N_h(f, g, u_0). \end{aligned}$$

This proves (4.15).

Without loss of generality assume that $n \geq m$; then from Lemma 4.4 we have

$$\begin{aligned}\sum_{i\neq 0}|z_i^n - z_i^m| &= \sum_{i\leq -1}|z_i^n - z_i^m| + \sum_{i\geq 1}|z_i^n - z_i^m| \leq L\sum_{i\neq 0}|u_i^n - u_i^m| \\ &\leq L\sum_{i\neq 0}\sum_{j=0}^{n-m+1}|u_i^{n-j} - u_i^{n-j-1}| \\ &\leq L|n-m|\sum_{i\neq 0}|u_i^1 - u_i^0| = \lambda L|n-m|N_h(f,g,u_0).\end{aligned}$$

This proves (4.16) and hence Lemma 4.7.

The following lemma is the analogue of Lemma 4.7 in terms of functions instead of point values.

LEMMA 4.8. *Let $u_0, v_0 \in L^\infty(\mathbb{R}, [s, S])$ such that $N(f, g, u_0) < \infty, N(f, g, v_0) < \infty$ are initial datas, and let u_h and v_h be the corresponding solutions obtained by the finite volume scheme* (3.6) *and defined as in* (3.7)*. Let $\{z_i^n\}$ defined as in* (4.14) *for u_0 and z_h be the corresponding function defined as in* (3.7)*. Then*

$$s \leq u_h(x,t) \leq S \qquad \forall \ (x,t) \in \mathbb{R}\times\mathbb{R}_+, \tag{4.17}$$

$$||z_h||_\infty \leq L, \qquad TV(z_h(\cdot,t)) \leq 2N_h(f,g,u_0), \tag{4.18}$$

$$\int_{\mathbb{R}}|u_h(x,t) - u_h(x,\tau)|dx \leq N_h(f,g,u_0)(2\Delta t + |t-\tau|), \tag{4.19}$$

$$\int_{\mathbb{R}}|z_h(x,t) - z_h(x,\tau)|dx \leq LN_h(f,g,u_0)(2\Delta t + |t-\tau|). \tag{4.20}$$

Moreover, for $a \leq b$ and $\tau < t$,

$$\int_a^b |u_h(x,t) - v_h(x,t)|dx \leq \int_{a-\frac{1}{\lambda}(t-\tau)}^{b+\frac{1}{\lambda}(t-\tau)} |u_h(x,\tau) - v_h(x,\tau)|dx + 4(S-s)h. \tag{4.21}$$

Proof. Inequalities (4.17) and (4.18) follow from (4.2) and (4.15). For inequality (4.19) let $t_n \leq t < t_{n+1}$ and $t_m \leq \tau < t_{m+1}$ so that

$$|n-m|\Delta t = |t_n - t_m| \leq |t_n - t| + |t-\tau| + |\tau - t_m| \leq 2\Delta t + |t-\tau|.$$

Hence from Lemma 4.2 we obtain

$$\begin{aligned}\int_{\mathbb{R}}|u_h(x,t) - u_h(x,\tau)|dx &= h\sum_{i\neq 0}|u_i^n - u_i^m| \leq h\sum_{i\neq 0}\sum_{j=0}^{n-m+1}|u_i^{n-j} - u_i^{n-j-1}| \\ &\leq h|n-m|\sum_{i\neq 0}|u_i^1 - u_i^0| \leq \frac{\Delta t|n-m|}{\lambda}\sum_{i\neq 0}|u_i^1 - u_i^0| \\ &\leq (2\Delta t + |t-\tau|)N_h(f,g,u_0).\end{aligned}$$

The proof of (4.20) follows from (4.19):

$$\begin{aligned}\int_{\mathbb{R}}|z_h(x,t) - z_h(x,\tau)|dx &\leq h\sum_{i\neq 0}|z_i^n - z_i^m| \leq Lh\sum_{i\neq 0}|u_i^n - u_i^m| \\ &\leq LN_h(f,g,u_0)(2\Delta t + |t-\tau|).\end{aligned}$$

We prove inequality (4.21) for $a < 0,\ b > 0$. The proofs are similar for the other cases. Let

$$\begin{aligned}&x_{i_0-3/2} < a \leq x_{i_0-1/2}, \qquad &&x_{j_0+1/2} \leq b < x_{j_0+3/2}, \\ &t_{n+1} \leq t < t_{n+2}, &&t_{n-p+1} \leq \tau < t_{n-p+2};\end{aligned}$$

so we have $x_{i_0-p-3/2} \le a - ph \le x_{i_0-p-1/2}$, $x_{j_0+p+1/2} \le b + \ ph < x_{j_0+p+3/2}$, and $t-\Delta t \le \tau + p\Delta t \le t + \Delta t$. From (4.17) $|u_h - v_h| \le (S-s)$; hence

$$\begin{aligned}\int_a^b |u_h(x,t)-v_h(x,t)|dx &= \int_a^{x_{i_0-1/2}} |u_h(x,t)-v_h(x,t)|dx \\ &\quad + \int_{x_{i_0-1/2}}^{x_{j_0+1/2}} |u_h(x,t)-v_h(x,t)|dx + \int_{x_{j_0+1/2}}^{b} |u_h(x,t)-v_h(x,t)|dx \\ &\le 2(S-s)h + h \sum_{\substack{i_0 \le i \le j_0 \\ i \neq 0}} |u_i^{n+1} - v_i^{n+1}|.\end{aligned}$$

