Portfolio Optimization with an Envelope-based Multi-objective Evolutionary Algorithm

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Abstract

The problem of portfolio selection is a standard problem in financial engineering and has received a lot of attention in recent decades. Classical mean-variance portfolio selection aims at simultaneously maximizing the expected return of the portfolio and minimizing portfolio variance. In the case of linear constraints, the problem can be solved efficiently by parametric quadratic programming (i.e., variants of Markowitz' critical line algorithm). However, there are many real-world constraints that lead to a non-convex search space, e.g. cardinality constraints which limit the number of different assets in a portfolio, or minimum buy-in thresholds. As a consequence, the efficient approaches for the convex problem can no longer be applied, and new solutions are needed.

In this paper, we propose to integrate an active set algorithm optimized for portfolio selection into a multi-objective evolutionary algorithm (MOEA). The idea is to let the MOEA come up with some convex subsets of the set of all feasible portfolios, solve a critical line algorithm for each subset, and then merge the partial solutions to form the solution of the original non-convex problem. We show that the resulting envelope-based MOEA significantly outperforms existing MOEAs.

1 Introduction

In finance, a portfolio is a collection of assets held by an institution or a private individual. The portfolio selection problem seeks an optimal way to distribute a given budget on a set of available assets.

The problem usually has two criteria: Expected return (mean profit) is to be maximized, while the risk is to be minimized. The optimal solution depends on the user's risk aversion. As it is often difficult to weigh the two criteria before the alternatives are known, it is common to search for the whole set of efficient or Pareto-optimal solutions, i.e. all solutions that can not be improved upon in both objectives simultaneously. This set of solutions is often called "efficient frontier", or "Pareto-optimal front". Another motivation for searching for the whole efficient frontier stems from the fact that in many applications, a set of solutions with different trade-offs with respect to the optimization criteria are desired, e.g. because a bank wants to offer its customers different choices depending on their risk aversion.

A natural measure for risk is the variance of the portfolio return. Mean-variance optimization is probably the most popular approach to portfolio selection. It was introduced more than 50 years ago in the pioneering work by Markowitz (1952, 1956). The basic assumptions in his theory are a rational investor with either multivariate normally distributed asset returns or, in the case of arbitrary returns, a quadratic utility function (Huang and Litzenberger, 1988). It has been shown that if these assumptions are valid, the optimal portfolio for the investor lies on the mean-variance efficient frontier.

The basic mean-variance portfolio selection problem we consider in this paper can be formalized as follows:

$$\min V(\boldsymbol{w}) = \boldsymbol{w}^T \boldsymbol{Q} \boldsymbol{w} \qquad \text{(minimize portfolio variance)} \tag{1a}$$

$$\max \boldsymbol{w}^T \boldsymbol{\mu} = E \qquad \text{(maximize expected return)} \tag{1b}$$

subject to

$$\boldsymbol{w}^T \boldsymbol{e} = 1 \qquad \text{(budget constraint)} \tag{1c}$$

$$0 \leq w_i \leq 1 \qquad i = 1, \ldots, N \tag{1d}$$

where N is the number of assets to choose from, $\boldsymbol{Q}$ denotes the covariance matrix of all investment alternatives, μ_i is the expected return of asset i, and $\boldsymbol{e}$ is the unit vector. The decision variables w_i determine what share of the budget should be distributed to asset i, and Eq.1c makes sure that the sum of all w_i is equal to one, i.e. that the whole budget is invested.

Computationally efficient algorithms that calculate a single point or even the whole efficient frontier of this basic portfolio selection problem have been discussed in a large number of publications. For a basic introduction to portfolio selection see e.g. Elton et al. (2003). The algorithm to compute the whole efficient frontier (critical line algorithm) is described in detail in Markowitz (1987). The result of this algorithm is a sequence of so-called **corner portfolios**. These corner portfolios define the complete efficient frontier as all other points on the frontier are convex combinations of the two adjacent corner portfolios.

However, these highly efficient algorithms can not take into account more complex constraints that lead to a non-convex search space, henceforth called non-convex constraints. Examples for such constraints which are commonly used in mutual fund management include:

1. **Cardinality constraints** restrict the number of securities which are allowed in the portfolio. Often, the maximal number of securities in the portfolio is limited to reduce the management cost e.g. for monitoring the performance of the companies in the portfolio.

2. The **5-10-40 constraint** is based on §60(1) of the German investment law (InvG, 2004). It defines an upper limit for each individual asset and for the sum of all "heavyweights" in the portfolio. More specifically, securities of the same issuer are allowed to amount to 5% of the net asset value of the mutual fund. They are allowed to amount to 10%, however, if the total share of all assets with a share between 5% and 10% is less than 40% of the net asset value.

3. **Buy-in thresholds** demand that if an asset is included in the portfolio, at least some amount l should be bought. The main reason for such a constraint is to reduce transaction costs. In principle, this constraint could also be handled by our approach in a straightforward manner, but is not considered further in this paper.

For convex portfolio selection problems, the critical line algorithm is able to generate the complete (continuous) efficient frontier. In the presence of non-convex constraints, most approaches found in the literature are based on the ϵ-constraint method (see e.g. Miettinen (1998)): Instead of optimizing both criteria simultaneously, they set the expected return to a constant (i.e. Eq. 1b becomes a constraint), and solve the resulting quadratic optimization problem to minimize the variance. Of course, this only generates a single solution on the efficient frontier, and the algorithm has to be applied many times with different expected returns to obtain a reasonable approximation of the true efficient frontier. Furthermore, as the solution of each quadratic sub-problem is computationally complex in the presence of non-convex constraints[1], usually heuristics are used to solve the individual sub-problems, which means that the generated solutions don't necessarily lie on the true efficient frontier. Some heuristic approaches like evolutionary algorithms allow to consider multiple objectives simultaneously, generating a set of solutions approximating the efficient frontier in one run. However, they are still point-based, generating a finite set of alternative portfolios.

In this paper, we propose to combine multi-objective evolutionary algorithms with an embedded critical line algorithm to solve complex portfolio selection problems with non-convex constraints. The basic idea is to let the EA handle the non-convex constraints and to basically divide the problem into a set of problems with convex constraints, which can be solved optimally with a critical line algorithm. The overall solution to the problem is then the combination of the solutions to the individual convex problems. As we will demonstrate, the approach significantly outperforms other state-of-the-art evolutionary algorithms. Furthermore, to our knowledge, this is the first approach to non-convex portfolio selection yielding a continuous solution set, as opposed to the discrete solution set generated by the point based approaches.

