

# Turbulence and Dispersion

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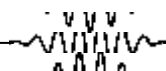
Flow of any fluid is characterized by speed  $V$ , the density of the fluid  $\rho$  a typical length of the object  $L$ , in or around which the flow occurs, and the viscosity of the fluid,  $\mu$ . These three can be combined into one dimensionless group, called Reynolds number,  $Re = \frac{\rho LV}{\mu}$ , and its magnitude determines many characteristics of flow. At low Reynolds numbers, usually achieved at low speeds, flow is smooth and orderly. Such flows are called laminar flows. When the Reynolds number is increased, the flow becomes irregular and random. This can be easily observed when a tap is cracked open, and when the opening is increased. The irregular phenomena is referred to as turbulence, and such flows are referred to as turbulent flows. In general, this transition from regular to random motion seems to occur, in a given equipment or apparatus, almost suddenly at a particular Reynolds number. Thus, in turbulent flow, even when all experimental conditions are kept constant, the velocity at any given point changes with time though it remains constant on the average. These phenomena can be seen in the snapshots of flow, 'made visible' by introducing marker particles, along with a time record of velocity at a point as shown in Figure 1. As it turns out, most important flows are turbulent! Motion of air in the atmosphere is turbulent, flow in rivers is turbulent, and in fact so are most water flows in pipes<sup>1</sup>. Despite a large number of people devoting time and effort, turbulence is not understood well. Motzart quotes Einstein: 'turbulence is the last unsolved problem of classical physics' [1].

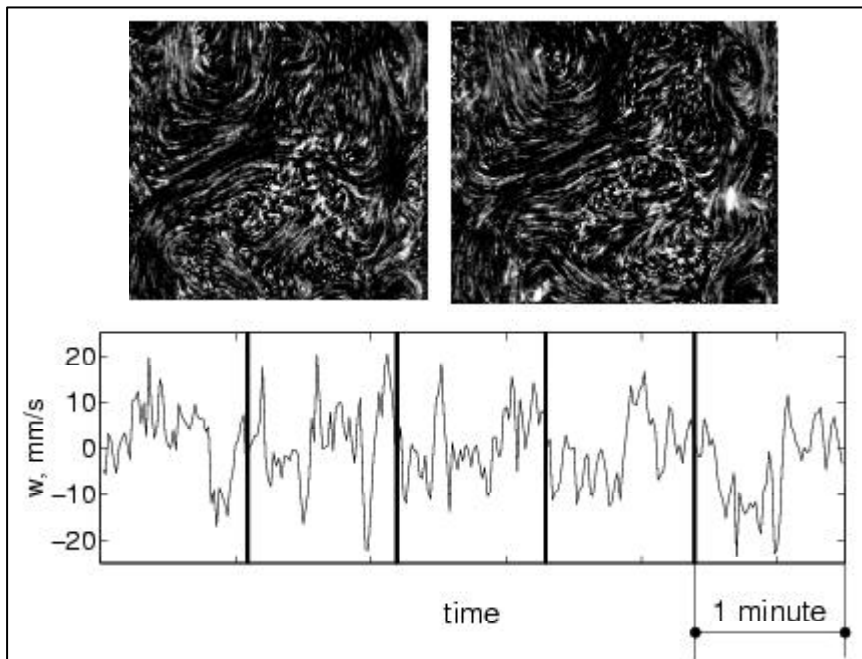
<sup>1</sup>Laminar flow may be roughly compared to the smooth, orderly traffic on a highway, and turbulent flow to the zig-zag chaotic traffic on city roads.

## Keywords

Turbulent diffusion, G I Taylor, atmospheric pollution.

There are many consequences of the random nature of motion, and one of them is the dispersion of matter. In





**Figure 1.** Pictures at the top show motion of marker particles in turbulent flow in a pipe and illustrate the random nature of flow. The time interval between the two frames is one second. The bottom frame shows the velocity observed at a point as a function of time. The velocity at a point varies around an average even when all conditions are kept constant [2].

