Emergence of quasi-units in the one dimensional Zhang model

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We study the Zhang model of sandpile on a one dimensional chain of length L, where a random amount of energy is added at a randomly chosen site at each time step. We show that in spite of this randomness in the input energy, the probability distribution function of energy at a site in the steady state is sharply peaked, and the width of the peak decreases as $L^{-1/2}$ for large L. We discuss how the energy added at one time is distributed among different sites by topplings with time. We relate this distribution to the time-dependent probability distribution of the position of a marked grain in the one dimensional Abelian model with discrete heights. We argue that in the large L limit, the variance of energy at site x has a scaling form $L^{-1}g(x/L)$, where $g(\xi)$ varies as $\log(1/\xi)$ for small ξ, which agrees very well with the results from numerical simulations.

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I. INTRODUCTION

After the pioneering work of Bak, Tang and Wiesenfeld (BTW) in 1987 [1], many different models for self-organized criticality have been studied in different contexts; for review see [2-5]. Of these, models in the general class known as Abelian distributed processors have been studied a lot, as they share an Abelian property that makes their theoretical study simpler [2]. The original sandpile model of Bak *et al.* [1], the Eulerian walkers model [8], and the Manna model [6] are all members of this class. Models which do not have the Abelian property have been studied mostly by numerical simulations. In this paper, we discuss the Zhang model [7], which does not have the Abelian property.

In the Zhang model, the amount of energy added at a randomly chosen site at each time step is not fixed, but random. In spite of this, the model in one dimension has the remarkable property that the energy at a site in the steady state has a very sharply peaked distribution in which the width of the peak is much less than the spread in the input amount per time step, and the width decreases with increasing system size L. This behavior was noticed by Zhang using numerical simulations in one and two dimension [7], and he called it the 'emergence of quasi-units' in the steady state of the model. He argued that for large systems, the behavior would be same as in the discrete model. Recently, A. Fey *et al.* [9] have proved that in one dimension, the variance of energy does go to zero as the length of the chain L goes to infinity, but they did not study how fast it decreases with L.

In this paper, we study this emergence of 'quasi-units' in one dimensional Zhang sandpile by looking at how the added energy is redistributed among different sites in the avalanche process. We show that the distribution function of the fraction of added energy at a site x' reaching a site x after t time steps following the addition is exactly equal to the probability distribution that a marked grain in the one-dimensional height type BTW model added at site x' reaches site x in time t. The latter problem has been studied recently [10]. We use this to show that the variance of energy asymptotically vanishes as $1/L$. We also discuss the spatial dependence of the variance along the system length. In the large L limit, the variance at site x has a scaling form $L^{-1}g(x/L)$. We determine an approximate form of the scaling function $g(\xi)$, which agrees very well with the results of our numerical simulations.

There have been other studies of the Zhang model earlier. Blanchard *et al.* [11] have studied the steady state of the model, and found that the distribution of energies even for the two site problem is very complicated, and has a multi-fractal character. In two dimensions, the distribution of energy seems to sharpen for larger L, but the rate of decrease of the width is very slow [12]. Most other studies have dealt with the question as to whether the critical exponents of the avalanche distribution in this model are the same as in the discrete Abelian model [13, 14]. A. Fey *et al.*'s results imply that the asymptotic behavior of the avalanche distribution in one dimension is indeed the same as in the discrete case, but the situation in higher dimension remains unclear [15, 16].

The plan of the paper is as follows. In Section II, we define the model precisely. In Section III, we show that the calculation of the way the energy added at a site is distributed among different sites by toppling is same as the calculation of the time-dependent probability distribution of the position of a marked grain in the discrete Abelian sandpile model. This correspondence is used in Section IV to determine the qualitative dependence of the variance of the energy variable at a site on its position x, and on the system size L. We propose a simple extrapolation form that incorporates this dependence. We

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check our theoretical arguments with numerical simulations in Section V. Section VI contains a summary and concluding remarks. A detailed calculation of the solution of an equation, required in Section IV, is added as an Appendix.