Our paper is structured as follows. First, we provide a brief overview on approaches to portfolio selection subject to non-convex constraints. A short introduction to multi-objective evolutionary algorithms is provided in Section 3, followed in Section 4 by a short description of a state-of-the-art point-based MOEA which will be used as a reference later on. Our approach, called envelope-based multi-objective evolutionary algorithm (E-MOEA) is presented in Section 5. Section 6 reports on the empirical evaluation of our approach. The paper concludes in Section 7 with a summary and some ideas for future work.

[1]In fact, some variants are NP-complete, e.g. the cardinality constrained portfolios if the maximum cardinality is a fraction of the total number of assets (Bienstock, 1996)

2 Related work

The work on non-convex portfolio selection problems can be roughly divided into mixed integer approaches and metaheuristics which will be discussed in turn.

2.1 Mixed Integer Approaches

By adding integer variables, the convex problem presented in the previous section can be extended to take non-convex constraints into account. For example, buy-in thresholds and cardinality constraints can be modeled as follows:

$$\min V(\boldsymbol{w}) = \boldsymbol{w}^T \boldsymbol{Q} \boldsymbol{w} \tag{2a}$$

subject to

$$\boldsymbol{w}^T \boldsymbol{\mu} = E \tag{2b}$$

$$\boldsymbol{w}^T \boldsymbol{e} = 1 \tag{2c}$$

$$\sum_{i=1}^{n} \rho_i \leq N \tag{2d}$$

$$l\rho_i \leq w_i \leq u\rho_i \qquad i = 1, \ldots, N \tag{2e}$$

$$\rho_i \in \{0, 1\} \qquad i = 1 \ldots N \tag{2f}$$

where ρ_i is the binary variable that specifies whether asset i should be included in the portfolio, and l and u are the lower and upper bounds for the weights if the corresponding asset is included.

The modelling of the 5-10-40 constraint as MIQP is not as straightforward, and requires to triple the number of variables:

$$\min V(\boldsymbol{w}) = \boldsymbol{w}^T \boldsymbol{Q} \boldsymbol{w} \tag{3a}$$

subject to

$$\boldsymbol{w}^T \boldsymbol{\mu} = E \tag{3b}$$

$$\boldsymbol{w}^T \boldsymbol{e} = 1 \tag{3c}$$

$$\boldsymbol{w} - 0.05\boldsymbol{\rho} \leq 0.05\boldsymbol{e} \tag{3d}$$

$$\boldsymbol{t} - 0.1\boldsymbol{\rho} \leq \boldsymbol{0} \tag{3e}$$

$$\boldsymbol{t} - \boldsymbol{w} \leq \boldsymbol{0} \tag{3f}$$

$$\boldsymbol{w} + 0.1\boldsymbol{\rho} - \boldsymbol{t} \leq 0.1\boldsymbol{e} \tag{3g}$$

$$\boldsymbol{t}^T \boldsymbol{e} \leq 0.4 \tag{3h}$$

$$\boldsymbol{w}, \boldsymbol{t} \geq \boldsymbol{0} \tag{3i}$$

$$\rho_i \in \{0, 1\} \qquad i = 1, \ldots, N \tag{3j}$$

Thereby, $\rho_i = 1$ implies that asset i is allowed up to 10%, while $\rho_i = 0$ restricts asset i to 5%. If $\rho_i = 0$, Inequality 3e requires $t_i = 0$, otherwise $t_i = w_i$ by Inequalities 3e, 3f, and 3g. Inequalities 3h and 3i restrict the "heavyweights" to 10% and their sum to 40%.

Once formulated appropriately, any of the available mixed integer quadratic programming (MIQP) solvers can be used to solve the model. Software packages that contain such a solver can be found at NEOS Guide (2006), although – compared to the number of available QP-solvers – there are by far fewer programs with the required capability. In any case, there is no guarantee for getting the optimal portfolio quickly. If there is not enough time to allow the algorithm to terminate regularly, the best valid portfolio calculated so far can be used instead — a normal practice when working with mixed integer solvers. Of course, this abandons the optimality guarantee.

For further information about the mixed integer approach to the cardinality constrained problem without using an external MIQP-solver, the reader is referred to Bienstock (1996). Bienstock examined the computational complexity of the problem, tested a self-developed branch-and-cut algorithm using disjunctive cuts, and discussed some of the implementation problems that occurred.

Jobst et al. (2001) use a branch-and-bound algorithm to solve the cardinality constrained problem with buy-in thresholds as well as an index tracking problem, where a portfolio subject to cardinality constraints should perform as closely as possible to an index in terms of mean and variance. As they calculate many points on the efficient frontier, the available time is too short for individually solving each MIQP to optimality. Therefore, they limit the number of nodes in the search tree of the branch-and-bound algorithm for each given expected return E. To speed up the algorithm further, a previously calculated solution for an adjacent point is used as a warm start solution for the new value of expected return.

Advantages of the mixed integer approaches are certainly that optimality is guaranteed if the algorithm terminates. On the other hand, running time is difficult to predict, and modelling complex constraints is not always straightforward. Finally, the efficient frontier can only be calculated pointwise, a mixed integer version of the critical line algorithm is not known.

2.2 Metaheuristics

Because of the potentially extensive running times of the mixed integer approaches and their complex implementation, many researchers have turned to metaheuristics. The term *metaheuristic* usually describes a generic optimization principle that is widely applicable to many different problem domains. Very often, the basic idea behind a metaheuristic is inspired by nature, be it a physical process like simulated annealing (SA) or biologically inspired algorithms like ant colony optimization (ACO) or evolutionary algorithms (EAs). Such metaheuristics do not guarantee optimality, but it is hoped that they find reasonable solutions quickly.

There are several publications discussing the use of metaheuristics to solve non-convex portfolio selection problems. Most of them apply the metaheuristic just to solve the MIQP with a given desired expected return, i.e. to generate a single point of the efficient frontier in mean-variance objective space.

Chang et al. (2000) compare an EA, tabu search, and SA to solve portfolio selection problems where each solution has to contain a predetermined number of assets. Because none of the approaches turned out to be a clear winner, they suggest to run all three and combine the results. Schaerf (2002) improves on the work of Chang et al. by testing several neighborhood relations for tabu search. Crama and Schyns (2003) apply simulated annealing to a portfolio selection problem with cardinality constraints as well as turnover and trading restrictions. They particularly focused on ways to handle constraints, partially by enforcing feasibility, partially by introducing penalties. A hybrid between SA and EA is proposed in Maringer and Kellerer (2003). In Maringer (2002), ACO is used only to determine the relevant assets for a cardinality-constrained portfolio selection problem, while the weights are determined with a conventional QP solver. The only authors who, to our knowledge, address the 5-10-40 constraint are Derigs and Nickel (2001). They use SA for optimization, but do not provide details on how they ensure feasibility of solutions. In Derigs and Nickel (2003), they expand their previous work, but the focus is put on developing a decision support system for portfolio selection. In Ehrgott et al. (2004), a problem with five objectives and non-convex constraints is considered, but the five objectives are accumulated into one based on multiattribute utility theory. Then local search, SA, tabu search and an EA are used to solve the resulting single-objective mixed-integer problem.

