INVERSE PROBLEMS FOR LORENTZIAN MANIFOLDS AND NON-LINEAR HYPERBOLIC EQUATIONS

YAROSLAV KURYLEV, MATTI LASSAS, GUNTHER UHLMANN

Abstract: *We study two inverse problems on a globally hyperbolic Lorentzian manifold* (M, g)*. The problems are:*

1. Passive observations in spacetime: Consider observations in an open set $V \subset M$*. The light observation set corresponding to a point source at* $q \in M$ *is the intersection of* V *and the light-cone emanating from the point* q*. Let* $W \subset M$ *be an unknown open, relatively compact set. We show that under natural causality conditions, the family of light observation sets corresponding to point sources at points* $q \in W$ *determine uniquely the conformal type of* W*.*

2. Active measurements in spacetime: We develop a new method for inverse problems for non-linear hyperbolic equations that utilizes the non-linearity as a tool. This enables us to solve inverse problems for non-linear equations for which the corresponding problems for linear equations are still unsolved. To illustrate this method, we solve an inverse problem for semilinear wave equations with quadratic non-linearities. We assume that we are given the neighborhood V *of the time-like path* μ *and the source-to-solution operator that maps the source supported on* V *to the restriction of the solution of the wave equation to* V*. When* M *is 4-dimensional, we show that these data determine the topological, differentiable, and conformal structures of the spacetime in the maximal set where waves can propagate from* μ *and return back to* μ*.*

Keywords: Inverse problems, Lorentzian manifolds, non-linear hyperbolic equations.

Contents

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1. Introduction and main results

We study the question of whether an observer in spacetime can determine the structure of the surrounding spacetime by doing measurements near its world line. We consider two kinds of problems: inverse problems for active measurements and for passive observations.

For active measurements, we consider the wave equation

$$\Box_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M, \qquad u(x) = 0, \quad \text{outside causal future of } \mathrm{supp}\,(f), \tag{1}$$

on the Lorentzian manifold (M, g), a future-pointing time-like path $\widehat{\mu} = \widehat{\mu}([-1, 1]) \subset M$ and an open neighborhood V of $\widehat{\mu}$. The wave equation is considered as a model problem for which we demonstrate the new techniques we develop. We will consider in a follow up paper the Einstein equations coupled with scalar fields with applications to general relativity and other physical models involving non-linear hyperbolic equations, see [46]. We assume that we can control sources supported in V and measure the physical fields in the same set V. Our aim is to determine the conformal class of the metric (or even the metric tensor in some cases) in a suitable larger set $J = J^+(p^-) \cap J^-(p^+)$, that is the set of the points that are in the causal future of the point $p^- = \widehat{\mu}(s_-)$ and in the causal past of the point $p^+ = \widehat{\mu}(s_+)$, where $-1 < s_- < s_+ < 1$, see Fig. 1(Left). We study the inverse problem for active measurements by considering the interaction of distorted plane wave packets (Fig. 1, Right) reducing the problem to the problem for passive observations.

The new method we introduce in this paper for inverse problems for non-linear hyperbolic equations utilises the non-linearity as a tool. This enables us to solve inverse problems for non-linear equations for which the corresponding problems for linear equations are still unsolved. Indeed, the existing uniqueness results for linear hyperbolic equations with vanishing initial data are limited to the time-independent or real-analytic coefficients, see e.g. [2, 7, 8, 19, 20, 42] since these results are based on Tataru's unique continuation theorem [70, 71]. Such unique continuation results have been shown to fail for general metric tensors which are not analytic in the time variable [1].

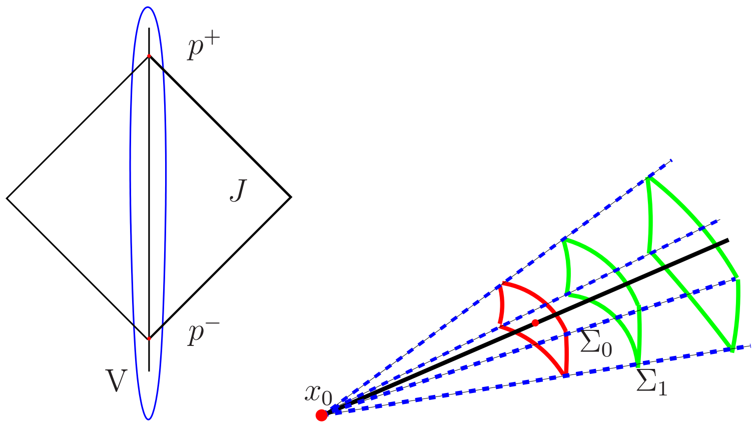

FIGURE 1. **Left:** *This is a schematic figure in $\mathbb{R}^{1+1}$. The black vertical line is a time-like path $\widehat{\mu}$ that contains the points p^- and p^+. The neighborhood V of $\widehat{\mu}$ is marked by a blue curve. The black "diamond" is the set $J = J(p^-, p^+) = J^+(p^-) \cap J^-(p^+)$.*

Right: *This is a schematic figure in the space $\mathbb{R}^3$. It describes the location of a distorted plane wave (or a piece of a spherical wave) u_1 at different time moments. This wave propagates near a light-like geodesic $\gamma_{x_0,\zeta_0}((0,\infty)) \subset \mathbb{R}^{1+3}$, $x_0 = (y_0, t_0)$ and is singular on a subset of a light cone emanating from x_0. The black line segment is the projection of $\gamma_{x_0,\zeta_0} \subset \mathbb{R}^{1+3}$ in to $\mathbb{R}^3$. The piece of the distorted plane wave is sent from the point y_0 at the time t_0 and at a later time moment t_1 the wave is singular on the red surface $\Sigma_0 \subset \mathbb{R}^3$. At later time moments, it is singular on the green surfaces, like on Σ_1.*

Our method to solve inverse problems for the non-linear wave equation with active measurements is to apply global Lorentzian geometry, our results on the inverse problem for passive measurements, and the results on the non-linear interaction of non-smooth waves having conormal singularities. There are many results on such non-linear interaction, starting from the studies of Bony [11], Melrose and Ritter [57, 58] and Rauch and Reed, [62]. However, these studies are different from the present paper that in these papers it is assumed that the geometrical setting of the interacting singularities, and in particular the locations and types of caustics, is a priori known. In inverse problems we study waves on an unknown manifold, so we do not know the underlying geometry and, therefore, the location of singularities of the fields. For example, the waves can have caustics that may even be of an unstable type. These produce further difficulties in the analysis of the non-linear interaction. To overcome these difficulties we use methods of the global Lorentzian geometry, results for the passive inverse problem, and the layer-stripping method. These make it possible to reconstruct the accessible part of the Lorentzian manifold step by step.

The inverse problem for passive observations means the reconstruction of a region W of a Lorentzian manifold from light observation sets $\mathcal{P}_V(q)$ corresponding to points $q \in W$. The light observation set $\mathcal{P}_V(q)$ is the intersection of set V and the future light cone $^+(q)$ emanating from the source point q. Physically, this corresponds to the case of a passive observer, who registers in the set V light (or a gravitational wave) coming from a source at q. Due to the existence of conjugate points (or physically speaking, gravitational lensing or Einstein rings) such observations can by strongly distorted. Under appropriate conditions, we first show that W can be reconstructed as a topological manifold from these data. After that, we show that the differentiable structure of W and the conformal class of $g|_W$ can be reconstructed.

1.1. **Inverse problem for passive observations.** To formulate the results, we first introduce some definitions. Let (M, g) be a n-dimensional Lorentzian manifold of signature $(1, n-1)$, $n \geq 3$. In this paper we assume that (M, g) is time-oriented so that we can define future and past pointing time-like and causal paths. We recall that a smooth path $\mu : (a, b) \to M$ is time-like if $g(\dot{\mu}(s), \dot{\mu}(s)) < 0$ for all $s \in (a, b)$. Also, μ is causal if $g(\dot{\mu}(s), \dot{\mu}(s)) \leq 0$ and $\dot{\mu}(s) \neq 0$ for all for all $s \in (a, b)$. For $p, q \in M$ we denote $p \ll q$ if $p \neq q$ and there is a future pointing time-like path from p to q. We denote $p < q$, if $p \neq q$ and there is a future pointing causal path from p to q and denote $p \leq q$ when either $p = q$ or $p < q$. The chronological future of $p \in M$ is the set $I^+(p) = \{q \in M;\ p \ll q\}$ and the causal future of p is the set $J^+(p) = \{q \in M;\ p \leq q\}$. Similarly, we introduce the chronological past, $I^-(p)$, and the causal past, $J^-(p)$, see [60]. Note that $I^\pm(p)$ are always open. For a set $A \subset M$ we denote $J^\pm(A) = \cup_{p \in A} J^\pm(p)$. We also denote $J(p, q) := J^+(p) \cap J^-(q)$ and $I(p, q) := I^+(p) \cap I^-(q)$.

By [9], a time-orientable Lorentzian manifold (M, g) is globally hyperbolic if and only if there are no closed causal paths in M and for all $q_1, q_2 \in M$ such that $q_1 < q_2$ the set $J(q_1, q_2) \subset M$ is compact. Roughly speaking the last property means that M has no naked singularities which one could reach by moving along a time-like path starting from a point q^- and ending in a point q^+. In particular, this condition is needed to make the hyperbolic equations on (M, g) well posed.

We assume throughout the paper that (M, g) is globally hyperbolic. In this case, $J^\pm(p)$ are closed and $\mathrm{cl}(I^\pm(p)) = J^\pm(p)$.

Let $L_pM = \{\xi \in T_pM \setminus \{0\};\ g(\xi, \xi) = 0\}$ be the set of light-like vectors in the tangent space T_pM. Also, $L_p^+M \subset L_pM$ and $L_p^-M \subset L_pM$ denote the future and the past light-like vectors in T_pM.

Let $\exp_q : T_qM \to M$ be the exponential map on (M, g). The geodesic starting at p in the direction $\xi \in T_pM \setminus \{0\}$ is the curve $\gamma_{p,\xi}(t) = \exp_p(\xi t)$, $t \geq 0$.

Let $\widehat{\mu} : [-1, 1] \to M$ be a C^∞-smooth future pointing time-like path and $V \subset M$ be an open connected neighborhood of $\widehat{\mu}([-1, 1])$.

1.1.1. *The set of earliest light observations.* Recall that $p^\pm = \widehat{\mu}(s_\pm)$ where $-1 < s_- < s_+ < 1$. Next define the light observation sets and consider in particular the case when $W \subset I^-(p^+) \setminus J^-(p^-)$ is a relatively compact open set, see Fig. 2 (Right).

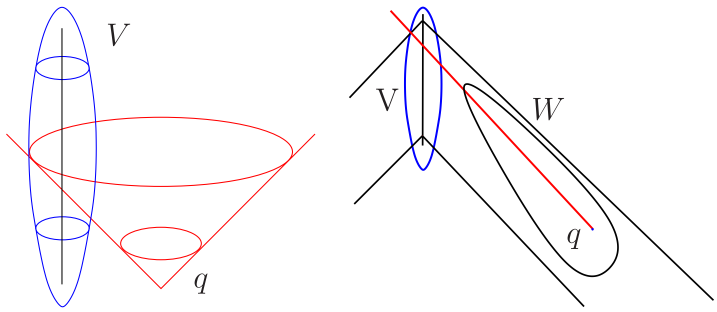

Figure 2. **Left:** *The future light cone $\mathcal{L}^+(q)$ from the point q is shown as a red cone. The point q is the tip of the cone. The set V, where observations are done, is shown in blue. The light observation set $\mathcal{P}_V(q)$ with a point source at q is the intersection $\mathcal{L}^+(q) \cap V$.* **Right:** *In Theorem 1.2, we consider a set $W \subset I^-(p^+) \setminus J^-(p^-)$. The boundary of W is shown in the figure as a black curve. The red line is a light ray from a point $q \in W$ that is observed in the blue set V. These observations are shown to determine W as a differentiable manifold and the conformal class of the metric on W.*

Definition 1.1. *(i) For $q \in M$, let*

$$^+(q) = \exp_q(L_q^+M) \cup \{q\} = \{\gamma_{q,\xi}(t) \in M;\ \xi \in L_q^+M,\ t \geq 0\} \subset M$$

be the future directed light-cone emanating from the point q.

The light observation set of q in the observation set V is

$$\mathcal{P}_V(q) = \mathcal{L}^+(q) \cap V \in 2^V.$$

(ii) The earliest light observation set of $q \in M$ in V is

$$\begin{aligned} \mathcal{E}_V(q) = \{x \in \mathcal{P}_V(q) :\ & \text{there are no } y \in \mathcal{P}_V(q) \text{ and} \\ & \text{future-pointing time-like path } \alpha : [0,1] \to V \\ & \text{such that } \alpha(0) = y \text{ and } \alpha(1) = x\} \subset V \end{aligned} \tag{2}$$

(iii) Let $W \subset M$ be open. The family of the earliest light observation sets with source points at W is

$$\mathcal{E}_V(W) = \{\mathcal{E}_V(q);\ q \in W\} \subset 2^V. \tag{3}$$

Note that $\mathcal{E}_V(W)$ *is defined as an unindexed set, that is, for an element* $\mathcal{E}_V(q) \in \mathcal{E}_V(W)$ *we do not know what is the corresponding point* q.

Above, $2^V = \{V';\ V' \subset V\}$ is the power set of V. Note that when the future directed path $\mu : [-1,1] \to V$ and the conformal type of (V,g), and therefore all time-like paths in V are known, the light observation set $\mathcal{P}_V(q)$ determines the earliest light observation set $\mathcal{E}_V(q)$, see (2).

Below, when $\Phi : V_1 \to V_2$ is a map, we say that the power set extension of Φ is the map $\widetilde{\Phi} : 2^{V_1} \to 2^{V_2}$ given by

$$\widetilde{\Phi}(V') = \{\Phi(z);\ z \in V'\}, \quad \text{for } V' \subset V_1. \tag{4}$$

Below, when we say that the set V is given as a differentiable manifold, we mean that we are given the set V and the local coordinate charts on it for which the corresponding transition maps are C^∞-smooth.

Inverse problem with passive observations: *We assume that we are given the set* V *as a differentiable manifold, the conformal class of the metric* $g|_V$ *on* V*, and the family of the earliest light observation sets* $\mathcal{E}_V(W) = \{\mathcal{E}_V(q) \subset V;\ q \in W\}$*, where* $W \subset I^-(p^+) \setminus J^-(p^-)$ *is a relatively compact open set. The inverse problem is whether these data determine the set* W *as a differentiable manifold and the conformal class of the metric* $g|_W$.

A map $\Psi : (V_1, g_1) \to (V_2, g_2)$ is a conformal diffeomorphism if $\Psi : V_1 \to V_2$ is a diffeomorphism and $g_1(x) = e^{2f(x)}(\Psi^* g_2)(x)$ for some scalar function $f(x)$. The following theorem implies that the family $\mathcal{E}_V(W)$ of the earliest light observation sets determines uniquely the conformal type of $(W, g|_W)$.

Theorem 1.2. *Let* (M_j, g_j)*,* $j = 1, 2$ *be two open,* C^∞*-smooth, globally hyperbolic Lorentzian manifolds of dimension* $n \geq 3$*,* $\widehat{\mu}_j : [-1,1] \to M_j$ *be smooth time-like paths, and* $p_j^\pm = \widehat{\mu}_j(s_\pm)$*. Let the observation sets* $V_j \subset M_j$ *be neighborhoods of* $\widehat{\mu}_j([-1,1])$ *and* $W_j \subset M_j$ *be relatively compact sets such that* $\overline{W}_j \subset J^-(p_j^+) \setminus I^-(p_j^-)$*. Let* $\mathcal{E}_{V_j}(W_j)$ *be the families of the earliest light observations sets with source points at* W_j*, see (3).*

Assume that there is a conformal diffeomorphism $\Phi : V_1 \to V_2$ *such that* $\Phi(\widehat{\mu}_1(s)) = \widehat{\mu}_2(s)$*,* $s \in [-1,1]$ *and*

$$\widetilde{\Phi}(\mathcal{E}_{V_1}(W_1)) = \mathcal{E}_{V_2}(W_2), \tag{5}$$

where $\widetilde{\Phi}$ *is the power set extension of* Φ*, see (4).*

Then there is a diffeomorphism $\Psi : W_1 \to W_2$ *such that the metric* $\Psi^* g_2$ *is conformal to* g_1 *and* $\Psi|_{W_1 \cap V_1} = \Phi|_{W_1 \cap V_1}$.

When M_j, $j = 1, 2$, have significant Ricci-flat parts, Theorem 1.2 can be strengthened.

Corollary 1.3. *Assume that* (M_j, g_j) *and* V_j*,* W_j*,* $j = 1, 2$ *satisfy the conditions of Theorem 1.2 with the resulting conformal map* Ψ :

$W_1 \to W_2$ as in Theorem 1.2. Moreover, assume that W_j are Ricci-flat and that $\Phi : V_1 \to V_2$ is an isometry. Also, assume that all topological components of W_j intersect V_j, $j = 1, 2$. Then the map Ψ is an isometry.

Idea of the proof of Theorem 1.2. The proof is given in Section 2 where the topological structure of W is reconstructed and in Section 5 where the differentiable structure of W and the conformal type of the metric $g|_W$ are reconstucted. We will define a suitable smaller observation set $U \subset V$ and consider the family $\mathcal{E}_U(W)$. The idea is to endow the set $\mathcal{E}_U(W) \subset 2^U$ with a Lorentzian manifold structure that makes it conformal to W. In other words, we consider the set $\mathcal{E}_U(W)$ as a manifold that is a "copy" of the manifold W. In the proofs we construct topological, differentialble and metric structures on $\mathcal{E}_U(W)$. To sketch the idea of the proof, we assume for simplicity that the manifold (M, g) has no conjugate points or cut points and that $U \cap W = \emptyset$. Then, any light-like geodesic segment γ_0 in the light-cone $^+(q)$, i.e., $\gamma_0 \subset {}^+(q)$, can be extended to a geodesic $\widetilde{\gamma}_0 \subset M$ that goes through the point q. Let us consider a light-like geodesic segment $\gamma_1 \subset U$ and define $\Theta(\gamma_1)$ to be the set of the elements $\mathcal{E}_U(q) \in \mathcal{E}_U(W)$ for which $\gamma_1 \subset \mathcal{E}_U(q)$. Then, when γ_1 is continued to a maximal geodesic $\widetilde{\gamma}_1 \subset M$, we have that $\Theta(\gamma_1)$ is the image of the geodesic segment $\widetilde{\gamma}_1 \cap W$ in the map $q \mapsto \mathcal{E}_U(q)$. This means that on the set $\mathcal{E}_U(W)$ we can see the images of a open family of light-like geodesics of M that intersect U. Using this we show that the map $\mathcal{E}_U : q \mapsto \mathcal{E}_U(q)$ is one-to-one and defines a homeomorphism $\mathcal{E}_U : W \mapsto \mathcal{E}_U(W)$. In this way we reconstruct the topological structure of W. The differentiable structure can be reconstructed by using the earliest observation time functions $f_a^+(q)$ on a time-like path $\mu_a \subset U$. The function $f_a^+(q)$ is equal to the smallest parameter value s for which $\mu_a(s)$ belongs in the light-cone $^+(q)$ emanating from $q \in W$. We show that that, for any $q_0 \in W$, there are $a_1, \dots, a_n$ such that $f_{a_j}^+(q)$, $j = 1, \dots, n$, can be used as local coordinates near q_0. Finally, we use the observation that for any $q \in W$ there is an open, conic set of directions $\xi \in L_q^+ M$ such that the geodesics $\gamma_{q,\xi}$ intersect the observation set U. As we can determine the images $\mathcal{E}_U(\gamma_{q,\xi} \cap W)$ of the geodesics $\gamma_{q,\xi}$ on the known manifold $\mathcal{E}_U(W)$, we can determine the image of the light-cones, $d\mathcal{E}_U(L_q^+ M)$, in the differential of the map $\mathcal{E}_U : q \mapsto \mathcal{E}_U(q)$. As this can be done for all $q \in W$, we see the images of the light-cones on the tangent bundle $T(\mathcal{E}_U(W))$ of the manifold $\mathcal{E}_U(W)$. Finally, we note that having in our possession light cones on $T(\mathcal{E}_U(W))$ we can determine the conformal class of the metric $g|_W$.

1.2. **Inverse problems for active measurements.** For active measurements, let (M, g) be a 4-dimensional Lorentzian manifold of signature $(1, 3)$.

1.2.1. *Inverse problem for the non-linear wave equation.* Several inverse problems encountered in applications are solved by constructing the coefficients of the equations using invariant techniques, e.g. using travel time coordinates. This is why many mathematical inverse problems are formulated in geometric terms, that is, on manifolds, see e.g. [2, 8, 17, 19, 25, 26, 51]. Even some linear inverse problem are not uniquely solvable. In fact, counterexamples for these problems have been based on the so-called transformation optics. This has led to models for fixed frequency invisibility cloaks, see e.g. [29, 30] and references therein.

Several physical models lead to non-linear differential equations. In small perturbations, these equations can be approximated by linear equations, and most of the previous results on hyperbolic inverse problems in the multi-dimensional case with vanishing initial data concern linear models. As noted above, the existing uniqueness results for linear hyperbolic equation with vanishing initial data are based on Tataru's unique continuation theorem [70, 71] that requires the coefficients to be constant or real-analytic in the time variable, see [1]. Also, the studies of inverse problems for hyperbolic equations with time-dependent coefficients that are based on other methods have been restricted to the case when only the lower order terms depend on time, see [66, 68], the monographs [61, 64] and the references therein.

Earlier studies on inverse problems for non-linear equations have concerned parabolic equations [38], elliptic equations [39, 41, 69], and 1-dimensional hyperbolic equations [59]. The present paper differs from the earlier studies in that in our approach we do not consider the non-linearity as a perturbation, whose effect is small, but as a tool that helps us solve the inverse problem for multidimensional non-linear wave equations with vanishing initial data and time-dependent coefficients. Indeed, it is the non-linearity that makes it possible to solve a non-linear inverse problem which linearized version is not yet solved. This is the key novel feature of this paper.

1.2.2. *Notations.* Let (M, g) be a C^∞-smooth $(1+3)$-dimensional globally hyperbolic Lorentzian manifold, where the metric signature of g is $(-,+,+,+)$.

By [10], the globally hyperbolic manifold (M, g) is isometric to a smooth manifold $(\mathbb{R}\times N, h)$, where N is a 3-dimensional manifold and the metric h has the form

$$h = -\beta(t,y)dt^2 + \kappa(t,y). \tag{6}$$

Here $\beta : \mathbb{R}\times N \to (0,\infty)$ is a smooth function and $\kappa(t,\cdot\,)$ is a Riemannian metric on N depending smoothly on $t \in \mathbb{R}$, and the submanifolds $\{t'\}\times N$ are C^∞-smooth Cauchy surfaces for all $t' \in \mathbb{R}$. Let us next identify these isometric manifolds, that is, we denote $M = \mathbb{R}\times N$.

Below, we will consider wave equation on the spacetime $(-\infty, T_0) \times N$, where $T_0 > 0$ is a fixed parameter, and consider solutions that vanish on $(-\infty, 0) \times N$.

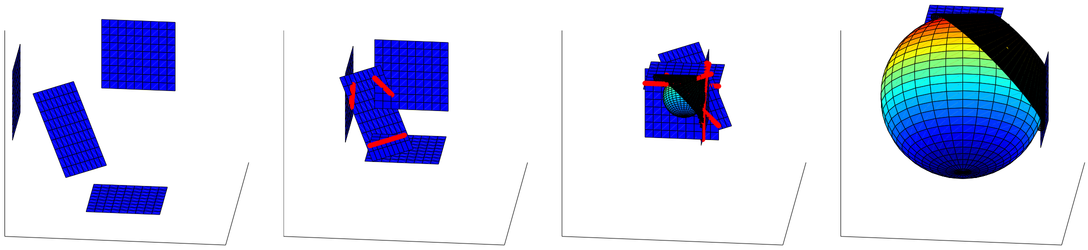

FIGURE 3. Four plane waves propagate in space. When the planes intersect, the non-linearity of the hyperbolic system produces new waves. The four figures show the waves before the interaction of the waves start, when 2-wave interactions have started, when all four waves have just interacted, and later after the interaction. **Left:** *Plane waves before interacting.* **Middle left:** *The 2-wave interactions (red line segments) appear but do not cause new propagating singularities.* **Middle right and Right:** *All plane waves have intersected and new waves have appeared. The three wave interactions cause new conic waves (black surface). Only one such wave is shown in the figure. The interaction of four waves causes a point source in spacetime that sends a spherical wave in all future light-like directions. This spherical wave is essential in our considerations. For an animation on these interactions, see the supplementary video* [76].

1.2.3. *Main result for the inverse problem for the non-linear wave equation.* Let $(M^{(j)}, g^{(j)})$, $j = 1, 2$ be two globally hyperbolic $(1+3)$ dimension Lorentzian manifolds that are isometric to manifolds $M^{(j)} = \mathbb{R} \times N^{(j)}$ having a Lorenzian metric of the form (6). Let $\widehat{\mu}_j = \widehat{\mu}_j([-1, 1]) \subset M^{(j)}$ be time-like paths, $V_j \subset M^{(j)}$ be an open, relatively compact, connected neighborhood of $\widehat{\mu}_j([-1, 1])$, and $p_j^+ = \widehat{\mu}_j(s_+)$, $p_j^- = \widehat{\mu}_j(s_-)$, where $-1 < s_- < s_+ < 1$. We assume that $V_j \subset (-\infty, T_0) \times N^{(j)}$.

Below, we sometimes drop the index j and denote $M^{(j)}$ by M, V_j by V, $g^{(j)}$ by g, a_j by a, etc.

Consider the non-linear wave equation

$$
\begin{aligned}
(7) \qquad & \square_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } (-\infty, T_0) \times N, \\
& u(x) = 0, \quad \text{for } x \in ((-\infty, T_0) \times N) \setminus J_g^+(\mathrm{supp}\,(f)),
\end{aligned}
$$

where $\mathrm{supp}\,(f) \subset V$, $x = (x^0, x^1, x^2, x^3) = (t, x') \in (-\infty, T_0) \times N$, and

$$
\square_g u = \sum_{p,q=0}^{3} (-\det(g))^{-1/2} \frac{\partial}{\partial x^p} \left((-\det(g))^{1/2} g^{pq} \frac{\partial}{\partial x^q} u(x) \right),
$$

$\det(g) = \det((g_{pq}(x))_{p,q=0}^3)$. Here, $f \in C_0^6(V)$ is a controllable source, and $a(x)$ is a nowhere vanishing C^∞-smooth function.

For any given $T_0 > 0$, the local existence results for the non-linear hyperbolic equations imply that there is $\varepsilon = \varepsilon(T_0, N, g, a, V) > 0$ such that if $f \in C_0^6(V)$ satisfies $\|f\|_{C^6(\overline{V})} < \varepsilon$, then the equation (7) has a unique solution $u \in C^2(M_0)$ see e.g. [13, 63]. Note that we do not consider here optimal results in terms of smoothness.

Definition 1.4. *Let $\mathcal{W} = \{f \in C_0^6(V);\ \|f\|_{C^6(\overline{V})} < \varepsilon\}$, where $\varepsilon > 0$ is so small that the equation (7) has a unique solution $u \in C^2((-\infty, T_0) \times N)$ for all $f \in \mathcal{W}$. The source-to-solution map $L_V : \mathcal{W} \to C(V)$, is the non-linear operator mapping the source f to the restriction of the corresponding solution of the wave equation u to the observation domain V, that is,*

$$L_V : f \mapsto u|_V, \quad f \in \mathcal{W} \subset C_0^6(V), \tag{8}$$

where u satisfies the wave equation (7) on $((-\infty, T_0) \times N, g)$.

Inverse problem with active measurements: *We assume that we are given the set V as a differentiable manifold and the source-to-solution map $L_V : f \mapsto u|_V$. The inverse problem with active measurements is whether these data determine the set $I(p^-, p^+) \subset M$ as a differentiable manifold and the conformal class of the metric $g|_{I(p^-,p^+)}$.*

The set $I(p^-, p^+)$ is the maximal set that one can reach by a causal curve that starts from p^- and ends to p^+.

Below, we return to consider two manifolds $(M^{(j)}, g^{(j)})$, $j = 1, 2$.We recall that $p_j^\pm = \widehat{\mu}_j(s_\pm)$ and denote $I(p_j^-, p_j^+) = I^+(p_j^-) \cap I^-(p_j^+)$ on $(M^{(j)}, g^{(j)})$ where the causality in $I(p_j^-, p_j^+)$ is defined using the metric $g^{(j)}$ and the path $\widehat{\mu}_j$.

Our main result for the the inverse problem for the non-linear wave equation is the following:

Theorem 1.5. *Let $(M^{(j)}, g^{(j)})$, $j = 1, 2$ be two smooth, globally hyperbolic Lorentzian manifolds of dimension $(1 + 3)$ that are represented in the form $M^{(j)} = \mathbb{R} \times N^{(j)}$ with a metric of the form (6).*

Let $\widehat{\mu}_j : [-1, 1] \to (-\infty, T_0) \times N^{(j)}$ be smooth time-like paths, $p_j^+ = \widehat{\mu}_j(s_+)$, $p_j^- = \widehat{\mu}_j(s_-)$, where $-1 < s_- < s_+ < 1$, and $V_j \subset M^{(j)}$ be neighborhoods of $\widehat{\mu}_j([-1, 1])$.

Let L_{V_j}, $j = 1, 2$ be the source-to-solution maps for wave equations (7) on manifolds $(M^{(j)}, g^{(j)})$ with nowhere vanishing C^∞-smooth functions $a_j : M^{(j)} \to \mathbb{R} \setminus \{0\}$, $j = 1, 2$, see (8).

Assume that there is a diffeomorphism $\Phi : V_1 \to V_2$ such that $\Phi(p_1^-) = p_2^-$ and $\Phi(p_1^+) = p_2^+$ and the source-to-solution maps satisfy

$$((\Phi^{-1})^* \circ L_{V_1} \circ \Phi^*) f = L_{V_2} f$$

for $f \in \mathcal{W}$, where $\mathcal{W}$ is a neighborhood of the zero function in $C_0^6(V_2)$.

Then there is a diffeomorphism $\Psi : I(p_1^-, p_1^+) \to I(p_2^-, p_2^+)$ and the metric $\Psi^* g^{(2)}$ is conformal to $g^{(1)}$ in $I(p_1^-, p_1^+) \subset M^{(1)}$, that is, there is $b : M^{(1)} \to \mathbb{R}_+$ such that $g^{(1)}(x) = b(x)(\Psi^* g^{(2)})(x)$ in $I(p_1^-, p_1^+)$. Moreover, $b(x) = 1$ for $x \in V_1$.

Later, in Remark 3.1 we will show that the set V and the map L_V determine the metric tensor $g|_V$ and coefficient $a|_V$ in V.

Outline of the proof of Theorem 1.5. The proof is given in Sections 2,3, and 4. In Sec. 2 we give preparatory geometrical results to estimate the locations of cut and conjugate points along geodesics and introduce the concepts of first observation time functions. In Sections 3 and 4 we use the non-linearity to reduce the studied inverse problem to an inverse source problem for a linear wave equation. In particular, we are interested in constructing "artificial point sources" in the spacetime. This is done by using the interaction of four distorted plane waves, see Fig. 3. Using the waves produced by such point sources we can determine the earliest light observation set and use Theorem 1.2, see Sec. 3-4.

As noted above, the interaction of the distorted plane waves are difficult to analyze if the waves have caustics. By using global Lorentzian geometry (in Sec. 2) we give in Sec. 4 give conditions that ensure that no caustics affect the earliest observations obtained from the interaction of four colliding, distorted plane waves when these waves are produced by appropriate sources and the collision of the waves is observed before a certain time. We use this in Sec. 4 to give a step-by-step construction of the earliest light observations corresponding to points q in the diamond set $I(p^-, p^+)$. After this the topological, differentiable, and conformal structures in $I(p^-, p^+)$ can be reconstructed using Theorem 1.2.

In this paper we present the complete proofs of the results, but mention for the convenience of the reader that extended versions of some technical computations discussed briefly in this paper and the follow up paper [46] can be found in the preprint [45].

1.3. Remarks and applications

Remark 1.1. The technique developed in the proof of Theorem 1.5 can be applied to many non-linear equations, including many semi-linear equations where the metric g depends on the solution. For example, in the follow up paper [46], we will show how the inverse problem for the coupled Einstein equations and scalar field equations can be solved the using methods developed in this paper.

The techniques considered in this paper can be used also to study inverse problems for non-linear hyperbolic systems encountered in applications and in problems encountered in mathematical physics. For instance, in medical imaging, in the the recently developed ultrasound elastography imaging technique the elastic material parameters are reconstructed by sending (s-polarized) elastic waves that are imaged using (p-polarized) elastic waves, see e.g. [33, 53]. This imaging method uses interaction of waves and is based on the non-linearity of the system.

Passive imaging problems similar to Thm. 1.2 are encountered in seismic imaging based on microseismic events, where one records waves coming from natural point sources that go off at unknown times [43].

Remark 1.2. Theorem 1.5 can in some cases be improved so that also the conformal factor of the metric tensor can be reconstructed. Indeed, Theorem 1.5 and Corollary 1.3 imply that if $W \subset I(p^-, p^+)$ is Vacuum, i.e., Ricci-flat, and all points $x \in W$ can be connected by a path $\alpha \subset W^{int}$ to points of V, then under the assumptions of Theorem 1.5, the whole metric tensor g in W can be reconstructed.

2. Earliest observation time functions

2.1. **Preliminary constructions.** Let (M, g) be a globally hyperbolic Lorentzian manifold of type $(1, n-1)$. As noted above, by [10], there is an isometry Φ from M to a manifold $M = \mathbb{R} \times N$ having the metric of the form (6). This isometry defines a smooth time function $\mathbf{t} : M \to \mathbb{R}$ by setting $\mathbf{t}(x) = t$ if $\Phi(x) \in \{t\} \times N$. We will use notation

$$M_0 = (-\infty, T_0) \times N. \tag{9}$$

In addition to the Lorentzian metric g, we introduce on M a smooth Riemannian metric g^+, that obtained by changing, in local coordinates, the sign of the negative eigenvalue of the Lorentzian metric g. We use the Sasaki distance induced by g^+ on TM.

For $W \subset M$, let $L^+W = \bigcup_{p\in W} L^+_p M \subset TM$ be the bundle of future pointing light-like vectors and $L^{*,+}W = \bigcup_{p\in W} L^{*,+}_p M \subset TM$ be the bundle of future pointing light-like co-vectors. Here, the covector $\eta \in T^*_x M$ is defined to be future pointing if the corresponding vector $\eta^\sharp = g^{jk}\eta_k \frac{\partial}{\partial x^k} \in T_x M$ is future pointing. The projection from the tangent bundle TM to the base point of a vector is denoted by $\pi : TM \to M$.

Let us consider points $x, y \in M$. For $x < y$, we define the time separation function $\tau(x, y) \in [0, \infty)$ to be the supremum of the lengths $L(\alpha) = \int_0^1 \sqrt{-g(\dot\alpha(s), \dot\alpha(s))}\, ds$ of the piecewise smooth causal paths $\alpha : [0, 1] \to M$ from x to y. If the condition $x < y$ does not hold, we define $\tau(x, y) = 0$. We note that $\tau(x, y)$ satisfies the reverse triangle inequality

$$\tau(x, y) + \tau(y, z) \le \tau(x, z), \quad \text{for } x \le y \le z. \tag{10}$$

As M is globally hyperbolic, the time separation function $(x, y) \mapsto \tau(x, y)$ is continuous in $M \times M$ by [60, Lemma 14.21]. By [60, Lemma 14.22], the sets $J^\pm(q)$ are closed. For $q < p$ there is a causal geodesic

$\gamma([0,1])$ with $\gamma(0)=q$ and $\gamma(1)=p$ such that $L(\gamma)=\tau(q,p)$, see [60, Lemma 14.19]. This geodesic, called a longest path from q to p, may not be unique.