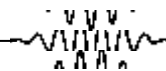
a seminal paper published [3] in 1921, G I Taylor made a significant contribution to the understanding of this phenomenon and this is the topic of the present article.

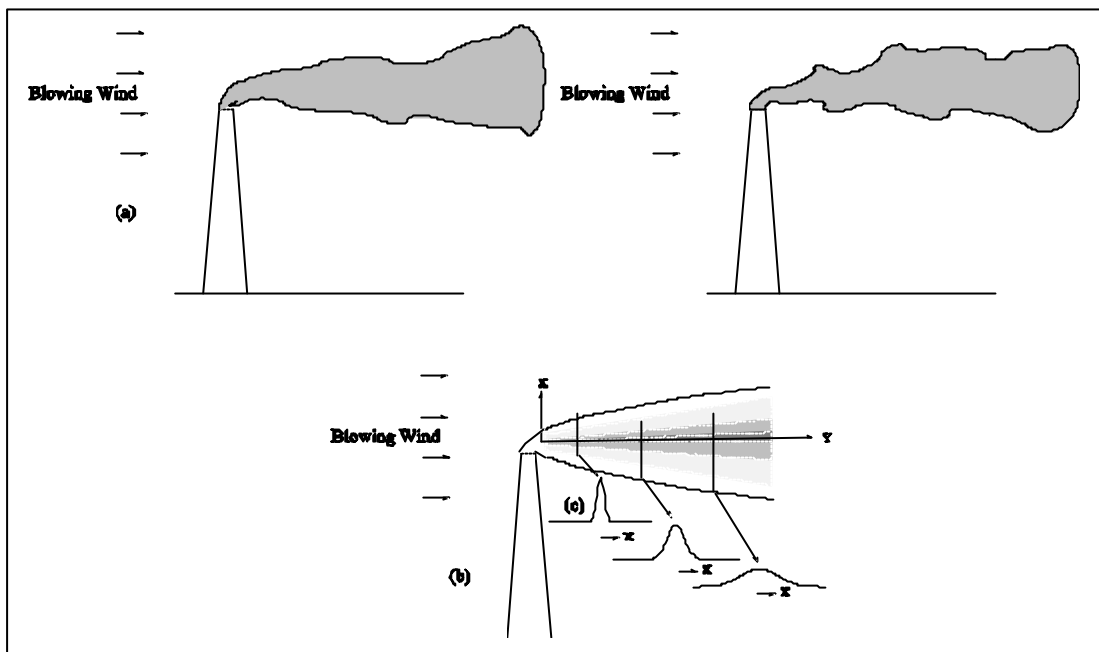
## Dispersion

Watch smoke flowing out of a chimney<sup>2</sup>. On a quiet day, it is seen that after rising for a small distance, the smoke stream bends and flows along with the wind. This is to be expected since the smoke particles are carried, or in technical terminology, convected by the wind in the direction of flow. This is called convective transfer. However, it is observed that smoke also spreads laterally as it flows. This is illustrated in Figure 2. Thus, the number of smoke particles per unit volume, or the concentration of smoke particles reduces and eventually, it is so dilute that the particles can not be seen<sup>3</sup>. The lateral spreading appears similar to diffusion. If a lump of sugar is placed in a beaker containing water, it will slowly dissolve and spread or diffuse into water. Eventually the concentration of sugar in water will be the

<sup>2</sup> Smoke is essentially a gaseous stream consisting of very tiny particles of soot. As they are tiny, they move with the velocity of the surrounding air. You can observe the main features of spreading of smoke if an 'agarbatti' is lit and put in gently blowing wind.

<sup>3</sup> As the jet of smoke flows out of the chimney, it can also suck some surrounding air into its interior. This is referred to as *entrainment*, and dilution can also occur due to it. But it is usually negligible since the velocity of smoke stream is small.