Eddelbüttel (1996) uses a QP solver within a genetic algorithm to solve the index-tracking problem. Here, the GA determines which assets should be included in the portfolio, while the optimal weights are calculated by the QP solver.

Instead of setting the expected return E as constant and then solving a separate single-objective optimization problem for several E as do all of the above methods, Streichert et al. (2003, 2004a,b), apply a multi-objective evolutionary algorithm (MOEA). MOEAs are variants of EAs that exploit the fact that EAs work with a population of solutions, and simultaneously search for a set of efficient alternatives in a multi-objective setting, see also the following section. Note that while this approach generates several solutions along the efficient frontier in one run, it is still point-based (i.e. it only generates a discrete set of solutions). Since we will use this algorithm for empirical comparison with our approach, it will be discussed in more depth below.

Other publications reporting on the use of MOEAs for portfolio selection include Lin and Wang (2002) who consider roundlots[2], and Fieldsend et al. (2004), who address the cardinality issue but consider the number of allowed assets, K, as a third objective to be minimized. Armananzas and Lozano (2005) apply multi-objective variants of greedy search, SA and ACO to the cardinality-constrained portfolio selection problem. Schlottmann and Seese (2005) consider credit portfolio optimization and use a gradient-based local search within their MOEA.

General advantages of metaheuristic approaches are certainly their flexibility. It would be

[2]In a portfolio selection problem with roundlots, an asset can only be bought in multiples of a minimum lot. This problem has been proven to be NP-complete.

straightforward e.g. to use alternative, risk measures. Also, the multi-objective versions allow to generate a whole set of solutions, approximating the efficient frontier with a single run.

3 Multi-objective evolutionary algorithms

Although a detailed introduction into multi-objective evolutionary algorithms (MOEAs) would exceed the scope of this paper, in this section we will briefly introduce the main ideas which are needed as foundation for the subsequent sections. For a more detailed introduction to MOEAs, the reader is referred to Deb (2001) or Coello et al. (2002). A survey on MOEA applications in finance can be found in Schlottmann and Seese (2004).

Evolutionary algorithms are stochastic iterative optimization heuristics inspired by natural evolution. Starting with a set of candidate solutions (population), in each iteration (generation), promising solutions are selected as potential parents (mating selection), and new solutions (individuals) are created by mixing information from the parents (crossover) and slightly modifying them (mutation). The resulting offspring are then inserted into the population, replacing some old or less fit solutions (environmental selection). By continually selecting good solutions for reproduction and then creating new solutions based on the knowledge represented in the selected individuals, the solutions "evolve" and become better and better adapted to the problem to be solved, just like in nature, where the individuals become better and better adapted to their environment through the means of evolution.

The basic operations of an evolutionary algorithm can be described as follows:

$t := 0$
initialize $P(0)$
evaluate $P(0)$
WHILE (termination criterion not fulfilled) DO
    copy selected individuals into mating pool: $M(t) := s(P(t))$
    crossover: $M'(t) := c(M(t))$
    mutation: $M''(t) := m(M'(t))$
    evaluate $M''(t)$
    update population: $P(t+1) := u(P(t) \cup M''(t))$

$t := t + 1$

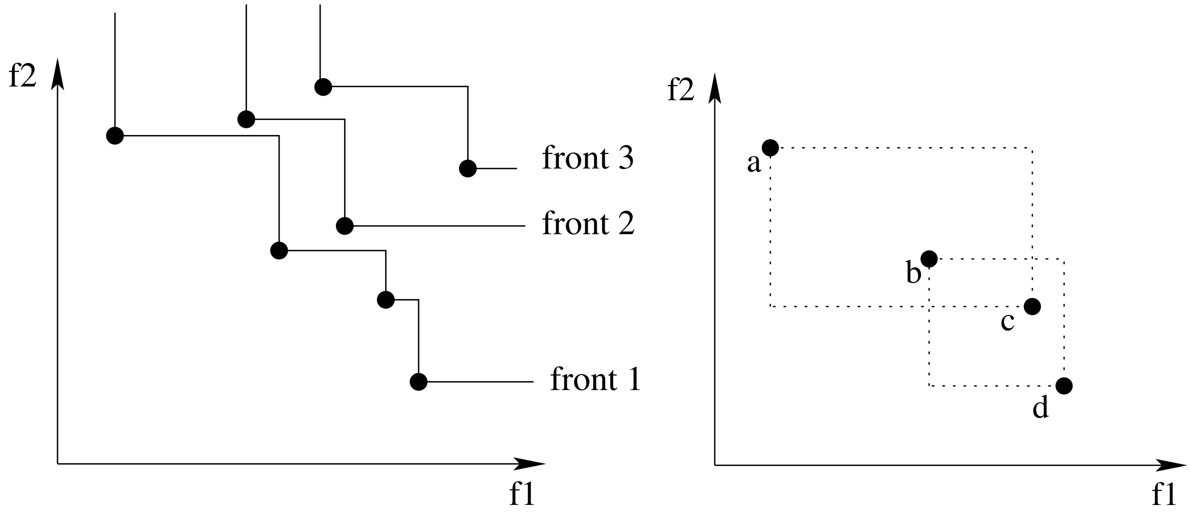
DONE

with t denoting the generation counter, $P(t)$ the population at generation t, and s, c, m, and u representing the different genetic operators. Evolutionary algorithms have proven successful in a wide variety of applications. For a more detailed introduction to EAs, the reader is referred to Eiben and Smith (2003).

Because EAs maintain a population of solutions throughout the run, they can also be used to simultaneously search for a set of solutions approximating the efficient frontier of a multi-objective problem.

The main difference between single objective EAs and MOEAs is the way they rank their solutions for selection. While in single objective EAs the ranking is unambiguously defined by the solution quality, this is not so straightforward in the case of multiple objectives. MOEAs have two goals: they want to drive the population towards the efficient frontier, while at the same time maintaining a diverse set of alternative solutions. To achieve the first goal, most MOEA implementations rely on the concept of dominance. Solution A dominates solution B if A is at least as good as B in all objectives, and better in at least one objective. A solution is called *non-dominated* with respect to a set of solutions if and only if it is not dominated by any other solution in that set.