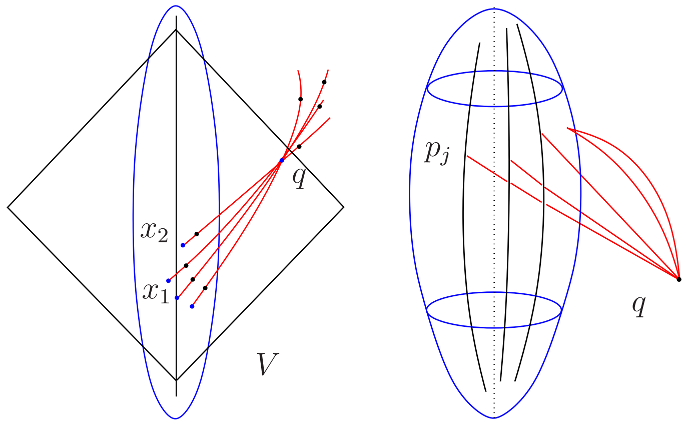

FIGURE 4. **Left:** *We do observations in the set V, marked by the blue boundary. This set contains the set U, defined in (12), that is a union of time-like paths. In the figure, the four light-like geodesics $\gamma_{x_j,\xi_j}([0,\infty))$, $j=1,2,3,4$ starting at the blue points $x_j\in U$ intersect at q before the first cut points of $\gamma_{x_j,\xi_j}([0,\infty))$, denoted by black points. The points $\gamma_{x_j,\xi_j}(t_0)$ are also shown as black points. We use interaction of waves to produce an artificial point source at q.*

Right: *The black curves are the time-like paths $\mu_a\subset U$, indexed by $a\in\mathcal{A}$ and the red curves are light-like geodesics from q, see Subsection 2.1.1 for the notation $\mathcal{A}$ and Definition 2.1 on the functions f_a^+. Some light rays from q intersect μ_a at the point $p_a=\mu_a(f_a^+(q))$, that is the first point of μ_a that is in the causal future of q. For any $q_0\in W$ we can find $a_j\in\mathcal{A}$, $j=1,2,\dots,n$ and a neighborhood of q_0 where the observation time functions $q\mapsto f_{a_j}^+(q)$ define a smooth coordinate system.*

When (x,ξ) is a non-zero vector, we define $\mathcal{T}(x,\xi)\in(0,\infty]$ to be the maximal value for which $\gamma_{x,\xi}:[0,\mathcal{T}(x,\xi))\to M$ is defined.

In addition to points $p^\pm=\widehat{\mu}(s_\pm)$, we use the points $p_{\pm2}=\widehat{\mu}(s_{\pm2})$ where $-1<s_{-2}<s_-$ and $s_+<s_{+2}<1$.

For $(x,\xi)\in L^+M$, we define the cut locus function

$$\rho(x,\xi)=\sup\{s\in[0,\mathcal{T}(x,\xi));\ \tau(x,\gamma_{x,\xi}(s))=0\},\tag{11}$$

c.f. [6, Def. 9.32]. The points $x_1=\gamma_{x,\xi}(t_1)$ and $x_2=\gamma_{x,\xi}(t_2)$, $t_1,t_2\in[0,t_0]$, $t_1<t_2$, are cut points on $\gamma_{x,\xi}([0,t_0])$ if $t_2-t_1=\rho(x_1,\xi_1)$ where $\xi_1=\dot\gamma_{x,\xi}(t_1)$. In particular, the point $p(x,\xi)=\gamma_{x,\xi}(s)|_{s=\rho(x,\xi)}$, if it exists, is called the first cut point on the geodesic $\gamma_{x,\xi}([0,\mathcal{T}(x,\xi)))$. Using [6, Thm. 9.33], we see that the function $\rho(x,\xi)$ is lower semi-continuous on a globally hyperbolic Lorentzian manifold (M,g).

Recall that $\gamma_{x,\xi}(t)$ is a conjugate point on $\gamma_{x,\xi}([0,\mathcal{T}(x,\xi)))$ if the differential of the map $\exp_x$ is not invertible at $t\xi$. By [6, Th. 9.15], on a globally hyperbolic manifold, $p(x,\xi)$ is either the first conjugate point along $\gamma_{x,\xi}$, or the first point on $\gamma_{x,\xi}$ where there is another light-like geodesic $\gamma_{x,\eta}$ from x to $p(x,\xi)$, $\eta \neq c\xi$.

Let us return to the longest paths. If $q < p$ but $\tau(q,p) = 0$, then there is a light-like geodesic $\gamma_{q,\xi}([0,t])$ from q to p so that there are no cut points on $\gamma_{q,\xi}([0,t))$, see [60, Thm. 10.51 and Prop. 14.19]. Note that if $\gamma_{q,\xi}([0,t])$ is a light-like geodesic from q to $p = \gamma_{q,\xi}(t)$ such that there are cut-points on the geodesic $\gamma_{q,\xi}([0,t))$, (10) and (11) yield $\tau(q,p) > 0$.

We say that a path $\alpha([t_1,t_2])$ is a pre-geodesic if $\alpha(t)$ is a C^1-smooth path such that $\dot\alpha(t) \neq 0$ on $t \in [t_1,t_2]$, and $\alpha([t_1,t_2])$ can be re-parametrized so that it becomes a geodesic. A conformal diffeomorphism preserves the light-like pre-geodesics by [6, Th. 9.17].

Moreover, it follows from [60, Prop. 10.46] that if q can be connected to p with a causal path which is not a light-like pre-geodesic then $\tau(q,p) > 0$. Let us apply this fact to a path from q to p which is the union of the future pointing light-like pre-geodesics $\gamma_{q,\eta}([0,t_0]) \subset M$ and $\gamma_{x_1,\theta}([0,t_1]) \subset M$, where $x_1 = \gamma_{q,\eta}(t_0)$, $p = \gamma_{x_1,\theta}(t_1)$ and $t_0, t_1 > 0$. Let $\xi = \dot\gamma_{q,\eta}(t_0)$. Then, if there is no $c > 0$ such that $\xi = c\theta$, or equivalently, the union of these geodesic is not a light-like pre-geodesics, we have $\tau(q,p) > 0$. In particular, this implies that there exists a time-like geodesic from q to p. In the following we call this kind of argument for a union of light-like geodesics a short-cut argument.

2.1.1. *Smaller observation domain U.* Next we define a domain $U \subset V$ that is a union of time-like paths.

We assume that we are given a family of future pointing, C^∞-smooth, time-like paths $\mu_a : [-1,1] \to V$, indexed by $a \in \overline{\mathcal{A}}$, where $\mathcal{A}$ is a connected metric space and the completion of $\mathcal{A}$, denoted by $\overline{\mathcal{A}}$, is compact. We assume that there is $\widehat a \in \mathcal{A}$ such that $\widehat\mu = \mu_{\widehat a}$. Also, we assume that $(a,s) \mapsto \mu_a(s)$ defines a continuous map $\overline{\mathcal{A}} \times [-1,1] \to M$ and an open map $\mathcal{A} \times [-1,1] \to M$. Then, we define the smaller observation domain $U \subset V$ to be the set

$$U = \bigcup_{a\in\mathcal{A}} \mu_a([-1,1]). \tag{12}$$

Note that as $(a,s) \mapsto \mu_a(s)$ in is a continuous and open map $\mathcal{A} \times [-1,1] \to M$, the set U is open.

Let $s_{-2} \in (-1, s_-)$, and $s_{+2} \in (s_+, 1)$. By replacing $\mathcal{A}$ in the formula (12) by a smaller neighborhood of $\widehat a$ we may assume for all $a \in \overline{\mathcal{A}}$ we have

$$\mu_a(s_{-2}) \in I^+(\mu_{\widehat a}(-1)) \cap I^-(p^-), \ \ \mu_a(s_{+2}) \in I^-(\mu_{\widehat a}(1)) \cap I^+(p^+). \tag{13}$$

When V is given as differentiable manifold and the conformal class of g is given, we may define a family of smooth time-like paths $\mu_a : [-1,1] \to$

V, $a \in \overline{\mathcal{A}}$ having the above properties and define the neighborhood U given in (13).

2.1.2. *Observation time functions.* Instead of the light observation sets we can consider the earliest observation time functions that we proceed to define.

Definition 2.1. *Let $a \in \overline{\mathcal{A}}$. For $x \in M$ we define $f_a^+(x), f_a^-(x) \in [-1,1]$ by setting*

$$\begin{aligned} f_a^+(x) &= \inf(\{s \in (-1,1);\ \tau(x,\mu_a(s)) > 0\} \cup \{1\}), \\ f_a^-(x) &= \sup(\{s \in (-1,1);\ \tau(\mu_a(s),x) > 0\} \cup \{-1\}). \end{aligned}$$

We call $f_a^+(x)$ the earliest observation time from the point x on the path μ_a. The functions $f_a^+ : M \to \mathbb{R}$, $a \in \overline{\mathcal{A}}$ are called the earliest observation time function on the path μ_a.

We will show that the map

$$\mathcal{F} : J^-(p^+) \setminus I^-(p^-) \to C(\overline{\mathcal{A}}), \quad \mathcal{F}(q) : a \mapsto f_a^+(q),$$

that maps a point q to the earliest observation times corresponding to the point, is a continuous function. We will prove the following proposition in Section 2.2.

Proposition 2.2. *Let (M,g) be an open, C^∞-smooth, globally hyperbolic Lorentzian manifold of dimension $n \geq 3$. Let $\mathcal{A}$ be a metric space which completion $\overline{\mathcal{A}}$ is compact, and $\mu_a : [-1,1] \to M$, $a \in \overline{\mathcal{A}}$ be C^∞-smooth, time-like paths. Let $p^- = \mu_{\widehat{a}}(s_-)$ and $p^+ = \mu_{\widehat{a}}(s_+)$ with $\widehat{a} \in \mathcal{A}$ and $-1 < s_- < s_+ < 1$. Also, assume that $(a,s) \mapsto \mu_a(s)$ defines a continuous map $\overline{A} \times [-1,1] \to M$ and an open map $\mathcal{A} \times [-1,1] \to M$.*

Let $W \subset M$ be open set such that $\overline{W} \subset J^-(p^+) \setminus I^-(p^-)$ is compact. Then $\mathcal{F} : W \to \mathcal{F}(W)$ is a homeomorphism. Here $F(W)$ has the metric induced by $C(\overline{\mathcal{A}})$.

In several geometric inverse problems [2, 42, 49], in order to reconstruct an unknown manifold W from a given data, one needs to construct a copy of the manifold. The importance of Proposition 2.2 lies in the fact that it can be used to construct a homeomorphic image of the original Lorentzian manifold W embedded in the known space $C(\overline{\mathcal{A}})$. After the homeomorphic image $\mathcal{F}(W)$ of the manifold is constructed, we can construct other structures on it, e.g., the differentiable coordinates and a metric tensor conformal to the original metric.

We need the following simple properties of the functions $f_a^\pm(x)$.

Lemma 2.3. *Let $a \in \overline{\mathcal{A}}$ and $q \in J^-(p^+) \setminus I^-(p^-)$. Then*
(i) We have that $s_{-2} \leq f_a^+(q) \leq s_{+2}$.
(ii) We have $\mu_a(f_a^+(q)) \in J^+(q)$ and $\tau(q, \mu_a(f_a^+(q))) = 0$. Moreover, the function $s \mapsto \tau(q, \mu_a(s))$ is continuous, non-decreasing on the interval $s \in [-1,1]$, and it is strictly increasing on $[f_a^+(q), 1]$.

(iii) Assume that $p \in U$. Then $p = \mu_a(f_a^+(q))$ with some $a \in \mathcal{A}$ if and only if $p \in \mathcal{P}_U(q)$ and $\tau(q,p) = 0$. Furthermore, these are equivalent to the fact that there are $\xi \in L_q^+M$ and $t \in [0, \rho(q,\xi)]$ such that $p = \gamma_{q,\xi}(t)$.

(iv) The function $\mathbf{F} : (a,q) \mapsto f_a^+(q)$ is continuous on $\overline{\mathcal{A}} \times (J^-(p^+) \setminus I^-(p^-))$.

For $q \in J^+(p^-) \setminus I^+(p^+)$ the claims analogous with (i)-(iv), with reversed causality, are valid for $q \mapsto f_a^-(q)$.

Proof. (i) This property follows from (13).

(ii) Since $J^+(q)$ is closed, $\mu_a(f_a^+(q)) \in J^+(q)$. The continuity of $s \mapsto \tau(q, \mu_a(s))$ follows from the continuity of $\tau(x,y)$ on $M \times M$.

If $\tau(q, \mu_a(f_a^+(q)))$ would be strictly positive, we would have $\mu_a(f_a^+(q)) \in I^+(q)$ and there would exist $s < f_a^+(q)$ such that $\mu_a(s) \in I^+(q)$. As this is not possible, we have $\tau(q, \mu_a(f_a^+(q))) = 0$.

Consider $s < s'$. Since μ_a is a time like-path, $\tau(\mu_a(s), \mu_a(s')) > 0$. Thus, when $s' > s \geq f_a^+(q)$, the inequality (10) yields $\tau(q, \mu_a(s)) < \tau(q, \mu_a(s'))$. For $s < f_a^+(q)$ we have $\mu_a(s) \notin J^+(q)$ and $\tau(q, \mu_a(s)) = 0$.

(iii) It is sufficient to prove the claim when $p \neq q$. First, assume that $p = \mu_a(f_a^+(q))$. Then $p \in J^+(q)$ and by (ii), we have $\tau(q,p) = 0$. The existence of the light-like geodesic follows from the above.

Second, assume that $p \in J^+(q)$ and $\tau(q,p) = 0$. This implies by [60, Prop. 14.19] that there exists a light-like geodesic $\gamma_{q,\xi}([0,t])$ from q to p. If $\gamma_{q,\xi}([0,t))$ would have a cut-point, then $\tau(q,p) > 0$ which is not possible. Thus, $t \in [0, \rho(q,\xi)]$.

Third, assume that $p = \gamma_{q,\xi}(t)$ with $\xi \in L_q^+M$ and $0 \leq t \leq \rho(q,\xi)$. Then $\tau(q,p) = 0$. Let $a \in \mathcal{A}$ and $s_0 \in [-1,1]$ be such that $p = \mu_a(s_0)$. As $q \notin I(p^-)$, using (13) we see that $q \notin I^-(\mu_a(s_{-2}))$ and hence $s_0 \geq s_{-2} > -1$. By (i), $\tau(q, \mu_a(s)) > 0$ for $s > f_a^+(q)$ and thus $s_0 \leq f_a^+(q) \leq s_{-2}$. However, $q \leq p = \mu_a(s_0)$ and $\tau(q, \mu_a(s)) > 0$ for $s \in (s_0, 1)$. Thus $s_0 \geq f_a^+(q)$. Hence, $s_0 = f_a^+(q)$ and $p = \mu_a(f_a^+(q))$.

(iv) Assume that $x_j \to x$ in $J^-(p^+) \setminus I^-(p^-)$ and $a_j \to a$ as $j \to \infty$. Let $s_j = f_{a_j}^+(x_j)$ and $s = f_a^+(x)$.

Since τ is continuous, for any $\varepsilon > 0$ we have $\lim_{j\to\infty} \tau(x_j, \mu_{a_j}(s + \varepsilon)) = \tau(x, \mu_a(s+\varepsilon)) > 0$. Then for j large enough $\mu_{a_j}(s+\varepsilon) \in I^+(x_j)$, so that $s_j \leq s + \varepsilon$. As $\varepsilon > 0$ is arbitrary, $\limsup_{j\to\infty} s_j \leq s$ Thus $\mathbf{F}$ is upper-semicontinuous.

Next, suppose $\liminf_{j\to\infty} s_j = \widetilde{s} < s$ and denote $\varepsilon = \tau(\mu_a(\widetilde{s}), \mu_a(s)) > 0$. Then by choosing a subsequence, we may assume that $\lim_{j\to\infty} s_j = \widetilde{s} < s$. By continuity of τ and (10),

$$\begin{aligned}
\lim_{j\to\infty} \tau(x_j, \mu_{a_j}(s)) &\geq \liminf_{j\to\infty} \tau(x_j, \mu_{a_j}(s_j)) + \tau(\mu_{a_j}(s_j), \mu_{a_j}(s)) \\
&\geq 0 + \tau(\mu_a(\widetilde{s}), \mu_a(s)) = \varepsilon,
\end{aligned}$$

and we obtain $\tau(x, \mu_a(s)) = \lim_{j\to\infty} \tau(x_j, \mu_{a_j}(s)) \geq \varepsilon$. This is not possible as $s = f_a^+(x)$. Hence $\liminf_{j\to\infty} s_j \geq s$ and $\mathbf{F}$ is also lower-semicontinuous. This proves (iv).

The analogous results for function f_a^- follow similarly by reversing the causality. □

Next we consider the earliest light observation sets in the observation domains U and V. We will show that without loss of generality we can take the neighborhood V of $\widehat{\mu}$ to be the set U defined in (12).

Lemma 2.4. *Let $q \in J^-(p^+) \setminus I^-(p^-)$.*
(i) The earliest light observation set of q in U has the form

$$\mathcal{E}_U(q)= \{\mu_a(f_a^+(q));\ a \in \mathcal{A}\}. \tag{14}$$

(ii) Assume that we are given the sets $\mathcal{E}_V(q)$ and U and the paths μ_a, $a \in \mathcal{A}$. These data determine the function $f_a^+(q)$ and moreover, the set $\mathcal{E}_U(q)$ by formula (14).

Proof. (i) Let $x \in \mathcal{E}_U(q) \subset U$. Then by Lemma 2.3 (i) there is $a \in \mathcal{A}$ such that $x = \mu_a(s)$ with $s \in [s_{-2}, s_{+2}]$ and $x \in \mathcal{P}_U(q)$.

Assume that $\tau(q,x) > 0$. Then $s > f_a^+(q)$. By Lemma 2.3 (iii), $y = \mu_a(f_a^+(q)) \in \mathcal{P}_U(q)$ and the time-like path $\mu_a([f_a^+(q), s]) \subset U$ connects $y \in \mathcal{P}_U(q)$ to x. This is not possible by the definition of $\mathcal{E}_U(q)$. This shows that $\tau(q,x) = 0$. As $x \in \mathcal{P}_U(q)$, by Lemma 2.3 (iii) we have $x \in \{\mu_a(f_a^+(q));\ a \in \mathcal{A}\}$.

On the other hand, assume that $x = \mu_a(f_a^+(q))$ with $a \in \mathcal{A}$. Then by Lemma 2.3 (iii), $x \in \mathcal{P}_U(q)$ and $\tau(q,x) = 0$. Then, if there would exist $y \in \mathcal{P}_U(q)$ that is connected to x with a future pointing time-like path, we would have $\tau(y,x) > 0$. Thus (10) implies that $\tau(q,x) \geq \tau(q,y)+\tau(y,x) > 0$. This shows that no such y can exist and $x \in \mathcal{E}_U(q)$. These prove (i).

(ii) As $q \in J^-(p^+) \setminus I^-(p^-)$, Lemma 2.3 and inequality (10) yield that the function $f_a^+(q)$ are determined by

$$f_a^+(q) = \inf\{s \in (-1,1);\ \mu_a(s) \in \mathcal{E}_V(q)\}. \tag{15}$$

□

Due to Lemma 2.4, without loss of generality we may in Theorem 1.2 consider the case when the set V is replaced by U. We will do so for the remaining of this paper.

2.2. **Observation time representation of a Lorentzian manifold.**

In this section our main goal is to prove Proposition 2.2.

2.2.1. *The direction set.*

Definition 2.5. *Let $q \in J^-(p^+) \setminus I^-(p^-)$. Let*

$$\begin{aligned} \mathcal{C}_U(q) = \{(y,\eta) \in L^+U \ &;\ y = \gamma_{q,\xi}(t) \in U,\ \eta = \dot\gamma_{q,\xi}(t), \\ &\quad \textit{with some } \xi \in L_q^+M,\ 0 \leq t \leq \rho(x,\xi)\}, \\ \mathcal{C}_U^{reg}(q) = \{(y,\eta) \in L^+U \ &;\ y = \gamma_{q,\xi}(t) \in U,\ \eta = \dot\gamma_{q,\xi}(t), \\ &\quad \textit{with some } \xi \in L_q^+M,\ 0 < t < \rho(x,\xi)\}. \end{aligned} \tag{16}$$

We say that $\mathcal{C}_U(q)$ is the direction set of q and $\mathcal{C}_U^{reg}(q)$ is the regular direction set of q.

Then, $\mathcal{E}_U(q) = \pi(\mathcal{C}_U(q))$. We denote $\mathcal{E}_U^{reg}(q) = \pi(\mathcal{C}_U^{reg}(q))$ where $\pi : TU \to U$ is the canonical projection, $\pi(y,\eta) = y$. We say that $\mathcal{E}_U^{reg}(q)$ is the regular earliest light observation set of q.

Note that $\mathcal{E}_U(q) = \{\mu_a(f_a^+(q));\ a \in \mathcal{A}\}$ and that the lower semicontinuity of $\rho(x,\xi)$ implies that $\mathcal{E}_U^{reg}(q) \subset U$ and $\mathcal{C}_U^{reg}(q) \subset TU$ are smooth $(n-1)$ and n dimensional submanifolds, respectively.

We need the following auxiliary result:

Lemma 2.6. (i) *Let $y \in U$, $\eta \in L_y^+M$, $r_1 > 0$, and $q \in W$ be such that $q \notin \gamma_{y,\eta}([-r_1,0])$ and $\gamma_{y,\eta}([-r_1,0]) \subset U$. Then $(y,\eta) \in \mathcal{C}_U(q)$ if and only if $\gamma_{y,\eta}([-r_1,0]) \subset \mathcal{E}_U(q)$.*

(ii) *Let $y \in U$, $\eta \in L_y^+M$, and $\widehat{t} > 0$ be the largest number such that the geodesic $\gamma_{y,\eta}((-\widehat{t},0])$ is defined and has no cut points. Then for $q \in W$ we have $q \in \gamma_{y,\eta}((-\widehat{t},0))$ if and only if $(y,\eta) \in \mathcal{C}_U^{reg}(q)$.*

Proof. (i) Suppose $(y,\eta) \in \mathcal{C}_U(q)$. Then $y \in \mathcal{E}_U(q)$ and $\tau(q,y) = 0$. Since $q \notin \gamma_{y,\eta}([-r_1,0]) \subset U$, there is $t > r_1$ such that $\gamma_{y,\eta}(-t) = q$ and for $\xi = \dot\gamma_{y,\eta}(-t)$ we have $\gamma_{y,\eta}([-r_1,0]) = \gamma_{q,\xi}([t-r_1,t]) \subset \mathcal{P}_U(q)$. If there would be $y_1 \in \gamma_{y,\eta}([-r_1,0])$ such that $y_1 \notin \mathcal{E}_U(q)$, it follows from (2) that there is $z \in \mathcal{P}_U(q)$ such that $z \ll y_1$. Then $q \le z \ll y_1 \le y$. These imply that $\tau(q,y) > 0$ and $y \in \mathcal{E}_U(q)$ which is not possible by Lemma 2.3 (iii) and Lemma 2.4. This shows that $\gamma_{y,\eta}([-r_1,0]) \subset \mathcal{E}_U(q)$.

On the other hand, assume that $\gamma_{y,\eta}([-r_1,0]) \subset \mathcal{E}_U(q)$. Then Lemma 2.3(ii) implies that $\tau(q,y) = 0$. Denote $y_1 = \gamma_{y,\eta}(-r_1)$. Since $y_1 \in \mathcal{E}_U(q)$ and $y_1 \neq q$, there is $\xi \in L_q^+M$ and $t_1 > 0$ such that $\gamma_{q,\xi}(t_1) = y_1$. Then the union of the geodesics $\gamma_{q,\xi}([0,t_1])$ and $\gamma_{y,\eta}([-r_1,0])$ form a causal path from q to y. Using short cut arguments, we see that if the union of these geodesics do not form one light-like pre-geodesic, we have $\tau(q,y) > 0$, that is not possible. Hence $\gamma_{y,\eta}([-r_1,0])$ lies in the continuation of $\gamma_{q,\xi}([0,t_1])$, that is, there is $t > 0$ such that $\gamma_{y,\eta}([-r_1,0]) \subset \gamma_{q,\xi}([0,t])$ and $y = \gamma_{q,\xi}(t)$. Then, there is $c > 0$ such that $\eta = c\dot\gamma_{q,\xi}(t)$. Moreover, if $\gamma_{q,\xi}([0,t))$ would contain cut points then [60, Prop. 10.46] implies that $\tau(q,y) > 0$. This would lead to a contradiction with $y \in \mathcal{E}_U(q)$. Hence, $\gamma_{q,\xi}([0,t))$ contains no cut points. Therefore, we have shown that $t \le \rho(q,\xi)$, $y = \gamma_{q,\xi}(t)$, and $\eta = c\dot\gamma_{q,\xi}(t)$. These imply that $(y,\eta) \in \mathcal{C}_U(q)$.

(ii) Let $(y,\eta) \in L^+U$ and $\widehat{t} > 0$ be as in the claim and $q \in W$.

First, assume that $q \in \gamma_{y,\eta}(-t_1)$, $t_1 \in (0,\widehat{t})$. Then, due to the symmetry of the cut points, $\tau(q,y) = 0$ and thus for $\xi = \dot\gamma_{y,\eta}(-t_1)$ we have $y = \gamma_{q,\xi}(t_1)$ and $t_1 < \rho(q,\xi)$. Thus $(y,\eta) \in \mathcal{C}_U^{reg}(q)$.

Second, assume that $(y,\eta) \in \mathcal{C}_U^{reg}(q)$. Again, we see that there is $t_1 > 0$ such that $q = \gamma_{y,\eta}(-t_1)$ and for $\xi = \dot\gamma_{y,\eta}(-t_1)$ we have $y = \gamma_{q,\xi}(t_1)$ and $t_1 < R_1 := \rho(q,\xi)$. Since ρ is a lower semi-continuous, we have that when $\varepsilon \in (0,(R_1-t_1)/2)$ is small enough, the point $x_1 = \gamma_{q,\xi}(-\varepsilon)$ and $\xi_1 = \dot\gamma_{q,\xi}(-\varepsilon)$ satisfy $\rho(x_1,\xi_1) > R_1-\varepsilon > t_1+\varepsilon$ and hence $\tau(x_1,y)=0$. This yields that $\widehat{t} > t_1$. Thus $q \in \gamma_{y,\eta}((-\widehat{t},0))$. □

Using this result we determine the direction sets $\mathcal{C}_U(q)$ from $\mathcal{E}_U(W)$:

Lemma 2.7. *Assume that we are given the conformal type of $(U, g|_U)$, the paths $\mu_a : [-1,1] \to U$, $a \in \mathcal{A}$, and the set $\mathcal{E}_U(W)$. Then*

(i) *For any $y \in U$, we can identify from the set $\mathcal{E}_U(W)$ the element $\mathcal{E}_U(q)$ for which $q = y$, if it exists. For such elements $L^+_y M \subset \mathcal{C}_U(q)$.*

(ii) *Let $q \in W$ and $(y,\eta) \in L^+U$. Then $(y,\eta) \in \mathcal{C}^{reg}_U(q)$ if and only if there exists a light-like pre-geodesic $\alpha([t_1,t_2]) \subset U$ such that $y = \alpha(t)$, $\eta = \dot\alpha(t)$, $t_1 < t < t_2$, and $\alpha([t_1,t_2]) \subset \mathcal{E}_U(q)$.*

(iii) *When $\mathcal{E}_U(q) \in \mathcal{E}_U(W)$ is given, one can determine the sets $\mathcal{C}_U(q)$, $\mathcal{C}^{reg}_U(q)$, and $\mathcal{E}^{reg}_U(q)$.*

Proof. (i) We observe that $q = y$ if and only if for $y \in \mathcal{E}_U(q)$ there are no $\eta \in L^+_y M$ and $t_0 > 0$ such that $\gamma_{y,\eta}([-t_0,0]) \subset \mathcal{E}_U(q)$. Claim (i) follows from this observation.

(ii) Let $q \in W$ and $\xi \in L^+_q W$ and $(y,\eta) = (\gamma_{q,\xi}(1), \dot\gamma_{q,\xi}(1))$. Using Definition 2.5 we see that $(y,\eta) \in \mathcal{C}^{reg}_U(q)$ if and only if $\gamma_{q,\xi}(1) \in U$ and $\rho(q,\xi) > 1$. This is equivalent to the fact that there are $t_1 \in (0,1)$ and $t_2 > 1$ such that $\gamma_{q,\xi}([t_1,t_2]) \subset U$ and $(\gamma_{q,\xi}(t_2), \dot\gamma_{q,\xi}(t_2)) \in \mathcal{C}_U(q)$. Also, by Lemma 2.6 (i) this is equivalent to the fact that there are $t_1 \in (0,1)$ and $t_2 > 1$ such that $\gamma_{q,\xi}([t_1,t_2]) \subset \mathcal{E}_U(q)$. This proves (ii).

(iii) Let $\mathcal{E}_U(q)$ be given. Since the conformal class of $g|_U$ is given, we can identify all light-like pre-geodesics in U. Thus by using (ii), we can verify for any $(y,\eta) \in L^+U$ whether it holds that $(y,\eta) \in \mathcal{C}^{reg}_U(q)$ or not. Thus we can determine the set $\mathcal{C}^{reg}_U(q)$. Then the set $\mathcal{C}_U(q)$ can be determined as the closure of the set $\mathcal{C}^{reg}_U(q)$ in TU. Finally, the set $\mathcal{E}^{reg}_U(q) = \pi(\mathcal{C}^{reg}_U(q))$ can be constructed using the map $\pi : TU \to U$. □

2.2.2. *Construction of W as a topological manifold.* For $q \in J^-(p^+) \setminus I^-(p^-)$ we define the continuous function $F_q : \overline{\mathcal{A}} \to \mathbb{R}$ by $F_q(a) = f^+_a(q)$. Also, we denote by $\mathcal{F} : J^-(p^+) \setminus I^-(p^-) \to C(\overline{\mathcal{A}})$ the function $\mathcal{F}(q) = F_q$, that maps q to the function $F_q : \mathcal{A} \to \mathbb{R}$.

By Lemma 2.4 , the set $\mathcal{E}_U(q)$ determines the restriction of $F_q = \mathcal{F}(q)$ in $\mathcal{A}$. As $F_q : \overline{\mathcal{A}} \to \mathbb{R}$ is continuous, this determines $F_q(a)$ for all $a \in \overline{\mathcal{A}}$. Also, $F_q = \mathcal{F}(q)$ determines $\mathcal{E}_U(q)$ via the formula (14).

Recall that W is open and relatively compact and $\overline{W} \subset J^-(p^+) \setminus I^-(p^-)$. Below, we consider the sets $\mathcal{F}(W) = \{\mathcal{F}(q);\ q \in W\} \subset C(\overline{\mathcal{A}})$ and $\mathcal{E}_U(W) = \{\mathcal{E}_U(q);\ q \in W\} \subset 2^U$ as two representations for W. We will construct the topological and differentiable structure of W using $\mathcal{F}(W)$ and the conformal class of the metric $g|_W$ using $\mathcal{E}_U(W)$. First, we consider the reconstruction of the topological type of W.

Now we are ready to prove Proposition 2.2.

Proof (of Prop. 2.2). Below, let $\overline{W} = \mathrm{cl}\,(W)$ be the closure of W in M, such that $\overline{W} \subset J^-(p^+) \setminus I^-(p^-)$. As $\overline{\mathcal{A}} \times \overline{W}$ is compact and thus $\mathbf{F} : \mathcal{A} \times \overline{W} \to \mathbb{R}$ is uniformly continuous by Lemma 2.3 (iv), the map $\mathcal{F} : \overline{W} \to C(\overline{\mathcal{A}})$ is continuous. Next we show that the map $\mathcal{F} : \overline{W} \to \mathcal{F}(\overline{W})$ is injective. Since $\mathcal{F}(q)$ determines the set $\mathcal{E}_U(q)$ uniquely, it is enough to show that the map $\mathcal{E}_U : \overline{W} \to \mathcal{E}_U(\overline{W})$ is injective. To prove this, we assume the opposite: Assume that there are $q_1 \neq q_2$ that satisfy $\mathcal{E}_U(q_1) = \mathcal{E}_U(q_2)$. By Lemma 2.7 (iii), this implies

$$\mathcal{C}_U(q_1) = \mathcal{C}_U(q_2). \tag{17}$$

Choose $a \in \mathcal{A}$ such that $q_i \notin \mu_a$, $i \in \{1, 2\}$. Let $(p, \eta) \in \mathcal{C}_U(q_i)$ with $p = \mu_a(f_a^+(q_i))$. Then there are $t_i > 0$ such that $q_i = \gamma_{p,\eta}(-t_i)$. Since $q_1 \neq q_2$, we have $t_1 \neq t_2$, and let us assume that $t_2 > t_1$. Then, we see there are $\xi_i \in L^+_{q_i}M$ such that

$$(p, \eta) = (\gamma_{q_i,\xi_i}(t_i), \dot\gamma_{q_i,\xi_i}(t_i)), \quad (q_1, \xi_1) = (\gamma_{q_2,\xi_2}(t_2 - t_1), \dot\gamma_{q_2,\xi_2}(t_2 - t_1)).$$

Since $\rho(q, \xi)$ is lower semicontinuous, for any $\delta_1 > 0$ there is $\delta_2 > 0$ such that $\rho(q_2, \xi_2') > \rho(q_2, \xi_2) - \delta_1$ when $\xi_2' \in T_qM$ satisfies $\|\xi_2' - \xi_2\|_{g^+} < \delta_2$. Choosing δ_1 and δ_2 to be sufficiently small, we have that there is $\xi_2' \in T_{q_2}M$ that is not parallel to ξ_2, $\|\xi_2' - \xi_2\| < \delta_2$, and $t_2' \in (t_2 - 2\delta_1, t_2 - \delta_1)$ such that $p' = \gamma_{q_2,\xi_2'}(t_2') \in U$, $p' \neq q_1$, and $t_2' < \rho(q_2, \xi_2')$. Thus for $\eta' = \dot\gamma_{q_2,\xi_2'}(t_2')$ we have $(p', \eta') \in \mathcal{C}_U(q_2)$. By (17), $(p', \eta') \in \mathcal{C}_U(q_1)$, and hence there is $t_1' > 0$ such that $q_1 = \gamma_{p',\eta'}(-t_1')$.

Observe that $\xi_1' = \dot\gamma_{p',\eta'}(-t_1')$ and ξ_1 are not parallel. We have that the union of the geodesic $\gamma_{q_2,\xi_2}([0, t_2 - t_1])$ and the geodesic $\gamma_{p',\eta'}([0, -t_1'])$, oriented in the opposite direction, form a causal path from q_2 to p' that is not a light-like pre-geodesic, and hence $\tau(q_2, p') > 0$. This is not possible as $p' \in \mathcal{E}_U(q_2)$. This contradiction proves that $\mathcal{E}_U : \overline{W} \to \mathcal{E}_U(\overline{W})$ is injective.