Using Lemma 4.4 it follows that

$$\begin{aligned}&\int_a^b |u_h(x,t)-v_h(x,t)|dx \\ &\quad \le 2(S-s)h + h \sum_{\substack{i_0-p \le i \le j_0+p \\ i \neq 0}} |u_i^{n+1-p} - v_i^{n+1-p}| \\ &\quad = 2(S-s)h + \int_{x_{i_0-p-1/2}}^{x_{j_0+p+1/2}} |u_h(x,\tau)-v_h(x,\tau)|dx \\ &\quad = 2(S-s)h + \int_{a-ph}^{b+ph} |u_h(x,\tau)-v_h(x,\tau)|dx \\ &\qquad - \int_{a-ph}^{x_{i_0-p-1/2}} |u_h(x,\tau)-v_h(x,\tau)|dx - \int_{x_{j_0+p+1/2}}^{b+ph} |u_h(x,\tau)-v_h(x,\tau)|dx \\ &\quad \le 2(S-s)h + \int_{a-\frac{t-\tau}{\lambda}}^{b+\frac{t-\tau}{\lambda}} |u_h(x,\tau)-v_h(x,\tau)|dx \\ &\qquad + \int_{a-\frac{t-\tau}{\lambda}}^{a-ph} |u_h(x,\tau)-v_h(x,\tau)|dx + \int_{b+ph}^{b+\frac{t-\tau}{\lambda}} |u_h(x,\tau)-v_h(x,\tau)|dx \\ &\quad \le 2(S-s)h + 2\left|\tfrac{t-\tau}{\lambda} - ph\right|(S-s) + \int_{a-\frac{t-\tau}{\lambda}}^{b+\frac{t-\tau}{\lambda}} |u_h(x,\tau)-u_h(x,\tau)|dx \\ &\quad \le 4(S-s)h + \int_{a-\frac{t-\tau}{\lambda}}^{b+\frac{t-\tau}{\lambda}} |u_h(x,\tau)-v_h(x,\tau)|dx\,.\end{aligned}$$

This completes the proof of Lemma 4.8.

4.3. Convergence of a subsequence to the weak solution. From hypotheses (H_1) and (H_2) we will construct a solution to the Riemann problem with undercompressive data which will enable us to prove that the solution satisfies the interface entropy condition (2.2). For $\alpha, \beta \in [s, S]$, let

$$v_0(x,\alpha,\beta) = \begin{cases} \alpha & \text{if} \quad x < 0, \\ \beta & \text{if} \quad x \ge 0\,. \end{cases}$$

Then we have the following lemma.

LEMMA 4.9. *Assume that f, g satisfy hypotheses* (H_1) *and* (H_2). *Let $\alpha, \beta \in [s, S]$ be such that $\alpha \le \theta_g$ and $\beta \ge \theta_f$. Let $v_h(x,t,\alpha,\beta)$ be the solution given by the finite volume scheme* (3.6) *with initial data $v_0(x,\alpha,\beta)$ and $\lambda M \le 1$. Assume that for a subsequence $h_k \to 0$, $v_{h_k}(x,t,\alpha,\beta) \to v(x,t,\alpha,\beta)$ on $L^\infty_{loc}(\mathbb{R}_+, L^1_{loc}(\mathbb{R}))$. Then*

$$\begin{aligned}\lim_{x\to 0^-} v(x,t,\alpha,\beta) &= \theta_g \quad \text{if} \quad \min\, g > \min f, \\ \lim_{x\to 0^+} v(x,t,\alpha,\beta) &= \theta_f \quad \text{if} \quad \min\, g \le \min f.\end{aligned} \tag{4.22}$$

Proof. We will prove the lemma for $\min g > \min f$. The other cases can be proved in a similar manner. Let $A \geq \theta_f$ be such that $f(A) = g(\theta_g)$ (see Figure 1). Since $\alpha \leq \theta_g,\ \beta \geq \theta_f$ it follows from (3.2) that

$$\overline{F}(\alpha,\beta) = \max\{g(\theta_g),\ f(\theta_f)\} = g(\theta_g). \tag{4.23}$$

Then if $\{v_i^n\}$ are the grid values corresponding to the initial data $v_0(x,\alpha,\beta)$,

$$v_i^1 = \begin{cases} \alpha - \lambda(g(\theta_g) - g(\alpha)) & \text{if} \quad i = -1, \\ \alpha & \text{if} \quad i \leq -2, \\ \beta - \lambda(f(\beta) - g(\theta_g)) & \text{if} \quad i = 1, \\ \beta & \text{if} \quad i \geq 2. \end{cases}$$

This implies that $v_{-1}^1 = \alpha - \lambda(g(\theta_g) - g(\alpha))$ and $v_{-1}^1 = \alpha + \lambda(g(\alpha) - g(\theta_g)) \leq \alpha + \lambda M|\alpha - \theta_g| < \alpha + \theta_g - \alpha = \theta_g$. Let $\beta \in [\theta_f, A]$; then $f(\beta) \leq f(A) = g(\theta_g)$, and hence $v_1^1 = \beta - \lambda(f(\beta) - g(\theta_g)) \geq \beta$ and $v_1^1 = \beta + \lambda(g(\theta_g) - f(\beta)) = \beta + \lambda(f(A) - f(\beta)) \leq \beta + (A - \beta) = A$. If $\beta \in [A, S]$, then $f(\beta) > f(A) = g(\theta_g)$, and hence $v_1^1 = \beta - \lambda(f(\beta) - g(\theta_g)) < \beta$ and $v_1^1 = \beta - \lambda(f(\beta) - f(A)) \geq \beta - \lambda M(\beta - A) \geq A$. Hence $\{v_i^1\}$ satisfies

$$\begin{aligned} \alpha &\leq v_{-1}^1 \leq \theta_g, \\ \beta &\leq v_1^1 \leq A \quad \text{if} \quad \beta \in [\theta_f, A], \\ A &\leq v_1^1 \leq \beta \quad \text{if} \quad \beta \in [A, S], \\ v_i^1 &= \begin{cases} \alpha & \text{if} \quad i \leq -2, \\ \beta & \text{if} \quad i \geq 2. \end{cases} \end{aligned} \tag{4.24}$$

Now we claim that $\{v_i^n\}$ satisfies

$$\begin{aligned} \alpha &\leq v_{-n}^n \leq v_{-n+1}^n \leq \cdots \leq v_{-1}^n \leq \theta_g, \\ \beta &\leq v_1^n \leq \cdots \leq v_n^n \leq A \quad \text{if} \quad \beta \in [\theta_f, A], \\ A &\leq v_1^n \leq \cdots \leq v_n^n \leq \beta \quad \text{if} \quad \beta \in [A, S], \\ v_i^n &= \begin{cases} \alpha & \text{if} \quad i \leq -n-1, \\ \beta & \text{if} \quad i \geq n+1. \end{cases} \end{aligned} \tag{4.25}$$

From (4.24) the claim is true for $n = 1$. Assume that it is true up to $n - 1$. Since $v_1^{n-1} \geq \theta_f$ and $v_{-1}^{n-1} \leq \theta_g$, hence as in (4.23) $\overline{F}(v_{-1}^n, v_1^n) = g(\theta_g)$. Hence by the same argument as in (4.24), it follows that $\alpha \leq v_{-1}^n \leq \theta_g,\ \beta \leq v_1^n \leq A$ if $\beta \in [\theta_f, A]$ and $A \leq v_1^n \leq \beta$ if $\beta \in [A, S]$. Now (4.25) follows since the scheme is monotone and consistent for $|i| \geq 2$. This proves (4.25).