II. DEFINITION AND PRELIMINARIES

We consider our model on a linear chain of size L. The sites are labelled by integers 1 to L and a real continuous energy variable is assigned to each site. Let $E(x,t)$ be the energy variable at site x at the *end* of the time-step t. We define a threshold energy value E_c, same for each site, such that sites with $E(x,t) \geq E_c$ are called unstable, and those with $E(x,t) < E_c$ are called stable. Starting from a configuration where all sites are stable, the dynamics is defined as follows.

(i) The system is driven by adding a random amount of energy at the *beginning* of every time-step at a randomly chosen site. Let the amount of energy added at time t be Δ_t. We will assume that all Δ's are independent, identically distributed random variables, each picked randomly from an uniform interval $1 - \epsilon \leq \Delta_t \leq 1 + \epsilon$. Let the site of addition chosen at time t be denoted by a_t.

(ii) We make a list of all sites whose energy exceeds or becomes equal to the critical value E_c. All these sites are relaxed in parallel by topplings. In a toppling, the energy of the site is equally distributed to its two neighbors and the energy at that site is reset to zero. If there is toppling at a boundary site, half of the energy at that site before toppling is lost.

(iii) We iterate Step (ii) until all topplings stop. This completes one time step.

This is the slow driving limit, and we assume that all avalanche activity stops before the next addition event. In this limit, the model is characterized by two parameters ϵ and E_c. In the limit $\epsilon = 0$, and $1 < E_c \leq 2$, the model reduces to the discrete case, where the behavior is well understood [17]. For non-zero but small ϵ, the behavior does not depend on the precise value of E_c. In fact, starting with a recurrent configuration of the pile, and adding energy at some chosen site, we get exactly the same sequence of topplings for a range of values of E_c [9]. To be precise, for any fixed initial configuration, and fixed driving sequence (of sites chosen for addition of energy), whether a site x topples at time t or not is independent of E_c, so long as we have $1 + \epsilon < E_c \leq 2 - 2\epsilon$. In the following, we assume for simplicity that $E_c = 3/2$, and $0 \leq \epsilon \leq 1/4$.

It was shown in [9] that in this case, the stationary state has at most one site with energy $E(x,t) = 0$ and all other sites have energy in the range $1 - \epsilon \leq E(x,t) \leq 1 + \epsilon$. The position of the empty site is equally distributed among all the lattice points. There are also some recurrent configurations in which all sites have energy $E(x,t) \geq 1 - \epsilon$. In such cases, we shall say that the site with zero energy is the site $L + 1$. Then, in the steady state, there is exactly one site with energy equal to 0, and the $L + 1$ different positions of the site are equally likely.

If E_c does not satisfy the inequality $1 + \epsilon < E_c \leq 2 - 2\epsilon$, this simple characterization of the steady state is no longer valid. However, our treatment can be easily extended to those cases. Since the qualitative behavior of the model is the same in all cases, we restrict ourselves to the simplest case here.

It is easy to see that the toppling rules are in general not Abelian. For example, start with a two site model in configuration $(1.6, 2.0)$ and $E_c = 1.5$. The final configuration would be $(1.4, 0)$, or $(0, 1.3)$, depending on whether the first or the second site is toppled initially. In our model, using the parallel update rule, the final configuration would be $(1.0, 0.8)$. A. Fey *et al.* [9] have shown that only in one dimension, for $1 + \epsilon < E_c$, the Zhang model has a restricted Abelian character, namely, that the final state does not depend on the order of topplings within an avalanche. However, topplings in two different avalanches do not commute.