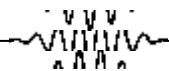




**Figure 2.** Two snap shots of smoke spreading from a chimney are shown in (a). The boundary is zig-zag, and dependent on time. However, if we superimpose many snap shots, which is equivalent to a time averaging process, the picture looks ordered and steady. This is illustrated in (b). It is found that the concentration of smoke particles in the center is more and decreases toward edge of the smoke stream. The averaged concentrations are plotted against distance from the center in graphs shown in (c) in the bottom frame. The plots look like those shown as lines. The three lines correspond to cross sections at different distances from the chimney. The vertical axis on these lines is the averaged concentration.

same everywhere in the beaker. The rate at which diffusion occurs is determined by the diffusion coefficient, which decreases as the molecular weight of the diffusing species increases. Similar to molecules, heat also diffuses in the presence of a temperature gradient. These processes are referred to as molecular diffusion and theories for describing them have been developed based on kinetic theory of gases.

There are some differences between diffusion of sugar in water and dispersion or spreading of smoke coming out of a chimney. While sugar is soluble in water, smoke is not soluble in air. The smoke particles are distinct



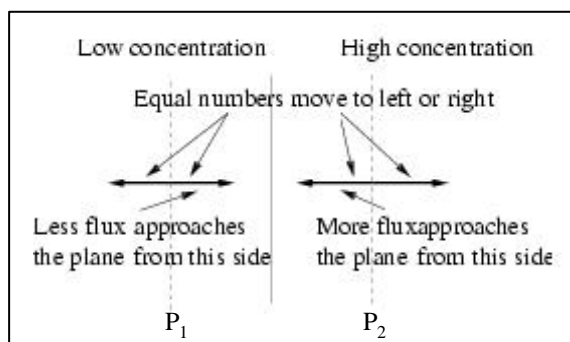
entities. This difference might appear to be only conceptual, since, on a molecular scale, water and sugar molecules are also distinct entities. Thus, if we apply molecular theories of diffusion to calculate the rate at which smoke diffuses into air, it turns out to be very small. For example, on a day when wind speed is 15 km/hr, smoke stream coming out of a chimney of 1 m diameter will spread and achieve a diameter<sup>4</sup> of about 2 m only after traveling a distance of 1250 km! In other words, smoke will hardly spread in lateral direction if molecular diffusion were to be the cause of spreading. This is contradictory to observations and smoke is seen to spread much faster than predicted by molecular diffusion.

<sup>4</sup> The method of making these estimates is considered later on.

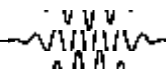
Taylor recognized that the rapidity of the dispersion of smoke particles in air is due to turbulence in the wind, and was able to explain quantitatively the rate of dispersion. A key to his explanation is his grasp of the similarity in the diffusion at the molecular scale and dispersion due to turbulence at the macroscopic scale, that is, at a scale visible to naked eye. To understand this, let us examine the ideas pertaining to molecular diffusion.

### Diffusion of Molecules

Molecules possess kinetic energy in proportion to the temperature. However, the motion is random, and at any given point, molecules move with equal probability in all directions. Thus, referring to Figure 3, equal



**Figure 3. Diffusion occurs by random motion of molecules. Molecules will on the average move in one direction only if concentration gradients exist.**



number of molecules will move to the left or right of plane  $P_1$ . Consider another plane  $P_2$  parallel to it but at a small distance away but where the concentration is higher than at  $P_1$ . Once again equal number of molecules will move to the left or right of this plane too. But more number of molecules will move from one plane to the other, that is, there is net movement of molecules in one direction, if the concentration of molecules at one plane is different from the other. It is this combination of random motion and difference in concentrations that causes 'diffusion' e.g., of sugar in water. The essential point is that diffusion occurs by random movement.