From (4.25) it follows that $v(x,t,\alpha,\beta)$ satisfies

$$\begin{aligned} \alpha &\leq v^-(t,\alpha,\beta) = \lim_{x\to 0^-} v(x,t,\alpha_1,\beta) \leq \theta_g, \\ \beta &\leq v^+(t,\alpha,\beta) = \lim_{x\to 0^+} v(x,t,\alpha,\beta) \leq A \quad \text{if} \quad \beta \in [\theta_f, A], \\ A &\leq v^+(t,\alpha,\beta) = \lim_{x\to 0^+} v(x,t,\alpha,\beta) \leq \beta \quad \text{if} \quad \beta \in [A, S]. \end{aligned}$$

From (4.25) and hypothesis (H_2) on the shape of f and g we observe that $\{v_i^{(n)}\}_{i\leq -1}$ is independent of β as long as $\beta \geq \theta_f$. Hence $v^-(t,\alpha,\beta)$ is independent of β, and hence $v^-(t,\alpha,\beta) = v^-(t,\alpha,\theta_f)$. Since $v^+(t,\alpha,\theta_f) \leq A$ and $g(v^-(t,\alpha,\beta)) = f(v^+(t,\alpha,\beta))$, hence $v^-(t,\alpha,\beta) = \theta_g$. This completes the proof of Lemma 4.9.

Proof of Theorem 3.2. Let $\lambda = \frac{\Delta t}{h} \le \frac{1}{M}$ be fixed. Since $N(f,g,u_0) < \infty$, then from Lemma 4.8 and by a standard argument there exists a subsequence $h_k \to 0$ such that z_{h_k} converges to z in $L^\infty(0,T,L^1_{loc}(\mathbb{R}))$ and for almost all fixed t, $z_{h_k}(.,t) \to z(.,t)$ in $L^1_{loc}(\mathbb{R})$. Let

$$u(x,t) = \begin{cases} \psi_2^{-1}(z(x,t)) & \text{if} \quad x > 0, \quad t > 0, \\ \psi_1^{-1}(z(x,t)) & \text{if} \quad x < 0, \quad t > 0. \end{cases}$$

Now for $x > 0, u_{h_k}(x,t) = \psi_2^{-1}(z_{h_k}(x,t))$ and for $x < 0, u_{h_k}(x,t) = \psi_1^{-1}(z_{h_k}(x,t))$ and ψ_1 and ψ_2 are continuous, and therefore for almost all t, $u_{h_k}(.,t) \to u(.,t)$ a.e. in $\mathbb{R}$. From (4.18), for a.e. t, $z(.,t) \in BV(\mathbb{R})$, and hence $z(x+,t), z(x-,t)$ exist for all $x \in \mathbb{R}$. This implies that $u(x+,t), u(x-,t)$ exist for all $x \in \mathbb{R}$ and a.e. t. We will complete the proof of the theorem in two steps.

Step 1. Let us prove that u is a weak solution of (1.2) satisfying the interior entropy condition (2.1). Remember that the scheme is not consistent. However, the proof follows almost as in the Lax–Wendroff theorem [10, Theorem 1.1].

Let $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)$, and let

$$\varphi_j^n = \varphi(x_j, t_n), \quad j \in Z \setminus (0), \quad n \ge 0\,.$$

Multiplying (3.6) by φ_j^n and summing over j and n we obtain

$$\begin{aligned} &h\sum_{n=1}^{\infty}\sum_{i\neq 0} u_i^n(\varphi_i^{n-1} - \varphi_i^n) + \Delta t \sum_{n=0}^{\infty}\sum_{-\infty}^{-1} G(u_{i-1}^n, u_i^n)(\varphi_{i-1}^n - \varphi_i^n) \\ &+\Delta t \sum_{n=0}^{\infty}\sum_{2}^{\infty} F(u_{i-1}^n, u_i^n)(\varphi_{i-1}^n - \varphi_i^n) + \Delta t \sum_{n=0}^{\infty} \overline{F}(u_{-1}^n, u_1^n)(\varphi_{-1}^n - \varphi_1^n) - h\sum_{i\neq 0} u_i^0 \varphi_i^0 = 0. \end{aligned}$$

Let

$$\begin{aligned} g_h(x,t) &= G(u_{i-1}^n, u_i^n), && i \le -1, \quad x \in (x_{i-1}, x_i], \quad t \in [n\Delta t, (n+1)\Delta t), \\ f_h(x,t) &= F(u_{i-1}^n, u_i^n), && i \ge 2, \quad x \in (x_{i-1}, x_i], \quad t \in [n\Delta t, (n+1)\Delta t), \\ \overline{F}_h(t) &= \overline{F}(u_{-1}^n, u_1^n), && t \in [n\Delta t, (n+1)\Delta t), \\ \overline{\varphi}_h(t) &= \varphi(\tfrac{h}{2}, n\Delta t) - \varphi(-\tfrac{h}{2}, n\Delta t), && t \in [n\Delta t, (n+1)\Delta t); \end{aligned}$$

then the above equalities read as

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{\Delta t}^{\infty} u_h(x,t)\left(\frac{\varphi_h(x,t) - \varphi_h(x,t-\Delta t)}{\Delta t}\right) dtdx \\ &+\int_{-\infty}^{x_{-1}}\int_0^{\infty} g_h(x,t)\left(\frac{\varphi_h(x+\frac{h}{2},t) - \varphi_h(x-\frac{h}{2},t)}{h}\right) dtdx \\ &+\int_{x_1}^{\infty}\int_0^{\infty} f_h(x,t)\left(\frac{\varphi_h(x+\frac{h}{2},t) - \varphi_h(x-\frac{h}{2},t)}{h}\right) dtdx \\ &+\int_0^{\infty} \overline{F}_h(t)\overline{\varphi}_h(t)dt + \int_{-\infty}^{\infty} u_h(x)\varphi_h(x)dx = 0\,. \end{aligned}$$

Let $h = h_k$ in the above equation, and by going to a subsequence if necessary, by using by the fact that $|\overline{F}(t)| \le \|F\|_\infty + \|G\|_\infty$ and by the dominated convergence theorem, it follows that as $k \to \infty, u_{h_k} \to u$ in $L^\infty_{loc}(\mathbb{R}, L^1_{loc}(\mathbb{R}))$ and the above equation gives that

$$\int_{-\infty}^{\infty}\int_0^{\infty}\left[u\frac{\partial\varphi}{\partial t} + (H(x)f(u) + (1-H(x))g(u))\frac{\partial\varphi}{\partial x}\right] dtdx + \int_{-\infty}^{\infty} u_0(x)\varphi(x,0)dx = 0,$$

where $H(x)$ is the Heaviside function. This proves that u is a weak solution.