III. THE PROPAGATOR, AND ITS RELATION TO THE DISCRETE ABELIAN MODEL

It is useful to look at the Zhang model as a perturbation about the $\epsilon = 0$ limit. For sufficiently small ϵ, given the site of addition and initial configuration, the toppling sequence is *independent* of ϵ. It is also independent of the amount of energy of addition Δ_t, and is same as the model with $\epsilon = 0$, which is the 1-dimensional Abelian sandpile model with integer heights (hereafter referred to simply as ASM, without further qualifiers). We decompose the energy variables as

$$E(x,t) = \mathrm{Nint}[E(x,t)] + \epsilon \eta(x,t), \qquad (1)$$

where Nint refers to the nearest integer value. Then the integer part of the energy evolves as in the ASM. We write

$$\Delta_t = 1 + \epsilon u_t, \quad \text{for all} \quad t. \qquad (2)$$

Here u_t is uniformly distributed in the interval $[-1, +1]$. The linearity of energy transfer in toppling implies that the evolution of the variables $\eta(x,t)$ is independent of ϵ. Thus, $\eta(x,t)$ is a linear function of u_t; the precise function depends on the sequence of topplings that took place. These are determined by the sequence of addition sites $\{a_t\}$ up to the time t, and the initial configuration C_0. These together will be called the evolution history of the system up to time t, and denoted by $\mathcal{H}_t$. We assume that at the starting time $t = 0$, the variables $\eta(x, t = 0)$ are zero for all x, and the initial configuration is a recurrent configuration C_0 of the ASM. Then, from the linearity of the toppling rules, we can write $\eta(x,t)$ as a linear function of $\{u_{t'}\}$ for $1 \leq t' \leq t$, and we can write

for a given history $\mathcal{H}_t$,

$$\eta(x,t|\{u_t\},\mathcal{H}_t) = \sum_{t'=1}^{t} G(x,t|a_{t'},t',\mathcal{H}_t)u_{t'}. \qquad (3)$$

This defines the matrix elements $G(x,t|a_{t'},t',\mathcal{H}_t)$. These can be understood in terms of the probability distribution of the position of a marked grain in the ASM as follows. Consider the motion of a marked grain in the one dimensional height type BTW model. We start with configuration C_0 and add grains at sites according to the sequence $\{a_t\}$. All grains are identical except the one added at time t', which is marked. In each toppling, the marked grain jumps to one of its two neighbors with equal probability. Consider the probability that the marked grain will be found at site x after a sequence of relaxation processes at time t. We denote this probability as $\mathrm{Prob}(x,t|a_{t'},t',\mathcal{H}_t)$. From the toppling rules in both the models, it is easy to see that

$$G(x,t|a_{t'},t',\mathcal{H}_t) = \mathrm{Prob}(x,t|a_{t'},t',\mathcal{H}_t). \qquad (4)$$

Averaging over different histories $\mathcal{H}_t$, we get the probability that a marked grain added at $x' = a_{t'}$ at time t' is found at a position x at time $t \geq t'$ in the steady state of the ASM. Denoting the latter probability by $\mathrm{Prob}_{\mathrm{ASM}}(x,t|x',t')$, we get

$$\overline{G(x,t|x' = a_{t'},t',\mathcal{H}_t)} = \mathrm{Prob}_{\mathrm{ASM}}(x,t|x',t'), \qquad (5)$$

where the over bar denotes averaging over different histories $\mathcal{H}_t$, consistent with the specified constraints. Here, the constraint is that $\mathcal{H}_t$ must satisfy $a_{t'} = x'$. At other places, the constraints may be different, and will be specified if not clear from the context.