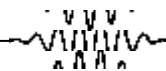
Let us consider the following experiment. Place a long and thin slice of a dye solution, containing  $N$  molecules of dye, in a large tank of water. A concentration difference is created as the dye is present only in the slice. Therefore, molecules will diffuse away from the slice both to its right and left. As motion is random, some will go far while some will move less. The dye will slowly spread. The spreading process is described by the following equation:

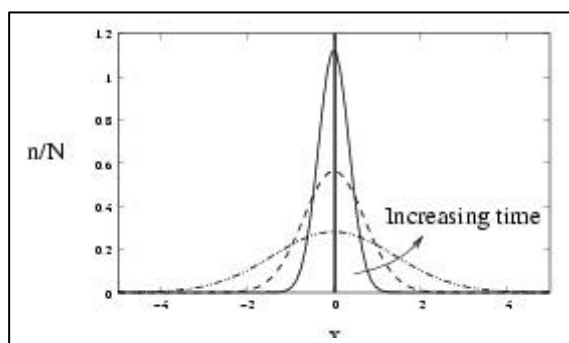
$$\frac{\partial n(x;t)}{\partial t} = D \frac{\partial^2 n(x;t)}{\partial x^2};$$

where  $n(x;t)$  is the concentration of dye molecules on a plane located (parallel to the slice) at  $x$  and at time  $t$ .  $D$  is the molecular diffusion coefficient, and we will see that its magnitude determines the rate of spreading of molecules<sup>5</sup>. Do not worry about the equation since it is not necessary to know how to solve it to understand the principles of diffusion or dispersion. The solution to the equation, for the problem of diffusion from a thin, large slice in a large volume of water, is given by

$$\frac{n(x;t)}{N} = \frac{1}{\sqrt{2\pi Dt}} e^{-\frac{x^2}{4Dt}}; \quad (1)$$

<sup>5</sup> It turns out that a similar equation also describes how heat diffuses (or is conducted) in various media.





**Figure 4.** Spreading of a slice of dye, shown by a thick vertical line, initially located at  $x = 0$ . Spreading occurs by molecular diffusion. Concentration of dye decreases away from the initial location and decreases with time everywhere.

Thus,  $n(x; t)$  number of molecules have moved a distance  $x$  from the origin, from where they started, into a small strip of thickness  $\pm x$ . The solution is plotted in Figure 4. From the figure, it is seen that more molecules are located near the center while increasingly small numbers are found away from the origin. This is to be expected. To move away by a significant distance in positive (or negative) direction, motion must be more coordinated in that direction. It is like finding 8 out of 10 tosses of a coin to be heads. In random motion, such an occurrence, though possible, is less probable.

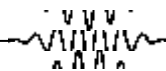
#### Diffusion Coefficient and Random Velocity

The diffusion coefficient has to be related to the random velocities to gain more insight. This is done in the following way. Different molecules will move different distances. The average can be found in the usual way. The number of molecules in a strip of unit area and thickness  $\pm x$  is given by  $n(x; t)\pm x$ . Thus, the average of the square of the distance traveled by molecules from the place they started, is given by

$$\overline{x^2} = \frac{\int_{-R_1}^{R_1} x^2 n(x; t) dx}{\int_{-R_1}^{R_1} n(x; t) dx} \quad (2)$$

The bar in the equation represents the average of the quantity beneath, with the average being taken by considering all the molecules. It can be calculated from (1) to be

$$\overline{x^2} = 2Dt \quad (3)$$



This equation can also be written, after differentiation with time, as

$$D = \frac{1}{2} \frac{d}{dt} \overline{x^2}$$

Carrying out the differentiation inside the integral and noting that  $v = dx/dt$ , we get

$$D = \overline{vx}$$

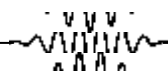
The velocity  $v$  is that of the molecules due to the random thermal motions. Though the velocity is randomly oriented, the distance moved is related to the velocity at least for some time, and hence the average value of  $vx$  will not be equal to zero. There is yet another way of writing the above. Since the molecules were initially placed at  $x = 0$ , the position of any molecule at time  $t$  is given by

$$x = \int_0^t v(t') dt'$$

As different molecules have different velocities, all molecules will not be at the same place, i.e., will not have the same value of  $x$ , at time  $t$ . Therefore the value of  $v(t)x$  will also be different for different molecules. The average value of  $v(t)x$  is equal to the diffusion coefficient. Thus,

$$D = \overline{v(t) \int_0^t v(t') dt'} = \int_0^t \overline{v(t)v(t')} dt' \quad (4)$$

The right hand side is referred to as velocity autocorrelation. The integrand is the product of velocity at the current time with the velocity at some earlier time. Molecules collide with each other and the collision changes the direction of the molecule's motion. Thus, one would expect that after some time, the molecule has 'forgotten' the direction of its motion. As a result, the integrand will become zero when the difference between  $t$  and  $t'$  becomes large. For the same reason, the above integral would also be independent of  $t$ , if  $t$  is sufficiently large.