Since $C(\overline{\mathcal{A}})$ is a Hausdorff space, $\overline{W}$ is a compact set, and the map $\mathcal{F} : \overline{W} \to \mathcal{F}(\overline{W})$ is continuous and injective, we have that $\mathcal{F} : \overline{W} \to \mathcal{F}(\overline{W})$ is a homeomorphism. Thus $\mathcal{F} : W \to \mathcal{F}(W)$ is a homeomorphism. □

2.2.3. *Estimates for the location of the first cut point.* We finish this section by auxiliary results that are needed in the proof of Theorem 1.5. Below, we use for a pair $(x, \xi) \in L^+M$ the notation

$$(x(t), \xi(t)) = (\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t)), \quad t \in \mathbb{R}_+. \tag{18}$$

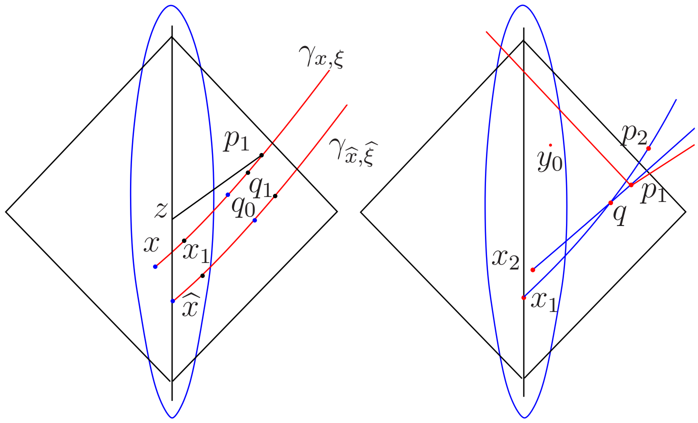

FIGURE 5. **Left:** *The figure shows the situation in Lemma 2.8. The point $\widehat{x} = \widehat{\mu}(r_1)$ is on the time-like path $\widehat{\mu}$ shown as a black line. The black diamond is the set $J^+(p^-) \cap J^-(p^+)$, (x, ξ) is a light-like direction close to $(\widehat{x}, \widehat{\xi})$, and $x_1 = \gamma_{x,\xi}(t_0) = x(t_0)$. The points $q_0 = \gamma_{x,\xi}(\rho(x, \xi))$ and $q_1 = \gamma_{x(t_0),\xi(t_0)}(\rho(x(t_0), \xi(t_0)))$ are the first cut point on $\gamma_{x,\xi}$ corresponding to the points x and x_1, respectively. The blue and black points on $\gamma_{\widehat{x},\widehat{\zeta}}$ are the corresponding cut points on $\gamma_{\widehat{x},\widehat{\zeta}}$. Also, $p_1 = \gamma_{x,\xi}(t_1)$ and $z = \widehat{\mu}(r_1)$, where $r_1 = f^-_{\widehat{a}}(p_1)$.* **Right:** *The figure shows the configuration in formulas (42) and (43) and in Theorem 3.3. We send light-like geodesics $\gamma_{x_j,\xi_j}([0, \infty))$ from x_j, $j = 1, 2, 3, 4$. The boundary $\partial\mathcal{N}(\vec{x}, \vec{\xi})$ is denoted by red line segments and $y_0 \in \mathcal{N}(\vec{x}, \vec{\xi})$. We assume the these geodesics intersect at the point q before their first cut points p_j.*

Later, we will consider waves sent from a point $x \in U$ that propagate near a geodesic $\gamma_{x,\xi}([0, \infty))$. These waves may have singularities near the conjugate points of the geodesic and due to this we analyze next how the conjugate points move along a geodesic when its initial point is moved from x to $\gamma_{x,\xi}(t_0)$. Below, let $T_+(x, \xi) = \sup\{t \geq 0\,;\ \gamma_{x,\xi}(t) \in J^-(\widehat{\mu}(1))\}$.

Lemma 2.8. *There are $\vartheta_1, \kappa_1, \kappa_2 > 0$ such that for all $\widehat{x} = \widehat{\mu}(r_0)$ with $r_0 \in [s_-, s_+]$, $\widehat{\xi} \in L^+_{\widehat{x}}M$, $\|\widehat{\xi}\|_{g^+} = 1$, $t_0 \in [\kappa_1, 4\kappa_1]$, and $(x, \xi) \in L^+M$ satisfying $d_{g^+}((x, \xi), (\widehat{x}, \widehat{\xi})) \leq \vartheta_1$ the following holds:*

(ii) If $0 < t \leq 5\kappa_1$, then $f^-_{\widehat{a}}(\gamma_{\widehat{x},\widehat{\xi}}(t)) = r_0$,

(ii) If $0 < t \leq 5\kappa_1$, then $\gamma_{x,\xi}(t) \in U$,

(iii) Assume that there exists t_1 that satisfies $t_0 + \rho(\gamma_{x,\xi}(t_0), \dot{\gamma}_{x,\xi}(t_0)) \leq t_1 < T_+(x, \xi)$ and let $p_1 = \gamma_{x,\xi}(t_1)$. Then $r_1 = f^-_{\widehat{a}}(p_1)$ satisfies $r_1 - r_0 > 2\kappa_2$.

Note that above in (iii) we can choose $t_1 = t_0 + \rho(\gamma_{x,\xi}(t_0), \dot{\gamma}_{x,\xi}(t_0))$ in which case p_1 is the first cut point q_1 of $\gamma_{x,\xi}([t_0, \infty))$, see Fig. 5(Left).

Proof. Let $B = \{(\widehat{x}, \widehat{\xi}) \in L^+M;\ \widehat{x} \in \widehat{\mu}([s_-, s_+]),\ \|\widehat{\xi}\|_{g^+} = 1\}$. Since B is compact, the positive and lower semi-continuous function

$\rho(x,\xi)$ obtains its minimum on B. This proves the claim (i) when $\kappa_1 \in (0, \frac{1}{5}\inf\{\rho(\widehat{x},\widehat{\xi});\ (\widehat{x},\widehat{\xi}) \in B\})$.

(ii) For $\vartheta > 0$ small enough, $K_\vartheta = \{(x,\xi) \in L^+M;\ d_{g^+}((x,\xi),B) \leq \vartheta\}$ is a compact subset of L^+U. Thus yields easily (ii) when ϑ_1 is small enough.

(iii) Let $\vartheta \in (0,\vartheta_1)$ be so small that $K_\vartheta \subset L^+U$ and

$$K_\vartheta^0 = \{(x,\xi) \in K_\vartheta;\ \rho(x(\kappa_1),\xi(\kappa_1)) + \kappa_1 \leq T_+(x,\xi)\}, \quad K_\vartheta^1 = K_\vartheta \setminus K_\vartheta^0.$$

Using [60, Lemma 14.13], we see that $T_+(x,\xi)$ is bounded in K_ϑ. Note that for $t_0 \geq \kappa_1$ and $a > t_0$ the geodesic $\gamma_{x,\xi}([t_0,a])$ can have a cut point only if $\gamma_{x,\xi}([\kappa_1,a])$ has a cut point and thus $t_0 + \rho(x(t_0),\xi(t_0)) \geq \kappa_1 + \rho(x(\kappa_1),\xi(\kappa_1))$. If $K_\vartheta^0 = \emptyset$, the claim is valid as the condition $p_1 \in J^-(\widehat{\mu}(1))$ does not hold for any $(x,\xi) \in K_\vartheta^1$. Thus it is enough to consider the case when $K_\vartheta^0 \neq \emptyset$.

Let

$$G_\vartheta = \{(x,\xi,t) \in K_\vartheta \times \mathbb{R}_+;\ \rho(x(\kappa_1),\xi(\kappa_1)) + \kappa_1 \leq t \leq T_+(x,\xi)\}.$$

As $\rho(x,\xi)$ is lower semi-continuous and $T_+(x,\xi)$ is upper semi-continuous and bounded, the sets K_ϑ^0 and G_ϑ are compact.

For $(x,\xi,t) \in G_\vartheta$, the geodesic $\gamma_{x,\xi}([\kappa_1,t])$ has a cut point. Thus for $y = \gamma_{x,\xi}(t)$, we have $\tau(x,y) > 0$. Hence, for $z = \widehat{\mu}(f_{\widehat{a}}^-(x))$, we have $\tau(z,y) \geq \tau(z,x) + \tau(x,y) \geq \tau(x,y) > 0$. This shows that $f_{\widehat{a}}^-(y) - f_{\widehat{a}}^-(x) > 0$. Since G_ϑ is compact and $f_{\widehat{a}}^-$ is continuous, $\varepsilon_1 := \inf\{f_{\widehat{a}}^-(\gamma_{x,\xi}(t)) - f_{\widehat{a}}^-(x);\ (x,\xi,t) \in G_\vartheta\} > 0$.

Then, if $\rho(x(\kappa_1),\xi(\kappa_1)) + \kappa_1 \leq t_1 < \mathcal{T}_+(x,\xi)$ and $p_1 = \gamma_{x,\xi}(t_1)$, we have that $r_1 = f_{\widehat{a}}^-(p_1)$ and $r_2 = f_{\widehat{a}}^-(x)$ satisfy $r_1 - r_2 \geq \varepsilon_1$.

As $f_{\widehat{a}}^-$ is continuous and $\widehat{\mu}([-1,1])$ is compact, we see that by making ϑ_1 smaller if necessary, we can assume that if $\widehat{x} = \widehat{\mu}(r_0) \in \widehat{\mu}$ and $d_{g^+}(x,\widehat{x}) \leq \vartheta_1$ then $|f_{\widehat{a}}^-(x) - f_{\widehat{a}}^-(\widehat{x})| < \varepsilon_1/2$. Let $\kappa_2 = \varepsilon_1/4$. Then $r_1 - r_2 \geq \varepsilon_1$ and $r_2 - r_0 = |f_{\widehat{a}}^-(x) - f_{\widehat{a}}^-(\widehat{x})| < \varepsilon_1/2$ imply that $r_1 - r_0 > \varepsilon_1/2 = 2\kappa_2$. This proves the claim. □

Finally, consider the case when (M_1,g_1) and (M_2,g_2) are two manifolds satisfying (5) with the sets W_1 and W_2 and a time-orientation preserving conformal diffeomorphism $\Phi : U_1 \to U_2$. Then, if U_1 is defined using paths $\mu_a^{(1)}(s)$, $a \in \overline{\mathcal{A}}$, $s \in [-1,1]$, by making $\mathcal{A}$ a smaller neighborhood of $\widehat{a}$ if necessary, we can use on U_2 the paths $\mu_a^{(2)}(s) = \Phi(\mu_a^{(1)}(s))$, $a \in \overline{\mathcal{A}}$. With such paths the sets $\mathcal{F}(W_1) \subset C(\overline{\mathcal{A}})$ on manifold M_1 and $\mathcal{F}(W_2) \subset C(\overline{\mathcal{A}})$ on manifold M_2 coincide.

3. Inverse problem for active measurements

In this section we start the proof of Theorem 1.5. Without loss of generality we may replace the set V where we do measurements by a smaller set U of the form (12). Also, by redefining the path $\widehat{\mu}_2$ in the claim of Theorem 1.5, we can assume that $\widehat{\mu}_2 = \Phi(\widehat{\mu}_1)$. Moreover, as the proof is constructive, and to simplify the notations, we do the constructions on just one Lorentzian manifold, (M, g) and assume that we are given the data

(19) the differentiable manifold U of the form (12), paths μ_a, $a \in \mathcal{A}$, and the source-to-solution map L_U.

Here, $L_U : f \mapsto u|_U$ is the source-to-observation map defined in (8) when the set $U \subset V$ is used as the measurement set. The choice of paths μ_a are discussed in Remark 3.1 below.

3.1. **Asymptotic expansion for the non-linear wave equation.** Let us consider the non-linear wave equation

$$\begin{aligned} &\square_g u + a u^2 = f, \quad \text{in } M_0 = (-\infty, T_0) \times N, \\ &u|_{(-\infty,0)\times N} = 0, \end{aligned} \tag{20}$$

where $a = a(x)$ is a smooth, nowhere vanishing function, $M_0 = (-\infty, T_0) \times N \subset M = \mathbb{R} \times N$, where (M, g) is a globally hyperbolic Lorentzian manifold. We denote by $\square_g^{-1}$ the causal inverse operator of $\square_g$.

When $B \subset N$ is compact and f in $C_0([0, T_0]; H_0^6(B)) \cap C_0^1([0, T_0]; H_0^5(B))$ is small enough, we see by using [63, Prop. 9.17], [34, Thm. III], or [13, App. III] that the equation (20) has a unique solution $u \in C([0, T_0]; H^5(N)) \cap C^1([0, T_0]; H^4(N))$. For a detailed analysis, see Appendix B in [45].

Let us consider the case when $f = \varepsilon f_1$ where $\varepsilon > 0$ is small. Then, we can write

$$u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$$

where w_j and the reminder term E_ε satisfy (see e.g. [13, App. III])

$$\begin{aligned} w_1 &= \square_g^{-1} f_1, \\ w_2 &= -\square_g^{-1}(a\, w_1\, w_1), \\ w_3 &= 2\,\square_g^{-1}(a\, w_1\, \square_g^{-1}(a\, w_1\, w_1)), \\ w_4 &= -\square_g^{-1}(a\, \square_g^{-1}(a\, w_1\, w_1)\, \square_g^{-1}(a\, w_1\, w_1)) \\ &\quad -4\,\square_g^{-1}(a\, w_1\, \square_g^{-1}(a\, w_1\, \square_g^{-1}(a\, w_1\, w_1))), \\ \|E_\varepsilon&\|_{C([0,T_0];H_0^4(N))\cap C^1([0,T_0];H_0^3(N))} \leq C(f_1)\, \varepsilon^5. \end{aligned} \tag{21}$$

In particular, we will consider sources f_1 for which the linearized term w_1 is a distorted plane wave.

Remark 3.1. The set U, given as differentiable manifold, and the source-to-solution map L_U determine the linearized source-to-solution map $L_U^{lin} : f_1 \mapsto \partial_\varepsilon(L_U(\varepsilon f_1))|_{\varepsilon=0}$. Furthermore, this map determines all pairs (f_1, w_1) such that $L_U^{lin}(f_1) = w_1$ and both f_1 and w_1 are compactly supported in U. Observer that then $\square_g w_1 = f_1$. In particular, for any $(x_0, \eta_0) \in T^*U$ there is a pair (f_1^τ, w_1^τ) such that $w_1^\tau(x) = e^{i\tau\phi(x)}\psi(x)$, where $\tau > 0$, $\phi, \psi \in C_0^\infty(U)$, $d\phi(x_0) = \eta_0$, in some neighborhood of x_0 we have $\psi = 1$. Then

$$g(\eta_0, \eta_0) = -\lim_{\tau\to\infty} \frac{\square_g w_1^\tau(x_0)}{\tau^2} = \lim_{\tau\to\infty} \frac{-f_1^\tau(x_0)}{\tau^2}.$$

This shows that U and L_U determine the metric tensor $g|_U$ in U. The set of pairs $(f_1, L_U f_1)$ that are in $C_0^\infty(U)^2$ coincide with the set of the pairs $\{(\square_g \phi + a\phi^2, \phi);\ \phi \in C_0^\infty(U)\}$. When $g|_U$ is known, these pairs determine $a|_U$. Hence, L_U determines also $a|_U$. Observe that by the same arguments, V and L_V determine the metric tensor $g|_V$ in V, too. Also, we note that when V and $g|_V$ are given, one can choose the time-like paths $\mu_a : [-1, 1] \to V$, $a \in \mathcal{A}$, appearing in (12), to be perturbations of the path $\widehat{\mu}([-1, 1])$ that depend smoothly on a parameter a in an open set. Thus the data (V, L_V) can be used to construct the paths μ_a and the set $U \subset V$ in (12).

3.2. **Linear wave equation and distorted plane waves.**

3.2.1. *Lagrangian distributions.* Let us recall the definition of the classical conormal and Lagrangian distributions that we will use below, see [27, 37, 56]. Let X be a manifold of dimension n and $\Lambda \subset T^*X \setminus \{0\}$ be a Lagrangian submanifold. Let $\phi(x, \theta)$, $(x, \theta) \in X \times \mathbb{R}^N$ be a non-degenerate phase function that locally parametrizes Λ near a point $(x_0, \xi_0) \in \Lambda$, i.e., in a conic neighborhood $\Gamma \subset T^*X \setminus \{0\}$ of (x_0, ξ_0), the submanifold Λ coincides with the set $\{(x, d_x\phi(x, \theta)) \in \Gamma;\ d_\theta\phi(x, \theta) = 0\}$. We say that a distribution $u \in \mathcal{D}'(X)$ is a classical Lagrangian distribution associated with Λ and denote $u \in \mathcal{I}^m(X; \Lambda)$, if in local coordinates $X : W \to \mathbb{R}^n$, u can be represented as an oscillatory integral,

$$u(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x, \theta)\, d\theta, \quad x \in W \tag{22}$$

where $a(x, \theta) \in S^\mu(W; \mathbb{R}^N)$ is a classical symbol of order $\mu = m+n/4-N/2$, see [27, 37, 56].

For classical Lagrangian distributions $u \in \mathcal{I}^m(X; \Lambda)$ one can define a principal symbol $\sigma_u^{(p)}(x_0, \zeta_0)$ of u, at $(x_0, \zeta_0) \in \Lambda$, that satisfies

$$\sigma_u^{(p)}(x_0, \zeta_0) \in S^{m+\frac{n}{4}}(\Lambda, \Omega^{1/2} \otimes L)/S^{m+\frac{n}{4}-1}(\Lambda, \Omega^{1/2} \otimes L),$$

where L is the Maslov-Keller line bundle and $\Omega^{1/2}$ are the half-densities on X, on details, see [31, Thm. 11.10]. We note that below we do computations using only principal symbols of conormal distributions considered below.

In particular, when $S \subset X$ is a submanifold, its conormal bundle $N^*S = \{(x, \xi) \in T^*X \setminus \{0\};\ x \in S,\ \xi \perp T_xS\}$ is a Lagrangian submanifold. If u is a Lagrangian distribution associated to Λ_1 where $\Lambda_1 = N^*S$, we say that u is a (classical) conormal distribution.

Let us next consider the case when $X = \mathbb{R}^n$, $(x^1, x^2, \dots, x^n)$ are the Euclidean coordinates and $x' = (x_1, \dots, x_{d_1})$, $S_1 = \{0\} \times \mathbb{R}^{n-d_1} = \{x' =$

$0\} \subset \mathbb{R}^n$ and $\Lambda_1 = N^*S_1$. Then $u \in \mathcal{I}^m(X;\Lambda_1)$ can be represented by (22) with $N = d_1$ and $\phi(x,\theta) = x'\cdot\theta$, that is,

$$u(x^1,\dots,x^n) = \int_{\mathbb{R}^{d_1}} e^{ix'\cdot\theta} a(x^1,\dots,x^n,\theta)\,d\theta. \tag{23}$$

For example, $\delta_{S_1}(x) \in \mathcal{I}^{-n/4+d_1/2}(\mathbb{R}^n; N^*S_1)$, where $\delta_{S_1}(x)$ denotes the Dirac delta distribution supported on S_1.

The principal symbol of a conormal distribution $u \in \mathcal{I}^m(\mathbb{R}^n; N^*S_1)$, represented in the form (23), can be identified with a function $c(x,\theta)$ that is μ-positive homogeneous in θ, such that $a(x,\theta)-(1-\phi(\theta))c(x,\theta) \in S^{\mu-1}(X;\mathbb{R}^{d_1})$ where $\phi \in C_0^\infty(\mathbb{R}^{d_1})$ is 1 in a neighborhood of zero. For a manifold X and a surface $S \subset X$, we can use this definition to define a principal symbol of a conormal distribution $u \in \mathcal{I}^m(X; N^*S)$ in local coordinates. On the invariant nature of this definition, see [36, Sec. 18.2].

Next we recall the definition of $\mathcal{I}^{p,l}(X;\Lambda_1,\Lambda_2)$, the space of the distributions u in $\mathcal{D}'(X)$ associated to two cleanly intersecting Lagrangian manifolds $\Lambda_1,\Lambda_2 \subset T^*X\setminus\{0\}$, see [15, 27, 56]. We recall that Λ_1 and Λ_2 intersect cleanly if $\Sigma = \Lambda_1\cap\Lambda_2$ is a smooth manifold and its tangent space satisfies $T_\lambda\Sigma = T_\lambda\Lambda_1 \cap T_\lambda\Lambda_2$ for all $\lambda \in \Sigma$. These classes have been widely used in the study of inverse problems, see [14, 21]. Let us start with the case when $X = \mathbb{R}^n$.

Let $(x^1,x^2,\dots,x^n)$ be the Euclidean coordinates in $\mathbb{R}^n$. Let $S_1,S_2 \subset \mathbb{R}^n$ be the linear subspaces of codimensions d_1 and d_1+d_2, respectively be such that $S_2 \subset S_1$. We use in $\mathbb{R}^n$ the Euclidean coordinates $(x^1,x^2,\dots,x^n) = (x',x'',x''')$ where $x' = (x_1,\dots,x_{d_1})$, $x'' = (x_{d_1+1},\dots,x_{d_1+d_2})$, $x''' = (x_{d_1+d_2+1},\dots,x_n)$ and assume that $S_1 = \{x'=0\}$, $S_2 = \{x'=x''=0\}$. Let us denote $\Lambda_1 = N^*S_1$, $\Lambda_2 = N^*S_2$. Then $u \in \mathcal{I}^{p,l}(\mathbb{R}^n; N^*S_1, N^*S_2)$ if and only if

$$u(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x'\cdot\theta'+x''\cdot\theta'')} a(x,\theta',\theta'')\,d\theta' d\theta'', \tag{24}$$

where the symbol $a(x,\theta',\theta'')$ belongs in the product type symbol class $S^{\mu_1,\mu_2}(\mathbb{R}^n;(\mathbb{R}^{d_1}\setminus 0)\times\mathbb{R}^{d_2})$ that is the space of functions $a \in C^\infty(\mathbb{R}^n\times\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ that satisfy

$$|\partial_x^\gamma\partial_{\theta'}^\alpha\partial_{\theta''}^\beta a(x,\theta',\theta'')| \le C_{\alpha\beta\gamma K}(1+|\theta'|+|\theta''|)^{\mu_1-|\alpha|}(1+|\theta''|)^{\mu_2-|\beta|} \tag{25}$$

for all $x \in K$, multi-indexes α,β,γ, and compact sets $K \subset \mathbb{R}^n$. Above, $\mu_1 = p+l-d_1/2+n/4$ and $\mu_2 = -l-d_2/2$.

When X is a manifold of dimension n and $\Lambda_1,\Lambda_2 \subset T^*X\setminus\{0\}$ are two cleanly intersecting Lagrangian manifolds, we define the class $\mathcal{I}^{p,l}(X;\Lambda_1,\Lambda_2) \subset \mathcal{D}'(X)$ to consist of locally finite sums of distributions of the form $u = Au_0$, where $u_0 \in \mathcal{I}^{p,l}(\mathbb{R}^n; N^*S_1, N^*S_2)$ and $S_1,S_2 \subset \mathbb{R}^n$ are the linear subspace of codimensions d_1 and d_1+d_2, respectively, such that $S_2 \subset S_1$, and A is a Fourier integral operator of order zero with a canonical relation Σ for which $\Sigma\circ(N^*S_1)' \subset \Lambda_1'$ and $\Sigma\circ(N^*S_2)' \subset \Lambda_2'$.

Here, for $\Lambda \subset T^*X$ we denote $\Lambda' = \{(x,-\xi) \in T^*X;\ (x,\xi) \in \Lambda\}$. The definition of $\mathcal{I}^{p,l}(X;\Lambda_1,\Lambda_2)$ is discussed in detail in [56], in particular the existence of the canonical relation Σ connecting the pair (Λ_1,Λ_2) of cleanly intersecting Lagrangians to the microlocal model (N^*S_1, N^*S_2) is proven in [56, Prop. 1.3]. When X and Y are manifolds and $\Sigma \subset T^*X \times T^*Y$ we use also the notation $\Sigma' = \{(x,\xi,y,-\eta);\ (x,\xi,y,\eta) \in \Sigma\}$.

In most cases below, $X = M$. We denote then $\mathcal{I}^p(M;\Lambda_1) = \mathcal{I}^p(\Lambda_1)$ and $\mathcal{I}^{p,l}(M;\Lambda_1,\Lambda_2) = \mathcal{I}^{p,l}(\Lambda_1,\Lambda_2)$. Also, $\mathcal{I}(\Lambda_1) = \cup_{p\in\mathbb{R}}\mathcal{I}^p(\Lambda_1)$.

By [27, 56], if R_1 and R_2 are pseudodifferential operators of order zero on M which are microlocally smoothing in a conic neighborhood of Λ_2 and Λ_1, respectively, we have

$$R_1 : \mathcal{I}^{p,l}(\Lambda_1,\Lambda_2) \to \mathcal{I}^{p+l}(\Lambda_1), \quad R_2 : \mathcal{I}^{p,l}(\Lambda_1,\Lambda_2) \to \mathcal{I}^{p}(\Lambda_2). \tag{26}$$

Thus the principal symbol of $u \in \mathcal{I}^{p,l}(\Lambda_1,\Lambda_2)$ is well defined on $\Lambda_1 \setminus \Lambda_2$ and $\Lambda_2 \setminus \Lambda_1$. We denote $\mathcal{I}(\Lambda_1,\Lambda_2) = \cup_{p,l\in\mathbb{R}}\mathcal{I}^{p,l}(\Lambda_1,\Lambda_2)$. We recall that $(x_0,\xi_0) \in T^*M$ belong in the wave front set $\mathrm{WF}\,(u)$ of a distribution $u \in \mathcal{D}'(M)$ if $u(x)$ is not C^∞ smooth near x_0 in the direction ξ_0 (see [18], Section 1.3 for the precise definition). For Lagrangian distributions $v \in \mathcal{I}^p(\Lambda_1)$ and $u \in \mathcal{I}^{p,l}(\Lambda_1,\Lambda_2)$ we have

$$\mathrm{WF}\,(v) \subset \Lambda_1, \quad \mathrm{WF}\,(u) \subset \Lambda_1 \cup \Lambda_2. \tag{27}$$

Below, when $\Lambda_j = N^*S_j$, $j = 1,2$ are conormal bundles of smooth cleanly intersecting submanifolds $S_j \subset M$ of codimension d_j, where $\dim(M) = n$, we use the traditional notations,

$$\mathcal{I}^\mu(S_1) = \mathcal{I}^{\mu+d_1/2-n/4}(N^*S_1), \quad \mathcal{I}^{\mu_1,\mu_2}(S_1,S_2) = \mathcal{I}^{p,l}(N^*S_1,N^*S_2), \tag{28}$$

where $p = \mu_1 + \mu_2 + d_1/2 - n/4$ and $l = -\mu_2 - d_2/2$, and call such distributions the conormal distributions associated to S_1 or product type conormal distributions associated to S_1 and S_2, respectively. By [27], $\mathcal{I}^\mu(X;S_1) \subset L^p_{loc}(X)$ for $\mu < -d_1(p-1)/p$, $1 \le p < \infty$. Further developments for the theory of the paired Lagrangian distributions are in [32, 28].

3.2.2. *Inverse of the linear wave operator.* Next we will shortly discuss how paired Lagrangian distributions are used in [56, 27] to study parametrices (and inverses) of real-principal type operators, in particular the wave operator $\square_g$ on a globally hyperbolic Lorentzian manifold (M,g). To consider the wave operator, recall that the characteristic variety of $\square_g$ is

$$\mathrm{Char}(\square_g) = \{(x,\xi) \in T^*M\backslash 0;\ p(x,\xi) = 0\},$$

where $p(x,\xi) = g^{jk}(x)\xi_j\xi_k$. For the wave operator, $\mathrm{Char}\,(\square_g)$ is the set of light-like co-vectors with respect to g. Also, a bicharacteristic of $\square_g$ is the integral curve of the Hamiltonian vector field of $p(x,\xi)$ in T^*M. For $(x,\xi) \in \mathrm{Char}\,(\square_g)$, we denote by $\Theta_{x,\xi} \subset T^*M$ the bicharacteristic of $\square_g$

that contains $(x, \xi) \in L^*M$. The bicharacteristics are closely related to light-like geodesics: We have $(y, \eta) \in \Theta_{x,\xi}$ if and only if there is $t \in \mathbb{R}$ such that for $v = \eta^\sharp$ and $w = \xi^\sharp$ we have $(y, v) = (\gamma_{x,w}(t), \dot\gamma_{x,w}(t))$ where $\gamma_{x,w}$ is a light-like geodesic with respect to the metric g with the initial data $(x, w) \in LM$. Here, we use the notations $(\xi^\sharp)^j = g^{jk}\xi_k$ and $(w^\flat)_j = g_{jk}w^k$.

Let $\Lambda_1 \subset T^*M$ be a Lagrangian manifold and consider the solution of $\square_g u_1 = f_1$ with a source $f_1 \in \mathcal{I}^m(\Lambda_1)$. When the characteristic variety $\mathrm{Char}(\square_g)$ intersects Λ_1, this gives rise to the propagation of singularities. Indeed, by Hörmander's theorem on propagation of singularities along bicharacteristics, [37, Theorem 26.1.4], see also [27, Prop. 2.1], the wave front set $\mathrm{WF}(u_1)$ of u_1 is contained in the union of Λ_1 and the bicharacteristics that contain points of the intersection $\mathrm{Char}(\square_g) \cap \Lambda_1$. When $\mathrm{Char}(\square_g)$ and Λ_1 intersect transversally, the union of these bicharacteristics is a Lagrangian manifold. This result was extended in [56, 32] where it was shown that the Schwartz kernel of the inverse of the wave operator is a distribution associated to two intersecting Lagrangian manifolds. Indeed, when (M, g) is a globally hyperbolic manifold, the operator $\square_g$ has a causal inverse operator $Q = \square_g^{-1}$, see e.g. [3, Thm. 3.2.11]. A geometric representation for its kernel is given in [48]. Below, we often use the same notation for the operator Q with its Schwartz kernel $Q(x, y)$. By [56], the Schwartz kernel Q satisfies $Q \in \mathcal{I}^{p,l}(\Delta'_{T^*M}, \Lambda_g)$, $p = -\frac{3}{2}$, $l = -\frac{1}{2}$. Here, $\Delta'_{T^*M} = N^*(\{(x, x);\ x \in M\})$, and $\Lambda_g \subset T^*M \times T^*M$ is the Lagrangian manifold associated to the canonical relation of the operator $\square_g$, that is,

$$\Lambda_g = \{(x, \xi, y, -\eta);\ (x, \xi) \in \mathrm{Char}\,(\square_g),\ (y, \eta) \in \Theta_{x,\xi}\}, \tag{29}$$

where $\Theta_{x,\xi} \subset T^*M$ is the bicharacteristic of $\square_g$ containing (x, ξ).

By [37, Thm. 26.1.14], $\square_g^{-1} : H^s_{comp}(M_0) \to H^{s+1}_{loc}(M_0)$ is a bounded. We will repeatedly use the fact (see [27, Prop. 2.1]) that if $F \in \mathcal{I}^p(\Lambda_0)$ is compactly supported and Λ_0 intersects $\mathrm{Char}(\square_g)$ transversally so that all bicharacterestics of $\square_g$ intersect Λ_0 only finitely many times, then $\square_g^{-1} F \in \mathcal{I}^{p-3/2,-1/2}(\Lambda_0, \Lambda_1)$ where $\Lambda_1' = \Lambda_g \circ \Lambda_0'$, that is,

$$\Lambda_1 = \{(x, -\xi);\ (x, \xi, y, -\eta) \in \Lambda_g,\ (y, \eta) \in \Lambda_0\}. \tag{30}$$

The manifold Λ_1 is called the flowout from $\Lambda_0 \cap \mathrm{Char}(\square_g)$ by the Hamiltonian vector field associated to $p(x, \xi)$.

3.2.3. *Distorted plane waves satisfying a linear wave equation.* Next we consider a distorted plane wave whose singular support is concentrated near a geodesic. These waves, sketched in Fig. 1(Right), propagate near the geodesic $\gamma_{x_0,\zeta_0}([0, \infty))$ and are singular on a surface $K(x_0, \zeta_0, s_0)$, defined below in (31). The surface $K(x_0, \zeta_0, s_0)$ is a subset of the light cone $^+(x_0)$ and the parameter s_0 gives a "width" of the singular support of the wave around $\gamma_{x_0,\zeta_0}([0,\infty))$. When $s_0 \to 0$, its singular support tends to the set $\gamma_{x_0,\zeta_0}([0,\infty))$. Next we will define these waves.

Let $x_0 \in U$, $\zeta_0 \in L^+_{x_0}M$ and $s_0 > 0$ and recall that g^+ is a Riemannian metric on M. Also, let

$$\mathcal{V}_{x_0,\zeta_0,s_0} = \{\eta \in T_{x_0}M : \|\eta - \zeta_0\|_{g^+} < s_0,\ \|\eta\|_{g^+} = \|\zeta_0\|_{g^+}\}$$

be a neighborhood of ζ_0 on a sphere.

We define the subset of the light cone, $K(x_0,\zeta_0,s_0) \subset M_0$ associated to the vector (x_0,ζ_0) and $x_0 \in U$ and parameter $s_0 \in \mathbb{R}_+$ by

$$K(x_0,\zeta_0,s_0) = \{\gamma_{x_0,\eta}(t) \in M_0;\ \eta \in \mathcal{W}_{x_0,\zeta_0,s_0},\ t \in (0,\infty)\}, \tag{31}$$

where $\mathcal{W}_{x_0,\zeta_0,s_0} = L^+_{x_0}M \cap \mathcal{V}_{x_0,\zeta_0,s_0}$, see Figure 1.

Let

$$\begin{aligned}\Sigma(x_0,\zeta_0,s_0) &= \{(x_0, r\eta^\flat) \in T^*M;\ \eta \in \mathcal{V}_{x_0,\zeta_0,s_0},\ r \in \mathbb{R}\setminus\{0\}\},\\ \Lambda(x_0,\zeta_0,s_0) &= \{(\gamma_{x_0,\eta}(t), r\dot\gamma_{x_0,\eta}(t)^\flat) \in T^*M;\ \eta \in \mathcal{W}_{x_0,\zeta_0,s_0},\\ &\qquad t \in (0,\infty),\ r \in \mathbb{R}\setminus\{0\}\}.\end{aligned} \tag{32}$$

Note that $\Lambda(x_0,\zeta_0,s_0)$ is the Lagrangian manifold that is the flowout from $\mathrm{Char}(\square_g) \cap \Sigma(x_0,\zeta_0,s_0)$ by the Hamiltonian vector field of associated to $p(x,\xi)$ in the future direction, see (30). Below, we will use sources $f \in \mathcal{I}^{n+1}(\Sigma(x_0,\zeta_0,s_0))$. An example of such sources are functions $A\delta_{x_0}$ where A is a psudodifferential operator miclocally supported near (x_0,ζ_0). For example, in local coordinates we can use $A = \phi_0(x)(1-\psi_0(D))\psi_1(D)$, where $\phi_0 \in C^\infty_0(\mathbb{R}^n)$ is supported near x_0, function $\psi_0 \in C^\infty_0(\mathbb{R}^n)$ is equal to 1 in the neighborhood of zero and $\psi_1 \in C^\infty(\mathbb{R}^n\setminus\{0\})$ is homogeneous function supported in a conic neighborhood of direction ζ_0. Function $A\delta_{x_0}$ can be considered as a "directed point source" that produces a wave $\square_g^{-1}(A\delta_{x_0})$ which singularities propagate along $\Lambda(x_0,\zeta_0,s_0)$. Outside x_0, such wave could be considered as a "piece of distorted plane wave", see Fig. 1. Note that $\Sigma(x_0,\zeta_0,s_0) \subset T^*M$ is a Lagrangian submanifold that is subset of the conormal bundle Σ_{x_0} of the point $\{x_0\}$, considered as a 0-dimensional submanifold of M, that is,

$$\Sigma_{x_0} = N^*(\{x_0\}) = T^*_{x_0}M \setminus \{0\}, \tag{33}$$

and hence, $f \in \mathcal{I}^{n+1}(\Sigma(x_0,\zeta_0,s_0))$ is a conormal distribution.