We shall denote the variance of a random variable ξ by $Var[\xi]$. From the definition in Eq. (1), it is easy to show that

$$Var[E(x,t)] = L/(L+1)^2 + \epsilon^2 Var[\eta(x,t)]. \qquad (6)$$

Different u_t are independent random variables, also independent of $\mathcal{H}_t$ and have zero mean. Let $Var[u_t] = \sigma^2$. For the case when u_t has a uniform distribution between -1 and $+1$, we have $\sigma^2 = 1/3$. Then, from Eq. (3), we get

$$Var[\eta(x,t)] = \sigma^2 \sum_{t'=1}^{t} \overline{G^2(x,t|a_{t'},t',\mathcal{H}_t)}. \qquad (7)$$

As $t \to \infty$, the system tends to a steady state, and the average in the right hand side of Eq. (7) becomes a function of $t - t'$. Also, for a given t', all values of $a_{t'}$ are equally likely. We define

$$F(x,\tau) \equiv \frac{1}{L} \lim_{t' \to \infty} \sum_{x'} \overline{G^2(x,t'+\tau|x',t',\mathcal{H}_t)}. \qquad (8)$$

Then, for large L, in the steady state (t large), the variance of energy at site x is $1/L + \epsilon^2 \Sigma^2(x)$, where

$$\Sigma^2(x) = \lim_{t \to \infty} Var[\eta(x,t)] = \sigma^2 \sum_{\tau=0}^{\infty} F(x,\tau). \qquad (9)$$

We define $\overline{\Sigma^2}$ to be the average of $\Sigma^2(x)$ over x.

$$\overline{\Sigma^2} = \frac{1}{L} \sum_x \Sigma^2(x). \qquad (10)$$

Evaluation of $G(x,t|x',t',\mathcal{H}_t)$ for a given history $\mathcal{H}_t$ and averaging over $\mathcal{H}_t$ is quite tedious for $t > 1$ or 2. For $\overline{G}$, the problem has been studied in the context of residence times of grains in sand piles, and some exact results are known in specific cases [10]. For $\overline{G^2}$, the calculations are much more difficult. However, some simplifications occur in large L limit. We discuss these in the next section.

IV. CALCULATION OF $\Sigma^2(x)$ IN LARGE-L LIMIT

In order to find the quantity $F(x,\tau)$ in Eq. (8), we have to average $G^2(x,t|x',t',\mathcal{H}_t)$ over all possible histories $\mathcal{H}_t$, which is quite difficult to evaluate exactly. However, we can determine the leading behavior of $F(x,\tau)$ in this limit.

We use the fact that the path of a marked grain in the ASM is a random walk [10]. Consider a particle that starts away from the boundaries, at $x' = \xi L$, with L large, and $0 < \xi < 1$. If it undergoes $r(\mathcal{H}_t)$ topplings between the time t' and $t = t' + \tau$ under some particular history $\mathcal{H}_t$, then its probability distribution is approximately a Gaussian, centered at x' with width $\sqrt{r}$. Then, we have

$$G(x,t|x',t',\mathcal{H}_t) \simeq \frac{1}{\sqrt{2\pi r(\mathcal{H}_t)}} \exp\left(-\frac{(x-x')^2}{2r(\mathcal{H}_t)}\right). \qquad (11)$$

Using this approximation for G, summing over x', we get

$$\sum_{x'} G^2(x,t|x',t',\mathcal{H}_t) \simeq \frac{1}{2\sqrt{\pi r(\mathcal{H}_t)}}. \qquad (12)$$

Thus, we have to calculate the average of $1/\sqrt{r(\mathcal{H}_t)}$ over different histories. Here $r(\mathcal{H}_t)$ was defined as the number of topplings undergone by the marked grain. Different possible trajectories of a marked grain, for a given history, do not have the same number of topplings. However, if the typical displacement of the grain is much smaller than its distance from the end, differences between these are small, and can be neglected. There are typically $\mathcal{O}(L)$ topplings per grain per avalanche in the model, and a grain moves a typical distance of $\mathcal{O}(\sqrt{L})$ in one avalanche. Then, we can approximate $r(\mathcal{H}_t)$ by $N(x')$, the number of topplings at x'.