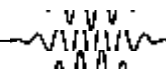
The velocities that appear in the above equation are those of molecules, and  $D$  is referred to as molecular diffusivity. Typically, the value is of the order of  $10^{15}$   $\text{m}^2/\text{sec}$  in gases. The diffusion coefficient of smoke particles, as they are heavier than molecules, will be smaller than this, but, even if this value of diffusion coefficient is used, the spreading of smoke is predicted to be extremely small as was indicated in the previous section. The reason for this is the very small value of the molecular diffusion coefficient. It is easy to see from (3) that the distance that molecules are able to move (on the average) in a given time  $t$  is  $\sqrt{2Dt}$ . Conversely, the time needed for molecules to move away by a distance  $a$  from the place where they have been placed is proportional to  $a^2/D$ . The time needed would obviously be smaller if the distance they have to move is small, and molecular diffusion is effective only over small length scales.

Let us recapitulate a little now. As mentioned earlier, spreading of smoke appears similar to molecular diffusion. However, if it were to occur by molecular diffusion spreading of smoke will be very slow compared to observations. Hence, it must occur by an entirely different process.

### Dispersion Due to Turbulence

Taylor recognized that just as diffusion of molecules occurs due to their random motion, dispersion must occur due to the random motions in turbulent flow. But what are the entities analogous to molecules? He imagined that fluid can be thought of as a collection of tiny packets of fluid or fluid particles<sup>6</sup>. It is these fluid particles that move randomly. Referring to Figure 1, though the main flow is in one direction, the fluctuating random velocities can also be in directions perpendicular to the main flow direction. These fluctuating velocities in the transverse directions will carry fluid particles, and any matter in it, like smoke, in directions perpendicular

<sup>6</sup> Each fluid particle will contain a huge number of molecules. The 'scale of a turbulent motion' is many orders greater than the scale of molecular motion.





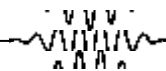
<sup>7</sup>The name 'eddy' arises since the velocity fluctuations create eddying motions.

to the main motion also. Smoke thus will disperse in the atmosphere and the rate at which it disperses can be calculated using an equation similar to the diffusion equation (1). To predict dispersion, the diffusivity that must be used in that equation is the 'diffusivity' due to turbulent motion (referred to as eddy diffusivity<sup>7</sup>,  $D_t$ ) and not the molecular diffusivity used in (1). Taylor exploited this similarity in the mechanisms of molecular diffusion and dispersion due to turbulence and proposed that the eddy diffusivity is related to the velocity fluctuations due to turbulence in the same way as molecular diffusivity is related to random molecular velocities. In other words, the eddy diffusivity is given by the same equation (4), except that the velocities are those due to turbulent motion. This equation is, to quote Taylor, "rather remarkable because it reduces the problem of diffusion ... to the consideration of a single quantity, namely, the correlation coefficient between the velocity of a particle at one instant and that at a (later) time".

<sup>8</sup>Presently, sophisticated techniques of imaging particles very frequently are available, and such averages can be experimentally measured in the laboratory. It is still not possible to do so at a larger scale.

<sup>9</sup>One common method to measure the rapidly varying fluid velocity is with a hot-wire probe. Typically a hot-wire probe consists of a heated  $5\mu\text{m}$  tungsten wire. The rate of heat lost by the wire to the surrounding fluid can be related to the instantaneous velocity of the fluid.