When $K^{reg} \subset K = K(x_0,\zeta_0,s_0)$ is the set of points x that have a neighborhood W such that $K \cap W$ is a smooth 3-dimensional submanifold, we have $N^*K^{reg} \subset \Lambda(x_0,\zeta_0,s_0)$. Note that if $(x,\xi) \in N^*K^{reg}$ then also $(x,-\xi) \in N^*K^{reg}$, and this is the reason why we used factor $r \in \mathbb{R}\setminus\{0\}$ in formula (32).

Lemma 3.1. *Let n be an integer, $s_0 > 0$, $K = K(x_0,\zeta_0,s_0)$, $\Lambda_1 = \Lambda(x_0,\zeta_0,s_0)$ and $\Sigma = \Sigma(x_0,\zeta_0,s_0)$. Let $(x,\xi) \in \Sigma \cap L^*M$, $v = \xi^\sharp \in L_xM$, $r \in \mathbb{R}$ and $y = \gamma_{x,v}(r)$ and $\eta = (\dot\gamma_{x,v}(r))^\flat$ be such that $x < y$.*

Assume that $f_1 \in \mathcal{I}^{n+1}(\Sigma)$ *is a compactly supported classical conormal distribution.*

Let us consider the restriction of $w_1 = \square_g^{-1} f_1$ *to* $M_0 \setminus \{x_0\}$*. Then* $w_1|_{M_0\setminus\{x_0\}} \in \mathcal{I}^{n-1/2}(M_0 \setminus \{x_0\}; \Lambda_1)$*.*

Let $\sigma^{(p)}_{f_1}(x,\xi)$ *be the principal symbol of* f_1 *at* (x,ξ) *and* $\sigma^{(p)}_{w_1}(y,\eta)$ *be the principal symbol of* w_1 *at* $(y,\eta) \in \Lambda_1$*. Then*

$$\sigma^{(p)}_{w_1}(y,\eta) = R(y,\eta,x,\xi)\sigma^{(p)}_{f_1}(x,\xi) \tag{34}$$

where $R = R(y,\eta,x,\xi)$ *is an invertible linear operator.*

Moreover, when the geodesic $\gamma_{x,v}([0,r])$ *has no cut points, the point* y *has a neighborhood* $V_0 \subset M$ *such that* $S_1 = {}^+(x) \cap V_0$ *is a smooth submanifold of codimension 1 and* $w_1|_{V_1} \in \mathcal{I}^n(S_1)$ *is a conormal distribution. Then,* R *can be considered as a non-zero complex number.*

Observe that in the claim of the lemma, $((x,\xi),(y,\eta)) \in \Lambda'_g$ and $(y,\eta) \in T^*M$ be on the same bicharacteristic of $\square_g$ as (x,ξ).

We call the solution w_1 a distorted plane wave associated to the submanifold $K(x_0,\zeta_0,s_0)$.

Proof. Recall that the Schwartz kernel Q of the causal inverse operator $Q = \square_g^{-1}$ satisfies $Q \in \mathcal{I}^{-3/2,-1/2}(\Delta'_{T^*M},\Lambda_g)$. As $f \in \mathcal{I}^{n+1}(\Sigma)$, [27, Prop. 2.1] and the definition (32) of Λ_1 imply that $w_1 = \square_g^{-1} f \in \mathcal{I}^{n+1-3/2,-1/2}(\Sigma,\Lambda_1)$. This yields that $w_1|_{M_0\setminus\{x_0\}} \in \mathcal{I}^{n+1-3/2}(\Lambda_1)$. This implies that the restriction $w_1|_V$ is a conormal distribution in $\mathcal{I}^n(S_1)$. Moreover, [27, Prop. 2.1] implies the formula (34) for the principal symbols, where R is obtained by solving an ordinary differential equation along a bicharacteristic curve. Making similar considerations for the adjoint of the $\square_g^{-1}$, i.e., considering the propagation of singularities using reversed causality, and by solving an ordinary differential equation along a bicharacteristic, we see that R is invertible.

Finally, when the geodesic $\gamma_{x,v}([0,r])$ has no cut points, y has a neighborhood V where the light cone is a smooth hypersurface. $\square$

3.3. **Microlocal analysis of the non-linear interaction of waves.** Next we consider the interaction of four C^k-smooth waves having conormal singularities on hypersurfaces, where $k \in \mathbb{Z}_+$ is sufficiently large. Interaction of such waves produces an artificial point source at the intersection point of the hypersurfaces. We will show that such artificial point sources can be created to arbitrary points of $I^+(p^-) \cap I^-(p^+)$. We use such artificial point sources on the unknown manifold (M,g) to create distorted spherical waves that determine the earliest light observations sets.

First considerations on the non-linear interaction of conormal waves, were done by Bony [11], Melrose and Ritter [57, 58] and Rauch and Reed, [62] for semilinear hyperbolic equations. In particular, they analyzed three conormal waves and showed that the interaction of three plane waves in three and higher dimensional spacetimes produces "virtual sources" that are singular on co-dimension 3 submanifolds. In the three dimensional spacetime such sources correspond to point sources. In [57, 58], the microlocal properties of non-linear waves are analyzed also for arbitrary many interacting waves when the interaction of the waves and the propagation of singularities take place on a union of finitely many submanifolds that form so-called characteristically complete variety of finite type (the geometrical restrictions caused by this assumption is discussed in detail in [57, Section 7]). Also, the appearance of the new wavefronts due to the interaction of non-linear terms at caustics or due to boundary and corner diffraction have been analyzed in [40, 55, 72, 73, 75]. The microlocal properties and regularity of the solutions of non-linear hyperbolic equations, that correspond to the interaction of several conormal waves, are analyzed in the monograph by Beals [4].

As discussed in the introduction, the focus of the above papers on the interaction of conormal singularities for non-linear hyperbolic equations is different from our paper as in those it is assumed that the geometrical setting of the interacting singularities is a priori known. In inverse problems, when we study waves on an unknown manifold, we do not know the geometry of the surfaces on which the waves are singular.

In this section we consider the interaction of waves in a subset of the spacetime where we are sure that the linearized waves have no caustics. However, caustics may appear in the interaction of waves and these waves may interact with the linearized waves. Later, in Section 4 we use global Lorentzian geometry to obtain a procedure that marches through the diamond set $J(p^-,p^+)$ by reconstructing it in small pieces. This will allow us to avoid difficulties associated with the appearance of caustics in the linearized waves.

3.3.1. *Forth order interaction of waves for the non-linear wave equation.* Next, we introduce a vector of four ε variables denoted by $\vec\varepsilon = (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in\mathbb{R}^4$. Let $s_0>0$. For the non-linear wave equation (7) we denote by $u_{\vec\varepsilon}$ its solution when the source $f_{\vec\varepsilon}$ is given by

$$f_{\vec\varepsilon} := \sum_{j=1}^4 \varepsilon_j f_j,\quad f_j\in\mathcal{I}^{n+1}(\Sigma(x_j,\zeta_j,s_0)), \tag{35}$$

and (x_j,ζ_j) are light-like vectors with $x_j\in U$. Moreover, we assume that the sources satisfy

$$\text{supp}\,(f_j)\cap J^+(\text{supp}\,(f_k))=\emptyset,\quad \text{for all } j\neq k, \tag{36}$$

$$J^+(W)\cap J^-(W)\subset U,\quad \text{where } W=\bigcup_{j=1}^4 \text{supp}\,(f_j).$$

The implies that the supports of the sources are causally independent.

The sources f_j give raise to the solutions of the linearized wave equations, which we denote by

(37) $u_j := \partial_{\varepsilon_j} u_{\vec\varepsilon}|_{\vec\varepsilon=0} = \square_g^{-1}(f_j) \in \mathcal{I}(M_0 \setminus \{x_j\}; \Lambda(x_j,\zeta_j,s_0)).$

In the following we use the notations $\partial_{\vec\varepsilon}^1 u_{\vec\varepsilon}|_{\vec\varepsilon=0} := \partial_{\varepsilon_1} u_{\vec\varepsilon}|_{\vec\varepsilon=0}$, $\partial_{\vec\varepsilon}^2 u_{\vec\varepsilon}|_{\vec\varepsilon=0} := \partial_{\varepsilon_1}\partial_{\varepsilon_2} u_{\vec\varepsilon}|_{\vec\varepsilon=0}$, $\partial_{\vec\varepsilon}^3 u_{\vec\varepsilon}|_{\vec\varepsilon=0} := \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec\varepsilon}|_{\vec\varepsilon=0}$, and

$$\partial_{\vec\varepsilon}^4 u_{\vec\varepsilon}|_{\vec\varepsilon=0} := \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec\varepsilon}|_{\vec\varepsilon=0}.$$

Below, for the non-linear wave equation, we denote the wave produced by the fourth order interaction of waves u_j by

$$\begin{aligned} \mathcal{U}^{(4)} &= \partial_{\vec\varepsilon}^4 u_{\vec\varepsilon}|_{\vec\varepsilon=0} = \square_g^{-1}\mathcal{S}, \quad \mathcal{S} = \sum_{\sigma\in\Sigma(4)} \mathcal{S}_\sigma, \\ \mathcal{S}_\sigma &= -\Big(a\,\square_g^{-1}(a\,u_{\sigma(4)}\,u_{\sigma(3)})\,\square_g^{-1}(a\,u_{\sigma(2)}\,u_{\sigma(1)}) \\ &\quad +4a\,u_{\sigma(4)}\,\square_g^{-1}(a\,u_{\sigma(3)}\,\square_g^{-1}(a\,u_{\sigma(2)}\,u_{\sigma(1)}))\Big), \end{aligned} \tag{38}$$

where $\Sigma(4)$ is the set of permutations σ of the set $\{1,2,3,4\}$, see (21).

3.3.2. *On the singular support of the non-linear interaction of three waves.* We will consider the case when we send distorted plane waves propagating on surfaces $K_j = K(x_j,\xi_j,s_0)$, $s_0>0$, cf. (31), and these waves interact.

Next we consider the geometry related to the three wave interactions of the waves. Let $\mathcal{X}((\vec x,\vec\xi),s_0) \subset L^*M$ be the set of all light-like co-vectors (x,ξ) that are in the normal bundles $N^*(K_{j_1}\cap K_{j_2}\cap K_{j_3})$ with some $1\le j_1<j_2<j_3\le 4$, that is, $\mathcal{X}=\mathcal{X}((\vec x,\vec\xi),s_0)$ is

$$\begin{aligned} \mathcal{X}((\vec x,\vec\xi),s_0) &= \bigcup_{1\le j_1<j_2<j_3\le 4} \mathcal{X}_{j_1j_2j_3}((\vec x,\vec\xi),s_0), \\ \mathcal{X}_{j_1j_2j_3}((\vec x,\vec\xi),s_0) &= \bigcup_{x\in K_{j_1}\cap K_{j_2}\cap K_{j_3}} (N_x^*K_{j_1}+N_x^*K_{j_2}+N_x^*K_{j_3})\cap L^*M_0. \end{aligned} \tag{39}$$

Moreover, we define

$$\begin{aligned} \mathcal{H}((\vec x,\vec\xi),s_0) &= \bigcup_{1\le j_1<j_2<j_3\le 4} \mathcal{X}_{j_1j_2j_3}((\vec x,\vec\xi),s_0), \\ \mathcal{H}_{j_1j_2j_3}((\vec x,\vec\xi),s_0) &= \{(y,\eta)\in T^*M_0;\ \text{there is } (x,\zeta)\in\mathcal{X}((\vec x,\vec\xi),s_0) \\ &\qquad \text{such that } x\le y \text{ and } (y,\eta)\in\Theta_{x,\zeta}\} \end{aligned} \tag{40}$$

and $\mathcal{Y}((\vec x,\vec\xi),s_0) = \pi(\mathcal{H}((\vec x,\vec\xi),s_0))$, where $\pi: T^*M_0\to M_0$ is the projection to the base space. Finally, let

$$\mathcal{X}(\vec x,\vec\xi) = \bigcap_{s_0>0} \mathcal{X}((\vec x,\vec\xi),s_0), \quad \mathcal{Y}(\vec x,\vec\xi) = \bigcap_{s_0>0} \mathcal{Y}((\vec x,\vec\xi),s_0). \tag{41}$$

The three wave interaction happens then on $\pi(\mathcal{X}((\vec x,\vec\xi),s_0))$ and, roughly speaking, this interaction sends singularities to $\mathcal{H}((\vec x,\vec\xi),s_0)$.

For instance, in Minkowski space, when three plane waves (whose singular supports are hyperplanes) collide, the intersections of the hyperplanes is a 1-dimensional space-like line $K_{123} = K_1 \cap K_2 \cap K_3$ in the 4-dimensional space-time. This corresponds to a virtual point source moving continuously in time and creates a "conical" wave that propagates near the surface $\mathcal{Y}((\vec{x},\vec{\xi}), s_0)$. To visualize this, see the supplementary video [76] and Figure 3 that display a conic waves produced by the interaction of three waves on K_{123}. The video shows also the spherical wave that is produced by the interaction of all four waves and that emanates from the intersection point $q \in K_1 \cap K_2 \cap K_3 \cap K_4$.

In this paper we do not analyze carefully the singularities produced by the three wave interaction near $\mathcal{Y}((\vec{x},\vec{\xi}), s_0)$. Our goal is to consider the singularities produced by the four wave interaction in the domain $M_0 \setminus \mathcal{Y}((\vec{x},\vec{\xi}), s_0)$. We consider also the limit when $s_0 \to 0$. Then, the exceptional set $\mathcal{Y}((\vec{x},\vec{\xi}), s_0)$ tends to a set $\mathcal{Y}(\vec{x},\vec{\xi})$ whose Hausdorff dimension is at most 2.

3.3.3. *Wave front set of the wave produced by the interaction of four waves.* Next we will consider $\mathrm{WF}(\mathcal{U}^{(4)})$ where $\mathcal{U}^{(4)}$ is the wave produced by the interaction of the four linearized waves corresponding to the sources $f_j \in \mathcal{I}^{n+1}(\Sigma(x_0,\zeta_0,s_0))$, $j \le 4$, see (38).

Definition 3.2. *We say that the geodesics corresponding to the vectors $(\vec{x},\vec{\xi}) = ((x_j,\xi_j))_{j=1}^4$ intersect and the intersection takes place at the point $q \in M_0$ if there are $t_j > 0$ such that $q = \gamma_{x_j,\xi_j}(t_j)$ for all $j = 1,2,3,4$. We say that the intersection of geodesics is regular if $t_j \in (0,\mathbf{t}_j)$, where $\mathbf{t}_j = \rho(x_j,\xi_j)$ and vectors $\dot\gamma_{x_j,\xi_j}(t_j) \in T_qM_0$, $j = 1,2,3,4$ are linearly independent.*

For $q \in M_0$, let Λ_q^+ be the Lagrangian manifold

$$\Lambda_q^+ = \{(y,\eta) \in T^*M_0 \quad ; \quad y = \gamma_{q,\zeta}(1),\ \eta^\sharp = r\dot\gamma_{q,\zeta}(1),\ \zeta \in L_q^+M_0,\ r \in \mathbb{R}\setminus 0\}.$$

Note that the projection $\pi(\Lambda_q^+)$ of Λ_q^+ on M_0 is the light cone ${}^+(q)$.

Next we consider $x_j \in U$ and $\xi_j \in L^+_{x_j}M_0$, such that $(\vec{x},\vec{\xi}) = ((x_j,\xi_j))_{j=1}^4$ satisfy, see Fig. 5(Right),

$$x_j \in U \quad \text{and} \quad x_j \notin J^+(x_k) \text{ for } j \neq k. \tag{42}$$

We denote

$$\mathcal{N}(\vec{x},\vec{\xi}) = M_0 \setminus \bigcup_{j=1}^4 J^+(\gamma_{x_j,\xi_j}(\mathbf{t}_j)), \quad \text{where } \mathbf{t}_j := \rho(x_j,\xi_j). \tag{43}$$

Note that two geodesics $\gamma_{x_j,\xi_j}([0,\infty))$ can intersect at most once in $\mathcal{N}(\vec{x},\vec{\xi})$. Below, let

$$K_j = K(x_j,\xi_j,s_0), \quad \Lambda_j = \Lambda(x_j,\xi_j,s_0), \tag{44}$$

cf. (31), (32), where $s_0 > 0$ is so small that one of the following two cases are satisfied:

(A) $(\cap_{j=1}^4 K_j) \cap \mathcal{N}(\vec{x}, \vec{\xi}) = \emptyset$,

or

(B) $(\cap_{j=1}^4 K_j) \cap \mathcal{N}(\vec{x}, \vec{\xi}) = \{q\}$, where
$q = \gamma_{x_j,\xi_j}(t_j)$ with $t_j > 0$ for all $j = 1, 2, 3, 4$,

and the intersection of any K_i and K_j with $i \neq j$ is transversal. In the case (B) all geodesics γ_{x_j,ξ_j}, $j = 1, 2, 3, 4$ intersect at a point q in $\mathcal{N}(\vec{x}, \vec{\xi})$ and in the case (A) the geodesics do not intersect.

Below, $\Sigma(4)$ is the set of permutations $\sigma : (1, 2, 3, 4) \to (1, 2, 3, 4)$.

Observe that in the set $\mathcal{N}(\vec{x}, \vec{\xi})$ the geodesics $\gamma_{x_j,\xi_j}([0, \infty))$ do not have conjugate points and thus the waves u_j do not have caustics in this set.

In the next theorem we consider the singularities of the wave $\mathcal{U}^{(4)}$, produced by the interaction of four waves u_j, outside a "small" set $\mathcal{Y}(\vec{x}, \vec{\xi}) \cup \bigcup_{j=1}^4 \gamma_{x_j,\xi_j}([0, \infty))$. Essentially, we show that no such singularities can be detected outside the causal future $J^+(q)$ of the point q where all four plane waves interact. Also, we show that in the case when the directions of the distorted planes at q are linearly independent and we are in a generic case, the singularities are observed in the set $\mathcal{E}_U^{reg}(q)$ that is the regular part of the boundary $\partial J^+(q)$, see Def. 2.5.

Remark 3.2. The non-linear interaction of waves may cause extraordinary singularities. For example, M. Beals showed in 1983 for the wave equation $\square u(t, y) + b(t, y)u(t, y)^3 = 0$ in $\mathbb{R}^4$ that there are solutions for which the singular support of the Cauchy data $(u|_{t=0}, \partial_t u|_{t=0})$ is the point $\{0\}$, but the singular support of u contains the entire solid cone $\{(t, y) \in \mathbb{R}^4;\ |y| < t\}$, see [4, Thm. 2.10] and [5]. This example has similarities to the above case (B) when the direction vectors $\dot{\gamma}_{x_j,\xi_j}(t_j)$, $j = 1, 2, 3, 4$ are not linearly independent. This happens e.g. when both the plane wave u_4 and the conic wave $w_{321} = \square_g^{-1}(au_3\, \square_g^{-1}(au_2u_1))$, produced by the interaction of three waves u_1, u_2 and u_3 (see Fig. 3), propagate along the same geodesic $\gamma_{x_4,\xi_4} \subset K_4$. In this case it may be that the wave front set of u_4 contains a point $(x, \zeta) \in N^*K_4$ and the wave front set of w_{321} contains the point $(x, -\zeta) \in N^*K_4$ with the opposite direction. In this case it is difficult to analyze the product u_4w_{321}. This difficulty, as well as the possible caustics of w_{321} (see Fig. 6), are the reasons why in the claim (ii) below we restrict ourselves to a geometrically nice case.

Theorem 3.3. *Let $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ be future pointing light-like vectors such that (42) is satisfied. Let $y_0 \in \mathcal{N}(\vec{x}, \vec{\xi}) \cap U$ be such that $y_0 \notin \mathcal{Y}(\vec{x}, \vec{\xi}) \cup \bigcup_{j=1}^4 \gamma_{x_j,\xi_j}([0, \infty))$, see (41) and (43).*

Assume that $s_0 > 0, K_j, \Lambda_j$ are as in (44) and assume that either the above condition (A) or (B) is satisfied. In the case (B), we consider $t_j > 0$ and $q \in M$ and co-vectors $b_j = (\dot\gamma_{x_j,\xi_j}(t_j))^\flat$.

Let $n \in \mathbb{Z}_+$ and $f_j \in \mathcal{I}^{-n+1}(\Sigma(x_j,\zeta_j,s_0))$, $j = 1,2,3,4$, be sources satisfying (36) and $u_j = \square_g^{-1} f_j$ and $\mathcal{U}^{(4)}$ be the wave produced by the 4th order interaction given in (38). When n is large enough and s_0 is small enough, the following holds:

(i) Assume that either (A) or (B) holds and that in the case (B) we have $y_0 \notin J^+(q)$. Then y_0 has a neighborhood W such that $\mathcal{U}^{(4)}|_W$ is C^∞-smooth.

(ii) Assume that (B) holds, $b_j \in T^*_q M$, $j = 1,2,3,4$ are linearly independent and $y_0 \in \mathcal{E}_U^{reg}(q)$, where $\mathcal{E}_U^{reg}(q)$ is the regular earliest light observation set of q, see Def. 2.5. Also, assume that $w_0 \in L^*_{y_0}M$ and $r \in \mathbb{R}$ are such that $\gamma_{y_0,w_0}(r) = q$ and denote $\eta = (\dot\gamma_{y_0,w_0^\sharp}(r))^\flat \in L^*_q M$.

Then the point y_0 has a neighborhood W such that $\mathcal{U}^{(4)}$ in W is a conormal distribution associated to $S = \mathcal{L}^+(q) \cap W$, that is, $\mathcal{U}^{(4)}|_W \in \mathcal{I}^m(S)$, with $m = -4n-4$. Moreover, let $\zeta_j \in N^*_q K_j$ be such that

$$\eta = \sum_{j=1}^{4} \zeta_j. \tag{45}$$

Note that the linear independence of b_j implies the uniqueness of representation (45). Then the principal symbol of $\mathcal{U}^{(4)}|_W \in \mathcal{I}^m(S)$, at the point (y_0,w_0), is

$$\sigma^{(p)}_{\mathcal{U}^{(4)}}(y_0,w_0) = R(y_0,w_0,q,\eta)a(q)^3 \mathcal{G}_{\mathbf{g}}(\vec\zeta) \prod_{j=1}^{4} \sigma^{(p)}_{u_j}(q,\zeta_j), \tag{46}$$

where $\vec\zeta = (\zeta_j)_{j=1}^4$, $R(y_0,w_0,q,\eta)$ is given in Lemma 3.1 and

$$\begin{aligned}\mathcal{G}_{\mathbf{g}}(\vec\zeta) \;=\; \sum_{\sigma\in\Sigma(4)} \Big(& \frac{C_1}{G(\zeta_{\sigma(1)}+\zeta_{\sigma(2)}) \cdot G(\zeta_{\sigma(1)}+\zeta_{\sigma(2)}+\zeta_{\sigma(3)})} \\ & + \frac{C_2}{G(\zeta_{\sigma(1)}+\zeta_{\sigma(2)}) G(\zeta_{\sigma(3)}+\zeta_{\sigma(4)})} \Big),\end{aligned} \tag{47}$$

where $G(\xi) = g(\xi,\xi)$ and C_1 and C_2 are non-zero constants.

Later, we will show that the function $\mathcal{G}_{\mathbf{g}}(\vec\zeta)$ is non-vanishing in a generic set.

Proof. Due to the general geometric setting on a globally hyperbolic manifold the proof is quite long and is divided to several parts.

1. Notations. As $y_0 \notin \mathcal{Y}(\vec x,\vec\xi) \cup \bigcup_{j=1}^4 \gamma_{x_j,\xi_j}([0,\infty))$, we can assume that $s_0 > 0$ is so small that $y_0 \notin \mathcal{Y}((\vec x,\vec\xi),s_0) \cup \bigcup_{j=1}^4 K(x_j,\xi_j,s_0)$, see (41) and (43), and that for all $i \neq j$, the surfaces $K(x_i,\xi_i,s_0)$ and $K(x_j,\xi_j,s_0)$ intersect transversally.

Below, we denote $\bar N_y^*K_j = N_y^*K_j \cup \{0\}$ and $\bar L_y^*M = L_y^*M \cup \{0\}$. Also, let $\mathcal{N} = \mathcal{N}(\vec x, \xi)$, $\mathcal{X} = \mathcal{X}((\vec x, \vec \xi), s_0)$, $\mathcal{H} = \mathcal{H}((\vec x, \vec \xi), s_0)$ and $\mathcal{Y} = \mathcal{Y}((\vec x, \vec \xi), s_0)$ be the sets given in (39)-(41).

We use the notations $K_j = K(x_j, \xi_j, s_0)$, $K_{12} = K_1 \cap K_2$, $K_{123} = K_1 \cap K_2 \cap K_3$, $\Lambda_{12} = N^*K_{12}$, etc.

Recall that g^+ is the Riemannian metric obtained by changing, in local coordinates, the sign of the negative eigenvalue of the Lorentzian metric g. On the g^+-unit sphere bundle S^*M we use the Sasaki metric determined by g^+. Below we say that a conic set $\mathcal{N}(\varepsilon) \subset T^*M$ is a conic ε-neighborhood of the set L^*M of the light-like co-vectors if $\mathcal{N}(\varepsilon) \cap S^*M$ is the ε-neighborhood of $L^*M \cap S^*M$ in the g^+-unit sphere bundle S^*M. Note that $(x, \xi) \in \mathcal{N}(\varepsilon)$ if and only if $(x, -\xi) \in \mathcal{N}(\varepsilon)$.

The Lorentzian volume on (M, g) at point x is denoted by dV_x.

Below, we will consider claims (i) and (ii) at the same time. To do that, we denote

$$\mathcal{N}_0 = \mathcal{N} \setminus J^+(q),\ \text{if (B) holds and } (b_j)_{j=1}^4 \text{ are linearly dependent}$$
$$\mathcal{N}_0 = \mathcal{N},\ \text{if (B) holds and } (b_j)_{j=1}^4 \text{ are linearly independent or (A) holds.}$$

We will assume below that $y_0 \in U \cap \mathcal{N}_0$.

2. Local coordinates. Recall that the intersection of the surfaces K_i and K_j with $i \neq j$ is transversal in $\mathcal{N}(\vec x, \vec \xi)$. To consider local coordinates, let us start with the observation that if three light-like vectors are not parallel, then those vectors are linearly independent, see [65, Cor. 1.1.5]. This implies that for any $p \in \mathcal{N}(\vec x, \vec \xi)$ and any three indexes $j_1, j_2, j_3 \in \{1, 2, 3, 4\}$ we can choose local coordinates $X : W \to \mathbb{R}^4$ so that $K_{j_i} \cap W \subset \{x \in W;\ X^{j_i}(x) = 0\}$ for $i = 1, 2, 3$ and we see that $K_{j_1} \cap K_{j_2} \cap K_{j_3}$ is a smooth path in the neighborhood W. In this case we say that $X : W \to \mathbb{R}^4$ are adapted to the surface K_{j_i}.

Also, in the case of claim (ii), at the point q we can use local coordinates $X : W \to \mathbb{R}^4$ such that the linearly independent co-vectors b_j, $j = 1, 2, 3, 4$ are the differentials of the coordinate functions at q and for all $j = 1, 2, 3, 4$ we have $K_j \cap W_0 = \{x \in W_0;\ X^j(x) = 0\}$. These coordinates are adapted to all K_j.

As in these set $\mathcal{N}$ the point q is the only possible point in $\cap_{j=1}^4 K_j$, the existence of the above coordinates imply that when (i_1, i_2, i_3, i_4) is any permutation of $\{1, 2, 3, 4\}$, then in the set $\mathcal{N}_0$, all possible intersections $K_{i_1 i_2 i_3} \cap K_{i_4}$ and $K_{i_1 i_2} \cap K_{i_3 i_4}$ and $K_{i_1 i_2} \cap K_{i_3}$ are transversal.

3. Testing when (y_0, w_0) is in the wave front set. Below, we consider $w_0 \in L^*_{y_0}M$. As Q is the causal inverse of the wave operator, we see using Hörmander's theorem on propagation of singularities along bicharacteristics, [37, Theorem 26.1.4], we see that if the point (y_0, w_0) is in $\mathrm{WF}(\mathcal{U}^{(4)})$ then either w_0 is not light-like and $(y_0, w_0) \in \mathrm{WF}(\mathcal{S})$, or, w_0 is light-like and there is $s \in \mathbb{R}$ such that $(\gamma_{y_0, w_0^\sharp}(s), \dot\gamma_{y_0, w_0^\sharp}(s)^\flat)$ is in

$\mathrm{WF}(\mathcal{S})$ and $\gamma_{y_0,w_0^\sharp}(s) \le y_0$. To apply this for the light-like singularities, we below consider a point $(x_0, \zeta_0) \in L^*M$ such that

$$(x_0, \zeta_0) = (\gamma_{y_0,w_0^\sharp}(s), \dot\gamma_{y_0,w_0^\sharp}(s)^\flat), \text{ such that } x_0 < y_0, \tag{48}$$

and study if (x_0, ζ_0) belongs in the wave front $\mathrm{WF}(\mathcal{S})$. Note that as $y_0 \in U \cap \mathcal{N}_0$, we have also $x_0 \in \mathcal{N}_0$.

To study the claim (ii), we see that when $s = r$, the point (x_0, ζ_0) coincides with (q, η). Also, to study the claim (i), we will study several cases when (x_0, ζ_0) will not be in $\mathrm{WF}(\mathcal{S})$ and use this to show that (y_0, w_0) does not belong in $\mathrm{WF}(\mathcal{U}^{(4)})$.

4. A neighborhood of light-like directions. We start with some auxiliary observations. First, note that as $y_0 \notin \mathcal{Y}$, formula (48) and definitions (39) and (41) imply that $(x_0, \zeta_0) \notin \mathcal{X}$.

Next we will choose a small parameter $\varepsilon_1 > 0$ that determines a conic neighborhood $\mathcal{N}(\varepsilon_1)$ of the set L^*M of light-like co-vectors. We consider separately two cases:

First, consider the case when

(49) the property (B) is valid, so that geodesics γ_{x_j,ξ_j} intersect at q, $b_j \in T_q^*M$, $j = 1, 2, 3, 4$ are linearly independent, there is $r \neq 0$ such that $\gamma_{y_0,w_0^\sharp}(r) = q$, and $x_0 = q$.

Then, denote $v_0 = \dot\gamma_{y_0,w_0^\sharp}(r) \in L_qM$ and $\eta = v_0^\flat \in L_q^*M$. Let $\zeta_j \in \bar N_q^* K_j$, $j = 1, 2, 3, 4$ be such that $\eta = \sum_{j=1}^4 \zeta_j$. As $y_0 \notin \mathcal{Y} \cup (\cup_{j=1}^4 K_j)$, we have $(q, \eta) \notin \mathcal{X} \cup (\cup_{j=1}^4 N^*K_j)$. This implies that $\zeta_j \neq 0$ for all $j = 1, 2, 3, 4$. Then, as η and ζ_j are light-like we have that $\eta - \zeta_j$ is not light-like as otherwise η and $\zeta_j \in N_q^*K_j$ would be parallel which is not possible. Hence, in the case (49) we can choose $\varepsilon_1 > 0$ be so small that we have

$$(q, \eta - \zeta_j) \notin \mathcal{N}(\varepsilon_1), \quad \text{for } j = 1, 2, 3, 4. \tag{50}$$

Second, in the case when condition (49) does not hold, we choose $\varepsilon_1 > 0$ to be an arbitrary positive number.

5. Decomposition of the operator $\square_g^{-1}$. Below we denote $Q = \square_g^{-1}$. We denote also the Schwartz kernel of Q by $Q(x, y)$. Let us next consider the map $Q : C_0^\infty(M_0) \to C^\infty(M_0)$. By [56], the Schwartz kernel satisfies $Q \in I(M_0 \times M_0; \Delta'_{T^*M_0}, \Lambda_g)$, see Sec. 3.2.1, and the canonical relation of the operator Q, denoted Λ'_Q, has the form $\Lambda'_Q = \Lambda'_g \cup \Delta_{T^*M_0}$, see [56]. Let ε_1 be as above, $\varepsilon_2 \in (0, \varepsilon_1)$ and $B_{\varepsilon_1,\varepsilon_2}$ be a pseudodifferential operator on M_0 which is microlocally a smoothing operator outside the conic ε_1-neighborhood $\mathcal{N}(\varepsilon_1) \subset T^*M_0$ of the set of the light-like covectors L^*M_0, and for which $(I - B_{\varepsilon_1,\varepsilon_2})$ is microlocally smoothing operator in the conic ε_2-neighborhood $\mathcal{N}(\varepsilon_2)$ of the bundle of the light-like co-vectors L^*M_0. Let us decompose the operator $Q = Q_1 + Q_2$ where $Q_1 = QB_{\varepsilon_1,\varepsilon_2}$ and $Q_2 = Q(I - B_{\varepsilon_1,\varepsilon_2})$.

The Schwartz kernel $Q_1(x,y)$ of the operator Q_1 satisfies $Q_1 \in \mathcal{I}(M_0\times M_0;\Delta'_{T^*M_0},\Lambda_g)$, similarly to Q. Moreover, the Schwartz kernel $Q_2(x,y)$ of the operator Q_2 satisfies $Q_2 \in \mathcal{I}(M_0\times M_0;\Delta'_{T^*M_0})$ and the operator Q_2 is a pseudodifferential operator that has the form

$$(Q_2v)(y) = \int_{M_0\times\mathbb{R}^4} e^{i\Psi_1(y,z,\xi)}\sigma_{Q_2}(y,z,\xi)v(z)\,dzd\xi, \tag{51}$$

where $\Psi_1(y,z,\xi)$ parametrises the diagonal Lagrangian manifold Δ'_{T^*M} and $\sigma_{Q_2}(z,y,\xi)\in S^{-2}_{cl}(M_0\times M_0;\mathbb{R}^4)$ is a classical symbol. When $X: W\to\mathbb{R}^4$ are local coordinates in an open set $W\subset M$, the restriction $Q_2: C_0^\infty(W)\to C^\infty(W)$, given by $v\mapsto Q_2v|_W$, can be written using the phase function $\Psi_1(y,z,\xi)=\sum_{j=1}^4(X^j(y)-X^j(z))\xi_j$ and symbol $\sigma_{Q_2}(z,y,\xi)\in S^{-2}_{cl}(W\times W;\mathbb{R}^4)$. It has the principal symbol

$$\sigma^{(p)}_{Q_2}(y,z,\xi)=\chi(z,y,\xi)\frac{1}{g^{jk}(y)\xi_j\xi_k},\quad y,z\in W, \tag{52}$$

where $\chi(z,y,\xi)\in C^\infty$ vanishes when $(y,\xi)\in T^*W$ is in some neighborhood of light-like co-vectors L^*M.

6. Products of u_j and the singular support of $\mathcal{S}$. In the computations below, we will represent the waves $u_j\in\mathcal{I}^n(K_j)=\mathcal{I}^{n-1/2}(N^*K_j)$ in the local coordinates $X: W\to\mathbb{R}^4$, $X(x)=(X^j(x))_{j=1}^4\in\mathbb{R}^4$, that are adapted to the surface K_j, as

$$u_j(x)=\int_{\mathbb{R}} e^{i\psi_j(x,\theta)}\sigma_{u_j}(x,\theta)d\theta,\quad \sigma_{u_j}(x,\theta)\in S^n_{cl}(W;\mathbb{R}), \tag{53}$$

where $\psi_j(x,\theta)=\theta\cdot X^j(x)$.