Let the number of topplings at x' at time steps $\tau = 0, 1, 2, \ldots$ be denoted by $N_0, N_1, N_2, \ldots$. Then, $N(x') = $

$N_0 + N_1 + N_2 + \cdots$. It can be shown that the number of topplings in different avalanches in the one dimensional ASM are nearly uncorrelated (In fact the correlation function between N_i and N_j varies as $(1/L)^{|i-j|}$.). By the central limit theorem for sum of weakly correlated random variables, the mean value of N grows linearly with τ, but the standard deviation increases only as $\sqrt{\tau}$. Then, for $\tau \gg 0$, the distribution is sharply peaked about the mean, and $\langle 1/\sqrt{N} \rangle \simeq 1/\sqrt{\langle N \rangle}$.

Clearly, for $\tau \gg 0$, $\langle N \rangle = \tau \bar{n}(x')$, where $\bar{n}(x')$ is the mean number of topplings per avalanche at x' in the ASM, given by

$$\bar{n}(x = \xi L) = L\xi(1-\xi)/2. \tag{13}$$

The upper limit on τ for the validity of the above argument comes from the requirement that the width of the Gaussian be much less than the distance from the boundary, (without any loss of generality, we can assume that $\xi < 1/2$, so that it is the left boundary), else we cannot neglect events where the marked grain leaves the pile. This gives $\sqrt{\tau \bar{n}(x)} \ll \xi L$, or equivalently, $\tau \ll \xi L$. Thus we get,

$$F(x,\tau) \simeq \frac{C_1}{L}[\tau L\xi(1-\xi)]^{-1/2}, \text{ for } 0 \ll \tau \ll \xi L, \tag{14}$$

where C_1 is some constant.

Also, we know that for $\tau \gg L$, the probability that the grain stays in the pile decays exponentially as $\exp(-\tau/L)$ [10]. Thus, $\overline{G}$, and also $\overline{G^2}$ will decay exponentially with τ, for $\tau \gg L$. Thus, we have, for some constants C_2 and a,

$$F(x,\tau) \simeq \frac{C_2}{L^2}\exp(-a\tau/L), \text{ for } \tau \gg L. \tag{15}$$

It only remains to determine the behavior of $F(x,\tau)$, for $\xi L \ll \tau \ll L$. In this case, in the ASM, there is a significant probability that the marked grain leaves the pile from the end. This results in a faster decay of G, and hence of F with time. We argue below that the behavior of the function $F(x,\tau)$ is given by

$$F(x,\tau) \sim \frac{C_3}{L\tau}, \text{ for } \xi L \ll \tau \ll L, \tag{16}$$

where C_3 is some constant. This can be seen as follows: Let us consider the special case when the particle starts at a site close to the boundary. Then $\bar{n}(x)$ is approximately a linear function of x for small x. Its spatial variation cannot be neglected, and Eq. (12) is no longer valid. We will now argue that in this case

$$\overline{G(x,t'+\tau|x',t')} \simeq x'\tau^{-2}\exp(-x/\tau), \text{ for } 0 \ll \tau \ll L. \tag{17}$$

The time evolution of $Prob_{ASM}(x,t|x',t')$ in Eq. (5) is well described as a diffusion with diffusion coefficient proportional to $\bar{n}(x)$ which is the mean number of topplings per avalanche at x in the ASM [10]. For understanding the long-time survival probability in this problem, we can

equivalently consider the problem in a continuous-time version: consider a random walk on a half line where sites are labelled by positive integers, and the jump rate out of a site x is proportional to x. A particle starts at site $x = x_0$ at time $t = 0$. If $P_j(t)$ is the probability that the particle is at j at time t, then the equations for the time-evolution of $P_j(t)$ are, for all $j > 0$,

$$\frac{d}{dt}P_j(t) = (j+1)P_{j+1}(t)+(j-1)P_{j-1}(t)-2jP_j(t). \tag{18}$$

The long time solution starting with $P_j(0) = \delta_{j,x_0}$ is

$$P_j(t) \simeq x_0 t^{-2}\exp(-j/t) \tag{19}$$

for $t \gg x_0$ and large j. The probability that the particle survives till time t decreases as $1/t$ for large t. We have discussed the calculation in the Appendix.