The power of Taylor's theory is that the eddy diffusivity can be measured by measuring the velocity autocorrelation. This can be done as follows. A marker particle can be introduced, and its velocity can be recorded as a function of time. A sufficiently long period from that record can be selected. The velocity at the end of the record can be multiplied by velocity at earlier times and the integral of the product can be evaluated. The process can be repeated and the average can be determined. The average then gives a measure of the eddy diffusivity. However this is quite difficult to actually carry out<sup>8</sup>. Taylor proposed, based on insights provided by observations, that the average of the autocorrelation of velocity measured at a point can be assumed to be the same as the average of the autocorrelation measured by following a large set of particles. (See Box 1). Whereas it is difficult to measure the velocity of a tagged particle, it is possible to measure the velocity at a point using a

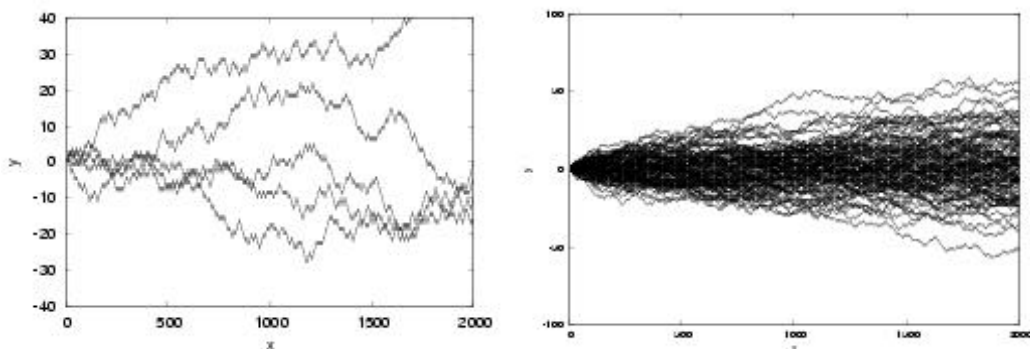


single velocity probe<sup>9</sup>. Thus all that has to be done is to measure velocity at a given point, multiply the current velocity with values at earlier times and evaluate the

#### Box 1. Simulation of Smoke Movement

Simple simulations can be used to demonstrate that random motion of smoke particles produces spreading in directions perpendicular to the direction of its main motion. Let a particle move with a constant velocity in the x direction. This simulates the steady non-random movement representing the wind. The random movement in the y direction can be simulated by moving particles in the positive or negative direction. Randomness is introduced by selecting the direction according to the result of tossing a coin: positive if heads turns up and negative if tails show up. To keep things simple, let us assume that the speed of motion is constant and that only the direction is random. Then in a small time interval, a smoke particle will move forward by distance  $\Delta x$  and by  $\pm \Delta y$ . By selecting  $\Delta x \gg \Delta y$ , the motion in the x direction becomes the strong wind, and the lateral movement represents dispersion. Now the paths of the particles can be traced by moving them by  $\Delta x$  and by  $\pm \Delta y$  as dictated by the toss. Let all particles start from the origin, the origin representing the chimney. Paths of the particles are plotted for 1000 steps. Figure A shows a few paths, and the stepping in the y direction can be seen to be random. Figure B shows 100 paths. The spreading in the y direction is apparent. The decrease in the intensity of the color also indicates that concentration decreases in the x or the down direction and also in the y direction away from the center.

Taylor's approximation of evaluating the eddy diffusivity is easily understood from this figure. We can take any one of the paths shown in Figures A and B and evaluate the integral of equation (4). The eddy diffusivity is obtained by taking several paths and evaluating the average of the values of the integral. Taylor suggested that this value will be equal to evaluating the autocorrelation by observing the velocity at any point (x; y) for a very long time. It is a plausible assumption since if the time is long enough, particles with all possible velocity histories would pass through a given point.