Next, let us consider two indexes $j,k\in\{1,2,3,4\}$, $j\neq k$, and use local coordinates $X: W\to\mathbb{R}^4$ that are adapted to the surfaces K_j and K_k. Recall that $\Lambda_j=N^*K_j$ and denote $\Lambda_{jk}=N^*(K_j\cap K_k)$. By [27, Lemma 1.2], the pointwise product satisfies $u_j\cdot u_k\in\mathcal{I}(\Lambda_j,\Lambda_{jk})+\mathcal{I}(\Lambda_k,\Lambda_{jk})$. Also, the Lagrangian manifolds Λ_j and Λ_k are invariant by the bicharacteristic flow in the future direction. By using [27, Prop. 2.2 and 2.3], we see that $Q(au_j\cdot u_k)\in\mathcal{I}(\Lambda_j,\Lambda_{jk})+\mathcal{I}(\Lambda_k,\Lambda_{jk})$ can be written as

$$G_{jk}(x):=Q(au_j\cdot u_k)=\int_{\mathbb{R}^2} e^{i\psi_{jk}(x,\theta,\theta')}\sigma_{G_{jk}}(x,\theta,\theta')d\theta d\theta', \tag{54}$$

where $x\in W$, $(\theta,\theta')\in\mathbb{R}^2$, $\psi_{jk}(x,\theta,\theta')=\theta X^j(x)+\theta' X^k(x)$ and $\sigma_{G_{jk}}(x,\theta,\theta')$ is a sum of product type symbols, see (25).

As $N^*(K_j\cap K_k)\setminus(N^*K_j\cup N^*K_k)$ consists of vectors which are non-characteristic for $\square_g$, the principal symbol $\sigma^{(p)}_{G_{jk}}(x,\theta,\theta')$ of G_{jk} on $N^*(K_j\cap K_k)\setminus(N^*K_j\cup N^*K_k)$ is given by

$$\begin{aligned}\sigma^{(p)}_{G_{jk}}(x,\theta,\theta')&=s(x,\theta,\theta')a(x)\sigma^{(p)}_{u_j}(x,\theta)\sigma^{(p)}_{u_k}(x,\theta'),\\ s(x,\theta,\theta')&=\frac{1}{g(\xi,\xi)},\quad \text{where } \xi=d_x\psi_{jk}(x,\theta,\theta')=\theta dX^j+\theta' dX^k\end{aligned} \tag{55}$$

and $g(\xi,\xi)=g^{jk}(x)\xi_j\xi_k$.

Let us next consider the singular supports of the functions S_σ given in (38). Let us start with the case when the permutation σ is the identity map.

As we showed above, for $i\neq j$ we have $Q(au_i\cdot u_j)\in \mathcal{I}(\Lambda_i,\Lambda_{ij})+\mathcal{I}(\Lambda_i,\Lambda_{ij})$, so that by (27), the wave front set of this function is a subset of $N^*K_i\cup N^*K_j\cup N^*K_{ij}$. Thus,

$$\text{singsupp}(Q(au_i\cdot u_j))\subset K_i\cup K_j. \tag{56}$$

Moreover, as $\text{WF}(u_3)\subset N^*K_3$ and the intersection of K_{12} and K_3 is transversal, the theorem for the wave front set of the pointwise product of distributions, [18, Thm. 1.3.6], yield that $F_{321}=au_3\cdot Q(au_2\cdot u_1)$ satisfies

(57)

$$\text{WF}(F_{321})\cap T^*_xM\subset \mathcal{P}^{(123)}_x=(\bigcup_j N^*_xK_j)\cup(\bigcup_{j,k} N^*_xK_{jk})\cup(\bigcup_{j,k,l} N^*_xK_{jkl}),$$

where $x\in M_0$ and $j,k,l\in\{1,2,3\}$ and we interpret $N^*_xK_j$ to be an empty set if $x\notin K_j$ etc. Thus, $\text{WF}(F_{321})\cap L^*M\subset \mathcal{X}\cup(\cup_{j=1}^4 N^*K_j)$. As Q is the causal inverse of the wave operator, we see by using Hörmander's theorem on propagation of singularities along bicharacteristics, [37, Theorem 26.1.4], that

$$\text{WF}(QF_{321})\subset \text{WF}(F_{321})\cup\mathcal{H}_{123}\cup(\bigcup_{j=1}^3 N^*K_j) \tag{58}$$

where $\mathcal{H}_{123}=\mathcal{H}_{123}((\vec x,\vec\xi),s_0)$, see (40). In particular, this implies that $\text{singsupp}(QF_{321})\subset\mathcal{Y}\cup(\cup_{j=1}^3K_j)$.

Formulas (56) and (58) give that for $\sigma=Id$ we have

$$\text{singsupp}\,(S_\sigma)\subset\mathcal{Y}\cup(\bigcup_{j=1}^4 K_j). \tag{59}$$

The same arguments yield that (59) holds for all permutations σ.

7. Decomposition of the source term $\mathcal{S}$. Below we will analyze the wave front set of the source $\mathcal{S}$ that is produced by the fourth order interaction. To this end, we use the decomposition $Q=Q_1+Q_2$, and write the source $\mathcal{S}$ in the form

$$\mathcal{S}=\mathcal{S}^{(1)}+\mathcal{S}^{(2)}+\mathcal{S}^{(3)},\quad \mathcal{S}^{(p)}=\sum_{\sigma\in\Sigma(4)}\mathcal{S}^{(p)}_\sigma,\ p\in\{1,2,3\} \tag{60}$$

where $\Sigma(4)$ is the set of permutations of the set $\{1,2,3,4\}$ and

$$\begin{aligned}\mathcal{S}^{(1)}_\sigma &= -4au_{\sigma(4)}\cdot Q_1(au_{\sigma(3)}\cdot Q(au_{\sigma(2)}\cdot u_{\sigma(1)})),\\ \mathcal{S}^{(2)}_\sigma &= -4au_{\sigma(4)}\cdot Q_2(au_{\sigma(3)}\cdot Q(au_{\sigma(2)}\cdot u_{\sigma(1)})),\\ \mathcal{S}^{(3)}_\sigma &= -aQ(au_{\sigma(4)}\cdot u_{\sigma(3)})\cdot Q(au_{\sigma(2)}\cdot u_{\sigma(1)}).\end{aligned} \tag{61}$$

Later, we consider the terms (61) with the permutation $\sigma = Id$. Note that the terms corresponding to the other permutations σ can be analyzed similarly by renumbering the indexes.

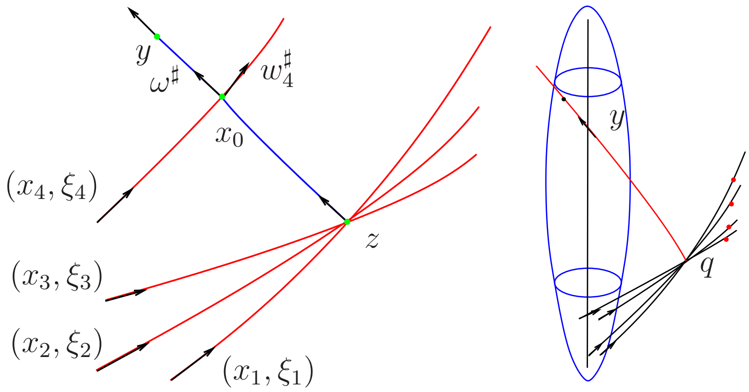

FIGURE 6. **Left:** *The figure shows the case when three geodesics intersect at z and the waves propagating near these geodesics interact and create a wave that hits to the fourth geodesic at the point x_0. The produced singularities propagate to the point $y \in \mathcal{Y}$. Note that z and x_0 may be conjugate points on the geodesic connecting them or the waves propagating from z to x_0 may have caustics.* **Right:** *Geodesics corresponding to directions (x_j,ξ_j), $j = 1,2,3,4$ intersect at the point q and b_j, $j = 1,2,3,4$ are linearly independent and the Condition I is valid for the point y with vectors (x_j,ξ_j) and with the parameter q. The red points are the conjugate points of $\gamma_{x_j,\xi_j}([0,\infty))$ and $\gamma_{q,w}([0,\infty))$.*

8. Analysis the wave front set of source $\mathcal{S}_\sigma^{(1)}$. In this step we consider the case when (x_0,ζ_0) is in the wave front set of source functions $\mathcal{S}_\sigma^{(1)}$, see Steps 2 and 5.

We start in the case when $\sigma = Id$. Then, $\mathcal{S}_{Id}^{(1)} = au_4 \cdot Q_1 F_{321}$. To analyze the product $u_4 \cdot Q_1 F_{321}$ in the set $\mathcal{N}_0$, we will first show that the wave front sets satisfy for $z \in \mathcal{N}_0$

(62) If $(z,\omega) \in \mathrm{WF}(Q_1 F_{321})$ and $(z,w_4) \in \mathrm{WF}(u_4)$ then $\omega + w_4 \neq 0$.

To show this, we assume the opposite, that there are $z_0 \in \mathcal{N}_0$ and

$$(z_0,\omega) \in \mathrm{WF}(Q_1 F_{321}) \quad \text{and} \quad (z_0,w_4) \in \mathrm{WF}(u_4) \subset N^*K_4,$$

such that $\omega + w_4 = 0$. We consider different cases for (z_0,ω) that are given by equations (57) and (58).

First, we consider the case when $(z_0,\omega) \in N^*K_{123}$. Since $z_0 \in K_4$, this yields $z_0 \in \cap_{j=1}^4 K_j$. Thus we are case (B) and $z_0 = q$. However, as $z_0 = q \in \mathcal{N}_0$, the vectors b_j, $j = 1,2,3,4$ are linearly independent, and it is not possible that $\omega = -w_4 \in N_q^*K_{123} \cap N_q^*K_4$. Thus we see that $(z_0,\omega) \notin N^*K_{123}$.

Second, we consider the case when $(z_0,\omega) \in N^*K_{j_1 j_2} \setminus (N^*K_{j_1} \cup N^*K_{j_2})$ where $j_1,j_2 \in \{1,2,3\}$, $j_1 \neq j_2$. Then, ω is not light-like and so it is not possible that $\omega = -w_4 \in N^*_{z_0}K_4$. Thus we conclude that $(z_0,\omega) \notin N^*K_{j_1j_2} \setminus (N^*K_{j_1} \cup N^*K_{j_2})$.

Third, we see that it is not possible that $\omega = -w_4 \in (\bigcup_{j=1}^3 N^*_{z_0}K_j) \cap N^*K_4$ as the surfaces K_i and K_j, $i \neq j$ intersect transversally.

Fourth, we consider the remaining case when $(z_0,\omega) \in \mathcal{H}_{123} \subset L^*M$. Then there is $(x,\zeta) \in \mathcal{X}_{123} \subset N^*K_{123}$, $x \in K_{123}$ such that $x \leq z_0$ and the bicharacteristic $\Theta_{x,\zeta}$ passes through (z_0,ω) and $\Theta_{x,\zeta} = \Theta_{z_0,\omega}$. Also, $(z_0,\omega) = (z_0,-w_4) \in L^*K_4 = \Lambda_4$. As $x \in J^+(x_1)$, we see using (42) that $x_4 \notin J^+(x)$. Thus, as $(x,\zeta) \in \Theta_{z_0,\omega} = \Theta_{z_0,-w_4}$, we have $(x,\zeta) \in \Lambda_4 = N^*K_4$ and $x \in K_4$. These imply that $x \in \cap_{j=1}^4 K_j$. Hence, the case (B) is valid and $x = q$. Moreover, as (x,ζ) is in the intersection of $N^*_qK_{123}$ and $N^*_qK_4$, we get that the vectors b_j, $j = 1,2,3,4$ are not linearly independent. As $x = q$ and $x \leq z_0$, this implies $z_0 \notin \mathcal{N}_0$, that not possible by our assumptions.

As none of the above four case is possible, we obtain that it is not possible that $\omega + w_4 = 0$. Hence, (62) is true.

Next we consider the question, can (x_0,ζ_0), given in (48), be in $\mathrm{WF}(\mathcal{S}^{(1)}_{id})$. Recall that ζ_0 is light-like.

Due to (62) we can use the formula for the wave front set of the pointwise product of distributions, [18, Thm. 1.3.6]. It implies that if $(x_0,\zeta_0) \in \mathrm{WF}(\mathcal{S}^{(1)}_{id})$, where $\mathcal{S}^{(1)}_{id} = au_4\, Q_1(F_{321})$, then there are

$$(x_0,\omega) \in \mathrm{WF}(Q_1F_{321}) \cup (M_0 \times \{0\}) \text{ and } (x_0,w_4) \in \mathrm{WF}(u_4) \cup (M_0 \times \{0\})$$

such that

$$\zeta_0 = \omega + w_4.$$

Let us use the fact that w_4 is light-like or zero and $\zeta_0 = \omega + w_4$ is light-like. If ω is light-like or zero, then ζ_0 has to be parallel either to ω or w_4. Then $(x_0,\zeta_0) \in \mathcal{H} \cup (\cup_{j=1}^4 N^*K_j)$, see (40) and (58). However, this is not possible since $y_0 \notin \mathcal{Y} \cup (\cup_{j=1}^4 K_j)$. Hence, ω is not light-like or zero.

Since $(x_0,\omega) \in \mathrm{WF}(Q_1F_{321})$ is not zero and we see by using the definition of Q_1 that

$$(x_0,\omega) \in \mathcal{N}(\varepsilon_1). \tag{63}$$

Also, as ω is not light-like and zero and $\zeta_0 = \omega + w_4$ is light-like we have $w_4 \neq 0$. Then $x_0 \in K_4$.

As the above implies that $\omega \in \mathcal{P}^{(123)}_{x_0} \setminus \{0\}$, we can consider separately the different cases given by definition of $P^{(123)}_{x_0}$ in (57).

First, as ω is not light-like, we have $\omega \notin N^*_{x_0}K_j$ for $j = 1,2,3$.

Second, if $\omega \in N^*_{x_0}K_{j_1j_2}$, with $j_1,j_2 \in \{1,2,3\}$, we have $x_0 \in K_{j_1j_2} \cap K_4$ and $(x_0,\zeta_0) = (x_0,\omega + w_4) \in N^*(K_{j_1j_2} \cap K_4)$. As ζ_0 is light-like, we have $(x_0,\zeta_0) \in \mathcal{X}$. This implies $y_0 \notin \mathcal{Y}$ which is not possible by our assumptions. Thus $\omega \notin N^*_{x_0}K_{j_1j_2}$.

Third, consider the case when $\omega \in N^*_{x_0}K_{123}$. Then $x_0 \in \cap_{j=1}^4 K_j$. Hence, (B) is valid and $x_0 = q$. Again, as $x_0 = q \in \mathcal{N}_0$, we see that the vectors b_j, $j = 1, 2, 3, 4$ have to be linearly independent. Then, the condition (49) is valid. Note that as $x_0 = q$, we have $\zeta_0 = \eta$. We recall that in the case (49) we chose $\varepsilon_1 > 0$ to be so small that (50) is valid.

As b_j, $j = 1, 2, 3, 4$ are linearly independent, η has a unique representation $\eta = \sum_{j=1}^4 \zeta_j$ where $\zeta_j \in N^*_q K_j$. Then, $\omega = \sum_{j=1}^3 \zeta_j$ and $w_4 = \zeta_4$. Then, by (50), we have that $(x_0, \omega) = (x_0, \eta - \zeta_4) \notin \mathcal{N}(\varepsilon_1)$. This is not possible since $(x_0, \omega) \in \mathcal{N}(\varepsilon_1)$ by (63). Hence, we have $(x_0, \zeta_0) \notin \mathrm{WF}(\mathcal{S}^{(1)}_\sigma)$.

Above, we have analyzed the case when the permutation σ is the identity. For other permutations $\sigma \in \Sigma(4)$, the same computations are valid with a renumbering of indexes, and we conclude that

$$(x_0, \zeta_0) \notin \mathrm{WF}(\mathcal{S}^{(1)}). \tag{64}$$

9. Analysis of the term $\mathcal{S}^{(2)}_\sigma$. Let start by analyzing the case when $\sigma = Id$. Recall that the wave front set of $F_{321} = au_3 \cdot Q(au_2 \cdot u_1)$ satisfies (57). Also, as Q_2 is a pseudodifferential operator,

$$\mathrm{WF}(Q_2 F_{321}) \subset \mathrm{WF}(F_{321}).$$

Recall, in the set $\mathcal{N}_0$ the intersections $K_4 \cap K_{123}$ and $K_{j_1 j_2} \cap K_{j_1}$ are transversal, where $j_1, j_2, j_3 \in \{1, 2, 3\}$. Thus as $\mathrm{WF}(u_4) \subset N^*K_4$ and $\mathcal{S}^{(2)}_{Id} = au_4 \cdot Q_2 F_{321}$, we can apply the formula for the wave front set of the pointwise product of distributions, [18, Thm. 1.3.6], and see for $x \in \mathcal{N}_0$ that

$$\mathrm{WF}(\mathcal{S}^{(2)}_{Id}) \cap T^*_x M \subset Z_x \tag{65}$$

where

$$Z_x = (\bigcup_j N^*_x K_j) \cup (\bigcup_{j,k} N^*_x K_{jk}) \cup (\bigcup_{j,k,l} N^*_x K_{jkl}) \cup (\oplus_{j=1}^4 N^*_x K_j) \tag{66}$$

and $j, k, l \in \{1, 2, 3, 4\}$. We interpret $N^*_x K_j$ to be an empty set if $x \notin K_j$ etc.

Consider light-like co-vector (x_0, ζ_0) given in (48). First, we consider the cases when (A) holds or (B) holds and $x_0 \neq q$. Then,

$$Z_{x_0} \subset (\bigcup_j N^*_{x_0} K_j) \cup (\bigcup_{j,k} N^*_{x_0} K_{jk}) \cup (\bigcup_{j,k,l} N^*_{x_0} K_{jkl}). \tag{67}$$

Then, if $(x_0, \zeta_0) \in Z_{x_0}$, formula (67) yields $(x_0, \zeta_0) \in \mathcal{X} \cup (\cup_{j=1}^4 N^*_{x_0} K_j)$. By (48), this yields that $y_0 \notin \mathcal{Y}((\vec{x}, \vec{\xi}), s_0) \cup (\cup_{j=1}^4 K_j)$ which is not possible by our assumptions. Thus in all the above mentioned cases $(x_0, \zeta_0) \notin \mathrm{WF}(\mathcal{S}^{(2)}_{Id})$. In particular, this holds under the assumption of claim (i). The similar analysis holds for a general permutation σ.

We have above considered the cases when (A) holds or (B) holds and $x_0 \neq q$. It remains to consider the case when (B) holds and $x_0 = q$. As $x_0 \in \mathcal{N}_0$, then the condition (49) has to be valid. Moreover, as $x_0 = q$, we have $\zeta_0 = \eta$, and

(68) there are $\zeta_j \in \bar N_q^* K_j$, $j = 1,2,3,4$, so that $\eta = \sum_{j=1}^4 \zeta_j \in T_q^* M$,

and such $\zeta_j \in \bar N_q^* K_j$ are uniquely determined by η. Recall that $(q,\eta) \notin \mathcal{X} \cup \bigcup_{j=1}^4 N^* K_j$. This implies $\zeta_j \neq 0$ for all $j = 1,2,3,4$, that is, $\zeta_j \in N_q^* K_j$. Also, note that as η and $\zeta_{\sigma(4)}$ are light-like vectors that are not parallel, we have that $\zeta_{\sigma(1)} + \zeta_{\sigma(2)} + \zeta_{\sigma(3)} = \eta - \zeta_{\sigma(4)}$ is not light-like or zero.

Next, in a neighborhood of the point q we use local coordinates $X : W_0 \to \mathbb{R}^4$ that are adapted to all surfaces K_j, see Step 2.

Let $R_{j,k} = R_{j,k}(x,D)$ be a pseudodifferential operator which symbol is one in a conic neighborhood $\Sigma_{j,k} \subset T^* M_0$ of $(q, \zeta_j + \zeta_k)$ and vanishes in a conic neighborhood of sets $N^* K_i$, $i = 1,2,3,4$. Also, let $R_{i,j,k} = R_{i,j,k}(x,D)$ be a pseudodifferential operator which symbol is one in a conic neighborhood in $\Sigma_{i,j,k} \subset T^* M_0$ of $(q, \zeta_i + \zeta_j + \zeta_k)$ and vanishes in a conic neighborhood of all sets $N^* K_{i_1}$ and $N^* K_{i_1 i_2}$. Moreover, let $R_{4321} = R_{4321}(x,D)$ be a pseudodifferential operator which symbol is one in a conic neighborhood of (q,η) and vanishes in a conic neighborhood of all sets $N^* K_{i_1}$, $N^* K_{i_1 i_2}$ and $N^* K_{i_1 i_2 i_3}$. We denote $R^\sigma_{jk} = R_{\sigma(j),\sigma(k)}$ and $R^\sigma_{ijk} = R_{\sigma(i),\sigma(j),\sigma(k)}$. Also, assume that the Schwartz kernels of these operators and that of R_{4321} are supported in $W_0 \times W_0$. Then we define

$$\begin{aligned}
\mathcal{S}^{(2),1}_\sigma &= -4 a u_{\sigma(4)} \cdot Q_2(a u_{\sigma(3)} \cdot R^\sigma_{21} Q(a u_{\sigma(2)} \cdot u_{\sigma(1)})),\\
\mathcal{S}^{(2),2}_\sigma &= -4 a u_{\sigma(4)} \cdot R^\sigma_{321} Q_2(a u_{\sigma(3)} \cdot R^\sigma_{21} Q(a u_{\sigma(2)} \cdot u_{\sigma(1)})),\\
\mathcal{S}^{(2),3}_\sigma &= -4 R_{4321}(a u_{\sigma(4)} \cdot R^\sigma_{321} Q_2(a u_{\sigma(3)} \cdot R^\sigma_{21} Q(a u_{\sigma(2)} \cdot u_{\sigma(1)}))).
\end{aligned}$$

Using (27) and [18, Thm. 1.3.6] we observe that then

$$\mathcal{S}^{(2)}_\sigma - \mathcal{S}^{(2),1}_\sigma = -4 a u_{\sigma(4)} \cdot Q_2(a u_{\sigma(3)} \cdot (I - R^\sigma_{21}) Q(a u_{\sigma(2)} \cdot u_{\sigma(1)})),$$

satisfies

$$\begin{aligned}
&\mathrm{WF}(\mathcal{S}^{(2)}_\sigma - \mathcal{S}^{(2),1}_\sigma) \cap T_q^* M \subset \\
&\quad \bar N_q^* K_{\sigma(4)} + \bar N_q^* K_{\sigma(3)} + ((\bar N_q^* K_{\sigma(2)} + \bar N_q^* K_{\sigma(1)}) \setminus \Sigma_{\sigma(2),\sigma(1)}).
\end{aligned}$$

As η is written in (68) in a unique way as a sum of vectors in $\bar N_q^* K_j$, $j = 1,2,3,4$, we see that $(q,\eta) \notin \mathrm{WF}(\mathcal{S}^{(2)}_\sigma - \mathcal{S}^{(2),1}_\sigma)$. Similarly,

$$(q,\eta) \notin \mathrm{WF}(\mathcal{S}^{(2),1}_\sigma - \mathcal{S}^{(2),2}_\sigma), \quad (q,\eta) \notin \mathrm{WF}(\mathcal{S}^{(2),2}_\sigma - \mathcal{S}^{(2),3}_\sigma).$$

Hence,

$$(q,\eta) \notin \mathrm{WF}(\mathcal{S}^{(2)}_\sigma - \mathcal{S}^{(2),3}_\sigma). \tag{69}$$

To simplify notations, we next consider the case when $\sigma = Id$. Using formulas (26) and (28), we see that $R^\sigma_{21} Q(a u_2 \cdot u_1) \in \mathcal{I}(K_{12})$. Then the results for the products of conormal distributions, [27, Lemma 1.1], imply $u_3 \cdot R^\sigma_{21} Q(au_2 \cdot u_1) \in \mathcal{I}(K_{12}, K_{123}) + \mathcal{I}(K_3, K_{123})$. By (26), we have $R^\sigma_{321}(u_3 R^\sigma_{21} Q(au_2 \cdot u_1)) \in \mathcal{I}(K_{123})$. Repeating the arguments, and computing above the orders of the distributions, we see that $\mathcal{S}^{(2),3}_\sigma \in \mathcal{I}^{-4n-3}(\Sigma_q)$, where $\Sigma_q = N^*\{q\} = T^*_q M \setminus \{0\}$, see (33). The other permutations can be analyzed in the same way.

9. Analysis of the term $\mathcal{S}^{(3)}_\sigma$. We start by considering the case when $\sigma = Id$. The term $\mathcal{S}^{(3)}_{Id}$ is the product of the terms $Q(au_2 \cdot u_1)$ and $Q(au_4 \cdot u_3)$. As above, we see that $Q(au_2 \cdot u_1) \in \mathcal{I}(\Lambda_1, \Lambda_{12}) + \mathcal{I}(\Lambda_2, \Lambda_{12})$. Similarly, $Q(au_4 \cdot u_3) \in \mathcal{I}(\Lambda_3, \Lambda_{34}) + \mathcal{I}(\Lambda_4, \Lambda_{34})$. Then (27) and [18, Thm. 1.3.6] yield that for $x \in \mathcal{N}_0$,

$$\mathrm{WF}(\mathcal{S}^{(3)}_{Id}) \cap T^*_x M \subset Z_x.$$

Consider next the cases when (A) holds or (B) holds and $x_0 \neq q$. Then, the same arguments that were used above to analyze the formula (65) yield that $(x_0, \zeta_0) \notin \mathrm{WF}(\mathcal{S}^{(3)}_{Id})$.

Again, as we have considered the cases when (A) holds or (B) holds and $x_0 \neq q$, it remains to consider the case (49). Then, as $x_0 = q$, we have $\zeta_0 = \eta$. We use the same notations as in Step 8.

Let $\mathcal{S}^{(3),0}_\sigma = \mathcal{S}^{(3)}_\sigma$ and

$$\begin{aligned}
\mathcal{S}^{(3),1}_\sigma &= -aQ(au_{\sigma(4)} \cdot u_{\sigma(3)}) \cdot R^\sigma_{21} Q(au_{\sigma(2)} \cdot u_{\sigma(1)}),\\
\mathcal{S}^{(3),2}_\sigma &= -aR^\sigma_{43} Q(au_{\sigma(4)} \cdot u_{\sigma(3)}) \cdot R^\sigma_{21} Q(au_{\sigma(2)} \cdot u_{\sigma(1)}),\\
\mathcal{S}^{(3),3}_\sigma &= -R_{4321}(aR^\sigma_{43} Q(au_{\sigma(4)} \cdot u_{\sigma(3)}) \cdot R^\sigma_{21} Q(au_{\sigma(2)} \cdot u_{\sigma(1)})).
\end{aligned}$$

Again, using (27) and [18, Thm. 1.3.6] we see that for $i = 0, 1, 2$

$$(q, \eta) \notin \mathrm{WF}(\mathcal{S}^{(3),i}_\sigma - \mathcal{S}^{(3),i+1}_\sigma).$$

so that

$$(q, \eta) \notin \mathrm{WF}(\mathcal{S}^{(3)}_\sigma - \mathcal{S}^{(3),3}_\sigma). \tag{70}$$

Using formula (26) and [27, Lemma 1.1], we see that $\mathcal{S}^{(3),3}_\sigma \in \mathcal{I}^{-4n-3}(\Sigma_q)$.

10. Proof of claim (i). Above we have shown in the case of claim (i) that $(x_0, \zeta_0) \notin \mathrm{WF}(\mathcal{S}^{(p)}_\sigma)$ for all $p = 1, 2, 3$, and hence (y_0, w_0) can not be in $\mathrm{WF}(\mathcal{U}^{(4)})$. As $y_0 \notin \cup_{j=1}^4 K_j$ and $w_0 \in L^*_{y_0} M$ can be arbitrary, (59) and Hörmander's theorem on propagation of singularities along bicharacteristics, [37, Theorem 26.1.4], prove the claim (i).

11. Proof of claim (ii). Assume that condition (49) is valid. Let

$$\mathcal{S}^{mod} = \sum_{\sigma \in \Sigma(4)} \mathcal{S}^{(2),mod}_\sigma + \mathcal{S}^{(3),mod}_\sigma, \quad \mathcal{S}^{(2),mod}_\sigma = \mathcal{S}^{(2),3}_\sigma,\ \mathcal{S}^{(3),mod}_\sigma = \mathcal{S}^{(3),3}_\sigma.$$

Since $\mathcal{S}^{mod} \in \mathcal{I}^{-4n-3}(\Sigma_q)$, we have $\mathrm{WF}(\mathcal{S}^{mod}) \subset T^*_q M$. In steps 7,8 and 9 we have shown that in the case when $(x_0, \zeta_0) \in \Theta_{y_0,w_0}$ is not equal to (q, η), we have $(x_0, \zeta_0) \notin \mathrm{WF}(\mathcal{S}^{(p)}_\sigma)$ for $p = 1, 2, 3$. Also, by (64), (69) and (70) we have $(q, \eta) \notin \mathrm{WF}(\mathcal{S}_\sigma - \mathcal{S}^{mod}_\sigma)$. These show that the bicharasteristic Θ_{y_0,w_0} does not intersect $\mathrm{WF}(\mathcal{S}-\mathcal{S}^{mod})$. Thus, using Hörmander's theorem on propagation of singularities along bicharacteristics, [37, Theorem 26.1.4], we see that (y_0,w_0) is not in the wave front set of the function $\mathcal{U}^{(4)} - Q\mathcal{S}^{mod} = Q(\mathcal{S}-\mathcal{S}^{mod})$. By replacing w_0 by $-w_0$, the arguments above show also that $(y_0,-w_0)$ is not in the wave front set of $Q(\mathcal{S}-\mathcal{S}^{mod})$. Since $y_0 \in \mathcal{E}_U^{reg}(q)$, see Def. 2.5, the light-like geodesic from q to y_0 has no cut points and there is only one light-like geodesics connecting q to y_0. Moreover, as $y_0 \notin \mathcal{Y}\cup\bigcup_{1\le j\le 4} K_j$, we obtain from (59) that the function $\mathcal{S}$ is smooth in a neighborhood of y_0. As $\mathcal{S}^{mod} \in \mathcal{I}^{-4n-3}(\Sigma_q)$, also $\mathcal{S}^{mod}$ is smooth in a neighborhood of y_0. Thus the above and [37, Theorem 26.1.4] show that $(q,w) \notin \mathrm{WF}(Q(\mathcal{S}-\mathcal{S}^{mod}))$ for all $w \in T^*_{y_0}M$. Hence, y_0 has a neighborhood W such that $Q(\mathcal{S}-\mathcal{S}^{mod})$ is C^∞-smooth in W.

By [27, Prop. 2.1], $Q: \mathcal{I}^{-4n-3}(\Sigma_q) \to \mathcal{I}^{-4n-3-3/2,-1/2}(\Sigma_q,\Lambda_q^+)$. This and (26) imply that y_0 as a neighborhood W such that $Q\mathcal{S}^{mod}|_W$ and thus $\mathcal{U}^{(4)}|_W$ are in $\mathcal{I}^{-4n-3-3/2}(\Lambda_q^+) = \mathcal{I}^m(S)$, where $S = {}^+(q)\cap W$. Here, S is a smooth surface as the light-like geodesic from q to y_0 does not have cut points.

We consider now the case when $x_0 = q$, so that $\zeta_0 = \eta$, and compute the principal symbol of $\mathcal{S}^{mod} \in \mathcal{I}^{-4n-3}(\Sigma_q)$. The principal symbol of $F^{mod} = R_{3,2,1}(au_3\cdot R_{2,1}Q(au_2\cdot u_1)) \in \mathcal{I}(K_{123})$ at $(q,\xi) \in N_q^*K_{123}$, where $\xi = \zeta_1+\zeta_2+\zeta_3$, is by (55)

$$\sigma^{(p)}_{F^{mod}}(q,\xi) = \frac{Ca(q)^2}{g(\zeta_1+\zeta_2,\zeta_1+\zeta_2)}\prod_{j=1}^{3}\sigma^{(p)}_{u_j}(q,\zeta_j),$$

and the principal symbol of Q_2 at (q,ξ) is given by (52). Using these, we obtain that the principal symbols of the sources $\mathcal{S}^{(2),mod}_\sigma$ and $\mathcal{S}^{(3),mod}_\sigma$, with $\sigma = Id$, at (q,η), are given by

$$\sigma^{(p)}_{\mathcal{S}^{(2),mod}_{Id}}(q,\eta) = \frac{Ca(q)^3}{g(\zeta_1+\zeta_2,\zeta_1+\zeta_2)\,g(\sum_{j=1}^3\zeta_j,\sum_{k=1}^3\zeta_k)}\prod_{j=1}^{4}\sigma^{(p)}_{u_j}(q,\zeta_j), \tag{71}$$

and

$$\sigma^{(p)}_{\mathcal{S}^{(3),mod}_{Id}}(q,\eta) = \frac{Ca(q)^3}{g(\zeta_1+\zeta_2,\zeta_1+\zeta_2)\,g(\zeta_3+\zeta_4,\zeta_3+\zeta_4)}\prod_{j=1}^{4}\sigma^{(p)}_{u_j}(q,\zeta_j), \tag{72}$$

where we recall that $\eta = \sum_{j=1}^4\zeta_j$, where $\zeta_j \in N_q^*K_j$.

Lemma 3.1 implies that the principal symbol of $\mathcal{U}^{(4)}$ at (y_0,w_0) is the product of a non-zero function $R(y_0,w_0,q,\eta)$ times the principal symbol of $\mathcal{S}^{mod}$ at (q,η). This and formulas (71) and (72), written for general permutation σ, yield formula (46). This proves the claim (ii). □

Next we will show that $\mathcal{G}_{\mathbf{g}}(\vec{\zeta})$ in (46) is not vanishing identically. This implies that at the point of interaction of the four waves a spherical wave is produced in a generic case.

3.3.4. *Non-vanishing of the function $\mathcal{G}_{\mathbf{g}}(\vec{\zeta})$ in a generic set.*

Proposition 3.4. *Let $\eta \in L_q^{+,*}M$ and*

$$\begin{aligned}\mathcal{Z}_0(\eta) \;&=\; \{(\zeta_j)_{j=1}^4 \in (L_q^*M)^4;\ (\zeta_j)_{j=1}^4 \textit{ are linearly independent}, \\ &\qquad \sum_{j=1}^4 \zeta_j = \eta,\ \textit{ and } \eta \neq \zeta_k \textit{ for all } k = 1,2,3,4\}.\end{aligned}$$

Then $\mathcal{Z}_0(\eta)$ is a real analytic manifold having several topological components and $\mathcal{G}_g(\vec{\zeta})$, given in (47), is non-vanishing for $\vec{\zeta} = (\zeta_j)_{j=1}^4$ in an open and dense subset of $\mathcal{Z}_0(\eta)$.

Proof. By its definition, $\mathcal{Z}_0(\eta)$ is a real analytic manifold having several topological components. In the proof, we use in T_q^*M a basis where the metric tensor g is the standard Minkowski metric diag $(-1,1,1,1)$. Also, without loss of generality we can assume that $\eta = (1,1,0,0)$. Moreover, we identify the space T_q^*M with $\mathbb{R}^4$. Also, note that for $(\zeta_j)_{j=1}^4 \in \mathcal{Z}_0(\eta)$ all ζ_j are non-zero and hence η does not belong in the span of any three co-vectors ζ_j, $j = 1,2,3,4$.