Using Eq. (5), we see that $\overline{G(j,t'+\tau|x_0,t')}$ scales as x_0/τ^2. It seems reasonable to assume that $\overline{G^2}$ will scale as $\overline{G}^2$. Then, each term in the summation for $F(x,\tau)$ in Eq. (8) scales as x_0^2/τ^4, and there are τ such terms, as the sum over x_0 has an upper cutoff proportional to τ, and so $F(x,\tau)$ varies as $1/\tau$ for $L \gg \tau \gg x_0$. This concludes the argument.

We can put these three limiting behaviors into a single functional form that interpolates between these, as

$$F(x,\tau) \simeq \frac{1}{L}\frac{K\exp(-a\tau/L)}{\tau + B\sqrt{\tau L\xi(1-\xi)}}, \tag{20}$$

where K, a and B are some constants. In Section V, we will see that results from numerical simulation are consistent with this phenomenological expression.

Using this interpolation form in Eq. (9), and converting the sum over τ to an integration over a variable $u = \tau/L$, we can write

$$\Sigma^2(x = \xi L) \simeq \frac{\sigma^2}{L}\int_0^\infty du\frac{K\exp(-au)}{u + B\sqrt{u\xi(1-\xi)}}. \tag{21}$$

This integral can be simplified by a change of variable $au = z^2$, giving

$$\Sigma^2(x = \xi L) \simeq \frac{K\sigma^2}{L}I\left(B'\sqrt{\xi(1-\xi)}\right), \tag{22}$$

where K, B' are constants, and $I(y)$ is a function defined by

$$I(y) = 2\int_0^\infty dz\frac{\exp(-z^2)}{z + y}. \tag{23}$$

It is easy to verify that $I(y)$ diverges as $\log(1/y)$ for small y. In particular, we note that the exponential term in the integral expression for $I(y)$ has a significant contribution only for z near 1. We may approximate this by dropping the exponential factor, and changing the upper limit of
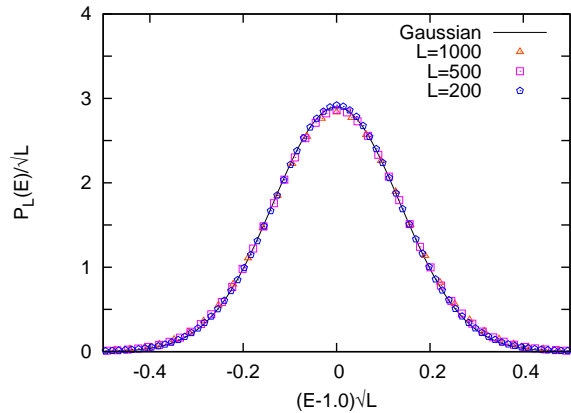
FIG. 1: (Color online) Scaling collapse of the probability distribution $\mathcal{P}_L(E)$ of energy per site in the steady state for different systems of size 200, 500 and 1000. The distribution is well described by a Gaussian of width 0.136.

the integral to 1. The resulting integral is easily done, giving

$$\Sigma^2(x = \xi L) \simeq \frac{K'\sigma^2}{L} \log\left(1 + \frac{1}{B'\sqrt{\xi(1-\xi)}}\right), \quad (24)$$

where K' is some constant. Averaging $\Sigma^2(x)$ over x, we get a behavior $\overline{\Sigma^2}(x) \simeq 1/L$. Of course, the answer is not exact, and one could have constructed other interpolation forms that have the same asymptotic behavior. We will see in the next Section that results from numerical simulations for $\Sigma^2(x)$ can be fitted very well to the phenomenological expression in Eq. (24).