One particularly popular MOEA variant is the non-dominated sorting genetic algorithm (NSGA-II), see Deb (2001). It ranks individuals according to two criteria. First, individuals are ranked according to non-dominancy: All non-dominated individuals are assigned to Front 1, then they are removed from the population, again all non-dominated solutions are determined and are assigned Front 2, etc. until the population is empty. The result of this process is illustrated in Figure 1(a). Within a front, solutions are ranked according to the crowding distance, which is defined as the circumference of the rectangle defined by their left and right neighbors, and infinity if there is no neighbor. This concept is illustrated in Figure 1(b). Individuals with high crowding distance are preferred, as they are in more isolated regions of the objective space. In the example in Figure 1(b), individuals a and d have the highest priority within the front, followed by individual b and then c because the rectangle defined by the respective left and right neighbor is larger for individual b.



(a) Non-dominance sorting to determine fronts

(b) Crowding distance calculation within a front

Figure 1: NSGA-II first ranks individuals according to non-dominance sorting (a) and within a front according to crowding distance (b).

In every iteration, NSGA-II generates p offspring solutions, where p is the population size. The old and offspring population are then combined, ranked according to the above two criteria, and the better half forms the new population. For mating selection, tournament selection is used which randomly draws two solutions from the population and chooses the one on the better front or, if they are from the same front, the one with the larger crowding distance.

4 A point-based multi-objective EA

Although there is, to our knowledge, no definite comparison of the different approaches discussed in Section 2, we consider the approach by Streichert et al. (2003, 2004a,b) as state-of-the-art metaheuristic. Not only is it one of the few papers applying MOEAs, but they have also tested and compared a number of variations regarding encoding and operators. We re-implemented their algorithm, and will use it as reference for empirical comparison with our newly developed envelope-based MOEA. The algorithm is described below. Note that some adaptations to the 5-10-40 constraint were necessary, as this constraint has not been considered in Streichert et al. (2003, 2004a,b).

As already mentioned in the introduction, the approach uses a MOEA for optimization. In particular, it is based on the standard non-dominated sorting genetic algorithm (NSGA-II and its predecessor NSGA) as described above.

As genetic representation, Streichert et al. (2003, 2004a,b) recommend to use a hybrid binary/real-valued encoding, which is also used by several other successful approaches like Chang et al. (2000). With this encoding, a solution is defined by a vector of continuous variables $\boldsymbol{c} = (c_1, \ldots, c_N)^T$ representing the weights of the individual assets. An additional vector of binary variables $\boldsymbol{k} = (k_1, \ldots, k_N)^T$ is used to indicate if the asset is included in the portfolio at all. The latter vector allows the EA to easily add or remove an asset by simply flipping the corresponding bit, and thus facilitates the handling of cardinality constraints.

For a portfolio selection problem that contains a cardinality constraint and buy-in thresholds, the decoding of the two vectors to get the actual portfolio works as follows (see e.g. Chang et al. (2000); Streichert et al. (2004a)):

1. All c_i are set to zero if $k_i = 0$.

2. If $\sum_{i=1}^{N} \text{sign}(k_i c_i)$ is more or less than the required cardinality, the solution is repaired by changing some elements of $\boldsymbol{c}$. For the maximum cardinality problem, elements are set to 0 in the order of increasing c_i (i.e. the assets with the smallest share are set to zero).

3. The vector $\boldsymbol{c}$ is normalized:
$$c_i' = \frac{c_i}{\sum c_i} \tag{4}$$

4. The final weight w_i is calculated:
$$w_i = \begin{cases} l + c_i' \left(1 - |\Upsilon| l\right) & \text{if } i \in \Upsilon \\ 0 & \text{otherwise} \end{cases} \tag{5}$$
where l is the minimum buy-in threshold and $|\Upsilon|$ denotes the number of elements in Υ.

The 5-10-40 constraint has not yet been considered by Streichert et al., and thus we had to adapt the decoding and repair mechanism. It works in 7 steps as follows.

1. All c_i are set to zero if $k_i = 0$.

2. The vector $\boldsymbol{c}$ is normalized:
$$w_i = \frac{c_i}{\sum c_i}$$

3. The surplus amount exceeding the 10% threshold is calculated, $\Omega = \sum_{w_i > 0.1} (w_i - 0.1)$. All $w_i > 0.1$ are set to 0.1.

4. the surplus Ω has to be redistributed to the weights below 10%. Each weight below 10% is raised by the amount $(0.1 - w_i) \cdot \frac{\Omega}{\Psi}$, where $\Psi = \sum_{0 < w_i < 0.1} (0.1 - w_i)$. If $\Psi < \Omega$, no weight will have a value above 10% after the first step.

5. In the group with more than 5%, we only accept the assets with the largest weights such that the sum is still less than or equal to 40%. All others are capped to 5% and the excess weight is distributed to the the other assets analogously to the above step.

6. If there is not enough room for all the excess weight to be redistributed among the assets with less than 5%, the remaining is used to fill up the assets between 5% and 10% up to 10% in order of decreasing weight.

7. The previous step may again lead to a violation of the 5-10-40 constraint. In this case, assets are removed from the 5-10% group in order of increasing weights, and the weight in excess of 5% is distributed to the other assets in the 5-10% group similar to the previous step.

Note that any valid portfolio for the problem with 5-10-40 constraint contains at least 16 different assets (4 times 10% plus 12 times 5%). If this is the case, the above decoding will result in a feasible portfolio. By appropriate mutation and crossover operators (see below) we make sure that the binary string has always at least 16 bits set to "1".

For both cardinality constraints and the 5-10-40 constraint, the weights after the above repair steps are written back into the weight-vector of the genotype and overwrite the original values. This allows the information gained by the repair mechanism to be inherited to the offspring (Lamarckism) and has been recommended in Streichert et al. (2004b).

For mutation, simple bit flip is used on the binary vector, and Gaussian mutation on the real-valued vector. For crossover, we use N-point crossover independently on both real-valued and binary vector. In Streichert et al. (2004a), this crossover operator was reported to be competitive to other, more complex crossover operators. We follow this suggestion for the cardinality constrained problem. For the 5-10-40 constraint, as explained above, we need at least 16 assets with $k_i = 1$ for a feasible solution. Therefore, for this constraint we modify the operators on the bit string as follows. The crossover operator first transfers all bits to the child where both parents are equal. The remaining bits are traversed in random order and randomly taken from either parent until the maximum number of zeros has been reached (i.e. $N - 16$ if N is the number of assets). Any remaining bits are set to 1. If after mutation the resulting bit string contains less than 16 ones, some of the performed 1 to 0 bit flips are reversed to make the string valid.

In the remainder of this paper, we will denote this point-based MOEA as P-MOEA.