Denote $\mathcal{B} = (L_q^{+,*}M)^4$. Let $\eta \in L_q^{+,*}M$ and

$$\begin{aligned}\mathcal{B}_0(\eta) \;&=\; \{(b_j)_{j=1}^4 \in L_q^{+,*}M;\ (b_j)_{j=1}^4 \text{ are linearly independent}, \\ &\qquad \text{there are } a_j \in \mathbb{R}\setminus\{0\} \text{ such that } \sum_{j=1}^4 a_j b_j = \eta \text{ and} \\ &\qquad \eta \neq a_j b_j,\ \text{ for all } j = 1,2,3,4\}.\end{aligned}$$

We observe from this that $\mathcal{B}_0(\eta)$ is a real analytic manifold that has several topological components but it is contained in the connected real-analytic manifold $\mathcal{B}$.

When $(b_j)_{j=1}^4 \in \mathcal{B}_0(\eta)$, let $\alpha_j = \alpha_j(\vec{b},\eta)$, $j = 1,2,3,4$ be such that

$$\eta = \sum_{j=1}^4 \alpha_j b_j.$$

Since b_j are linearly independent, $\alpha_j(\vec{b},\eta)$ are uniquely determined. Considering b_j as elements of $\mathbb{R}^4$ and using Cramer's rule, we obtain

$$\alpha_1(\vec{b},\eta) = \frac{\det(\eta,b_2,b_3,b_4)}{\det(b_1,b_2,b_3,b_4)}, \quad \alpha_2(\vec{b},\eta) = \frac{\det(b_1,\eta,b_3,b_4)}{\det(b_1,b_2,b_3,b_4)}.$$

Similar formulas hold for $\alpha_j(\vec{b},\eta)$ with $j = 3,4$. Using these formulas, we define

$$F(\vec{b},\eta) = (\alpha_1(\vec{b},\eta)b_1, \alpha_2(\vec{b},\eta)b_2, \alpha_3(\vec{b},\eta)b_3, \alpha_4(\vec{b},\eta)b_4).$$

Note that $\zeta_j = \alpha_j b_j$ and $\eta = \sum_{j=1}^4 \zeta_j$ so that for all $\sigma \in \Sigma(4)$,

$$\zeta_{\sigma(1)} + \zeta_{\sigma(2)} + \zeta_{\sigma(3)} = \eta - \zeta_{\sigma(4)}. \tag{73}$$

Then, if $\zeta_{\sigma(1)} + \zeta_{\sigma(2)} + \zeta_{\sigma(3)}$ would be light-like or zero then $\eta - \zeta_{\sigma(4)}$ would be light-like or zero that is possible only if $\eta = \zeta_{\sigma(4)}$ and this can not happen since $(b_j)_{j=1}^4 \in \mathcal{B}_0(\eta)$. We recall also that the inner product of two light-like vectors is zero if and only if the vectors are parallel. Then, we consider the function $\mathcal{G}_{\mathbf{g}}(\vec{\zeta})$ given in (47) with $\vec{\zeta} = F(\vec{b}, \eta)$. We denote $\widetilde{\mathcal{G}}(\vec{b}, \eta) = \mathcal{G}_{\mathbf{g}}(F(\vec{b}, \eta))$ and observe that

$$\widetilde{\mathcal{G}}(\vec{b}, \eta) = P(\vec{b}, \eta)/Q(\vec{b}, \eta)$$

where $\vec{b} \mapsto P(\vec{b}, \eta)$ and $\vec{b} \mapsto Q(\vec{b}, \eta)$ are real analytic functions defined on the whole set $\mathcal{B}$ and $Q(\vec{b}, \eta) \neq 0$ for $\vec{b} \in \mathcal{B}_0(\eta)$.

Let us next show that $\widetilde{\mathcal{G}}(\vec{b}, \eta)$ obtains a non-zero finite value at some $(\vec{b}, \eta)$. Let

$$\begin{aligned} &b_1 = (1+\rho_1^2, 1-\rho_1^2, 2\rho_1, 0), \quad b_2 = (1+\rho_2^2, 1-\rho_2^2, 0, 2\rho_2), \\ &b_3 = (1+\rho_3^2, 1-\rho_3^2, 2\rho_3, 0), \quad b_4 = (1+\rho_4^2, 1-\rho_4^2, 0, 2\rho_4), \\ &\eta = (1,1,0,0), \end{aligned} \tag{74}$$

where $\rho_j \in (0,1)$ are small parameters. Below in this proof, we use the parameters ρ_j given by

$$\rho_4 = \rho_3^{100}, \ \rho_3 = \rho_2^{100}, \ \text{and } \rho_2 = \rho_1^{100}. \tag{75}$$

We denote $\vec{\rho} \to 0$ when $\rho_1 \to 0$ and ρ_2, ρ_3, and ρ_4 are defined using (75). Note that $\rho_4 < \rho_3 < \rho_2 < \rho_1$.

The vectors b_k are light-like. For small ρ_1 we have $\vec{b} = (b_1, b_2, b_3, b_4) \in \mathcal{B}_0(\eta)$ and

$$g(\eta, b_j) = -2\rho_j^2, \quad g(b_k, b_j) = -2(\rho_k^2 + \rho_j^2 + O(\rho_k\rho_j)), \tag{76}$$

for $j, k = 1, 2, 3, 4$.

Below, we denote $\alpha_j = \alpha_j(\vec{b}, \eta)$ and see that

$$\begin{aligned} &\alpha_1 = \frac{\rho_2\rho_4}{\rho_1^2}(1 + O(\rho_1)), \quad \alpha_2 = -\frac{\rho_4}{\rho_2}(1 + O(\rho_1)), \\ &\alpha_3 = -\frac{\rho_2\rho_4}{\rho_1\rho_3}(1 + O(\rho_1)), \quad \alpha_4 = 1 + O(\rho_1). \end{aligned} \tag{77}$$

Then $\widetilde{\mathcal{G}}(\vec{b}, \eta) = \sum_{\sigma\in\Sigma(4)} H^{-1}(M_\sigma^1 + M_\sigma^2)$, where

$$M_\sigma^1 = \frac{C_1 \alpha_{\sigma(3)}}{g(\eta, b_{\sigma(4)}) g(b_{\sigma(1)}, b_{\sigma(2)})} \quad \text{and} \quad M_\sigma^2 = \frac{C_2}{g(b_{\sigma(3)}, b_{\sigma(4)}) g(b_{\sigma(1)}, b_{\sigma(2)})},$$

and $H = \prod_{j=1}^4 \alpha_j$.

When $\sigma(4) = 4$, see that the leading order asymptotics of M^1_σ is given by

$$
\begin{aligned}
&M^1_{Id} \sim C_0 \frac{1}{\rho_4^2}\frac{1}{\rho_1^2}\frac{\rho_2\rho_4}{\rho_1\rho_3} = C_0\rho_4^{-1}\rho_3^{-1}\rho_2^{+1}\rho_1^{-3},\\
&M^1_{(1,3,2,4)} \sim C\frac{1}{\rho_4^2}\frac{1}{\rho_1^2}\frac{\rho_4}{\rho_2} = C\rho_4^{-1}\rho_3^{0}\rho_2^{-1}\rho_1^{-2},\\
&M^1_{(3,2,1,4)} \sim C\frac{1}{\rho_4^2}\frac{1}{\rho_2^2}\frac{\rho_2\rho_4}{\rho_1} = C\rho_4^{-1}\rho_3^{0}\rho_2^{-1}\rho_1^{-2},\\
&M^1_{(3,1,2,4)} \sim C\frac{1}{\rho_4^2}\frac{1}{\rho_1^2}\frac{\rho_4}{\rho_2} = C\rho_4^{-1}\rho_3^{0}\rho_2^{-1}\rho_1^{-2},\\
&M^1_{(2,3,1,4)} \sim C\rho_4^{-1}\rho_3^{0}\rho_2^{-1}\rho_1^{-2},\\
&M^1_{\sigma_1} = M^1_{id},
\end{aligned}
\tag{78}
$$

where $\sigma_1 = (2,1,3,4)$ and $C_0 \neq 0$.

Using the formulas for α_j given by Cramer's rule, we have that the terms M^1_σ with $\sigma(4) \neq 4$ do not contain the factor ρ_4^j with $j \le -1$. Also M^2_σ do not contain the factor ρ_4^j with $j \le -1$. Hence, $M^1_{\sigma_1} = M^1_{id}$ has the strongest asymptotics when $\vec\rho \to 0$. More precisely, for all $\sigma \notin \{Id, \sigma_1\}$, we have that $M^1_\sigma/M^1_{Id} \to 0$ as $\vec\rho \to 0$. Also, for all σ we have that $M^2_\sigma/M^1_{Id} \to 0$ as $\vec\rho \to 0$. As $C_0 \neq 0$, this implies that

$$
\widetilde{\mathcal{G}}(\vec b, \eta)/M^1_{Id} \to 2 \text{ as } \vec\rho \to 0. \tag{79}
$$

Recall that $\widetilde{\mathcal{G}}(\vec b, \eta)$ is a quotient of two real analytic functions $P(\vec b, \eta)$ and $Q(\vec b, \eta)$. Since for the vectors given in (74) with small ρ_1 we have $\vec b \in \mathcal{B}_0(\eta)$, we see using (78) and (79) that there is a point $\vec b \in \mathcal{B}_0(\eta) \subset \mathcal{B}$ for which $Q(\vec b, \eta) \neq 0$ and $P(\vec b, \eta) \neq 0$, that is, $Q(\vec b, \eta)$ and $P(\vec b, \eta)$ are not identically vanishing. As $\mathcal{B}$ is a connected, real-analytic manifold, we get that the functions $P(\vec b, \eta)$ and $Q(\vec b, \eta)$ are not identically vanishing in any open subset of $\mathcal{B}$. As $\mathcal{B}_0(\eta) \subset \mathcal{B}$ is open and dense, this implies that $P(\vec b, \eta)$ and $Q(\vec b, \eta)$, and thus also $\widetilde{\mathcal{G}}(\vec b, \eta)$, are non-vanishing in an open and dense subset of $\mathcal{B}_0(\eta)$, too. Since $F_\eta : \mathcal{B}_0(\eta) \to \mathcal{Z}_0(\eta)$, $F_\eta(\vec b) = F(\vec b, \eta)$, is an open, continuous and surjective map, we conclude that $\mathcal{G}_{\mathbf{g}}(\vec\zeta)$ is non-vanishing in an open and dense subset of $\mathcal{Z}_0(\eta)$. □

3.4. **Detection of singularities.** We use now the above results to detect in the set U the singularities that are produced by the interaction of four waves.

First we show that for all $q \in I^+(p^-) \cap I^-(p^+)$ there are $(\vec x, \vec\xi)$ such that q is the intersection point of the geodesics corresponding to $(\vec x, \vec\xi)$.

Lemma 3.5. *Let $q \in I^+(p^-) \cap I^-(p^+)$. Then*

(i) There are $(z, \zeta) \in L^+M$ and $0 < r < \rho(z, \zeta)$ such that $z \in I^+(p^-) \cap U$ and $q = \gamma_{z,\zeta}(r)$.

(ii) In any neighborhood of (z,ζ) *there are* $(x_j,\xi_j)\in L^+U$, $j=1,2,3,4$ *such that the geodesics corresponding to* $(\vec x,\vec\xi)$ *intersect regularly at* q*, see Def. 3.2. Moreover, we have that* $q\in\mathcal{N}(\vec x,\vec\xi)$*, where* $\mathcal{N}(\vec x,\vec\xi)$ *is defined in (43), and the points* x_j *satisfy the condition (42).*

Proof. (i) In the case when $q\in\widehat\mu$, let $\zeta_0\in L^+_q(M)$, and let $r_0>0$ be so small that the geodesic $\gamma_{q,\zeta_0}([-r_0,0])\subset U\cap I^+(p^-)$ has no cut points. Then, we define $z=\gamma_{q,\zeta_0}(-r_0)$ and $\zeta=\gamma_{q,\zeta_0}(-r_0)$.

In the case when $q\notin\widehat\mu$, let $z_1=\widehat\mu(f^-_{\widehat a}(q))\in U\cap I^+(p^-)$. By Lemma 2.3 (iii) there is $\zeta_1\in L^+_{z_1}M$ such that γ_{z_1,ζ_1} is one of the longest light-like geodesics connecting y and q, $q=\gamma_{z_1,\zeta_1}(r_1)$ where $0<r_1\le\rho(z_1,\zeta_1)$. Then, let $r_2>0$ be so small that $\gamma_{z_1,\zeta_1}([0,r_2])\subset U$, and let $z=\gamma_{z_1,\zeta_1}(r_2)$ and $\zeta=\gamma_{z_1,\zeta_1}(r_2)$.

In both cases $(z,\zeta)\in L^+U$ has the properties required in (i).

(ii) Note that $q=\gamma_{z,\zeta}(r)\in I^-(p^+)$ with $0<r\le\rho(z,\zeta)$. Let $\rho_0\in(0,r)$ be such that $\gamma_{z,\zeta}([0,\rho_0])\subset U$. Let $W\subset TM$ be a neighborhood of (z,ζ) such that $\pi(W)\subset U$.

Let $\theta=-\dot\gamma_{z,\zeta}(r)\in T_qM$. Then the geodesic $\gamma_{z,\zeta}([\rho_0,r])=\gamma_{q,\theta}([0,r-\rho_0])$ has no cut points. Thus, consider four geodesics that emanate from q to the past, in the light-like direction $\eta_1=\theta$ and in the light-like directions $\eta_j\in T_qM$, $j=2,3,4$ that are close to the direction θ, such that $(\eta_j)_{j=1}^4$ are linearly independent. Let $\gamma_{q,\eta_j}(r_j)$ be the intersection points of γ_{q,η_j} with the surface $\{x\in M;\ \mathbf{t}(x)=c_0\}$, on which the time function $\mathbf{t}(x)$ has the constant value $c_0:=\mathbf{t}(z)$, see subsection 2.1. Let $x_j=\gamma_{q,\eta_j}(r_j)$ and $\xi_j=-\dot\gamma_{q,\eta_j}(r_j)$. When η_j are sufficiently close to η_1, such $r_j=r_j(\eta_j)$ exist by the inverse function theorem, we have that $\rho(x_j,\xi_j)>r_j$, (x_j,ξ_j) are in W. Then the obtained points $((x_j,\xi_j))_{j=1}^4$ satisfy condition (42) and geodesics corresponding to $((x_j,\xi_j))_{j=1}^4$ intersect regularly at $q\in\mathcal{N}(\vec x,\vec\xi)$. □

We say that the *interaction condition* (I) is satisfied for $y\in U$ with light-like vectors $(\vec x,\vec\xi)=((x_j,\xi_j))_{j=1}^4$ and parameters (q,w,t), if

(I) *There exist* $q\in\bigcap_{j=1}^4\gamma_{x_j,\xi_j}((0,\mathbf{t}_j))$*, where* $\mathbf{t}_j=\rho(x_j,\xi_j)$*, and* $w\in L^+_qM_0$ *and* $t\ge 0$ *such that* $y=\gamma_{q,w}(t)$.

Below, in TM_0 we use the Sasaki metric corresponding to the Riemannian metric g^+. Moreover, let $B_j\subset U$ be open sets such that, cf. (36), we have

$$\overline B_j\subset U \text{ and } B_j\cap J^+(B_k)=\emptyset \text{ for all } j\ne k. \tag{80}$$

Next we formulate a condition (D) that is valid when we can detect singularities at a point $y\in U$. We say that a function $v(x)$ is C^∞-smooth at y if there is an open neighborhood W of y such that $v|_W\in C^\infty(W)$.

We define that point $y \in U$ satisfies the singularity *detection condition* (D) with light-like directions $(\vec{x}, \vec{\xi})$ and $\widehat{s} > 0$ if

(**D**) *For any $s_0, s_1 \in (0, \widehat{s})$ and $j = 1,2,3,4$ and sufficiently large n there exist (x_j', ξ_j') in the s_1-neighborhood of (x_j, ξ_j), open sets $B_j \subset B_{g^+}(x_j, s_1)$ satisfying (80), and source functions $f_j \in \mathcal{I}^{n+1}(\Sigma(x_j', \xi_j', s_0))$, supp$(f_j) \subset B_j$, such that the following holds: When $u_{\vec{\varepsilon}}$ is the solution of the non-linear wave equation (20) with the source $f_{\vec{\varepsilon}} = \sum_{j=1}^4 \varepsilon_j f_j$, then the function $\partial_{\vec{\varepsilon}}^4 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ is not C^∞-smooth at y.*

Lemma 3.6. *Let $(\vec{x}, \vec{\xi})$, and $\mathbf{t}_j$ with $j = 1,2,3,4$ satisfy (42)-(43). Assume that $y \in \mathcal{N}(\vec{x}, \vec{\xi}) \cap U$ satisfies $y \notin \mathcal{Y}(\vec{x}, \vec{\xi}) \cup \bigcup_{j=1}^4 \gamma_{x_j,\xi_j}(\mathbb{R})$. Then*

(i) If the geodesics corresponding to $(\vec{x}, \vec{\xi})$ either do not intersect in $\mathcal{N}(\vec{x}, \vec{\xi})$, or the geodesics intersect at a point $q \in \mathcal{N}(\vec{x}, \vec{\xi})$ and $y \notin J^+(q)$, then y does not satisfy condition (D) with $(\vec{x}, \vec{\xi})$ and any $\widehat{s} > 0$.

(ii) Assume $y \in U$ satisfies condition (I) with $(\vec{x}, \vec{\xi})$ and the parameters q, w, and $0 < t < \rho(q, w)$. Then y satisfies condition (D) with $(\vec{x}, \vec{\xi})$ for any sufficiently small $\widehat{s} > 0$.

(iii) Using the source-to-solution operator L_V one can determine whether the condition (D) is valid for the given $y \in U$, $(\vec{x}, \vec{\xi})$ and $\widehat{s}$.

Proof. (i) Assume that $y \in U$ satisfies the conditions stated in (i) and let s_0, s_1, (x_j', ξ_j'), B_j, and $f_j \in \mathcal{I}^{n+1}(\Sigma(x_j', \xi_j', s_0))$ be as in condition (D). When s_0 and s_1 are small enough, we see that $y \in \mathcal{N}(((x_j', \xi_j'))_{j=1}^4)$ and if the geodesics corresponding to $((x_j', \xi_j'))_{j=1}^4$ intersect at a point $q' \in \mathcal{N}(((x_j', \xi_j'))_{j=1}^4)$, then $y \notin J^+(q')$. Hence, the point y, the vectors (x_j', ξ_j') and the sources f_j satisfy the assumptions of the claim (i) in Thm. 3.3.

Let $u_{\vec{\varepsilon}}$ be the solution of (20) with $f_{\vec{\varepsilon}} = \sum_{j=1}^4 \varepsilon_j f_j$ and $\mathcal{U}^{(4)} = \partial_{\vec{\varepsilon}}^4 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$. Then Theorem 3.3 (i) implies that $\mathcal{U}^{(4)}$ is C^∞-smooth at y. Thus y does not satisfy condition (D).

(ii) Let $y \in U$ satisfy condition (I) with $(\vec{x}, \vec{\xi})$ and the parameters q, w, and $0 < t < \rho(q, w)$, so that $\gamma_{q,w}(t) = y$. Let $s_0, s_1 > 0$. Note that then $y \in \mathcal{E}_U^{reg}(q)$. Let $\eta = w^\flat \in L_q^{*,+} M_0$. For $j = 1,2,3,4$, let $t_j > 0$ be such that $\gamma_{x_j,\xi_j}(t_j) = q$. Let $b_j = \dot{\gamma}_{x_j,\xi_j}(t_j)^\flat$. We can make an arbitrarily small perturbation to the co-vectors b_j to obtain co-vectors $\widehat{b}_j \in L_q^{*,+} M$ such that $(\widehat{b}_j)_{j=1}^4$ are linearly independent and η is not in the space spanned by any three of the vectors b_j, $j = 1,2,3,4$. Then, there are $\alpha_j \in \mathbb{R} \setminus \{0\}$ such that $\widehat{\zeta}_j = \alpha_j \widehat{b}_j$ satisfy $\sum_{j=1}^4 \widehat{\zeta}_j = \eta$. Then, $(\widehat{\zeta}_j)_{j=1}^4 \in \mathcal{Z}_0(\eta)$. Furthermore, by using Prop. 3.4, we see that there are arbitrarily small perturbations $\zeta_j' \in L_q^* M$ of the co-vectors $\widehat{\zeta}_j$, such that $(\zeta_j')_{j=1}^4 \in \mathcal{Z}_0(\eta)$ and $\mathcal{G}_{\mathbf{g}}((\zeta_j')_{j=1}^4) \neq 0$. Let $b_j' = \frac{1}{\alpha_j} \zeta_j'$,

$j = 1,2,3,4$ and $x_j' = \gamma_{q,(b_j')^\sharp}(-t_j)$ and $\xi_j' = \dot\gamma_{q,(b_j')^\sharp}(-t_j)$. When the perturbations above are small enough, we have that $x_j' \in B_{g^+}(x_j, s_1/2)$ and the points x_j' satisfy conditions (42) and $\rho(x_j',\xi_j') > t_j$. Then, using Lemma 3.1 we obtain that when $n \in \mathbb{Z}_+$ is sufficiently large, there are $f_j \in \mathcal{I}^{n+1}(\Sigma(x_j',\xi_j',s_0))$ for which the principal symbol of $\Box_g^{-1} f_j$ at (q,ζ_j') are non-vanishing and f_j are supported in sets $B_j \subset B_{g^+}(x_j,s_1)$ satisfying (80). By Thm. 3.3 (ii), the function $\mathcal{U}^{(4)}$ is not C^∞-smooth at y. Hence, as $s_0, s_1 > 0$ are above arbitrary, we conclude that condition (D) is valid.

(iii) The non-linear source-to-solution map L_U determines the functions $\mathcal{U}^{(4)} = \partial_{\vec\varepsilon}^4 u_{\vec\varepsilon}|_{\vec\varepsilon=0}$ in U. This yields (iii). □

4. Determination of the earliest light observation sets

In this section we reduce the proof of Theorem 1.5 to proving Theorem 1.2 that is proven later in Section 5. Below, we assume that we are given $(U, g|_U)$ and the source-to-solution map L_U.

4.1. Surfaces of the earliest singularities.

Next we consider the determination of the the earliest light observation sets $\mathcal{E}_U(q)$, see Def. 1.1. To this end we need the following notation:

Definition 4.1. *For a closed set $S \subset U$, we define the earliest points of set S on the path $\mu_a = \mu_a([-1,1])$ to be*

$$\text{(81)} \qquad \begin{aligned} &\mathbf{e}_a(S) = \{\mu_a(\inf\{s \in [-1,1];\ \mu_a(s) \in S\})\}, \ \textit{if } \mu_a \cap S \neq \emptyset,\\ &\mathbf{e}_a(S) = \emptyset, \quad \textit{if } \mu_a \cap S = \emptyset. \end{aligned}$$

Note that by the above definition, $\mu_a(f_a^+(q)) = \mathbf{e}_a(\mathcal{P}_U(q))$ for $q \in J(p^-,p^+)$.

Our next aim is to consider the global problem of constructing the set of the earliest light observations in U of all points $q \in J(p^-,p^+)$. To this end, we need to handle the technical problem that in the set $\mathcal{Y}(\vec x,\vec\xi)$, see (41), we have not analyzed if we observe singularities. Moreover, we have not analyzed the wave $\mathcal{U}^{(4)}$ in the set $J^+(q)$, if the geodesics intersect at q and the velocity vectors of the geodesics at the point q are not linearly independent, or in the set $M_0 \setminus \mathcal{N}(\vec x,\vec\xi)$ that is the causal future of first conjugate points of the geodesics γ_{x_j,ξ_j}. As discussed in Remark 3.2, the waves created by the non-linear interaction can be very complicated in these sets. To avoid these difficulties, we make the following definition

Definition 4.2. *Let $(\vec x,\vec\xi) = ((x_j,\xi_j))_{j=1}^4$ be a collection of light-like vectors with $x_j \in U$. We define*

$$\mathcal{S}(\vec x,\vec\xi) = \{y \in U \ \ ; \ \ \textit{there is } \widehat s > 0 \textit{ such that the condition (D) is valid for } y,\ (\vec x,\vec\xi) \textit{ and } \widehat s\}.$$

Moreover, let $\mathcal{S}_H(\vec{x},\vec{\xi})$ be the set of such points $y_0 \in U$ that for every neighborhood $W \subset U$ of y_0 the Hausdorff dimension of the intersection $W \cap \mathcal{S}(\vec{x},\vec{\xi})$ is at least 3. Note that $\mathcal{S}_H(\vec{x},\vec{\xi})$ is closed in the relative topology of U. We denote (see (81) and Def. 1.1)

$$\mathcal{S}_e(\vec{x},\vec{\xi}) = \bigcup_{a\in\mathcal{A}} \mathbf{e}_a(\mathcal{S}_H(\vec{x},\vec{\xi})). \tag{82}$$

We call $\mathcal{S}_e(\vec{x},\vec{\xi})$ the surface of the earliest stable singularities produced by the interaction of four waves.

Lemma 4.3. *The path $\widehat{\mu} : [-1,1] \to U$, the manifold $(U, g|_U)$ and L_U determine the set $\mathcal{S}_e(\vec{x},\vec{\xi})$ for all $(\vec{x},\vec{\xi}) \in (L^+U)^4$. Moreover, these data determines the sets $\mathcal{E}_U(p)$ for all $p \in U$, the causality relation $R_U^< = \{(p_1,p_2) \in U \times U :\ p_1 < p_2\}$, where $<$ is the causality relation of (M_0, g), and the set*

$$G_{\widehat{\mu}} = \{(x,\xi) \in L^+M_0;\ x \in U, \gamma_{x,\xi}(\mathbb{R}_+) \cap \widehat{\mu} \neq \emptyset\}. \tag{83}$$

Proof. The first claim follows from Lemma 3.6 and Def. 4.2.

The derivative of the map L_U at zero is the map $DL_U|_0 : f \mapsto Qf|_U$, defined for the distributions f supported in U. This map coincides with the source-to solution map for the linearized wave equation. By [35, Thm. 8.1.4], for any $(x,\xi) \in L^*M$ there is a distribution f_1 such that $\mathrm{WF}(f_1)$ is the half-line $\{(x, s\xi) \in T^*M;\ s > 0\}$. Then by using Hörmander's theorem on propagation of singularities along bicharacteristics, [37, Theorem 26.1.4], we see that the singular support of $DL_U|_0(f_1)$ is $\gamma_{x,\xi}([0,\infty)) \cap U$. Thus, we can determine the set $G_{\widehat{\mu}}$. Also, for $p \in U$ we can find the sets $\mathcal{L}^+(p)$ and $\mathbf{e}_a(\mathcal{L}^+(p))$ and the values $f_a^+(p)$ for $a \in \mathcal{A}$. These determine the sets $\mathcal{E}_U(p)$ and $J^+(p) \cap U$ for all $p \in U$. The latter determines all pairs $(p_1,p_2) \in U^2$ such that $p_1 < p_2$. □

Next we show that if the geodesics emanating from $(\vec{x},\vec{\xi})$ intersect before their first cut points at q_0 then $\mathcal{S}_e(\vec{x},\vec{\xi})$ coincides with the set $\mathcal{E}_U(q_0)$, see Def. 1.1.

If the set $\cap_{j=1}^4 \gamma_{x_j,\xi_j}([0,\infty))$ is non-empty we denote its earliest point by $Q(\vec{x},\vec{\xi})$. If such intersection point does not exists, we define $Q(\vec{x},\vec{\xi})$ to be the empty set. Next we consider the relation of $\mathcal{S}_e(\vec{x},\vec{\xi})$ and $\mathcal{E}_U(q)$ for $q = Q(\vec{x},\vec{\xi})$.

Lemma 4.4. *Let $(\vec{x},\vec{\xi}) = ((x_j,\xi_j))_{j=1}^4 \in (L^+U)^4$ be such that the points x_j satisfy (42). Let $\mathcal{N} = \mathcal{N}(\vec{x},\vec{\xi})$ be the set defined in (43). Then*

(i) Assume that $y \in \mathcal{N} \cap U$ satisfies the condition (I) with $(\vec{x},\vec{\xi})$ and parameters q, w, and t such that $0 \le t \le \rho(q,w)$. Then $y \in \mathcal{S}_H(\vec{x},\vec{\xi})$.

(ii) Assume that the geodesics corresponding to $(\vec x,\vec \xi)$ *either do not intersect in* $\mathcal{N}$ *or those intersect at* q *and* $y\not\in J^+(q)$. *Then* $y\not\in \mathcal{S}_H(\vec x,\vec \xi)$.

(iii) The sets $\mathcal{S}_e(\vec x,\vec \xi)$ *satisfy*

$$\begin{aligned}&\mathcal{S}_e(\vec x,\vec \xi)=\mathcal{E}_U(q)\subset \mathcal{N}, &&\text{if } Q(\vec x,\vec \xi)\not =\emptyset \text{ and } q=Q(\vec x,\vec \xi)\in \mathcal{N},\\ &\mathcal{S}_e(\vec x,\vec \xi)\subset M_0\setminus \mathcal{N}, &&\text{if } Q(\vec x,\vec \xi)\cap \mathcal{N}=\emptyset.\end{aligned}$$

Proof. (i) Consider sets $\mathcal{X}(\vec x,\vec \xi)$ and $\mathcal{Y}(\vec x,\vec \xi)$ defined in (41). When $p\in \mathcal{N}$ and $(p,\zeta)\in \mathcal{X}(\vec x,\vec \xi)$, we see that there are $1\le i_1<i_2<i_3\le 4$ such that $p=\gamma_{x_{i_k},\xi_{i_k}}(t_{i_k})$ with some $t_{i_k}>0$, that is, p is an intersection point of some three geodesics. When $v_k=(\dot \gamma_{x_{i_k},\xi_{i_k}}(t_{i_k}))^\flat$, $k=1,2,3$, we see that $\mathcal{X}(\vec x,\vec \xi)\cap T^*_pM=\{v=\sum_{k=1}^3 a_kv_k\in T^*_pM\setminus \{0\};\ g(v,v)=0\}$. Then, we see that $\mathcal{X}(\vec x,\vec \xi)\cap T^*_pM$ is a union of two 2-dimensional cones. This implies that the Hausdorff dimension of the set $\mathcal{Y}(\vec x,\vec \xi)\cap \mathcal{N}$ is at most 2.

Assume first that the point y satisfies conditions in (i), $y\not \in \mathcal{Y}(\vec x,\vec \xi)$ and $t<\rho(q,w)$. Then $y\in \mathcal{E}_U^{reg}(q)$, see (16), and y has a neighborhood $W\subset U$ such that $\mathcal{E}_U^{reg}(q)\cap W$ is a smooth 3-dimensional submanifold. Then the assumptions of Lemma 3.6 (ii) are valid for all points $y'\in \mathcal{E}_U^{reg}(q)\cap W$, and Lemma 3.6 (ii) implies that $y\in \mathcal{S}_H(\vec x,\vec \xi)$.

Consider next a general point y satisfying the assumptions in (i) and let $q=Q(\vec x,\vec \xi)$. Then $y\in \mathcal{E}_U(q)$. Recall that $\rho(x,\xi)$ is lower semi-continuous. Then the set $\mathcal{E}_U^{reg}(q)\setminus \mathcal{Y}(\vec x,\vec \xi)$ is dense in $\mathcal{E}_U(q)$ and we have that y is a limit point of points $y_n\in \mathcal{E}_U^{reg}(q)\setminus \mathcal{Y}(\vec x,\vec \xi)$. As $\mathcal{S}_H(\vec x,\vec \xi)$ is closed in the relative topology of U, this yields that $y\in \mathcal{S}_H(\vec x,\vec \xi)$.

(ii) In the case when the geodesics corresponding to $(\vec x,\vec \xi)$ do intersect at $q\in \mathcal{N}$, denote $\mathcal{N}_1=\mathcal{N}\setminus J^+(q)$. Also, in the case when the geodesics corresponding to $(\vec x,\vec \xi)$ do not intersect in $\mathcal{N}$, denote $\mathcal{N}_1=\mathcal{N}$. Then by Lemma 3.6 (i), condition (D) is not valid for any point in the set $\mathcal{N}_1\setminus (\mathcal{Y}(\vec x,\vec \xi)\cup \bigcup_{j=1}^4\gamma_{x_j,\xi_j}(\mathbb{R}))$. As the Hausdorff dimension of the set $(\mathcal{Y}(\vec x,\vec \xi)\cup \bigcup_{j=1}^4\gamma_{x_j,\xi_j}(\mathbb{R}))\cap \mathcal{N}$ is at most 2, $\mathcal{N}_1$ does not intersect $\mathcal{S}_H(\vec x,\vec \xi)$. This yields the claim (ii).

(iii) Suppose $q=Q(\vec x,\vec \xi)\in \mathcal{N}$ and $y\in \mathcal{E}_U(q)\setminus \bigcup_{j=1}^4\gamma_{x_j,\xi_j}([0,\infty))$. Let $\gamma_{q,\eta}([0,l])$ be a light-like geodesic that is one of the longest causal geodesics from q to y. Then $l\le \rho(q,\eta)$. Let $p_j=\gamma_{x_j,\xi_j}(\mathbf{t}_j)$, $\mathbf{t}_j=\rho(x_j,\xi_j)$, be the first cut point on the geodesic $\gamma_{x_j,\xi_j}([0,\infty))$. To show that y is in $\mathcal{N}$, we assume the opposite, $y\not\in \mathcal{N}$. Then for some j there is a causal geodesic $\gamma_{p_j,\theta_j}([0,l_j])$ from p_j to y. Now we can use a short-cut argument: Let $q=\gamma_{x_j,\xi_j}(t')$. As $q\in \mathcal{N}$, we have $t'<\mathbf{t}_j$. Moreover, as $y\not \in \gamma_{x_j,\xi_j}([0,\infty))$, the union of the geodesic $\gamma_{x_j,\xi_j}([t',\mathbf{t}_j])$ from q to p_j and $\gamma_{p_j,\theta_j}([0,l_j])$ from p_j to y does not form a light-like geodesic and thus $\tau(q, y) > 0$. As $y \in \mathcal{E}_U(q)$, this is not possible. Hence $y \in \mathcal{N}$. Thus by (i), $y \in \mathcal{S}_H(\vec{x}, \vec{\xi})$ and hence $\mathcal{E}_U(q) \setminus (\bigcup_{j=1}^4 \gamma_{x_j,\xi_j}([0,\infty))) \subset \mathcal{S}_H(\vec{x}, \vec{\xi})$. Since the set $\mathcal{E}_U(q) \setminus (\bigcup_{j=1}^4 \gamma_{x_j,\xi_j}([0,\infty)))$ is dense in the closed set $\mathcal{E}_U(q) \subset U$, the above shows that $\mathcal{E}_U(q) \subset \mathcal{S}_H(\vec{x}, \vec{\xi})$. Also by (ii), $\mathcal{S}_H(\vec{x}, \vec{\xi}) \subset J^+(q)$. Using Lemma 2.4, Def. 4.1, and (82), we conclude that $\mathcal{S}_e(\vec{x}, \vec{\xi}) = \mathcal{E}_U(q)$.

On the other hand, if $Q(\vec{x}, \vec{\xi}) \cap \mathcal{N} = \emptyset$, we can apply (ii) for all $y \in \mathcal{N} \cap U$ and obtain that $\mathcal{S}_H(\vec{x}, \vec{\xi}) \cap \mathcal{N} = \emptyset$. This and (82) prove (iii). □

Next we show the sets $\mathcal{S}_e(\vec{x}, \vec{\xi})$, where $(\vec{x}, \vec{\xi}) \in (L^+U)^4$, determine the family of the earliest light observation sets. As we believe that this type of result is useful for inverse problems for a wide range of non-linear partial differential equations, we formulate this in more general terms using two geometric properties, denoted below by (P1) and (P2), and the first cut points $\gamma_{x_j,\xi_j}(\rho(x_j, \xi_j))$ of the geodesics $\gamma_{x_j,\xi_j}([0,\infty))$. We note that Theorem 4.5 below is also valid for Lorentzian manifolds of dimension $n \geq 3$.