**Figure. (A) shows paths of five particles to clearly show the stepping nature. (B) has paths of 100 particles. (Courtesy: M V Ramana Reddy)**

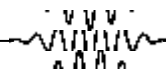


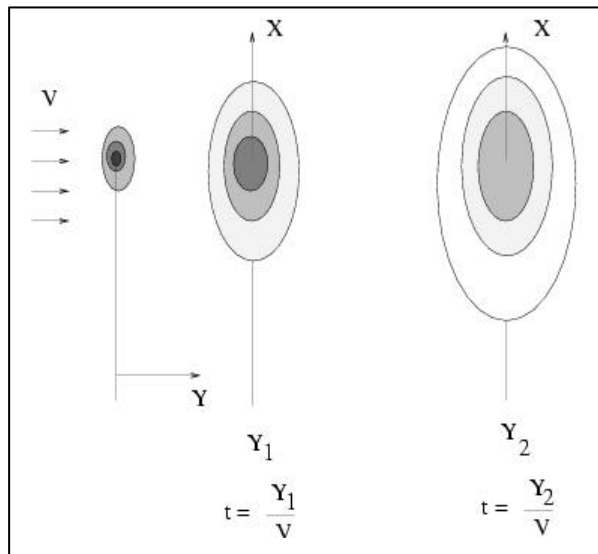
integral of the product. Taylor showed that this works quite well under appropriate conditions.

Unlike at molecular level, the fluctuations in velocity in turbulent flow are large. But similar to the molecular diffusion coefficient, which depends on the temperature or the root mean square of speed of molecular motion, the eddy diffusion coefficient also depends upon the magnitude of the mean velocity. Typical numbers are of the order of  $10 \text{ m}^2/\text{sec}$  in atmosphere at a wind speed of about  $5 \text{ m}/\text{sec}$ . Thus, eddy diffusivity is about a million times larger than molecular diffusivity. In other situations, the eddy diffusivities can be larger than molecular diffusivities by a factor of a thousand. Thus, dispersion due to turbulence is very much faster than the spread due to molecular diffusion.

One more link is needed to connect spreading of smoke in atmosphere, where wind is present, to the diffusion equation (1), which describes the rate of diffusion of a slab in a stationary medium. If smoke comes out of a chimney continuously into the atmosphere with wind blowing at a constant speed, it spreads into a diverging jet-like shape and remains unchanged with time on the average, as shown in Figure 2(b). Consider such a situation. A parcel of smoke comes out of the chimney and flows away from it and parallel to the ground (that is in the Y direction of Figure 2) with the same velocity as the wind speed. Since the speed is constant, the parcel will have to travel for time  $t = Y/V$  to move away from the chimney by a distance Y. This is illustrated in Figure 5. The extent to which the concentration of smoke from this parcel has decreased from its center at such a time can be calculated from the solution to the diffusion equation (1). Thus knowing wind speed, we can calculate the extent of spreading at different distances away from the chimney<sup>10</sup>. Returning to our earlier calculation of spreading of smoke, typical values of eddy diffusivity

<sup>10</sup> Advanced readers will recognize this to be the solution of the convective diffusion equation with constant velocity.

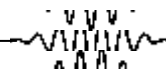




**Figure 5. Spreading of a small packet of smoke as it travels away from the chimney. The time it takes to move away by distance  $Y$  is related to the wind velocity. Due to dispersion, concentration decreases away from the center and the diameter of the smoke stream increases. But the diameter continues to increase as it travels further. Hence, concentration in the strip decreases from the value it had when it came out as it moves away from the chimney.**

will predict that, at a wind speed of 5 m/sec, spreading will occur in about 1 km to such an extent that smoke can hardly be seen. This is very close to observed values. As mentioned earlier, in the absence of turbulence, smoke will not spread laterally, but will be carried only in the direction of wind.