In order to prove the interior entropy condition, for $l \in \mathbb{R}$ define

$$A(a,b) = F(a \wedge l,\ b \wedge l) - F(a \vee l,\ b \vee l), \quad B(a,b) = G(a \wedge l,\ b \wedge l) - G(a \vee l,\ b \vee l),$$
$$A^n_{j+1/2} = A(u^n_j, u^n_{j+1}), \quad B^n_{j+1/2} = B(u^n_j,\ u^n_{j+1}). \tag{4.26}$$

Then as in [4, 10], for $|i| \geq 2$, u^n_i satisfies

$$|u^{n+1}_i - l| \leq |u^n_i - l| - \lambda(A^n_{i+1/2} - A^n_{i-1/2}) \quad \text{if } i \geq 2, \tag{4.27}$$

$$|u^{n+1}_i - l| \leq |u^n_i - l| - \lambda(B^n_{i+1/2} - B^n_{i-1/2}) \quad \text{if } i \leq -2. \tag{4.28}$$

Let $0 \leq \varphi \in C^1_0(\mathbb{R}_+ \times \mathbb{R}_+)$. Then there exists $\alpha > 0$ such that supp $(\varphi) \subset \{(x,t); x > \alpha,\ t > \alpha\}$. Hence for h_k small, $\varphi_{h_k}(x,t) = 0$ for $x \leq x_4,\ t \geq \Delta t$. Let $l \in \mathbb{R}$, A, and $A_{i+1/2}$ be defined as in (4.26). Let $A_h(x,t) = A^n_{i+1/2}$ for $x_i \leq x < x_{i+1},\ t_n \leq t < t_{n+1}$. Then multiplying (4.27) by φ^n_i and summing we obtain

$$\int_0^\infty \int_0^\infty |u_{h_k} - l| \left(\frac{\varphi_{h_k}(x,t) - \varphi_{h_k}(x, t - \Delta t)}{\Delta t} \right) + \int_0^\infty \int_{x_3}^\infty A_{h_k}(x,t) \left(\frac{\varphi(x + \frac{h_k}{2}, t) - \varphi(x - \frac{h_k}{2}, t)}{h_k} \right) dxdt \geq 0\,.$$

Now letting $h_k \to 0$ yields $\int_0^\infty \int_0^\infty [\,|u - l| \frac{\partial \varphi}{\partial t} + (f(u) - f(l)) sign(u - l) \frac{\partial \varphi}{\partial x}]\, dx\, dt \geq 0$, and similarly for $x < 0$. Hence u satisfies the interior entropy condition (2.1), and this complete the proof of Step 1.

Step 2. We will now show that if u is the weak solution constructed in Step 1 for some $h_k \to 0$, and assuming that the set of discontinuities of u is a discrete set of Lipschitz curves $\{\Gamma_j\}$, then u satisfies the interface entropy condition (2.2), and the solution thus obtained is unique.

The main ingredient to prove this is the choice of the solution constructed in Lemma 4.9. Without loss of generality we can assume that $\min g \geq \min f$. Since $x \to z(x,t)$ is TV bounded, hence $z(0+,t)$ and $z(0-,t)$ exist. This implies that $u^+(t)$ and $u^-(t)$ exist. Suppose that u does not satisfy the interface entropy condition (2.2). Then $meas\,\{L \setminus U\} \neq 0$. Since for $t \in L \setminus U$ $u^+(t) > \theta_f, u^-(t) < \theta_g$, from hypothesis ($H_2$) we obtain, for almost all $t \in L \setminus U$ $u^-(t) < S, u^+(t) > s$ and

$$meas\,\{t \in L; u^-(t) < S, u^+(t) > s\} \neq 0.$$

Hence from the hypothesis on u and (H_2) we can choose $t_0 \in L \setminus U,\ \alpha, \beta, \varepsilon\ \in \mathbb{R}_+$ such that they satisfy

$$t_0 = n_k \Delta t,\ u_{h_k}(x,t_0) \to u(x,t_0) \text{ in } L^1_{loc}(\mathbb{R}) \text{ and } u^+(t_0) > \theta_f,\ u^-(t_0) < \theta_g, \tag{4.29}$$

u is continuous in $[-\beta, 0) \times [t_0 - \alpha, t_0 + \alpha]$ and $(0, \beta] \times [t_0 - \alpha, t_0 + \alpha]$,

$$\begin{array}{lll} u^-(t_0) - \varepsilon \leq u(x,t) \leq u^-(t_0) + \varepsilon < \theta_g & \text{in} & [-\beta, 0) \times [t_0 - \alpha, t_0 + \alpha], \\ \theta_f < u^+(t_0) - \varepsilon \leq u(x,t) \leq u^+(t_0) + \varepsilon & \text{in} & (0, \beta] \times [t_0 - \alpha,\ t_0 + \alpha]. \end{array} \tag{4.30}$$

On $\mathbb{R} \times \{t_0\}$, define the functions

$$\begin{aligned} V_k(x,t_0) &= u_{h_k}(x,t_0) \\ V_{k,\varepsilon}(x,t_0) &= \begin{cases} u_{h_k}(x,t_0) & \text{if } \ |x| \geq \beta, \\ \max(u_{h_k}(x,t_0), u^-(t_0) - \varepsilon) & \text{if } \ -\beta \leq x \leq 0, \\ \max(u_{h_k}(x,t_0), u^+(t_0) - \varepsilon) & \text{if } \ 0 \leq x \leq \beta, \end{cases} \\ V_0 &= \begin{cases} \max(s, u^-(t_0) - \varepsilon) & \text{if } \ x \leq 0, \\ u^+(t_0) - \varepsilon & \text{if } \ x > 0. \end{cases} \end{aligned} \tag{4.31}$$

From (4.29) and (4.30) it follows that as $k \to \infty$, for almost every $x \in \mathbb{R}$

$$V_k(x, t_0) \to u(x, t_0), \quad V_{k,\varepsilon}(x, t_0) \to u(x, t_0). \tag{4.32}$$