Theorem 4.5. *(i) Assume that for all* $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4 \in (L^+U)^4$, *such that* $(x_j)_{j=1}^4$ *satisfy (42), there are sets* $S_0(\vec{x}, \vec{\xi}) \subset U$ *that have the following properties:*

(P1) If there is $q \in J^-(p^+)$ *such that* $q = \gamma_{x_j,\xi_j}(t_j)$ *with* $t_j \in (0, \rho(x_j, \xi_j))$, *for all* $j = 1, 2, 3, 4$, *then* $S_0(\vec{x}, \vec{\xi}) = \mathcal{E}_U(q)$,

(P2) If there are no such $q \in J^-(p^+)$, *then* $S_0(\vec{x}, \vec{\xi}) \subset M \setminus \mathcal{N}$, *where* $M \setminus \mathcal{N} = \cup_{j=1}^4 J^+(\gamma_{x_j,\xi_j}(\rho(x_j, \xi_j)))$.

Assume that we are given $(U, g|_U)$, *the causality relation* $R_U^<$ *in* U, *the set* $G_{\widehat{\mu}}$, *see (83), and the family* $\{S_0(\vec{x}, \vec{\xi});\ (\vec{x}, \vec{\xi}) \in (L^+U)^4\}$.

Then these data determine uniquely the family $\{\mathcal{E}_U(q);\ q \in I^+(p^-) \cap I^-(p^+)\}$ *of the earliest light observation sets.*

(ii) The properties (P1) and (P2) are valid for all set $\mathcal{S}_e(\vec{x}, \vec{\xi})$ *such that* $(\vec{x}, \vec{\xi}) \in (L^+U)^4$ *and* $(x_j)_{j=1}^4$ *satisfy (42).*

We call the sets $S_0(\vec{x}, \vec{\xi})$ the generalized observations, and emphasise that for such a set we do not a priori know it is of the type considered in (P1) or (P2). In particular, we do not know a prior if a given set $S_0(\vec{x}, \vec{\xi})$ corresponds to the interaction of waves that has started to happen before or after the conjugate points of the geodesics $\gamma_{x_j,\xi_j}([0,\infty))$. By claim (ii), the observations related to wave equation are an example of generalized observations

The proof of the claim (ii) of Theorem 4.5 is obtained immediately from Lemma 4.4 (iii).

We prove the claim of Theorem 4.5 (i) in the next subsection. To give the idea, before proving Theorem 4.5 (i) for general globally hyperbolic manifolds, we consider a simpler case where the proof of Theorem 4.5 is easier.

Proof of Theorem 4.5 (i) in a special case. Let us consider a special case when the light-like geodesics in $I(p^-,p^+)=I^+(p^-)\cap I^-(p^+)$ do not contain cut points. Then, we consider all $(\vec x,\vec \xi)\in (L^+U)^4$ such that $(x_j)_{j=1}^4$ satisfy conditions (42) and $x_j\in I^+(p^-)\cap U$. When there are no cut points, all intersection points of geodesics in $I(p^-,p^+)$ are automatically in the set $\mathcal N(\vec x,\vec \xi)$, see (43). Then (P1) and (P2) imply that $S_0(\vec x,\vec \xi)=\mathcal E_U(q)$ if the geodesics corresponding to $(\vec x,\vec \xi)$ intersect at some point $q\in I(p^-,p^+)$. Moreover, $\mathcal S_e(\vec x,\vec \xi)$ is an empty set if no such intersection point exists in $I^+(p^-)$.

Consider a point $q\in I^+(p^-,p^+)$. By Lemma 3.5 there are $(\vec x,\vec \xi)$ satisfying (42) such that $x_j\in I^+(p^-)\cap U$ and that the corresponding geodesics intersect at q. Also, we have that $S_0(\vec x,\vec \xi)$ intersects the set $U\cap I^-(p^+)$. Then, let us consider the family

$$\{S_0(\vec x,\vec \xi)\ \ ;\ \ (\vec x,\vec \xi)\in (L^+U)^4 \text{ are such that } x_j\in I^+(p^-)\cap U \\ \text{satisfy conditions (42) and } S_0(\vec x,\vec \xi)\cap (U\cap I^-(p^+))\neq\emptyset\}.$$

The above yields that this family coincides with the family $\{\mathcal E_U(q);\ q\in I^+(p^-,p^+)\}$ of the earliest light observation sets. This completes the proof of Theorem 4.5 in the special case when the light-like geodesics in $I(p^-,p^+)$ do not contain cut points. □

Theorem 4.5 reduces the active inverse problem considered in Theorem 1.5 to the passive inverse problem. Later, in Section 5 we finish the proof on the uniqueness for the inverse problem with passive observations. In the rest of this section we will prove Theorem 4.5 for general globally hyperbolic spacetimes and remark that a reader who is interested just in spacetimes having no cut points can move to Section 5.

4.2. **Determination of the earliest light observation set.** In this subsection we consider the proof of Theorem 4.5 (i) in the general case. The proof will be quite technical due to the reason that above we have analyzed interaction of waves that propagate near light-like geodesics and intersect before the first conjugate to cut points of the geodesics. If the geodesics intersect after or at the conjugate points, the waves may have caustics and the waves produced by the non-linear interaction may be very complicated. However, we do not know the manifold (M,g) and thus we do not a priori know when the geodesics have conjugate points. Therefore, we have to determine from our observations when we are sure that the interaction of the waves has taken place before the the conjugate points. Then we can remove from our data all observations that may be caused by caustics.

Proof of Theorem 4.5 (i). Below, we use the numbers $\vartheta_1, \kappa_1, \kappa_2 > 0$ appearing in Lemma 2.8 and denote $t_0 = 4\kappa_1$.

First, consider the set

$$\mathcal{K}_{t_0} = \{x \in M \quad ; \quad x = \gamma_{\widehat{x},\xi}(r),\ \widehat{x} = \widehat{\mu}(s),\ s \in [s^-, s^+],\ \xi \in L^+_{\widehat{x}}M,\ \|\xi\|_{g^+} = 1,\ r \in [0, 2t_0)\} \subset U. \tag{84}$$

that is the closure of a small neighborhood of $\widehat{\mu}$. As we will consider waves propagating near geodesics $\gamma_{\widehat{x},\widehat{\zeta}}([t_0, \infty))$ where $\widehat{x} \in \widehat{\mu}$, we have to consider the earliest light observation sets corresponding to the points $\gamma_{\widehat{x},\widehat{\zeta}}([0, t_0))$ separately. This is why the set $\mathcal{K}_{t_0}$ is introduced.

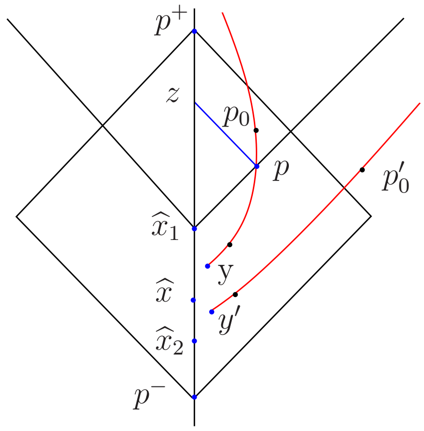

FIGURE 7. The blue points on $\widehat{\mu}$ are $\widehat{x}_1 = \widehat{\mu}(s_1)$, $\widehat{x}_2 = \widehat{\mu}(s_2)$, and $\widehat{x} = \widehat{\mu}(s)$. The blue points y and y' are close to $\widehat{x}$. The boundary of $J^+(\widehat{x}_1)$ is marked by black. We consider the geodesics $\gamma_{y,\zeta}([0,\infty))$ and $\gamma_{y',\zeta'}([0,\infty))$. These geodesics corresponding to the cases when the geodesic $\gamma_{y,\zeta}([0,\infty))$ enters in $J^-(p^+) \cap J^+(\widehat{x}_1)$, and the case when the geodesic $\gamma_{y',\zeta'}([0,\infty))$ does not enter this set. The point p_0 is the cut point of $\gamma_{y,\zeta}([t_0,\infty))$ and p_0' is the cut point of $\gamma_{y',\zeta'}([t_0,\infty))$. At the point $z = \widehat{\mu}(\mathbb{S}(y,\zeta,s_1))$ we observe for the first time on the geodesic $\widehat{\mu}$ that the geodesic $\gamma_{y,\zeta}([0,\infty))$ has entered $J^+(\widehat{x}_1)$. The entering in the set $J^+(\widehat{x}_1)$ happens at the point p.

As we are given $(U, g|_U)$ and the causality relation $R_U^<$ in U, we can determine the subset $\mathcal{K}_{t_0} \cap J^+(\widehat{\mu}(s))$ for all $s \in [s_-, s_+]$. As this set is a subset of U, by Lemma 4.3 we can determine the earliest light observation sets in $\mathcal{E}_U(\mathcal{K}_{t_0} \cap J^+(\widehat{\mu}(s)))$ for all $s \in [s_-, s_+]$. Here, recall that for a set $W \subset M$, we denote $\mathcal{E}_U(W) = \{\mathcal{E}_U(q);\ q \in W\} \subset 2^U$. Also, we may assume below that ϑ_1 is so small that $\gamma_{y,\zeta}([0, t_0]) \cap J(p^-, p^+) \subset \mathcal{K}_{t_0}$ when $y \in J(p^-, p^+)$, $d_{g^+}(y, \widehat{\mu}) < \vartheta_1$ and $\zeta \in L_y^+M$, $\|\zeta\|_{g^+} \le 1 + \vartheta_1$.

Let $s_0 \in [s_-, s_+]$ be so close to s_+ that $J^+(\widehat{\mu}(s_0)) \cap J^-(p^+) \subset \mathcal{K}_{t_0}$. Then the data given in the claim determine $\mathcal{E}_U(J^+(\widehat{\mu}(s_0)) \cap J^-(p^+))$.

4.2.1. *Determination of the time when a geodesic is observed to enter in to the already reconstructed set.* Let us describe the rough idea of the construction that we do next: We will consider the point $p = \gamma_{y,\zeta}(r)$ where a geodesic $\gamma_{y,\zeta}$ enters for the first time (see Fig. 7) in the "already reconstructed" set $J^+(\widehat{x}_1)\cap J^-(p^+)$ or exits the set $J^-(p^+)$. In particular we consider the time $s = \mathbb{S}(y,\zeta,s_1)$ where the light coming from the point p is observed on $\widehat{\mu}$ for the first time. The time $\mathbb{S}(y,\zeta,s_1)$ will be essential for us as we are sure that before this time we do not observe on $\widehat{\mu}$ any strange signals that caustics may have produced. The idea to find $\mathbb{S}(y,\zeta,s_1)$ is that when $r' > r$ is close to r, the observations from an artificial point sources produced at the point $\gamma_{y,\zeta}(r')$ coincide with some of the observations that we have made earlier. Next we present details of this construction.

We recall the notation that $\widehat{\mu} = \mu_{\widehat{a}}$ and we use $s_+ < s_{+2} < 1$ such that $p^+ = \widehat{\mu}(s_+)$.

Next we consider $\vartheta \in (0,\vartheta_1)$, the points $x_j \in U$ and the directions $\xi_j \in L^+_{x_j}M_0$, denoted by $(\vec{x},\vec{\xi}) = ((x_j,\xi_j))_{j=1}^4$, that satisfy, see Fig. 5(Right),

$$(i)\ \gamma_{x_j,\xi_j}([0,t_0]) \subset U,\ \gamma_{x_j,\xi_j}(t_0) \notin J^+(\gamma_{x_k,\xi_k}(t_0)),\ \text{for } j \neq k,$$
$$(ii)\ \text{For all } j,k \leq 4,\ d_{g^+}((x_j,\xi_j),(x_k,\xi_k)) < \vartheta,$$
$$(iii)\ \text{There is } \widehat{x} \in \widehat{\mu} \text{ such that for all } j \leq 4,\ d_{g^+}(\widehat{x},x_j) < \vartheta. \tag{85}$$

Definition 4.6. *Let $s_- \leq s_2 \leq s < s_1 \leq s_+$ satisfy $s_1 < s_2 + \kappa_2$, $\widehat{x}_j = \widehat{\mu}(s_j)$, $j = 1,2$, and $\widehat{x} = \widehat{\mu}(s)$, $\widehat{\zeta} \in L^+_{\widehat{x}}U$, $\|\widehat{\zeta}\|_{g^+} = 1$. Let $(y,\zeta) \in L^+U$ be in ϑ_1-neighborhood of $(\widehat{x},\widehat{\zeta})$ such that $y \in J^+(\widehat{x}_2)$ and the geodesic $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat{\mu}$. Define*

$$r_1(y,\zeta,s_1) = \inf\{r > 0;\ \gamma_{y,\zeta}(r) \in J^+(\widehat{\mu}(s_1))\},$$
$$r_2(y,\zeta) = \inf\{r > 0;\ \gamma_{y,\zeta}(r) \in M \setminus I^-(\widehat{\mu}(s_{+2}))\},$$
$$r_0(y,\zeta,s_1) = \min(r_2(y,\zeta), r_1(y,\zeta,s_1)).$$

When $\gamma_{y,\zeta}(\mathbb{R}_+)$ intersects $J^+(\widehat{\mu}(s_1)) \cap J^-(p^+)$ we define

$$\mathbb{S}(y,\zeta,s_1) = f^+_{\widehat{a}}(q_0), \tag{86}$$

where $q_0 = \gamma_{y,\zeta}(r_0)$ and $r_0 = r_0(y,\zeta,s_1)$. In the case when $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $J^+(\widehat{\mu}(s_1)) \cap J^-(p^+)$, we define $\mathbb{S}(y,\zeta,s_1) = s_+$.

We note that above $r_2(y,\zeta)$ is finite by [60, Lemma 14.13]. Note that by the assumptions of the claim, we can check if $\gamma_{x,\xi}\cap\widehat{\mu} = \emptyset$ for a given (x,ξ).

Definition 4.7. *Let $0 < \vartheta < \vartheta_1$ and $\mathcal{D}_\vartheta(y,\zeta)$ be the set of $(\vec{x},\vec{\xi}) = ((x_j,\xi_j))_{j=1}^4$ that satisfy the conditions in the formula (85) and $(x_1,\xi_1) = (y,\zeta)$. We say that the set $S \subset U$ is a repeated observation associated* *to the geodesic* $\gamma_{y,\zeta}$ *if there is* $\widehat{\vartheta} > 0$ *such that for all* $\vartheta \in (0, \widehat{\vartheta})$ *there are* $(\vec{x}, \vec{\xi}) \in \mathcal{D}_\vartheta(y, \zeta)$ *such that* $S = S_0(\vec{x}, \vec{\xi})$.

Next we use a step-by-step construction: We consider $s_1 \in (s_-, s_+)$ and assume that we are given $\mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$ with $\widehat{x}_1 = \widehat{\mu}(s_1)$. Then, let $s_2 \in (s_1 - \kappa_2, s_1)$. Our next aim is to find the earliest light observation sets $\mathcal{E}_U(J^+(\widehat{x}_2) \cap J^-(p^+))$ with $\widehat{x}_2 = \widehat{\mu}(s_2)$.

Let us consider now four light-like future pointing directions (x_j, ξ_j), $j = 1, 2, 3, 4$, and use below the notation $(\vec{x}(t), \vec{\xi}(t))$ defined in (18).

Lemma 4.8. *Assume that* $\max(s_-, s_1 - \kappa_2) \leq s < s_1 < s^+$ *and let* $\widehat{x} = \widehat{\mu}(s)$, $\widehat{x}_1 = \widehat{\mu}(s_1)$, *and* $\widehat{\zeta} \in L^+_{\widehat{x}} M$, $\|\widehat{\zeta}\|_{g^+} = 1$. *Moreover, let* (y, ζ) *be in a* ϑ_1*-neighborhood of* $(\widehat{x}, \widehat{\zeta})$. *Assume also that the geodesic* $\gamma_{y,\zeta}(\mathbb{R}_+)$ *does not intersect* $\widehat{\mu}$. *Then,*
(A) The cut point $p_0 = \gamma_{y(t_0),\zeta(t_0)}(\mathbf{t}_*)$, $\mathbf{t}_* = \rho(y(t_0), \zeta(t_0))$ *of the geodesics* $\gamma_{y(t_0),\zeta(t_0)}([0, \infty))$, *if it exists, satisfies either*

(i) $p_0 \notin J^-(\widehat{\mu}(s_{+2}))$,

or

(ii) $r_0 = r_0(y, \zeta, s_1) < r_2(y, \zeta)$ *and* $p_0 \in I^+(\widehat{x}_1)$.

(B) There is $\vartheta_2(y, \zeta, s_1) \in (0, \vartheta_1)$ *such that if* $0 < \vartheta < \vartheta_2(y, \zeta, s_1)$, $(\vec{x}, \vec{\xi}) \in \mathcal{D}_\vartheta(y, \zeta)$, *and the geodesics* $\gamma_{x_j,\xi_j}([0, \infty))$, $j \in \{1, 2, 3, 4\}$, *has a cut point* $p_j = \gamma_{x_j(t_0),\xi_j(t_0)}(\rho(x_j(t_0), \xi_j(t_0)))$, *then the following holds:*

If either the point p_0 *does not exist or it exists and (i) holds then* $p_j \notin J^-(p^+)$. *On the other hand, if* p_0 *exists and (ii) holds, then* $f^+_{\widehat{a}}(p_j) > f^+_{\widehat{a}}(q_0)$, *where* $q_0 = \gamma_{y,\zeta}(r_0(y, \zeta, s_1))$.

Note that $f^+_{\widehat{a}}(q_0) = \mathbb{S}(y, \zeta, s_1)$.

Proof. (A) Assume that (i) does not hold, that is, $p_0 = \gamma_{y,\zeta}(t_0 + \mathbf{t}_*) \in J^-(\widehat{\mu}(s_{+2}))$. By Lemma 2.8 (iii) we have $f^-_{\widehat{a}}(p_0) > s + 2\kappa_2 \geq s_1$ that yields $p_0 \in I^+(\widehat{x}_1)$. Thus, the geodesic $\gamma_{y(t_0),\zeta(t_0)}([0, \rho(y(t_0), \zeta(t_0)))$ intersects $J^+(\widehat{x}_1) \cap J^-(\widehat{\mu}(s_{+2}))$. Hence the alternative (ii) holds with $0 < r_0 < r_2(y, \zeta)$ and moreover, $r_0 < t_0 + \rho(y(t_0), \zeta(t_0))$.

(B) If (i) holds, the claim follows since the function $(x, \xi) \mapsto \rho(x, \xi)$ is lower semi-continuous and $(x, \xi, t) \mapsto \gamma_{x,\xi}(t)$ is continuous.

In the case (ii), we saw above that $r_0 < t_0 + \rho(y(t_0), \zeta(t_0))$. Let $q_0 = \gamma_{y,\zeta}(r_0)$. Then by using a short cut argument and the fact that $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat{\mu}$ we see similarly to the above that $f^+_{\widehat{a}}(p_0) > f^+_{\widehat{a}}(q_0) = \mathbb{S}(y, \zeta, s_1)$. Since the function $(x, \xi, t) \mapsto f^+_{\widehat{a}}(\gamma_{x,\xi}(t))$ is continuous and $t \mapsto f^+_{\widehat{a}}(\gamma_{x,\xi}(t))$ is non-decreasing, and the function $(x, \xi) \mapsto \rho(x, \xi)$ is lower semi-continuous, we have that the function $(x, \xi) \mapsto f^+_{\widehat{a}}(\gamma_{x(t_0),\xi(t_0)}(\rho(x(t_0), \xi(t_0))))$ is lower semi-continuous, and the claim follows. □

We recall that below we assume that the set $\mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$ is already constructed.

Definition 4.9. *Let $s_- \le s_2 \le s < s_1 \le s_+$ satisfy $s_1 < s_2 + \kappa_2$, $\widehat{x}_j = \widehat{\mu}(s_j)$, $j = 1, 2$, and $\widehat{x} = \widehat{\mu}(s)$, $\widehat{\zeta} \in L^+_{\widehat{x}}U$, $\|\widehat{\zeta}\|_{g^+} = 1$. Also, let $(y, \zeta) \in L^+U$ be in ϑ_1-neighborhood of $(\widehat{x}, \widehat{\zeta})$ and $\mathcal{R}(y, \zeta, s_1)$ be the set of the repeated observations $S \subset U$ associated to the geodesic $\gamma_{y,\zeta}$ such that $S \in \mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$. Moreover, define $\mathbb{S}^{obs}(y, \zeta, s_1)$ to be the infimum of $s' \in [-1, s_+]$ such that $\widehat{\mu}(s') \in S \cap \widehat{\mu}$ with some $S \in \mathcal{R}(y, \zeta, s_1)$. If no such s' exists, we define $\mathbb{S}^{obs}(y, \zeta, s_1) = s^+$.*

Let us next consider $(\vec{x}, \vec{\xi}) \in \mathcal{D}_\vartheta(y, \zeta)$ where $0 < \vartheta < \vartheta_2(y, \zeta, s_1)$. Here, $\vartheta_2(y, \zeta, s_1)$ is defined in Lemma 4.8. Assume that for some $j = 1, 2, 3, 4$ we have that $\rho(x_j(t_0), \xi_j(t_0)) < \mathcal{T}(x_j(t_0), \xi_j(t_0))$, see Sec. 2.1, and consider the cut point $p_j = \gamma_{x_j(t_0),\xi_j(t_0)}(\rho(x_j(t_0), \xi_j(t_0)))$. Then either the case (i) or (ii) of Lemma 4.8 holds. If (i) holds, by Lemma 4.8 (B), p_j satisfies $p_j \notin J^-(p^+)$ and thus $f^+_{\widehat{a}}(p_j) > s_+ \ge \mathbb{S}(y, \zeta, s_1)$. If (ii) holds, there exists $r_0 = r_0(y, \zeta, s_1) < r_1(y, \zeta)$ such that $q_0 = \gamma_{y,\zeta}(r_0) \in J^+(\widehat{x}_1)$ and $f^+_{\widehat{a}}(p_j) > f^+_{\widehat{a}}(q_0) = \mathbb{S}(y, \zeta, s_1)$. Thus in both cases (i) and (ii) we have

$$f^+_{\widehat{a}}(p_j) > \mathbb{S}(y, \zeta, s_1). \tag{87}$$

We consider next a point $q = \gamma_{y,\zeta}(r) \in J^-(p^+)$, where $t_0 < r \le r_0 = r_0(y, \zeta, s_1)$. By Lemma 2.8 (iii), the geodesic $\gamma_{y,\zeta}([t_0, r])$ has no cut points.

By Lemma 3.5, we see that when $\vartheta_3 \in (0, \vartheta_2(y, \zeta, s_1))$ is small enough, for all $\vartheta \in (0, \vartheta_3)$, there is $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4 \in \mathcal{D}_\vartheta(y, \zeta)$ such that the geodesics corresponding to $(\vec{x}, \vec{\xi})$ intersect at q. As the set (U, g) is known, that for sufficiently small ϑ one can verify using $(U, g|_U)$ and the causality relation in U, if the given vectors $(\vec{x}, \vec{\xi})$ satisfy $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4 \in \mathcal{D}_\vartheta(y, \zeta)$. Also, note that as then $\vartheta < \vartheta_2(y, \zeta, s_1)$, the inequality (87) yields $\widetilde{x} = \widehat{\mu}(\mathbb{S}(y, \zeta, s_1)) \in \mathcal{N}(\vec{x}, \vec{\xi})$ and thus $q \in J^-(\widetilde{x}) \subset \mathcal{N}(\vec{x}, \vec{\xi})$. Then property (P2) implies that $S_0(\vec{x}, \vec{\xi}) = \mathcal{E}_U(q)$.

As $\vartheta \in (0, \vartheta_3)$ above can be arbitrarily small, we see for any $q = \gamma_{y,\zeta}(r) \in J^-(p^+)$ where $t_0 < r \le r_0 = r_0(y, \zeta, s_1)$, we have

$$\begin{aligned} &S = \mathcal{E}_U(q) \text{ is a repeated observation associated to } \gamma_{y,\zeta} \text{ and} \\ &S \cap \widehat{\mu} = \{\widehat{\mu}(\widehat{s})\},\ \widehat{s} := f^+_{\widehat{a}}(q) \le \mathbb{S}(y, \zeta, s_1). \end{aligned} \tag{88}$$

Lemma 4.10. *Assume that $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat{\mu}$. Then we have $\mathbb{S}^{obs}(y, \zeta, s_1) = \mathbb{S}(y, \zeta, s_1)$.*

Proof. Let us first prove that $\mathbb{S}^{obs}(y, \zeta, s_1) \ge \mathbb{S}(y, \zeta, s_1)$. To this end, let $s' < \mathbb{S}(y, \zeta, s_1)$ and $x' = \widehat{\mu}(s')$. Assume $S \in \mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$ is a repeated observation associated to the geodesic $\gamma_{y,\zeta}$ and $S \cap \widehat{\mu} = \{\widehat{\mu}(s')\}$. Let $q \in J^+(\widehat{x}_1) \cap J^-(p^+)$ be such that $S = \mathcal{E}_U(q)$.

Then for arbitrarily small $0 < \vartheta < \vartheta_2(y,\zeta,s_1)$ there is $(\vec x,\vec \xi) \in \mathcal{D}_\vartheta(y,\zeta)$ satisfying $S_0(\vec x,\vec \xi) = S$. Let $\mathcal{N} = \mathcal{N}(\vec x,\vec \xi)$. Then by (87), we have $x' \in \mathcal{N}$.

If the geodesics corresponding to $(\vec x,\vec \xi)$ intersect at some point $q' \in J^-(x')$, then by (P1) we have $S = \mathcal{E}_U(q')$. Recall that $\widehat\mu = \mu_{\widehat a}$. Then $\widehat\mu(s') = \widehat\mu(f^+_{\widehat a}(q'))$ implying that $f^+_{\widehat a}(q') = s'$. Moreover, we have then that $\mathcal{E}_U(q) = \mathcal{E}_U(q')$ and Proposition 2.2 yields $q' = q$. Since $(\vec x,\vec \xi) \in \mathcal{D}_\vartheta(y,\zeta)$ implies $(x_1,\xi_1) = (y,\zeta)$, we see that $q' \in \gamma_{y,\zeta}([t_0,\infty))$. As $q \in J^+(\widehat x_1)$, we have that $q = q' \in \gamma_{y,\zeta}([t_0,\infty)) \cap J^+(\widehat x_1) = \gamma_{y,\zeta}([r_0(y,\zeta,s_1),\infty))$. However, then $f^+_{\widehat a}(q') \geq \mathbb{S}(y,\zeta,s_1) > s'$ and thus $S \cap \widehat\mu = \mathcal{E}_U(q) \cap \widehat\mu$ can not be equal to $\{\widehat\mu(s')\}$.

On the other hand, if the geodesics corresponding to $(\vec x,\vec \xi)$ do not intersect at any point in $J^-(x') \subset \mathcal{N}$, then either they intersect in some $q_1' \in (M \setminus J^-(x')) \cap \mathcal{N}$, they do not intersect at all, or they intersect at $q_2' \in M \setminus \mathcal{N}$. In the first case, $S = \mathcal{E}_U(q_1')$ do not satisfy $S \cap \widehat\mu \in \widehat\mu((-1,s'))$. In the other cases, property (P2) yields $S_0(\vec x,\vec \xi) \cap \mathcal{N} = \emptyset$. As $x' = \widehat\mu(s') \in \mathcal{N}$, we see that $S \cap \widehat\mu$ can not be equal to $\{\widehat\mu(s')\}$. Since above $s' < \mathbb{S}(y,\zeta,s_1)$ is arbitrary, this shows that $\mathbb{S}^{obs}(y,\zeta,s_1) \geq \mathbb{S}(y,\zeta,s_1)$.

Let us next show that $\mathbb{S}^{obs}(y,\zeta,s_1) \leq \mathbb{S}(y,\zeta,s_1)$. Assume the opposite. Then, if $\mathbb{S}(y,\zeta,s_1) = s_+$, we see by Def. 4.9 that $\mathbb{S}^{obs}(y,\zeta,s_1) = \mathbb{S}(y,\zeta,s_1)$ which leads to a contradiction. However, if $\mathbb{S}(y,\zeta,s_1) < s_+$, by Def. 4.6, we have (87). This implies the existence of $q_0 = \gamma_{y,\zeta}(r_0)$, $r_0 = r_0(y,\zeta,s_1)$ such that $q_0 \in J^+(\widehat x_1) \cap J^-(p^+)$ and by (88), $S = \mathcal{E}_U(q_0)$ is a repeated observation associated to the geodesic $\gamma_{y,\zeta}$. By Lemma 4.8 (ii), $\mathbb{S}(y,\zeta,s_1) = f^-_{\widehat a}(q_0)$ which implies, by Def. 4.9, that $\mathbb{S}^{obs}(y,\zeta,s_1) \leq \mathbb{S}(y,\zeta,s_1)$. Thus, $\mathbb{S}^{obs}(y,\zeta,s_1) = \mathbb{S}(y,\zeta,s_1)$. □

The above means that the function $\mathbb{S}(y,\zeta,s_1) = \mathbb{S}^{obs}(y,\zeta,s_1)$ can be reconstructed from the data given in the claim.

4.2.2. *Construction of the family of the earliest light observation sets.* Next we will collect together all $\mathcal{E}_U(q)$ where q is in an appropriate geodesic segment.

Lemma 4.11. *Let $s_- \leq s_2 \leq s < s_1 \leq s_+$ with $s_1 < s_2 + \kappa_2$, let $\widehat x_j = \widehat\mu(s_j)$, $j = 1,2$, and $\widehat x = \widehat\mu(s)$, $\widehat\zeta \in L^+_{\widehat x}U$, $\|\widehat\zeta\|_{g^+} = 1$. Let $(y,\zeta) \in L^+U$ be in the ϑ_1-neighborhood of $(\widehat x,\widehat\zeta)$ such that $y \in J^+(\widehat x_2)$. Assume that $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat\mu$. Then, under the assumptions of Theorem 4.5 we can determine the family $\{\mathcal{E}_U(q);\ q \in G_0(y,\zeta,s_1)\}$, where $G_0(y,\zeta,s_1) = \{q \in \gamma_{y,\zeta}([t_0,\infty)) \cap (I^-(p^+) \setminus J^+(\widehat x_1))\}$.*

Proof. Let $s' = \mathbb{S}(y,\zeta,s_1)$, $x' = \widehat\mu(s')$, and Σ be the set of all repeated observations S associated to the geodesic $\gamma_{y,\zeta}$ such that S intersects $\widehat\mu([-1,s'))$.

Let $q = \gamma_{y,\zeta}(r) \in G_0(y,\zeta,s_1)$. Since $\gamma_{y,\zeta}(\mathbb{R}_+)$ does not intersect $\widehat\mu$, using a short cut argument for the geodesics from q to $q_0 = \gamma_{y,\zeta}(r_0(y,\zeta,s_1))$

and from q_0 to x', we see that $f^-_{\widehat a}(q) < s'$. Then, $q \in I^-(p^+) \setminus J^+(\widehat x_1)$ and $r < r_0(y, \zeta, s_1)$, and we see using (88) that $S = \mathcal{E}_U(q)$ is a repeated observation associated to the geodesic $\gamma_{y,\zeta}$ and $S \cap \widehat\mu = \{\widehat\mu(f^-_{\widehat a}(q))\}$ with $f^-_{\widehat a}(q) < s'$. Thus $\mathcal{E}_U(q) \in \Sigma$ and we conclude that $\mathcal{E}_U(G_0(y, \zeta, s_1)) \subset \Sigma$.

Next, suppose $S \in \Sigma$. Then there is $\widehat\vartheta \in (0, \vartheta_2(y, \zeta, s_1))$ such that for all $\vartheta \in (0, \widehat\vartheta)$ there is $(\vec x, \vec\xi) \in \mathcal{D}_\vartheta(y, \zeta)$ so that $S_0(\vec x, \vec\xi) = S$. Observe that by (87) we have $\widehat\mu([-1, s')) \subset J^-(x') \subset \mathcal{N}(\vec x, \vec\xi)$.

First, consider the case when the geodesics corresponding to $(\vec x, \vec\xi)$ do not intersect at any point in $I^-(x')$. Then properties (P1) and (P2) yield that $S_0(\vec x, \vec\xi)$ is either empty or does not intersect $I^-(x')$. Thus $S \cap I^-(x') = S_0(\vec x, \vec\xi) \cap I^-(x')$ is empty and S does not intersect $\widehat\mu([-1, s'))$. Hence S cannot be in Σ.

Second, consider the case when the geodesics corresponding to $(\vec x, \vec\xi)$ intersect at some point $q \in I^-(x') \subset \mathcal{N}(\vec x, \vec\xi)$. Then, property (P1) yields $S = \mathcal{E}_U(q)$. Since $(\vec x, \vec\xi) \in \mathcal{D}_\vartheta(y, \zeta)$ implies $(x_1, \xi_1) = (y, \zeta)$, the intersection point q has a representation $q = \gamma_{x_1,\xi_1}(r)$. As $q \in I^-(x')$, this yields $q \in G_0(y, \zeta, s_1)$ and $S \in \mathcal{E}_U(G_0(y, \zeta, s_1))$. Hence $\Sigma \subset \mathcal{E}_U(G_0(y, \zeta, s_1))$.

Combining the above arguments, we conclude that $\Sigma = \mathcal{E}_U(G_0(y, \zeta, s_1))$. As Σ is determined by the given data, the claim follows. □

Now we can complete the proof of Theorem 4.5. Let $B(s_2, s_1)$ be the set of all (y, ζ, t) such that there are $\widehat x = \widehat\mu(s)$, $s \in [s_2, s_1)$, $\widehat x_j = \widehat\mu(s_j)$, $j = 1, 2$, and $\widehat\zeta \in L^+_{\widehat x}U$, $\|\widehat\zeta\|_{g^+} = 1$ so that $(y, \zeta) \in L^+U$ in ϑ_1-neighborhood of $(\widehat x, \widehat\zeta)$, $y \in J^+(\widehat x_2)$, and $t \in [t_0, r_0(y, \zeta, s_1)]$. Moreover, let $B_0(s_2, s_1)$ be the set of all $(y, \zeta, t) \in B(s_2, s_1)$ such that $t < r_0(y, \zeta, s_1)$ and $\gamma_{y,\zeta}(\mathbb{R}_+) \cap \widehat\mu = \emptyset$. Using Lemma 4.11 we then can determine the collection $\Sigma_0(s_2, s_1) := \{\mathcal{E}_U(q);\ q = \gamma_{y,\zeta}(t),\ (y, \zeta, t) \in B_0(s_2, s_1)\}$. We denote also $\Sigma(s_2, s_1) := \{\mathcal{E}_U(q);\ q = \gamma_{y,\zeta}(t),\ (y, \zeta, t) \in B(s_2, s_1)\}$.

Recall that the sets $\mathcal{E}_U(q) \subset U$, where $q \in J := J^-(p^+) \cap J^+(p^-)$, can be identified with the continuous function, $F_q : \overline{\mathcal{A}} \to \mathbb{R}$, $F_q(a) = f^+_a(q)$, c.f. (13). When we endow the set $C(\overline{\mathcal{A}})$ of continuous maps $\overline{\mathcal{A}} \to \mathbb{R}$ with the topology of uniform convergence, Lemma 2.3 yields that $F : q \mapsto F_q$ is continuous map $F : J \to C(\overline{\mathcal{A}})$. By Proposition 2.2, $F : J \to F(J)$ is homeomorphism. Next, we identify $\mathcal{E}_U(q)$ and F_q. Also, on the space $\mathcal{E}_U(J)$ we will use the topology that makes the map $\mathcal{E}_U \circ F^{-1} : F(J) \to \mathcal{E}_U(J)$ a homeomorphism.