V. NUMERICAL RESULTS

We have tested our non-rigorous theoretical arguments against results obtained from numerical simulations. In Fig. 1, we have plotted the probability distribution $\mathcal{P}_L(E)$ of energy at a site, averaged over all sites. We used $L = 200$, 500 and 1000, and averaged over 10^8 different configurations in the steady state. We plot the scaled distribution function $\mathcal{P}_L(E)/\sqrt{L}$ versus the scaled energy $(E - \bar{E})\sqrt{L}$. A good collapse is seen, which verifies the fact that the width of the peak varies as $L^{-1/2}$.

The dependence of the variance of $E(x, t)$ on x is plotted in Fig. 2 for systems of length 200, 300 and 400. The data was obtained by averaging over 10^8 avalanches. We plot $(L + \lambda)\Sigma^2(x)/\sigma^2$ versus x_{eff}/L_{eff}, where x_{eff} differs from x by an amount δ to take into account the corrections due to end effects. Then, for consistency, L is replaced by $L_{eff} = L + 2\delta$. For the specific choice of $\lambda = 5 \pm 1$ and $\delta = 1.0 \pm 0.2$, we get a good collapse of the curves for different L. We also show a fit to the proposed
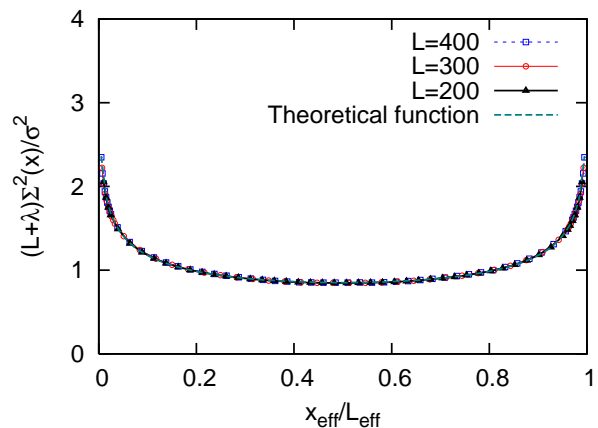

FIG. 2: (Color online) Scaling collapse of $\Sigma^2(x)/\sigma^2$ at site x for systems of different length L.
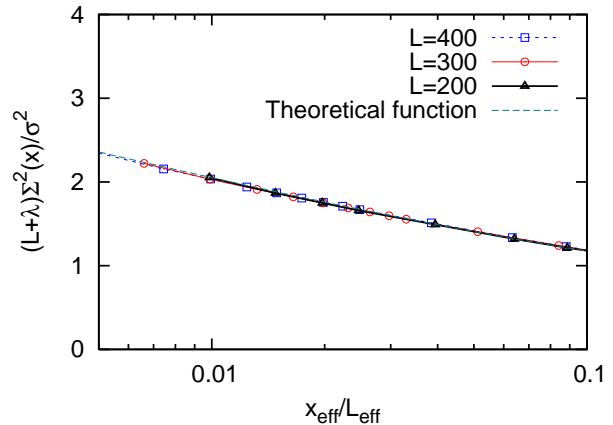

FIG. 3: (Color online) The same plot in Fig. 2 resolved more at the left boundary of the model and taking x axis in log scale.

interpolation form in Eq. (24), with $K' = 1.00 \pm 0.01$ and $B' = 1.5 \pm 0.2$. We see that the fit is very good.

In order to check the logarithmic dependence of $\Sigma^2(x)$ on x for small x, we re-plot the data in Fig. 3 using logarithmic scale for x. We get a good collapse of the data for different L, supporting our proposed dependence in Eq. (24).

VI. CONCLUDING REMARKS

To summarize, we have studied the emergence of quasi-units in the one-dimensional Zhang sandpile model. The variance of energy variables in the steady state is governed by the balance between two competing processes. The randomness in the drive i.e., the energy of addition, tends to increase the variance in time. On the other hand, the topplings of energy variables tend to equalize the ex-

cess energy by distributing it to the nearby sites. There are on an average $O(L^2)$ topplings per avalanche. Hence, in one dimension there are, on an average, $O(L)$ topplings per site per avalanche. For large system size, the second process dominates over the first and the variance becomes low. We have shown that the variance vanishes as $1/L$ with increasing system size and the probability distribution of energy concentrates around a non-random value which depends on the energy of addition. We have also proposed a functional form for the spatial dependence of variance of energy which incorporates the correct limiting behaviors, and matches very well with the numerical data.