5 Envelope-based multi-objective evolutionary algorithm

In this section, we will present our new envelope-based multi-objective evolutionary algorithm (E-MOEA) for portfolio selection problems. It combines the efficiency of the critical line algorithm for calculating the whole continuous front with the ability of multi-objective evolutionary algorithms to take complex constraints into account and to generate multiple solutions within a single run. The main idea of our paper is to use the MOEA to define suitable convex subsets of the original search space, run the critical line algorithm on every subset, and then recombine the partial solutions to form the complete front.

A single solution (individual of the MOEA) defines a convex subset by specifying how the non-convex constraints are to be handled. In case of a cardinality constraint, a solution defines which assets are allowed a weight greater than zero. The corresponding convex problem is just the standard problem which contains only those variables not forced to zero. Note that, in particular if the allowed cardinality is much smaller than the total number of available assets, this means that the generated sub-problem is much smaller than the original problem, which results in a tremendous reduction of the running time of the critical line algorithm.

For the case of the 5-10-40 constraint, a solution defines which assets are allowed up to 10% and hence have to be included in the 40% constraint. All other assets are restricted to at most 5%.

For each subset, the critical line algorithm can be used to efficiently calculate the whole efficient frontier of the corresponding standard mean-variance portfolio selection problem. For more details to the critical line algorithm, the reader is referred to Markowitz (1987). Because we apply the critical line algorithm to every individual generated by the EA, an efficient implementation is crucial. We use a modified variant of the algorithm described by Best, M. J. and Kale, J. K.

(2000). For an in-depth discussion of implementation intricacies see Stein et al. (2006).

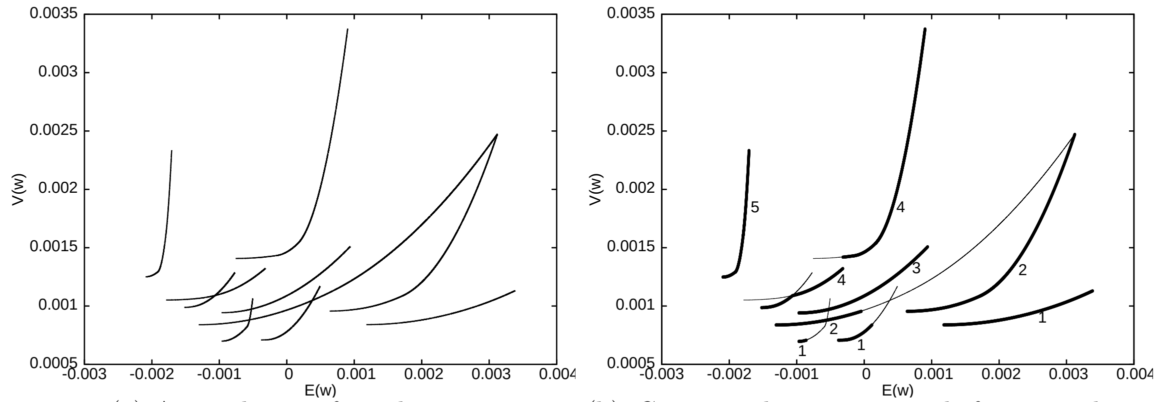
The result of the critical line algorithm is a front in the mean-variance space which is efficient for the sub-problem, but not necessarily for the overall problem with non-convex constraints. We call such a partial front an **envelope**. The EA is now used to find a collection of such envelopes which together form a solution to the overall problem.

For this purpose, we use a multi-objective EA based also on the general framework of NSGA-II (Deb, 2001). But instead of a solution being represented by a single point in the mean-variance space, now every solution is represented by an envelope in the mean-variance space. An exemplary population is depicted in Figure 2(a). The example shows that situations where envelopes entirely dominate other envelopes occur very rarely. Instead, at many points, envelopes intersect with other envelopes. Even without intersections, many envelopes have dominated and non-dominated parts. Thus, we had to adapt the non-dominated sorting and crowding distance calculation to work with envelopes. The basic idea can be described as follows.

We are interested in the non-dominated part of the set union of all envelopes. Let us denote the non-dominated part of the unit set of all envelopes as (first) **aggregated front**. Following the idea of non-dominated sorting, we assign Rank 1 to all individuals contributing at least partially to the first aggregated front. Then, we iteratively remove these individuals/envelopes from the population, and determine the aggregated front of the remaining individuals, assigning them the next higher rank, etc. The resulting ranking and the generated aggregated fronts are depicted in Figure 2(b).

It is clear that different individuals contribute differently to an aggregated front. Some may contribute only a small segment of the front, some may contribute large segments. Also, some parts of the aggregated front may be represented by several individuals. We use this information to rank the individuals within a front (substituting the crowding distance sorting in NSGA-II). For this purpose, we determine for each individual the length of the contributed segment of the aggregated front[3]. Parts common to several individuals are shared among those individuals. For example, if the part contributed solely by an individual i has length 5, and a part with length 4 is shared with another individual j, the overall contribution of individual i is $5 + 4/2 = 7$. Within a

[3]Although in principle it would be possible to calculate the true length of a segment, for reasons of simplicity we approximated the length by the Euclidean distance between the end points. Another possible criterion would have been the reduction in hypervolume if the individual is removed.



(a) A population of envelopes.

(b) Corresponding aggregated fronts and assigned ranks.

Figure 2: Ten randomly initialized envelopes and the five corresponding aggregated fronts for the max-card problem with k = 4 for the Nikkei dataset.

front, individuals contributing more are considered more important. Parts of the efficient frontier not belonging to the aggregated front are not taken into account. The actual implementation of the above envelope-based non-dominated sorting is rather tricky, the reader is referred to Scheckenbach (2006).

5.1 Representation and genetic operators

As discussed above, the EA is only responsible for handling the non-convex constraints, the appropriate weights are then determined by a critical line algorithm. Thus, in principle, a simple binary encoding would be sufficient. However, we wanted to feed back some information from the critical line algorithm to the evolutionary algorithm. For this reason, we are using a permutation encoding. Then, for the maximum cardinality constraint, simply the first K assets are used in the portfolio. For the 5-10-40 constraint, the first K assets are considered potential heavyweights, with a share of at most 10%[4] and inclusion in the 40% constraint, while all others are restricted to less than 5%. The parameter K here is variable and also part of the solution encoding.

After the critical line algorithm has been applied, the permutation is sorted with respect to the average weight an asset had in all corner portfolios that are part of the aggregated front. Thus, an

[4]Note that the weight of these assets can be set below 5% by the critical line algorithm, although they are still included in the 40% constraint. This helps in the sorting, as such an asset is then moved behind more important assets that are set to 5%, leading to a smaller chance to be included again after mutation.

asset which received a high weight will appear early on the permutation, and subsequently have a higher probability to be among the first K after crossover and mutation.