Using standard results of differential topology, we have that any neighborhood of $(y, \zeta) \in L^+U$ contains $(y', \zeta') \in L^+U$ such that the geodesic $\gamma_{y',\zeta'}([0, \infty))$ does not intersect $\widehat\mu$. Since $(y, \zeta) \mapsto r_0(y, \zeta, s_1)$ is lower semicontinuous, this implies that $\Sigma_0(s_2, s_1)$ is dense in $\Sigma(s_2, s_1)$. Hence we obtain the closure $\overline{\Sigma}(s_2, s_1)$ of $\Sigma(s_2, s_1)$ as the limits points of $\Sigma_0(s_2, s_1)$.

Then, we obtain the set $\mathcal{E}_U(J^+(\widehat{\mu}(s_2)) \cap J^-(p^+))$ as the union $\overline{\Sigma}(s_2, s_1) \cup \mathcal{E}_U(J^+(\widehat{\mu}(s_1)) \cap J^-(p^+)) \cup \mathcal{E}_U(\mathcal{K}_{t_0} \cap J^+(\widehat{\mu}(s_2)))$, see (84).

Let $s_0, \dots, s_K \in [s_-, s_+]$ be such that $s_j > s_{j+1} > s_j - \kappa_2$ and $s_K = s_-$. Then, by iterating the above construction so that the values of the parameters s_1 and s_2 are replaced by s_j and s_{j+1}, respectively, we can construct the set $\mathcal{E}_U(J^+(\widehat{\mu}(s_-)) \cap J^-(\widehat{\mu}(s_+)))$.

Moreover, similarly to the above construction, we can find the sets $\mathcal{E}_U(J^+(\widehat{\mu}(s')) \cap J^-(\widehat{\mu}(s'')))$ for all $s_- < s' < s'' < s_+$, and taking their union, we construct the set $\mathcal{E}_U(I(\widehat{\mu}(s_-), \widehat{\mu}(s_+))$. This proves Theorem 4.5 (i). As the claim (ii) was already proven earlier, this finishes the proof of Theorem 4.5. □

After Theorem 1.2 is proven, Theorem 1.5 will follow from Lemma 4.3 and Theorem 4.5. Thus we consider next the proof of Theorem 1.2 and return later to the proof of Theorem 1.5.

5. Solution of the inverse problem for passive observations

In this section we prove Theorem 1.2. Earlier we have shown that the conformal type of $(U, g|_U)$ and the family $\mathcal{E}_U(W)$ determine the family of the earliest observation times $\mathcal{F}(W) \subset C(\overline{\mathcal{A}})$. In the proof of Proposition 2.2 we have shown that the topology of $C(\overline{\mathcal{A}})$ induces on the set $\mathcal{F}(W)$ a topology that makes it homeomorphic to W. In this section we construct on $\mathcal{F}(W)$ smooth coordinates and show that then $\mathcal{F}(W)$ is diffeomorphic to W. After this we construct a metric on $\mathcal{F}(W)$ that makes it conformal to $(W, g|_W)$.

The proof is constructive and to simplify the notations, we do the constructions on just one Lorentzian manifold, (M, g) and assume that we are given the data

$$\begin{aligned} &\text{the differentiable manifold } U, \text{ the conformal class of } g|_U, \\ &\text{the paths } \mu_a : [-1, 1] \to U,\ a \in \mathcal{A}, \text{ and the set } \mathcal{E}_U(W), \end{aligned} \tag{89}$$

where W is a relatively compact open set such that $\overline{W} \subset I^-(p^+) \setminus J^-(p^-)$.

In this section we do not use the functions f_a^- and denote

$$f_a(x) = f_a^+(x).$$

Also we use the notations defined in Section 2.

5.1. Construction of the differentiable structure.

Let us next consider the set $\mathcal{Z} = \{(q, p) \in W \times U;\ p \in \mathcal{E}_U^{reg}(q)\}$. For every $(q, p) \in \mathcal{Z}$ there is a unique $\xi \in L_q^+ M$ such that $\gamma_{q,\xi}(1) = p$ and $\rho(q, \xi) > 1$. We will denote $\Theta(q, p) = (q, \xi)$ that defines a map by $\Theta : \mathcal{Z} \to L^+W$. Below, let $\mathcal{W}_\varepsilon(q_0, \xi_0) \subset TM$ be an ε-neighborhood of (q_0, ξ_0) with respect to the Sasaki-metric induced by g^+ on TM.

Lemma 5.1. *Let* $(q_0, p_0) \in \mathcal{Z}$ *and* $(q_0, \xi_0) = \Theta(q_0, p_0)$. *When* $\varepsilon > 0$ *is small enough, the map*

$$X : \mathcal{W}_\varepsilon(q_0, \xi_0) \to M \times M, \quad X(q, \xi) = (q, \exp_q(\xi)) \tag{90}$$

is open and defines a diffeomorphism $X : \mathcal{W}_\varepsilon(q_0, \xi_0) \to \mathcal{U}_\varepsilon(q_0, p_0) := X(\mathcal{W}_\varepsilon(q_0, \xi_0))$. *When* ε *is small enough,* Θ *coincides in* $\mathcal{Z} \cap \mathcal{U}_\varepsilon(q_0, p_0)$ *with the inverse map of* X. *Moreover,* $\mathcal{Z}$ *is a* $(2n-1)$*-dimensional manifold and the map* $\Theta : \mathcal{Z} \to L^+M$ *is* C^∞*-smooth.*

Proof. We start the proof with some technical considerations.

Let $p_{+2} = \widehat{\mu}(s_{+2})$, $s_+ < s_{+2} < 1$. For $(x, \xi) \in L^+M$, $x \in J^-(p^+)$, the value $T_{+2}(x, \xi) = \sup\{t \geq 0\ ;\ \gamma_{x,\xi}(t) \in J^-(p_{+2})\}$, is finite by [60, Lemma 14.13]. Since $J_2 = J^-(p_{+2})$ is closed and $\gamma_{x,\xi}$, $(x, \xi) \in L^+M$, are future-pointing paths, we have that $T_{+2} : L^+J_2 \to \mathbb{R}$ is upper semicontinuous. As $W \subset J^-(p_{+2})$ is relatively compact, the set

$$K = \{(x, \xi) \in L^+M;\ x \in \mathrm{cl}\,(W),\ \|\xi\|_{g^+} = 1\} \tag{91}$$

is compact and there is $c_0 \in \mathbb{R}_+$ such that $T_{+2}(x, \xi) \leq c_0$ for all $(x, \xi) \in K$.

Let us now start the proof of the claim. Since the geodesic $\gamma_{q_0,\xi_0}([0, 1])$ does not contain cut points and thus conjugate points, we see that when $\varepsilon > 0$ is small enough, the set $\mathcal{U}_\varepsilon(q_0, p_0) = X(\mathcal{W}_\varepsilon(q_0, \xi_0)) \subset M \times M$ is open and the map $X : \mathcal{W}_\varepsilon(q_0, \xi_0) \to \mathcal{U}_\varepsilon(q_0, p_0)$ has a C^∞-smooth inverse map $X^{-1} : \mathcal{U}_\varepsilon(q_0, p_0) \to \mathcal{W}_\varepsilon(q_0, \xi_0)$. Thus $X : \mathcal{W}_\varepsilon(q_0, \xi_0) \to \mathcal{U}_\varepsilon(q_0, p_0)$ is a diffeomorphism. Note that $X^{-1}(q_0, p_0) = (q_0, \xi_0)$.

First, we prove that $\Theta : \mathcal{Z} \to L^+W$ is continuous. If $\Theta : \mathcal{Z} \to L^+W$ would not be continuous at $(q_0, p_0) \in \mathcal{Z}$, there would exists a sequence $(q_k, p_k) \in \mathcal{Z}$ converging to (q_0, p_0) as $k \to \infty$, such that $\Theta(q_k, p_k) \in L^+M$ does not converge to $(q_0, \xi_0) = \Theta(q_0, p_0)$.

Since $p_k \in J^-(p_{+2})$ and the function T_{+2} is bounded by $c_0 \in \mathbb{R}_+$ in the set K given in (91), the sequence $\|\Theta(q_k, p_k)\|_{g^+}$ is uniformly bounded. By considering a subsequence we may assume that $\Theta(q_k, p_k) \to (q_0, \eta) \in L^+M$ as $k \to \infty$ and $\eta \neq \xi_0$. In this case the geodesics $\gamma_{q_0,\xi_0}([0, 1])$ and $\gamma_{q_0,\eta}([0, 1])$ would be two light-like geodesics connecting q_0 to p_0 so that $\rho(q_0, \xi_0) \leq 1$. This would be in contradiction with the assumption that $p_0 \in \mathcal{E}_U^{reg}(q_0)$. This shows that $\Theta : \mathcal{Z} \to L^+W$ is continuous at (q_0, p_0).

Let $\varepsilon_1 \in (0, \varepsilon)$ and $\mathcal{Z} \cap \mathcal{U}_{\varepsilon_1}(q_0, p_0)$ be a neighborhood of (q_0, p_0) in the relative topology of $\mathcal{Z} \subset W \times U$. When ε_1 is small enough, we have $\Theta(\mathcal{Z} \cap \mathcal{U}_{\varepsilon_1}(q_0, p_0)) \subset \mathcal{W}_\varepsilon(q_0, \xi_0)$. Then for $(q, p) \in \mathcal{Z} \cap \mathcal{U}_{\varepsilon_1}(q_0, p_0)$ and $(q, \xi) = \Theta(q, p) \in \mathcal{W}_\varepsilon(q_0, \xi_0)$ we have $\exp_q(\xi) = p$, and hence $X(\Theta(q, p)) = (q, p)$. Since $\Theta(q, p) \in \mathcal{W}_\varepsilon(q_0, \xi_0)$, we have $\Theta(q, p) = X^{-1}(q, p)$. Therefore for $(q, p) \in \mathcal{Z} \cap \mathcal{U}_{\varepsilon_1}(q_0, p_0)$ the function $\Theta : \mathcal{Z} \cap \mathcal{U}_{\varepsilon_1}(q_0, p_0) \to TM$ coincides with the smooth function $X^{-1} : \mathcal{Z} \cap \mathcal{U}_{\varepsilon_1}(q_0, p_0) \to TM$. Given that $(q_0, p_0) \in \mathcal{Z}$ is arbitrary, this shows that $\mathcal{Z}$ is a $(2n-1)$-dimensional manifold and $\Theta : \mathcal{Z} \to L^+M$ is C^∞-smooth. □

Proposition 5.2. *Let $q_0 \in I^-(p^+) \setminus J^-(p^-)$ and $(q_0, p_j) \in \mathcal{Z}$, $j = 1, 2, \dots, n$ and $\xi_j \in L^+_{q_0}M$ be such that $\gamma_{q_0,\xi_j}(1) = p_j$. Assume that ξ_j, $j = 1, 2, \dots, n$ are linearly independent. Then, if $a_j \in \mathcal{A}$ and $\vec{a} = (a_j)_{j=1}^n$ are such that $p_j \in \mu_{a_j}$, there is a neighborhood $V_1 \subset M$ of q_0 such that the corresponding observation time functions*

$$\mathbf{f}_{\vec{a}}(q) = (f_{a_j}(q))_{j=1}^n$$

define C^∞-smooth coordinates in V_1. Moreover, $\nabla f_{a_j}|_{q_0}$, the gradient of f_{a_j} with respect to q at q_0 satisfies $\nabla f_{a_j}|_{q_0} = c_j \xi_j$ for some $c_j \neq 0$.

Proof. Let $(q_0, p_0) \in \mathcal{Z}$ and $\xi_0 \in L^+_{q_0}M$ such that $\gamma_{q_0,\xi_0}(1) = p_0$. Moreover, let $\varepsilon > 0$ be so small that the map $X : \mathcal{W}_\varepsilon(q_0, \xi_0) \to \mathcal{U}_\varepsilon(q_0, p_0)$ has a C^∞-smooth inverse, see (90). We denote this inverse map by

$$X^{-1}(q, p) = (q, \xi(q, p)). \tag{92}$$

Recall that we denote $\mathcal{W} = \mathcal{W}_\varepsilon(q_0, \xi_0)$ and $\mathcal{U} = \mathcal{U}_\varepsilon(q_0, p_0)$.

We associate with any $(q, p) \in \mathcal{W}$ the energy $E(q, p) = E(\gamma_{q,\xi(q,p)}([0, 1]))$ of the geodesic segment $\gamma_{q,\xi(q,p)}([0, 1])$ from p to q. Here, the energy of a piecewise smooth path $\alpha : [0, l] \to M$ is defined by

$$E(\alpha) = \frac{1}{2} \int_0^l g(\dot\alpha(t), \dot\alpha(t))\, dt.$$

Observe that the sign of $E(q, p)$ depends on the causal nature of $\gamma_{q,p}$. In particular, $E(q, p) = 0$ if and only if $\xi(q, p)$ is light-like. Moreover, since X^{-1} is C^∞-smooth on $\mathcal{U}$, also $E(q, p)$ is C^∞-smooth in $\mathcal{U}$.

Let us return to consider $(q_0, p_0) \in \mathcal{Z}$ and let $a_0 \in \mathcal{A}$ be such that $p_0 \in \mu_{a_0}$. Then $p_0 = \mu_{a_0}(s_0)$ with $s_0 = f_{a_0}(q)$.

Let $V_0 \subset W$ be an open neighborhood of q_0 and $t_1, t_2 \in (s_{-2}, s_{+2})$, $t_1 < s_0 < t_2$ be such that $V_0 \times \mu_{a_0}([t_1, t_2]) \subset \mathcal{U}$. Then for $q \in V_0$ and $s \in (t_1, t_2)$ the function $\mathbf{E}_{a_0}(q, s) := E(q, \mu_{a_0}(s))$ is well defined and smooth. Using the first variation formula for $\mathbf{E}_{a_0}(q, s)$, see e.g. [60, Prop. 10.39], we obtain

$$\frac{\partial \mathbf{E}_{a_0}(q_0, s)}{\partial s}\Big|_{s=s_0} = g\left(\eta, \dot\mu_{a_0}(f_{a_0}(q_0))\right), \quad \nabla \mathbf{E}_{a_0}(q, s_0)\Big|_{q=q_0} = -\xi_0, \tag{93}$$

where $\xi_0 = \xi(q_0, p_0)$ and $\eta = \dot\gamma_{q_0,\xi_0}(1)$, see (92). Since $\dot\mu_{a_0}(s)$ is time-like and future-pointing and η is light-like and future-pointing, $\frac{\partial \mathbf{E}_{a_0}}{\partial s}(q_0, s_0) < 0$.

It follows from the implicit function theorem that there is an open neighborhood $V_{a_0} \subset V_0$ of q_0 and a smooth function $q \mapsto s(q, a_0)$ defined for $q \in V_{a_0}$ such that $s(q_0, a_0) = f_{a_0}(q_0)$ and $\mathbf{E}_{a_0}(q, s(q, a_0)) = 0$. Then

$q \mapsto s(q, a_0)$ and $q \mapsto f_{a_0}(q)$ coincide in V_{a_0}, and it follows from (93) that

$$\nabla f_{a_0}(q)\Big|_{q=q_0} = \frac{1}{c(q_0, a_0)}\xi_0,\ c(q_0, a_0) = \frac{\partial \mathbf{E}_{a_0}}{\partial s}(q_0, s))\Big|_{s=f_{a_0}(q_0)}. \tag{94}$$

Next we choose $p_1, p_2, \dots, p_n \in \mathcal{E}_U^{reg}(q_0)$ and let $\xi_1, \dots, \xi_n \in L^+_{q_0}(M)$ be such that $p_i = \gamma_{q_0,\xi_i}(1)$. We assume that $\xi_1, \dots, \xi_n \in L^+_{q_0}(M)$ are linearly independent. Moreover, let $a_j \in \mathcal{A}$ be such that $p_j \in \mu_{a_j}$ and $\vec a = (a_j)_{j=1}^n$. Finally, we denote by $q \mapsto s(q, a_j)$ the above constructed smooth functions that are defined in some neighborhoods $V_{a_j} \subset W$ of q_0

Let $V_{\vec a} = \bigcap_{j=1}^n V_{a_j}$ and consider the map

$$\mathbf{f}_{\vec a} : V_{\vec a} \to \mathbb{R}^n, \quad \mathbf{f}_{\vec a}(q) = (f_{a_1}(q), \dots, f_{a_n}(q)).$$

It follows from (94) that the map $\mathbf{f}_{\vec a}$ has an invertible differential at q_0 and, therefore, the function $\mathbf{f}_{\vec a} : V_{\vec a} \to \mathbb{R}^n$ defines a C^∞-smooth coordinate system in some neighborhood of q_0. □

5.1.1. *Properties of the C^0 and C^∞ smooth coordinates.*

Definition 5.3. *Let $\vec a = (a_j)_{j=1}^n \in \mathcal{A}^n$, $\mathcal{O} \subset \mathcal{F}(W)$ be an open set, $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$, and $\mathbf{s}_{\vec a} = \mathbf{f}_{\vec a} \circ \mathcal{F}^{-1}$. We say that $(\mathcal{O}, \mathbf{s}_{\vec a})$ are C^0-observation coordinates on $\mathcal{F}(W)$ if the map $\mathbf{s}_{\vec a} : \mathcal{O} \to \mathbb{R}^n$ is an open and injective map. Also, we say that $(\mathcal{O}, \mathbf{s}_{\vec a})$ are C^∞-observation coordinates on $\mathcal{F}(W)$ if $\mathbf{s}_{\vec a} \circ \mathcal{F} : \mathcal{F}^{-1}(\mathcal{O}) \to \mathbb{R}^n$ are C^∞-smooth local coordinates on $W \subset M$, see Fig. 1(Right).*

Note that, by the invariance of domain theorem, the above continuous map $\mathbf{s}_{\vec a} : \mathcal{O} \to \mathbb{R}^n$ is open if it is injective. Even though for a given $\vec a \in \mathcal{A}^n$ there are several sets $\mathcal{O}$ for which $(\mathcal{O}, \mathbf{s}_{\vec a})$ form C^0-observation coordinates, to clarify the notations, we sometimes denote the coordinates $(\mathcal{O}, \mathbf{s}_{\vec a})$ by $(\mathcal{O}_{\vec a}, \mathbf{s}_{\vec a})$.

Since $\mathcal{F} : W \to \mathcal{F}(W)$ is a homeomorphism, we can determine all C^0-observation coordinates on $\mathcal{F}(W)$ using data (89). Next we will consider $\mathcal{F}(W)$ as a topological manifold endowed with the C^0-observation coordinates and denote $\mathcal{F}(W) = \widetilde W$. We denote the points of this manifold by $\widetilde q = \mathcal{F}(q)$. Next we construct a differentiable structure on $\widetilde W$ that is compatible with that of W.

5.1.2. *Construction of the C^∞ smooth coordinates.*

Lemma 5.4. *Assume that we are given data (89). Then for any C^0-observation coordinates $(\mathcal{O}_{\vec a}, \mathbf{s}_{\vec a})$ with $\vec a \in \mathcal{A}^n$ we can determine if $(\mathcal{O}_{\vec a}, \mathbf{s}_{\vec a})$ are C^∞-observation coordinates on $\widetilde W$. Moreover, for any $\widetilde q \in \widetilde W$ there exists C^∞-observation coordinates $(\mathcal{O}_{\vec a}, \mathbf{s}_{\vec a})$ such that $\widetilde q \in \mathcal{O}_{\vec a}$.*

Proof. Let $q \in W$. We say that $p \in \mathcal{E}_U(q)$ and $a \in \mathcal{A}$ are associated if $p \in \mu_a$. Next, consider $p \in \mathcal{E}_U^{reg}(q)$ and $a \in \mathcal{A}$ that are associated. Note that then $q \notin \mu_a$. By (94), the function $f_a(q)$ satisfies

$$\nabla f_a(q) = c(q,a)\,\xi(q,y), \quad c(q,a) \neq 0,$$

where $y = \mu_a(f_a^+(q))$, see (92). Let

$$K(q) = \{(\xi_j)_{j=1}^n;\ \xi_j \in L_q^+ M,\ \rho(q,\xi_j) > 1,\ \gamma_{q,\xi_j}(1) \in U\}$$

and $H : K(q) \to U^n$ be the map $H((\xi_j)_{j=1}^n) = (p_j)_{j=1}^n$, where $p_j = \gamma_{q,\xi_j}(1)$. Then $p_j \in \mathcal{E}_U^{reg}(q)$ and $\xi_j = \Theta(q,p_j)$. Given that ρ is lower semi-continuous, we have that $K(q) \subset (L_q^+ M)^n$ is open. Clearly, H is C^∞-smooth. Since $\Theta : \mathcal{Z} \to L^+W$ is continuous and injective, we see that $H : K(q) \to H(K(q)) = (\mathcal{E}_U^{reg}(q))^n$ is a homeomorphism. We denote below $Y(q) = (\mathcal{E}_U^{reg}(q))^n$. Note that for all $\widetilde{q} \in \widetilde{W}$ the data (89) determine the set $Y(q) \subset U^n$, where $q = \mathcal{F}^{-1}(\widetilde{q})$.

Let us consider the set

$$K_0(q) = \{(\xi_j)_{j=1}^n \in K(q);\ \xi_j,\ j = 1,\dots,n \text{ are linearly independent}\}.$$

Clearly, the set $K_0(q)$ is dense and open in $K(q)$, and hence $Y_0(q) := H(K_0(q))$ is open and dense in $Y(q)$.

Let $(\mathcal{O}_{\vec{a}}, \mathbf{s}_{\vec{a}})$, $\vec{a} \in \mathcal{A}^n$ be C^0-observation coordinates on $\widetilde{W}$, $\widetilde{q} \in \mathcal{O}_{\vec{a}}$, and $q = \mathcal{F}^{-1}(\widetilde{q})$. Also, let $(p_j)_{j=1}^n \in Y(q)$ be such that the p_j's are associated with a_j. Similarly, let $(\mathcal{O}_{\vec{b}}, \mathbf{s}_{\vec{b}})$, $\vec{b} \in \mathcal{A}^n$ be another C^0-observation coordinates on $\widetilde{W}$ such that $\widetilde{q} \in \mathcal{O}_{\vec{b}}$, and let $(z_j)_{j=1}^n \in Y(q)$ be such that z_j are associated with b_j. Note that then $p_j = \mathcal{E}_{a_j}^{reg}(q)$ and $z_j = \mathcal{E}_{b_j}^{reg}(q)$.

In the case when $z_j)_{j=1}^n \in Y_0(q)$, q has a neighborhood $V_1 \subset W$ in which the function $\mathbf{f}_{\vec{b}} : V_1 \to \mathbb{R}^n$ give C^∞-smooth local coordinates. Thus, if $(z_j)_{j=1}^n \in Y_0(q)$, then it holds that $(p_j)_{j=1}^n \in Y_0(q)$ if and only if

(*i*)The functions $s_{a_j} \circ \mathbf{s}_{\vec{b}}^{-1}$, $j = 1,2,\dots,n$ are C^∞-smooth at $\mathbf{s}_{\vec{b}}(\widetilde{q})$ and the Jacobian determinant $\det(D(\mathbf{s}_{\vec{a}} \circ \mathbf{s}_{\vec{b}}^{-1}))$ at $\mathbf{s}_{\vec{b}}(\widetilde{q})$ is non-zero.

Denote $\vec{p} = (p_j)_{j=1}^n \in Y(q)$, and define $\mathcal{X}_{\vec{p}} \subset Y(q)$ to be the set of those $(z_j)_{j=1}^n \in Y(q)$, for which there are $\vec{b} \in \mathcal{A}^n$ and C^0-observation coordinates $(\mathcal{O}_{\vec{b}}, \mathbf{s}_{\vec{b}})$ such that $\widetilde{q} = \mathcal{F}(q) \in \mathcal{O}_{\vec{b}}$, z_j are associated with b_j for $j = 1,2,\dots,n$, and the condition (i) is satisfied. If $\vec{p}$ is in $Y_0(q)$, we see that $Y_0(q) \subset \mathcal{X}_{\vec{p}}$. On the other hand, if $\vec{p}$ is not in $Y_0(q)$, we have $Y_0(q) \cap \mathcal{X}_{\vec{p}} = \emptyset$. Since the set $Y_0(q)$ is open and dense in $Y(q)$, we observe that $\vec{p} \in Y_0(q)$ if and only if the interior of set $\mathcal{X}_{\vec{p}}$ is a dense subset of $Y(q)$. This in particular implies that using the data (89) we can determine whether $(p_j)_{j=1}^n$ is in $Y_0(q)$ or not. As we can do the above considerations for all $\widetilde{q} \in \mathcal{O}_{\vec{a}}$, we get that the C^0-observation coordinates $(\mathcal{O}_{\vec{a}}, \mathbf{s}_{\vec{a}})$, $\vec{a} \in \mathcal{A}^n$ are C^∞-observation coordinates on $\widetilde{W}$ if and only if for all $\widetilde{q} \in \mathcal{O}_{\vec{a}}$, $q = \mathcal{F}^{-1}(\widehat{q})$, and $p_j = \mu_{a_j}(f^+_{a_j}(q))$, $j = 1, 2, \dots, n$ we have $(p_j)_{j=1}^n \in Y_0(q)$. Thus we can determine all C^0-observation coordinates $(\mathcal{O}_{\vec{a}}, \mathbf{s}_{\vec{a}})$ on $\widetilde{W}$ that are C^∞-observation coordinates. Moreover, since for all $q \in W$ the set $Y_0(q)$ is non-empty, we obtain that any $\widetilde{q} = \mathcal{F}(q) \in \widetilde{W}$ belongs in the domain some C^∞-observation coordinates. □

We endow $\widetilde{W} = \mathcal{F}(W)$ with the differentiable structure provided by all C^∞-observation coordinates on $\widetilde{W}$. By Lemma 5.4 and [60, Lemma 1.42] the C^∞-observation coordinates make $\widetilde{W}$ a differentiable manifold and its the differentiable structure is uniquely determined. Since the differentiable structure of W is determined by the functions $f_{\vec{a}}$ that are C^∞-smooth local coordinates, we have using Def. 5.3 that the map

$$\mathcal{F} : W \to \widetilde{W} = \mathcal{F}(W) \tag{95}$$

is a diffeomorphism.

5.2. **Construction of the conformal type of the metric.** Let us denote by $\widetilde{g} = \mathcal{F}_* g$ the metric on $\widetilde{W} = \mathcal{F}(W)$ that makes $\mathcal{F}$ an isometry. Next we show that the set $\mathcal{F}(W)$, the paths μ_a and the conformal class of the metric g on U determine the conformal class of $\widetilde{g}$ on $\widetilde{W}$.

Lemma 5.5. *The data (89) determines a metric G on $\widetilde{W} = \mathcal{F}(W)$ that is conformal to $\widetilde{g}$ and the time orientation on $\widetilde{W}$ that makes $\mathcal{F} : W \to \widetilde{W}$ a causality preserving map.*

Proof. Let $(\mathcal{O}_{\vec{a}}, \mathbf{s}_{\vec{a}})$ be C^∞-observation coordinates on $\widetilde{W}$. Then by (94) the co-vectors $-ds_{a_1}|_{\widetilde{q}}$ and $-ds_{a_2}|_{\widetilde{q}}$ are non-parallel future-pointing light-like co-vectors. Thus their sum determines a future-pointing time-like co-vector field on $\mathcal{O}_{\vec{a}}$. Using a suitable partition of unity we can construct a future-pointing time-like co-vector field X on $\widetilde{W}$.

Let $(\mathcal{O}_{\vec{a}}, \mathbf{s}_{\vec{a}})$ be C^∞-observation coordinates on $\widetilde{W}$. Let $\widetilde{q} \in \mathcal{O}_{\vec{a}}$ and $q \in W$ be such that $\widetilde{q} = \mathcal{F}(q)$. Using the data (89), the function $F_q = \mathcal{F}(q) : \mathcal{A} \to \mathbb{R}$, and the formula (14, we can determine the set $\mathcal{E}_U(q) \subset U$. By Prop. 2.7 (iii), this further determines the set $\mathcal{C}_U^{reg}(q)$.

Then, let us fix a point $\widetilde{q} = \mathcal{F}(q) \in \mathcal{O}_{\vec{a}}$. Let $(y, \eta) \in \mathcal{C}_U^{reg}(q)$ and let $\widehat{t} > 0$ be the largest number such that the geodesic $\gamma_{y,\eta}((-\widehat{t}, 0]) \subset M$ is defined and has no cut points. For $q \in W$, Proposition 2.6 (ii) yields that $q \in \gamma_{y,\eta}((-\widehat{t}, 0))$ if and only if $(y, \eta) \in \mathcal{C}_U^{reg}(q)$. Hence (y, η) and the data (89) determine the set

$$\beta = \{\widetilde{q} \in \mathcal{O}_{\vec{a}};\ \widetilde{q} = \mathcal{F}(q),\ \mathcal{C}_U^{reg}(q) \ni (y, \eta)\} = \mathcal{F}(\gamma_{y,\eta}((-\widehat{t}, 0))) \cap \mathcal{O}_{\vec{a}}.$$

This implies that on $\mathcal{O}_{\vec{a}} \subset \widetilde{W}$ we can find the image, in the map $\mathcal{F}$, of the light-like geodesic segment $\gamma_{y,\eta}((-\widehat{t}, 0)) \cap \mathcal{F}^{-1}(\mathcal{O}_{\vec{a}})$ that contains $q = \gamma_{y,\eta}(-t_1)$. Let $\alpha(s)$, $s \in (-s_0, s_0)$ be a smooth path on $\mathcal{O}_{\vec{a}}$ such that $\partial_s \alpha(s)$ does not vanish, $\alpha((-s_0, s_0)) \subset \beta$, and $\alpha(0) = \widehat{q}$. Such smooth path $\alpha(s)$ can be obtained e.g. by parametrizing β by arc-length with respect to some auxiliary smooth Riemannian metric on $\mathcal{O}_{\vec{a}}$. Then $\widetilde{\xi} = \partial_s\alpha(s)|_{s=0} \in T_{\widehat{q}}\widetilde{W}$ has the form $\widetilde{\xi} = c\mathcal{F}_*(\dot{\gamma}_{y,\eta}(t_1))$ where $c \neq 0$. Since we can do the above construction for all points $(y,\eta) \in \mathcal{C}_U^{reg}(q)$, we determine in the tangent space $T_{\widehat{q}}\widetilde{W}$ the set $\Gamma = \mathcal{F}_*(\{c\xi \in L_qM;\ \exp_q(\xi) \in \mathcal{E}_U^{reg}(q),\ c \in \mathbb{R},\ c \neq 0\})$, that is an open, non-empty subset of the light cone at $\widehat{q}$ associated to the metric $\widetilde{g}$. Let us now consider the set Γ in the coordinates of $T_{\widehat{q}}\widetilde{W}$ associated to $\mathbf{s}_{\vec{a}}$. Since the light cone is determined by a quadratic equation in the tangent space, having an open set Γ of the light cone we can uniquely determine the whole light cone. Using this construction with all points $\widehat{q} \in \mathcal{O}_{\vec{a}}$, we can determine all light-like vectors in the tangent space $T_{\widehat{q}}\mathcal{O}_{\vec{a}}$ for all $\widehat{q} \in \mathcal{O}_{\vec{a}}$. The collections of light-like vectors at tangent spaces of $\widetilde{W}$ determine uniquely the conformal class of the tensor $\widetilde{g} = \mathcal{F}_*g$ in the manifold $\widetilde{W}$, see [6, Thm. 2.3] (or [6, Lemma 2.1] for a constructive procedure).

The above shows that the data (89) determines the conformal class of the metric tensor $\widetilde{g}$. In particular, we can construct a metric G on $\widetilde{W}$ that is conformal to $\widetilde{g}$ and satisfies $G(X,X) = -1$. □

We have shown that the data (89) determine the topological and the differentiable structures on $\widetilde{W} = \mathcal{F}(W)$ and a metric G on it that makes the map $\mathcal{F} : (W, g|_W) \to (\widetilde{W}, G)$ a diffeomorphism and a conformal map. Moreover, we determine the time-orientation on $\widetilde{W}$ that makes $\mathcal{F}$ a causality preserving map.

Finally, by Prop. 2.7 *(i)*, for any $y \in U$ we can verify if $y = q \in W$ and find the corresponding element $\mathcal{F}(q) \in \mathcal{F}(W)$. Thus we can find the set $\mathcal{F}(W \cap U)$ and the map $\mathcal{F}^{-1} : \mathcal{F}(W \cap U) \to W \cap U$. This yields the claim (ii) of Thm. 1.2. Thus Theorem 1.2 is proven. □

5.2.1. *Construction of the conformal factor in the vacuum spacetime.*

Proof of Corollary 1.3. By Theorem 1.2, there is a conformal diffeomorphism $\Psi : (W_1, g^{(1)}) \to (W_1, g^{(2)})$. By our assumptions, $\Phi : (V_1, g^{(1)}) \to (V_1, g^{(2)})$ is an isometry, the Ricci curvature of $g^{(j)}$ is zero in W_j, and any point $x_1 \in W_1$ is connected to some point $y_1 \in V_1 \cap W_1$ with a piecewise smooth path $\mu_{y_1,x_1}([0,1]) \subset W_1$, $\mu_{y_1,x_1}(0) = y_1$. Note that then $\Psi(\mu_{x_1,y_1}([0,1])) \subset W_2$ connects $x_2 = \Psi(x_1)$ to $y_2 = \Psi(y_1)$.

To simplify notations we denote $\widehat{g} = g^{(1)}$ and $g = \Psi^* g^{(2)}$. Since Ψ is conformal, there is $f : W_1 \to \mathbb{R}$ such that $\widehat{g} = e^{2f}g$ on W_1, and as $\Phi : V_1 \to V_2$ is an isometry, $f = 0$ in V_1. By [**?**, formula (2.73)],the Ricci tensors $\mathrm{Ric}_{jk}(g)$ of g and $\mathrm{Ric}_{jk}(\widehat{g})$ of $\widehat{g}$ satisfy on W_1

$$\begin{aligned} 0 = \mathrm{Ric}_{jk}(\widehat{g}) &= \mathrm{Ric}_{jk}(g) - 2\nabla_j\nabla_k f + 2(\nabla_j f)(\nabla_k f) \\ &\quad -(g^{pq}\nabla_p\nabla_q f + 2g^{pq}(\nabla_p f)(\nabla_q f))g_{jk} \end{aligned}$$

where $\nabla = \nabla^g$. For the scalar curvature this yields

$$0 = e^{2f}\widehat{g}^{pq}\mathrm{Ric}_{pq}(\widehat{g}) = g^{pq}\mathrm{Ric}_{pq}(g) - 3g^{pq}\nabla_p\nabla_q f.$$

Combining the above with the fact that $\mathrm{Ric}_{jk}(g) = 0$, we obtain

$$\nabla_j\nabla_k f-(\nabla_j f)(\nabla_k f)+g^{pq}(\nabla_p f)(\nabla_q f)g_{jk}=0.$$

This equation gives a system of first order ordinary differential equations for the vector field $Y = \nabla f$ along $\mu_{y_1,x_1}([0,1])$ with initial value $Y(y_1) = \nabla f(y_1) = 0$, that has the unique solution $Y = 0$. As $f(y_1) = 0$, we obtain $f(\mu_{y_1,x_1}(t)) = 0$ for $t \in [0,1]$. Since all points $x \in W_1$ are connected in W_1 to the set V_1 by piecewise smooth paths, this shows that $f = 0$. □

Finally, we are ready to complete the proof of the main theorem for active measurements.

Proof. (of Theorem 1.5) Theorem 1.5 follows from Theorem 4.5 and Theorem 1.2. □

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