With t_0 as the initial time and $h_k,\ \lambda = \frac{\Delta t}{h_k}$, as the grid lengths, let $\tilde{V}_{h_k}, \tilde{V}_{h_{k,\varepsilon}}$, and W_{h_k} be the respective solutions calculated with the finite volume scheme (3.6) for $t \geq t_0$ and associated with $V_k, V_{k,\varepsilon}, V_0$ as initial data at $t = t_0$. Since $V_k, V_{k,\varepsilon}$, and V_0 are such that $N(f, g, V_k), N(f, g, V_{k,\varepsilon}), N(f, g, V_0)$ are bounded, one can extract a subsequence still denoted by h_k such that $\tilde{V}_{h_k}, \tilde{V}_{h_{k,\varepsilon}}, W_{h_k}$ converge to u (since $u_{h_k} = \tilde{V}_{h_k}$), v, w a.e., respectively. Letting $h_k \to 0$ in (4.21) for any $a > 0$, $t > t_0$, we have

$$\int_{-a}^{a} |u(x,t) - v(x,t)| dx \leq \int_{-a-t_0/\lambda}^{a+t_0/\lambda} |u(x,t_0) - v(x,t_0)| dx = 0\,.$$

Hence $u \equiv v$.

From (4.31), $V_0(x, t_0) \leq V_{k,\varepsilon}(x, t_0)$ for $x \in [-\beta, \beta]$. Hence by monotonicity of the scheme (see (ii) of Lemma 4.2), $W_{h_k}(x, t) \leq \tilde{V}_{k,\varepsilon}(x, t) + o(\Delta t)$ for $-\beta + \frac{t-t_0}{\lambda} \leq x \leq \beta - \frac{t-t_0}{\lambda}$, and hence for a.e. (x, t) with $t > t_0$, $-\beta + \frac{t-t_0}{\lambda} \leq x \leq \beta - \frac{t-t_0}{\lambda}$,

$$w(x,t) \leq u(x,t).$$

From this inequality, Lemma 4.9, and (4.30) we have for a.e. $t \in (t_0,\ \min(t_0 + \lambda\beta, t_0 - \alpha))$

$$\theta_g = w^-(t) \leq u^-(t) \leq u^-(t_0) + \varepsilon < \theta_g,$$

which is a contradiction. Hence u satisfies the interface entropy conditions.

Let u, v be two limit points of the scheme $\{u_h\}$ such that u and v have a discrete set of Lipschitz curves as discontinuities. From Steps 1 and 2, u and v satisfy the entropy conditions (2.1) and (2.2), and hence from Lemma A.1, for $b > \overline{M}t$,

$$\int_{-b+\overline{M}t}^{b+\overline{M}t,} |u(x,t) - v(x,t)| dx \leq \int_{-b}^{b} |u(x,0) - v(x,0)| dx = 0\,.$$

Hence $u \equiv v$. This proves Step 2.

Furthermore, let u and v be the weak solutions of (1.2), constructed in Steps 1 and 2 for the initial data u_0 and v_0, respectively. Then by taking $a = -\infty$, $b = +\infty$, and letting $h \to 0$ in (4.21) we obtain

$$\int_{\mathbb{R}} |u(x,t) - v(x,t)| dx \leq \int_{\mathbb{R}} |u_0(x) - v_0(x)| dx\,.$$

Finally, if $f \equiv g$, then by Lemma 4.1 $\overline{F}(a, b) = F(a, b)$, and hence the scheme is Godunov's scheme. Now Theorem 3.2 follows from Steps 1 to 2.

Note that the scheme defined in (3.6) using the interface flux $\overline{F}$ gives a much stronger bound, i.e., $E \leq 0$. This helps us to extend the result for a flux $F(x, u)$ having more discontinuities in the space variable, as stated in the next remark.

Remark 4.10. The above analysis can be extended readily to the equation

$$u_t + f(k(x), u)_x = 0,$$

where $f(a, b) \in C^1(\mathbb{R} \times \mathbb{R})$ and k is a piecewise smooth function satisfying

(i) $f(a, s) = f(b, s), \quad f(a, S) = f(b, S)$ for all $a, b \in \mathbb{R}$,

(ii) for all a the function $u \longrightarrow f(a, u)$ satisfies (H_2).

Remark 4.11. In a forthcoming paper we will extend the above analysis to all E-schemes, including Engquist–Osher, Lax–Friedrich, etc. The case of the upstream mobility is considered in the next section, where it is compared with scheme (3.6).

5. Two-phase flow in porous media. Capillary-free two-phase incompressible flow in a porous medium with a rock type changing at $x = 0$ is modelled by (1.3), (1.4), where u is the saturation of one of the two phases, say phase 1. Equations (1.3) represent conservation of phase 1 inside each rock type, and (1.4) ensures conservation of the same phase at the interface between the two rock types. The functions f and g are the Darcy velocities (divided by the porosity) of phase 1 in each rock type, and they have the form

$$\begin{aligned} f = f_1 &= \frac{1}{\phi}\frac{\lambda_1}{\lambda_1+\lambda_2}[q+(c_1-c_2)\lambda_2] \quad \text{for } x>0,\\ g = g_1 &= \frac{1}{\phi}\frac{\mu_1}{\mu_1+\mu_2}[q+(c_1-c_2)\mu_2] \quad \text{for } x<0, \end{aligned} \tag{5.1}$$

where ϕ is the porosity of the rock and q, a constant in space, is the total Darcy velocity, that is, the sum of the Darcy velocities of the two phases, $q = \phi(f_1 + f_2) = \phi(g_1+g_2)$. The Darcy velocities (divided by the porosity) of phase 2 denoted by f_2, g_2 are given by

$$f_2 = \frac{1}{\phi}\frac{\lambda_2}{\lambda_1+\lambda_2}[q+(c_2-c_1)\lambda_1], \quad g_2 = \frac{1}{\phi}\frac{\mu_2}{\mu_1+\mu_2}[q+(c_2-c_1)\mu_1].$$

The quantities λ_1, μ_1 and λ_2, μ_2 are the effective mobilities of the two phases. They are functions of u satisfying the following properties:

$$\begin{aligned} &\lambda_1, \mu_1 \text{ are increasing functions of } u, \quad \lambda_1(s) = \mu_1(s) = 0,\\ &\lambda_2, \mu_2 \text{ are decreasing functions of } u, \quad \lambda_2(S) = \mu_2(S) = 0. \end{aligned} \tag{5.2}$$

The gravity constants c_1, c_2 of the phases are proportional to their density.