As genetic operators, we use the uniform order based crossover and swap mutation. For the latter, an asset that belongs to the first K is swapped with an arbitrary other asset. For the 5-10-40 constraint, the parameter K is modified by adding a Gaussian number with mean 0 and standard deviation P_m. The size is then rounded and capped if necessary.

To further improve the efficiency of our algorithm, we introduced two additional concepts: duplicate elimination and a variable population size. In duplicate elimination, we remove individuals that share a part of an aggregated front with another individual, but which are nowhere better than the other individual. The variable population size allows us to increase the number of individuals in the population if the current first aggregated front consists of more individuals than would fit into the population. Keeping a fixed population size would then mean to delete a valuable part of the solution. Note that because our approach is envelope-based, it requires a much smaller population size than point-based approaches anyway. The alternative to an adaptive population size, namely to work with an equally large population as the point-based approaches from the beginning, would have slowed down convergence unnecessarily. Independent of the population size, the number of offspring generated in every iteration remains constant and equal to the original population size.

6 Empirical evaluation

6.1 Benchmark problems and performance measure

Many authors test their approaches on the publicly available benchmarks provided in the OR-library (Beasley, 2006). We will also use some of these benchmarks, namely

- P1 The smallest problem, the Hang Seng benchmark consisting of 31 assets.
- P2 The S&P benchmark consisting of 98 assets
- P3 The largest problem in the OR-library, the Nikkei benchmark consisting of 225 assets.

Because we want to show that our algorithm also scales well, we additionally tested it on

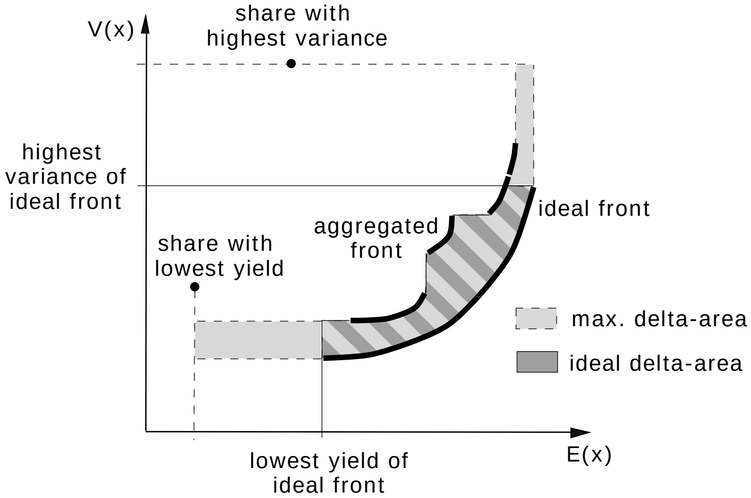


Figure 3: Ideal and maximum Δ-area.

P4 a benchmark with 500 assets, generated according to a method described in Hirschberger et al. (ming) and generously provided by Ralph Steuer.

For the cardinality constrained problems, we set the maximum cardinality to $K = 4$ for P1 and P2, and to $K = 8$ for P3 and P4.

Measuring performance in a multi-objective setting is difficult, because it requires the comparison of frontiers (not only solutions). A number of possible performance measures are discussed e.g. in Zitzler et al. (2003). In the following, we will judge a generated front by its deviation from the ideal front, which is defined as the efficient frontier of the problem without non-convex constraints[5]. This ideal front is an upper bound on the performance and can be computed efficiently with the critical line algorithm. To measure the deviation, we calculate the area between the resulting front and the ideal front. One difficulty with area-based methods is to define the maximum variance and minimum return boundaries to calculate the area, see Figure 3. If these values are set far apart, extreme portfolios have a very large impact on solution quality. If they are set too close, some parts of the front may be cut off. Since the appropriate borders are not clear, we report on two values here: The area using the maximum variance and minimum yield portfolios from the ideal front (ideal delta-area), and the maximum variance and minimum yield from any asset in the available universe.

[5]For the problem with 5-10-40 constraint, all assets are restricted by 10%.

6.2 Parameter settings

For P-MOEA, we use the same parameter settings as in Streichert et al. (2004a), i.e. a population size of 250 and tournament size of 8. For the bit string, bit flip mutation with mutation probability for each bit 2/(number of assets) and N-point crossover with probability 1.0 are applied. For the real-valued string, mutation is done by adding a value from a Normal distribution with $\sigma = 0.05$ to each weight, crossover is again N-point crossover with probability 1.0.

For E-MOEA, the following parameter settings have been chosen without much testing. The initial population size is set to 30, and 30 individuals are generated in every iteration.

The mutation and crossover operators have been described above. Probability to swap each of the first K assets for the cardinality problem is $1/K$, for the 5-10-40 constrained problem, probability to swap any of the heavyweights is $1/7$.

As discussed before, P-MOEA requires a significantly larger population size, as each individual only represents a single point, as opposed to a whole envelope as in E-MOEA.

All reported results are averages over 30 runs. Experiments were conducted on a PC with AMD Sempron 1.6 GHz processor and 1 GB RAM. Maximum allowed running time for the problem with 5-10-40 constraint was set to 500, 1000, 2000, and 4000 CPU seconds for P1, P2, P3, and P4, respectively. Since the cardinality constrained problem seemed easier, we allowed only half the running time for each problem.

6.3 Test results

The results on all 4 benchmark problems are summarized in Tables 1 to 4. Table 1 reports on the max-delta-area, i.e. the area between obtained efficient frontier and ideal front with wide margins, for the cardinality constrained problem at the end of the run. The same information, but with respect to ideal-delta-area, is provided in Table 2. As can be seen, E-MOEA significantly outperforms P-MOEA on all benchmark problems. In terms of the ideal-delta-area, the relative performance of P-MOEA is somewhat better, indicating that it particularly has a problem in finding the portfolios with high expected return or low variance.

The results for the problem with 5-10-40 constraint look similar (see Tables 3 and 4), although the differences between P-MOEA and E-MOEA are generally smaller.

Typically obtained efficient frontiers for P1 and P3 with cardinality constraint are depicted in

Table 1: Max-delta-area at the end of the run for E-MOEA and P-MOEA for all four test problems with cardinality constraint, average ± std. error, all values in terms of 10^{-6}.