An interesting question is whether one can extend these arguments to the two-dimensional Zhang model. In this case, there are several peaks in the distribution of energies at a site, but there are some numerical evidences for the sharpening of the peaks as the system size is increased. However, as the number of topplings per site varies only as $\log L$, the width is expected to decrease much more slowly with L, and the fluctuation effects can be much stronger. This remains an open question for further study.

APPENDIX

Here we discuss the solution of the Eq. (18) for the starting values given by

$$P_j(t=0) = \alpha^{j-1}. \tag{A.1}$$

We start with an ansatz $P_j(t) = b_t \exp(-a_t j)$, where both a_t and b_t are functions only of t. This form satisfies the Eq. (18) for all j, $t > 0$, if a_t and b_t satisfy

$$\frac{da_t}{dt} = 2 - e^{a_t} - e^{-a_t}, \tag{A.2}$$

$$\frac{db_t}{dt} = b_t(e^{-a_t} - e^{a_t}). \tag{A.3}$$

To solve the Eq. (A.2), we first make a change of variable $z = e^{-a_t}$. In terms of z, the equation becomes $dz/dt = (1-z)^2$, which can be easily integrated to give

$$e^{-a_t} = \frac{t+A-1}{t+A}, \tag{A.4}$$

where A is an integration constant. To satisfy the initial condition in Eq. (A.1), we choose

$$A = (1-\alpha)^{-1}. \tag{A.5}$$

Similarly, to solve the equation for b_t, we use the form of e^{-a_t} given in Eq. (A4) and get

$$\frac{db_t}{dt} = b_t \frac{1 - 2(t+A)}{(t+A)(t+A-1)}. \tag{A.6}$$

This can be integrated to give

$$b_t = \frac{B}{(t+A)(t+A-1)}, \tag{A.7}$$

where B is an integration constant. Then the probability can be written as

$$P_j(t) = B\frac{(t+A-1)^{j-1}}{(t+A)^{j+1}}. \tag{A.8}$$

To satisfy the initial condition at $t = 0$, we choose the integration constant $B = (1-\alpha)^{-2}$. Then, with these values of A and B, we have the solution for all j, $t > 0$, given by

$$\begin{aligned} P_j(t) &= \frac{[(1-\alpha)t + \alpha]^{j-1}}{[(1-\alpha)t + 1]^{j+1}} \\ &= \phi_j(\alpha, t), \quad \text{say.} \end{aligned} \tag{A.9}$$

Now, as $\phi_j(\alpha, t)$ satisfies the Eq. (18),

$$\psi_{j,n}(\alpha, t) = \frac{1}{(n-1)!} \frac{\partial^{n-1}\phi_j(t)}{\partial\alpha^{n-1}} \tag{A.10}$$

will also satisfy the equation for any natural number n. In addition,

$$\psi_{j,n}(\alpha = 0, t = 0) = \delta_{j,n}. \tag{A.11}$$

Hence, we see that the solution of the Eq. (18), starting with $P_j(t) = \delta_{j,n}$ at $t = 0$ is

$$P_j(t) = \psi_{j,n}(\alpha = 0, t) = \frac{1}{(n-1)!} \frac{\partial^{n-1}\phi_j(\alpha, t)}{\partial\alpha^{n-1}} \Big|_{\alpha=0}, \tag{A.12}$$

for all j, $t > 0$, where $\phi_j(\alpha, t)$ is given in Eq. (A.9) and n is any natural number.

It can be shown that for large t and j, the solution asymptotically becomes $P_j(t) = nt^{-2}\exp(-j/t)$.

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