In such a context the flux functions f and g satisfy hypotheses (H_1), (H_2), or (H_3), and Theorems 2.1, and 3.2 apply, provided that an appropriate CFL condition is satisfied. In numerical computations one can, of course, use the numerical fluxes F, G defined in (3.1) inside the rock types and $\overline{F}$, defined in (3.2) at the interface.

However, petroleum engineers have designed, from simple physical considerations, another numerical flux called the upstream mobility flux. It is an ad hoc flux for two-phase flow in porous media which corresponds to an approximate solution to the Riemann problem. It is given by the following formula:

$$\begin{aligned} &F^{UM}(a,b) = \frac{1}{\phi}\frac{\lambda_1^*}{\lambda_1^*+\lambda_2^*}[q+(c_1-c_2)\lambda_2^*],\\ &\lambda_\ell^* = \begin{cases} \lambda_\ell(a) & \text{if } q+(c_\ell-c_i)\lambda_\ell^* > 0, \quad i=1,2, \quad i\neq\ell,\\ \lambda_\ell(b) & \text{if } q+(c_\ell-c_i)\lambda_\ell^* \le 0, \quad i=1,2, \quad i\neq\ell, \end{cases} \qquad \ell = 1,2, \end{aligned} \tag{5.3}$$

and similarly for G^{UM} associated with g. As we can see, the flux is calculated using the mobilities of the phases which are upstream with respect to the flow of the phases. When the two phases are flowing in the same direction, the Godunov flux and the upstream mobility flux give the same answer and coincide with standard upstream weighting, but they differ when the phases are flowing in opposite directions. This flux has been shown to have all the desired properties for convergence of the associated finite difference scheme [26, 2] in the case of a flux function which does not vary with space (one rock type).

The generalization of the upstream mobility flux to the case of two rock types is straightforward, and at the interface the corresponding flux is

$$
\begin{aligned}
&\overline{F}^{UM}(a,b) = \frac{1}{\phi}\,\frac{\lambda_1^*}{\lambda_1^*+\lambda_2^*}[q+(c_1-c_2)\lambda_2^*],\\
&\lambda_\ell^* = \begin{cases} \mu_\ell(a) & \text{if } q+(c_\ell-c_i)\lambda_\ell^*>0, \quad i=1,2, \quad i\neq \ell,\\ \lambda_\ell(b) & \text{if } q+(c_\ell-c_i)\lambda_\ell^*\leq 0, \quad i=1,2, \quad i\neq\ell, \end{cases} \quad \ell=1,2.
\end{aligned} \tag{5.4}
$$

This upstream mobility flux at the interface satisfies the consistency condition of Lemma 4.1:

$$\overline{F}^{UM}(s,s) = f(s) = g(s) = 0, \quad \overline{F}^{UM}(S,S) = f(S) = g(S) = \frac{q}{\phi}.$$

6. Numerical experiments. We consider an idealized experiment in which two phases of different densities are flowing in a vertical closed core. This core is made of two rock types, the top part being associated with the flux function g and the bottom part associated with the function f defined in (5.1). The data associated with the problem are as follows:

$$
\begin{aligned}
&\phi=1, \quad q=0, \quad c_1=2, \quad c_2=1,\\
&s=0., \quad S=1., \quad \lambda_1=10u^2, \quad \lambda_2=20(1-u)^2, \quad \mu_1=50u^2, \quad \mu_2=5(1-u)^2,
\end{aligned}
$$

which gives the flux function f and g represented in Figure 3. Note that we are in the case where f and g satisfy hypothesis (H_3). Phase 1 is the heavy phase, and it moves downwards while phase 2, the light phase, moves upward.

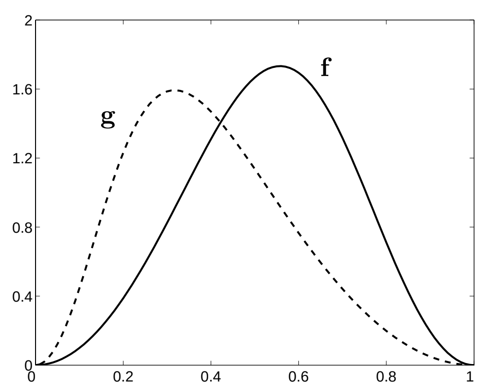

Fig. 3. *The flux functions at the interface for the numerical experiments.*

We present here two simulations which differ by the initial condition. In the first case we start with discontinuous data $u_0(x) = 1$ if $x < 0$, $u_0(x) = 0$ if $x > 0$; that is, at initial time the core is saturated with the heavy fluid (phase 1) in the upper half and with the light fluid (phase 2)in the lower half. The calculated solution is shown in Figure 4. In the second case we start with a constant initial data $u_0 = .5$ which corresponds to a situation where the two phases are "mixed." In this case the solution is shown in Figure 5.

In all figures the top part of the core is on the left of the picture and the bottom part is on the right. As expected, observe as time goes on the heavy fluid moving

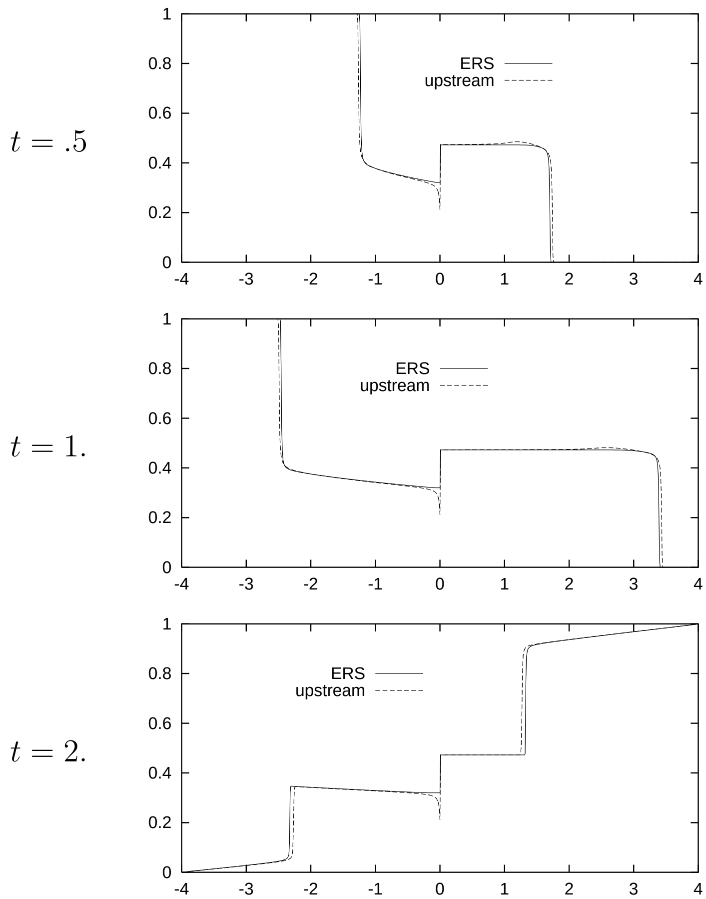

FIG. 4. *Finite difference solutions calculated with numerical flux* (3.1), (3.3) *(ERS) and with the upstream mobility flux* (5.3), (5.4) *at different times for a discontinuous initial data (*$h = 1/100$*).*