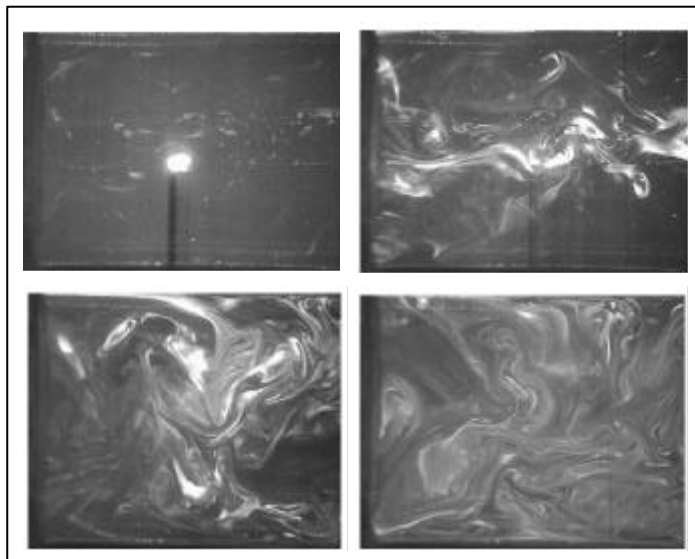
In the atmosphere, there are several complications that introduce inaccuracies into Taylor's theory. The assumption of constant wind speed is not very correct. Further, the atmosphere can be unstable and that means the characteristics of turbulence under different stability conditions would be different. This in turn means that the eddy diffusivity will be different under different atmospheric conditions. Lastly, the assumption of equating the autocorrelation at one point to the autocorrelation that should be evaluated by moving with the fluid is also not correct. However, the basic framework of Taylor's theory is still used by introducing appropriate eddy diffusivity values, which are correlated to different atmospheric conditions. Interested readers can refer to the book by Scorer [4] for qualitative description and the book by Panofsky and Dutton [5] for quantitative information.



## Other Applications of Taylor's Theory

It turns out that Taylor's theory, developed in the context of dispersion of particles like smoke, has application to soluble materials like sugar as well! Suppose a concentrated solution of sugar is placed (very gently to prevent any motion) on top of a more dilute solution. Sugar will diffuse from the concentrated solution to the dilute one, and the two solutions will 'mix', but very slowly. If the two solutions are stirred vigorously, then, turbulence is created. The random motion breaks up the two solutions into blobs, and blobs of one solution are drawn into the midst of the other. This phenomena can be illustrated by introducing a fluorescent dye into a turbulent flow and photographing it. A sequence of such photographs is shown in Figure 6. Continued stirring breaks up these blobs into smaller ones and thus they get dispersed into each other. This is exactly the process of turbulent dispersion and Taylor's theory can be used to predict the rate of this process. In the end, when the blobs become tiny, molecular diffusion is able to achieve homogeneity on a molecular scale in reasonable time periods. Thus, we are able to smell flowers

**Figure 6. Dispersion of dye injected at a point in turbulent flow [2]. The images are eight seconds apart. Notice that blobs form and break up. The blobs move away or spread. The blobs also become thinner and thinner. When they become sufficiently thin, molecular diffusion evens out the concentration.**



due to this kind of process whereby packets of air mixed with aroma are created and dispersed into the atmosphere. Had it not been for this process, even a °ower placed very near can not be smelled!

Similar is the case with e<sup>2</sup> uents coming out of factories, which can be toxic. Here the question is how tall a chimney should be so that the harmful e<sup>2</sup> uents are sufficiently diluted by the time the expanding stream hits the ground. For example, such questions had to be answered in determining how far the Mathura refinery should be situated from Taj Mahal to prevent gas e<sup>2</sup> uents being let out of chimneys from the refinery from corroding<sup>11</sup> the monument. It turns out that this process of turbulent dispersion plays a very important role in fast reactions like combustion where it is necessary to mix air and fuel in proper proportions quickly to burn the fuel efficiently. Taylor's ideas, no doubt refined by recent research, play an important role in answering these difficult questions, if not exactly, with strong scientific basis.

<sup>11</sup> Amount of effluents emitted also needs to be known to provide the answer.

### Suggested Reading

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Although aeroplanes have not had so detestable an effect on our country as automobiles, the activities of travel agents are forcing them along the same destructive path.

– G I Taylor