	P1 ($K = 4$)	P2 ($K = 4$)	P3 ($K = 8$)	P4 ($K = 8$)
P-MOEA	1.1613 ± 0.0159	2.7787 ± 0.0521	9.3292 ± 0.2287	4124.1363 ± 123.6716
E-MOEA	0.2275 ± 0	0.8048 ± 0.00003	0.0561 ± 0.0052	5.9063 ± 0.2939

Table 2: Ideal-delta-area at the end of the run for E-MOEA and P-MOEA for all four test problems with cardinality constraint, average ± std. error, all values in terms of 10^{-6}.

	P1 ($K = 4$)	P2 ($K = 4$)	P3 ($K = 8$)	P4 ($K = 8$)
P-MOEA	1.0605 ± 0.0160	1.6981 ± 0.0175	2.1250 ± 0.0189	125.1611 ± 1.1580
E-MOEA	0.1371 ± 0	0.5222 ± 0.00003	0.0123 ± 0.0003	2.4398 ± 0.09637

Table 3: Max-delta-area at the end of the run for E-MOEA and P-MOEA for all four test problems with 5-10-40 constraint, average ± std. error, all values in terms of 10^{-6}.

	P1	P2	P3	P4
P-MOEA	1.3274 ± 0.0002	1.5107 ± 0.0197	1.7339 ± 0.0138	446.7284 ± 11.4963
E-MOEA	1.3019 ± 0.0007	1.2416 ± 0.0001	1.6511 ± 0.0047	344.7926 ± 3.8489

Table 4: Ideal-delta-area at the end of the run for E-MOEA and P-MOEA for all four test problems with 5-10-40 constraint, average ± std. error, all values in terms of 10^{-6}.

	P1	P2	P3	P4
P-MOEA	0.2188± 0.00002	0.1028 ± 0.0009	0.0807 ± 0.0003	7.6607 ± 0.1116
E-MOEA	0.2023 ± 0.00001	0.0631 ± 0.0001	0.0700 ± 0.00006	4.9756 ± 0.0290

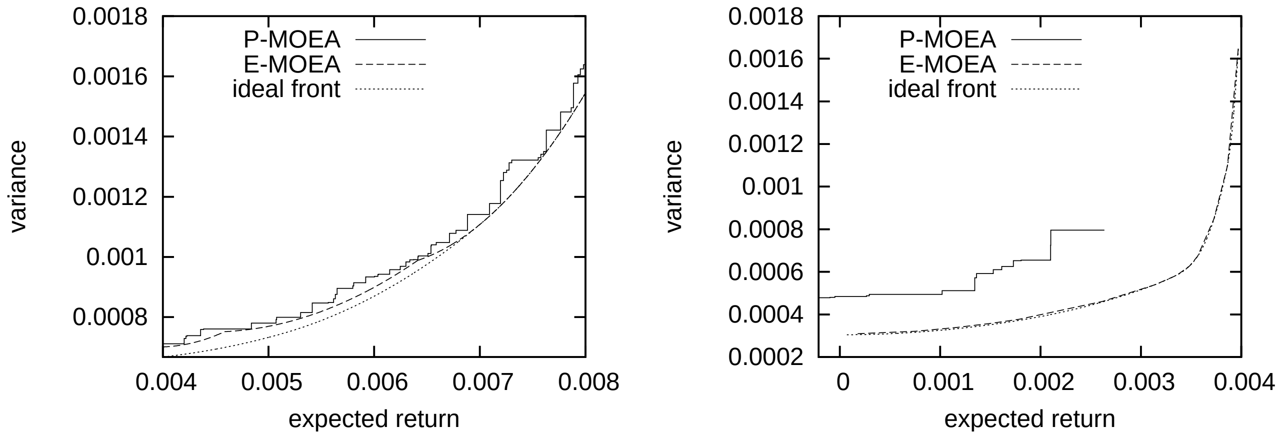


Figure 4: Typical fronts obtained on test problem P1 (left) and test problem P3 (right) with cardinality constraint.

Figure 4. As can be seen, for the small problem (P1), both algorithms perform quite well. In fact, the figure zooms in on only a part on the front, as on a plot of the whole front, the differences would be hard to see. Still, E-MOEA clearly outperforms P-MOEA and is indistinguishable from the ideal front over large parts. For the larger problem, P-MOEA does not seem to be able to come close to the performance of E-MOEA. In particular in the area of higher returns, there are clear deficiencies. It seems that P-MOEA does not scale very well to larger problem sizes[6]. One reason may be that there are usually only few assets with a high return, and exactly those have to be combined in the portfolio to obtain an overall high return. Identifying the high-return assets out of a large set may prove difficult for the P-MOEA. E-MOEA on the other hand finds solutions hardly distinguishable from the ideal front also for the larger problems.

For the 5-10-40 constraint, the fronts obtained by P-MOEA and E-MOEA are much closer to each other, and further away from the ideal front. Still, the front obtained by E-MOEA dominates P-MOEA's front basically everywhere. Note that again for visibility, the plot for the larger problem only shows a segment of the overall front.

When looking at the obtained solution quality in terms of max-delta-area over running time, it is clear that the advantage of E-MOEA over P-MOEA is significant throughout the run. Figure 6(a) looks at P3 with cardinality constraint. Clearly, E-MOEA starts out much better, and converges much faster than P-MOEA (for the small problem, E-MOEA even found the best so-

[6]In Streichert et al. (2004a,b, 2003), the algorithm was only tested on small problems with up to 81 assets.

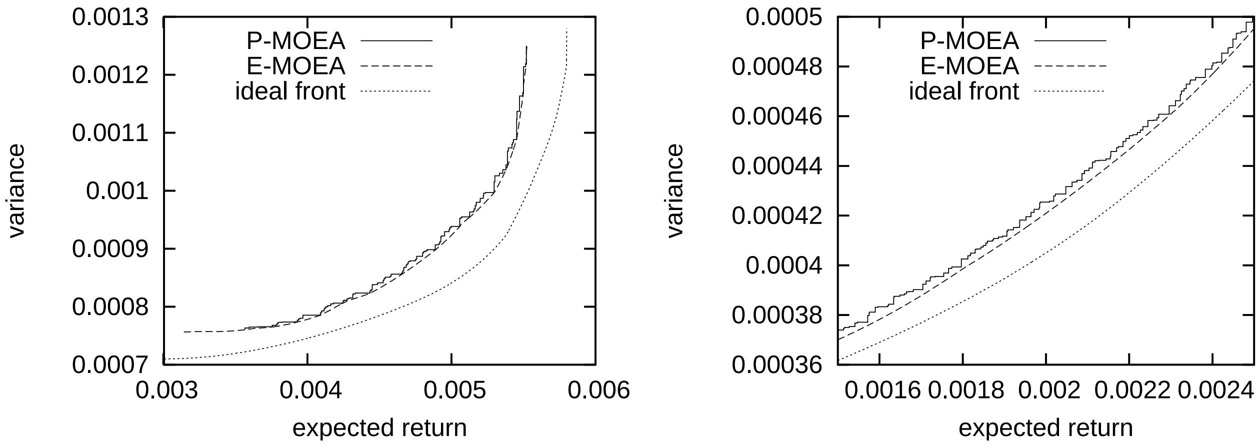


Figure 5: Typical fronts obtained on test problem P1 (left) and test problem P3 (right) with 5-10-40 constraint.

lution within 5 out of the allowed 250 seconds in every single run). Note that we plot against running time. Because E-MOEA has to run a critical line algorithm for every individual, it can only evaluate about 13500 individuals during the 1000 seconds allowed, while P-MOEA generates and evaluates approx. 1,945,000 individuals in the same time frame.