downward which is represented by its saturation u decreasing on the left and increasing on the right. Obviously, in the case of the continuous initial data we reach earlier the stationary state where the heavy phase occupies the bottom half of the core ($u(x) = 0$ if $x < 0, u(x) = 1$ if $x > 0$). However, one can observe the complexity of the solution, which presents several shocks.

In Figures 4 and 5 we compare the finite difference solutions calculated when using the numerical flux based on the exact Riemann solver (ERS) (3.1), (3.3) and the one calculated when using the upstream mobility flux (5.3), (5.4). We can observe that the latter is doing very well even in these complex situations. However, small differences can be seen. In particular, a small boundary layer appears on the left side of the interface. For these numerical examples these differences vanish when $h \to 0$ if they are measured in the L_1 norm.

Finally, in Figure 6 we present the solution given by the numerical flux which was presented in [3, 8, 13, 6] (ERS-NIF) and which is not valid when the flux functions intersect in the undercompressive case, which is our situation. The picture in Figure 6 is to be compared with the bottom picture in Figure 4. As expected, this numerical flux is not able to capture the complexity of the solution.

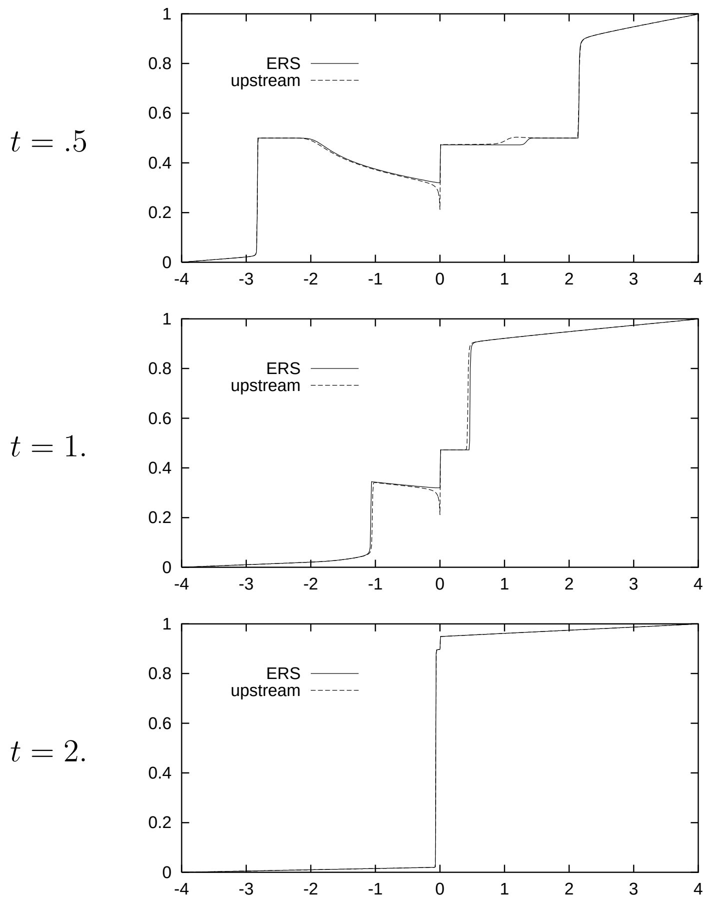

FIG. 5. *Finite difference solutions calculated with numerical flux* (3.1), (3.3) *(ERS) and with the upstream mobility flux* (5.3), (5.4) *at different times for a constant initial data (*$h = 1/100$*).*

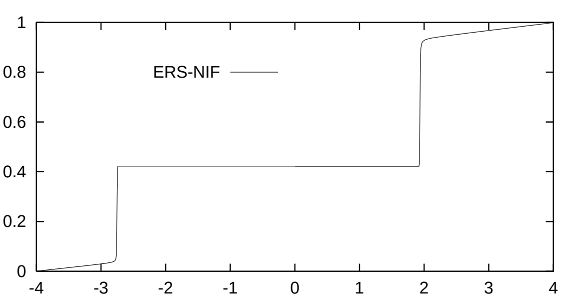

FIG. 6. *Finite difference solution calculated when using the numerical flux for nonintersecting fluxes.*

7. Conclusion. The calculation of the solutions of conservation laws with a flux function discontinuous in space needs appropriate numerical methods. We presented a Godunov method which uses an exact Riemann solver, and we proved convergence of the corresponding numerical scheme. We compared numerically with the upstream mobility numerical flux used for multiphase flow in porous media, showing that the latter still works well in the case of a discontinuous flux function. A consequence of the proof of the convergence of the numerical scheme is an existence and uniqueness of the solution to the continuous problem.

Appendix A. End of the proof of existence and uniqueness theorem, Theorem 2.1. In this appendix we terminate the proof of Theorem 2.1 for nonconvex functions as in Figure 1. Existence was a consequence of the convergence theorem, Theorem 3.2, and to prove uniqueness we need to show that all solutions of (1.2) satisfying entropy conditions (2.1) and (2.2) can be represented by an L^1-contractive semigroup. The proof is as in [1], so we sketch only the proof. The main idea of this proof goes back to Kruzkov [19].

LEMMA A.1. *Let $u, v \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ with $s \le u, v \le S$ be two solutions of* (1.2) *with initial data $u_0, v_0 \in L^\infty(\mathbb{R})$, respectively. Assume the following:*

(i) *For almost every t, $u(x+,t), v(x+,t), u(x-,t)$, and $v(x-,t)$ exist.*

(ii) *The set of discontinuities of u and v is a discrete set $\{\Gamma_j\}_{j\in N}$ of Lipschitz curves.*

(iii) *u and v satisfies the entropy conditions* (2.1) *and* (2.2).

Then for any $\overline{M} \ge M$, $a < 0, b > 0$, $b - a \ge 2\overline{M}t$ the function

$$t \mapsto \int_{a+\overline{M}t}^{b-\overline{M}t} |u(x,t) - v(x,t)|dx$$

is nonincreasing.

Proof. The first three steps are exactly as in Kruzkov's proof (see [12, p. 24]), and the interface entropy condition (2.2) is used to prove Step 4.

Step 1. Let $l \in \mathbb{R}$, $\varphi_l(\theta) = |\theta - l|$, $\tilde{f}(\theta, l) = (f(\theta) - f(l))$ sign $(\theta - l)$. Let $0 \le \rho \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$. Let $\Gamma_j^+ = \Gamma_j \cap \{(x,t) : \ x > 0, t > 0\}$ and $\nu^j = (\nu_1^j, \nu_2^j)$ be the a.e. normal to Γ_j^+. Then by integration by parts and using the interior entropy condition (2.1) we obtain

$$\begin{aligned}\int_0^\infty \int_0^\infty (\varphi_l(u(x,t))\frac{\partial \rho}{\partial t} + \tilde{f}\,(u(x,t),l)\,\frac{\partial \rho}{\partial x}dxdt \\ = \sum_{j=1}^\infty \int_{\Gamma_j^+} ([\varphi_l(u)]\nu_1^j + [\tilde{f}(u,l)]\nu_2^j)\rho d\sigma - \int_0^\infty \tilde{f}(u^+(t),l)\rho(0,t)dt \\ \ge -\int_0^\infty \tilde{f}(u^+(t),l)\rho(0,t)dt,\end{aligned} \tag{A.1}$$

where $[\varphi_l(u)] = \varphi_l(u^-) - \varphi_l(u^+)$, the jump across of Γ_j^+, $[\tilde{f}(u,l)] = \tilde{f}(u^-,l) - \tilde{f}(u^+,l)$, the jump across of Γ_j^+, and $u^+(t) = u(0+,t)$.

Step 2. Let $A(x,t,y,s) = \frac{f(u(x,t)) - f(v(y,s))}{u(x,t) - v(y,s)}$, $\alpha \in C_0^1((-1,0) \times (-1,0))$ with $\int_{\mathbb{R}^2} \alpha(z)dz = 1$ and $\beta \in C_0^1(\overline{\mathbb{R}}_+ \times \mathbb{R}_+)$. Let $\varepsilon_1 > 0, \varepsilon_2 > 0$, and define

$$\rho_\varepsilon(x,t,y,\tau) = \frac{1}{\varepsilon_1 \varepsilon_2}\alpha\left(\frac{x-y}{\varepsilon_1}, \frac{t-\tau}{\varepsilon_2}\right)\beta(y,s)\,.$$

Now taking $l = v(y,\tau)$ and $\rho = \rho_\varepsilon(x,t,y,\tau)$ in (A.1) we integrate with respect to $(y,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+$. Then using symmetry and letting $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ we obtain

$$\int_0^\infty \int_0^\infty |u(x,t) - v(x,t)| \left\{\frac{\partial \beta}{\partial t} + A(x,t,x,t)\frac{\partial \beta}{\partial x}\right\} dxdt \ge -\int_0^\infty \tilde{f}(u^+(t), v^+(t))\beta(0,t)dt.$$

Step 3. Let $b \geq 0$ and χ_ε be a decreasing smooth function in $(0,\infty)$ converging to $\chi_{[0,b]}$ as $\varepsilon \to 0$. Let $0 \leq \varphi \in C_0^1(\mathbb{R}_+)$, and let $\beta(x,t) = \chi_\varepsilon(|x| + \overline{M}t)\varphi(t)$ in the above equation. Letting $\varepsilon \to 0$ we can write

$$\int_0^\infty \varphi'(t) \int_0^{b-\overline{M}t} |u(x,t) - v(x,t)|dt \geq -\int_0^{b/\overline{M}} \tilde{f}(u^+(t), v^+(t))\varphi(t)dt\,.$$

Similarly for $x \leq 0$,

$$\int_0^\infty \varphi_1(t) \int_{a+\overline{M}t}^0 |u(x,t) - v(x,t)|dt \geq \int_0^{b/\overline{M}} \tilde{g}(u^-(t), v^-(t))\varphi(t)dt.$$

Adding both inequalities we obtain

$$\int_0^\infty \varphi'(t) \int_{a+\overline{M}t}^{b-\overline{M}t} |u(x,t) - v(x,t)|dx \geq \int_0^{b/\overline{M}} (\tilde{g}(u^-(t), v^-(t)) - \tilde{f}(u^+(t), v^+(t))\varphi(t))dt.$$

Step 4. So far, all the above steps are standard, and now we will make use of the interface entropy condition (2.2) to prove Lemma A.1. In order to prove the lemma it is sufficient to show that for almost all $t, I(t) \geq 0$, where

$$\begin{aligned} I(t) = \tilde{g}(u^-(t), v^-(t)) - \tilde{f}(u^+(t), v^+(t)) = |u^-(t) - v^-(t)|\frac{g(u^-(t)) - g(v^-(t))}{u^-(t) - v^-(t)} \\ -|u^+(t) - v^+(t)|\frac{f(u^+(t)) - f(v^+(t))}{u^+(t) - v^+(t)}. \end{aligned}$$

Without loss of generality, we can assume that $u^+(t) > v^+(t)$. If $f(u^+(t)) \leq f(v^+(t))$, then $I(t) \geq 0$. Hence let $f(u^+(t)) > f(v^+(t))$. Since $u^+(t) > v^+(t)$, from hypothesis (H_2) we have $u^+(t) \in (\theta_f, S]$. From the interface entropy condition (2.2) either $u^+(t) = u^-(t) = S$ or $u^-(t) \in (\theta_g, S]$. In the first case, $I(t) = 0$. In the latter case from the Rankine–Hugoniot condition, $g(u^-(t)) > g(v^-(t))$ and from hypothesis (H_2) $u^-(t) > v^-(t)$, and hence $I(t) = 0$. This completes the proof of (A.2) and of Lemma A.1.

Lemma A.1 implies that

$$\int_{a+\overline{M}t}^{b-\overline{M}t} |u(x,t) - v(x,t)|dx \leq \int_a^b |u(x,0) - v(x,0)|dx.$$

Letting $a \to -\infty$, $b \to +\infty$ we obtain the L^1 contractivity and terminate the proof of Theorem 2.1.

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