For the problem P3 with 5-10-40 constraint, E-MOEA also starts better than P-MOEA, then P-MOEA quickly catches up only to fall behind again (see Figure 6(b)). The difference in the number of individuals evaluated is even more striking here than for P3 with cardinality constraint, because the 5-10-40 constraint does not allow to remove a large fraction of the assets for the critical line algorithm. While E-MOEA can generate only about 3000 individuals in the given 2000 seconds, P-MOEA generates approx. 3,475,000.

One explanation for the superiority of E-MOEA, besides being envelope-based, is certainly its in-built weight optimization by the critical line algorithm. This effect is visible in Figure 7, which compares the solution quality or randomly generated solutions with both, the P-MOEA and the E-MOEA. Clearly, E-MOEA has a much better start, as the majority of randomly generated envelopes is clearly better than the majority of randomly generated portfolios.

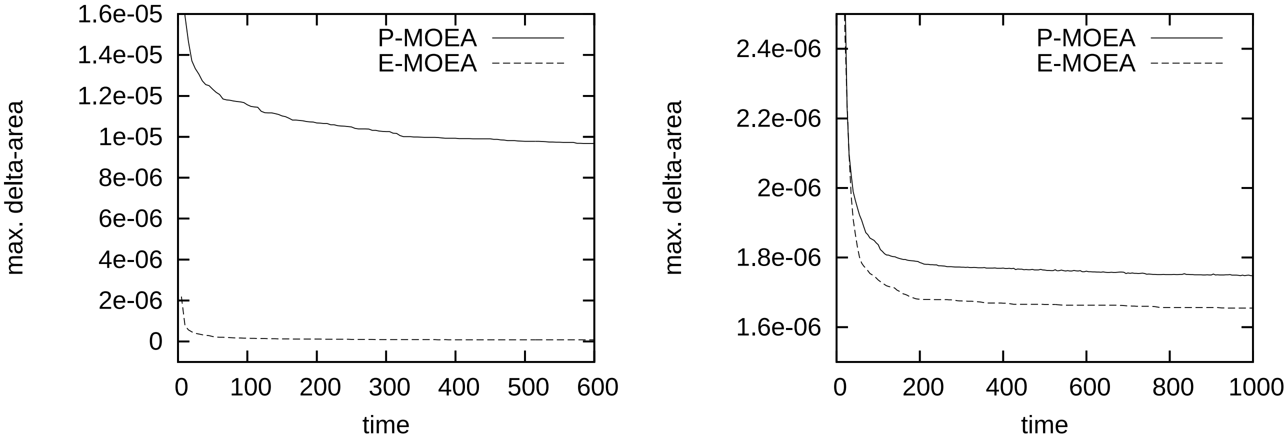


Figure 6: Convergence curves for P3 with cardinality constraint (left) and 5-10-40 constraint (right).

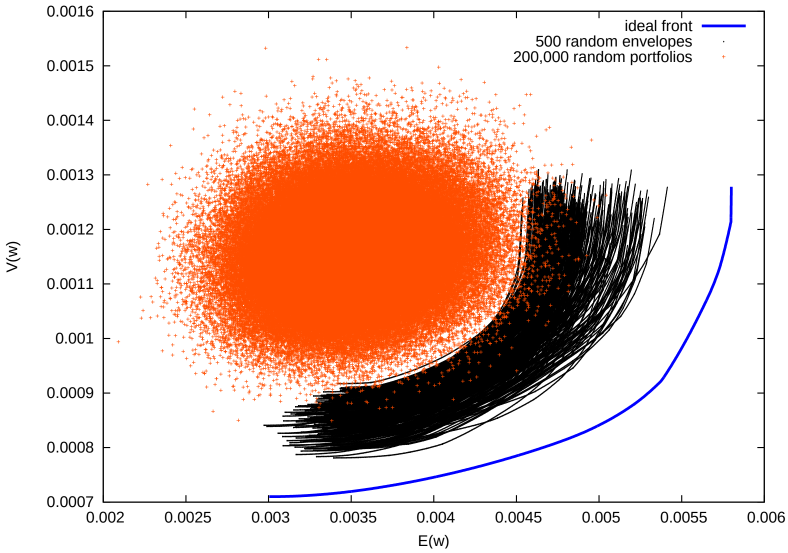


Figure 7: Randomly initialized populations of envelopes and portfolios for the 5-10-40 problem and the Hang Seng Dataset.

7 Conclusion

The critical line algorithm is a very efficient algorithm to calculate the whole efficient frontier for a standard mean-variance portfolio selection problem. However, this method is no longer applicable if practical real-world constraints such as maximum cardinality constraint, buy-in thresholds, or the 5-10-40 rule from the German investment law have to be considered, because such constraints render the search space non-convex. Researchers have therefore resorted to solving these problems point-based, approximating the efficient frontier by iteratively solving a sub-problem with a fixed expected return, for many different settings of the expected return. Because even the sub-problems are rather complex, often meta-heuristics are used.

In this paper, we have proposed a new envelope-based multi-objective evolutionary algorithm (E-MOEA), which is a combination of a multi-objective algorithm with an embedded algorithm for parametric quadratic programming. The task of the MOEA is to define a set of convex subsets of the search space. Then, for each subsets the critical line algorithm can efficiently generate an efficient frontier which we call envelope. The combination of all generated envelopes then forms the overall solution to the problem.

To our knowledge, our approach is the first metaheuristic approach which is not point-based, but which is capable of generating a continuous front of alternatives for portfolio selection problems with non-convex constraints. Compared with a state-of-the-art point-based MOEA, E-MOEA was shown to find significantly better frontiers in a shorter time.

As future work, we are planning to integrate a more intelligent mutation operator that uses shadow prices to influence mutation probabilities. Also, the ideas of envelope-based MOEAs should be transferred to other applications. In particular, we are planning to consider portfolio re-balancing problems that include consideration of fixed and variable transaction costs.

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