Steady State and Relaxation Spectrum of the Oslo Rice-pile Model

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Abstract

We show that the one-dimensional Oslo rice-pile model is a special case of the abelian distributed processors model. The exact steady state of the model is determined. We show that the time evolution operator $\mathcal{W}$ for the system satisfies the equation $\mathcal{W}^{n+1} = \mathcal{W}^n$ where $n = L(L+1)/2$ for a pile with L sites. This is used to prove that $\mathcal{W}$ has only one eigenvalue 1 corresponding to the steady state, and all other eigenvalues are exactly zero. Also, all connected time-dependent correlation functions in the steady state of the pile are exactly zero for time difference greater that n. Generalization to other abelian critical height models where the critical thresholds are randomly reset after each toppling is briefly discussed.

1   Introduction

In their pioneering work in 1987, Bak et al proposed the well-known sandpile model as a paradigm for self-organized criticality [1]. Since then, many different variants of sandpiles have been studied. These include models where the height of the pile is a real variable [2], different toppling rules [3], with preferred direction [4], stochastic topplings [5], models with fixed energy [6], etc.. While there is a fair amount of understanding by now about many of these models, the understanding of different universality classes of critical

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behavior possible, and of relationship to the critical steady states seen in other driven systems remains unsatisfactory [7].

The static properties of the BTW sandpile model can be related to the equilibrium properties of $q \to 0$ limit of the q-state Potts model and the statistics of spanning trees [8]. The directed model [4] can be related to the well-known voter and the Takayasu aggregation models [9]. However, the exponents characterizing the avalanche distributions in sandpile models are not determined completely by the critical exponents of the $q = 0$ Potts model. We have argued that the generic behavior of sandpile like models is in the universality class of directed percolation [10]. However, the critical behavior of special models, like the BTW or the Manna models, is certainly not in this class. Paczuski and Boettcher have given arguments relating the exponents of sandpile model to that of growing interfaces [11]. Generically, the growing surfaces are expected to be in the universality class of Kardar-Parisi-Zhang type models, not of directed percolation. It seems desirable to have a closer look at the relation between sandpile models and growing surfaces. This relationship is most clearly seen [12] in the simple Oslo rice-pile model [13].

The Oslo rice-pile is one of the simplest of models of self-organized criticality and seems to be able to describe quite well the real experiments on rice-piles [14]. It shows non-trivial avalanche exponents even in one dimension. There are some exact results known about this model [15], and the values of avalanche exponents are known quite accurately [16]. The damage propagation in the model has also been studied [17]. It seems worthwhile to try to see if the model can be solved exactly. While attempts in this direction have not been successful so far, it does seem to have a rather unusual mathematical structure. In this paper, I will try to outline some of these properties, and hope that this will encourage further work.

2   Definition of the Model

The Oslo rice-pile model is a one-dimensional cellular automaton model with stochastic toppling rules. It is defined as follows [13]: We consider a line of L sites, labelled by integers 1 to L. At each site i, there is a non-negative integer height variable h_i, called the height of the pile at that site. We start with a configuration in which all heights are zero. At each site i, there is also a variable σ_i^c, called the critical threshold, which takes values 2 or 1 with probabilities p and $q = (1 - p)$ respectively, independently at each site.

We shall call $h_i - h_{i+1}$ as the local slope of the pile at site i, and denote it by σ_i. We define $h_{L+1} = 0$, so that the local slope at the site $i = L$ is equal to h_L. We call a site i stable if $\sigma_i \leq \sigma_i^c$. A configuration is said to be stable, if all sites in the configuration are stable.

If a site i is unstable, it relaxes by sending one grain to site on its right. In this process, h_i will decrease by 1, and h_{i+1} will increase by 1. If a configuration has more than one unstable sites, then they are updated in parallel. (We will show later that actually the order of topplings does not matter). After each toppling at a site, the critical threshold at that site is reset, and takes a new random value 1 or 2 with probabilities q and p respectively.

The system is driven by adding a grain at the leftmost site $i = 1$. This increases the local height by 1. If this leads to an unstable configuration, it is relaxed by topplings till a stable configuration is reached. And then we again add a grain at $i = 1$.

If we evolve the system like this, it reaches a self-organized critical state at large times, where the distribution of avalanches shows a power law tail with an L-dependent cutoff.

3   The Abelian Property

It is convenient to specify the configurations in terms of local slopes σ_i. A particular configuration $\{h_i\}$ corresponds to a unique set of slope variables $\{\sigma_i\}$, and vice versa. In terms of the slope variables, the evolution rules of the rice-pile become the following:

i) If a particle is added from the left, σ_1 increases by 1.

ii)a. If $\sigma_i > \sigma_i^c$ with $2 \leq i \leq L - 1$, then σ_i decreases by 2, and σ_{i-1} and σ_{i+1} increase by 1.

　b. If $\sigma_1 > \sigma_1^c$, then, σ_1 decreases by 2, and σ_2 increases by 1.

　c.If $\sigma_L > \sigma_L^c$, then, σ_L decreases by 1, σ_{L-1} increases by 1.

iii) After a toppling at site i, σ_i^c is reset to a new randomly chosen value.

It is interesting that these rules are same as for a one-dimensional critical height model, where the critical threshold is a random variable, and its value at a site is reset randomly after each toppling at that site. This correspondence to an *abelian* critical height model is not possible in higher dimensions, or if grains in the Oslo model are added at all sites, and not only at the left end.

It is convenient to work with this one-dimensional critical height model, in which the "particles" are the "slopes variables" of the original model. To avoid confusion, in this paper we shall call the rice-particles of the original model "grains", and use the word "particles" to refer to the slope variables. Thus, in a single toppling event at site i, $(2 \leq i \leq L-1)$, a grain is transferred from i to $(i+1)$, but two particles are moved from i, one each to the sites $(i-1)$ and $(i+1)$. The topplings at the boundary sites can be similarly described.

As another simplification, we need not keep track of the critical thresholds, if we adopt the following rule for relaxation *whenever a new particle is added to a site* : If, after addition, the new height is > 2, the site topples; if the height is 2, it topples only with probability $(1-p)$; and if the new height is 1, it does not topple. Clearly, the actual evolution is same as under the original definition.

This model, then is a special case of the general Abelian distributed processors model defined earlier [9]. We consider each site as a finite-state automaton, with a local pseudo random number generator (PRNG), and a single integer giving the number of particles at the site. If an added particle makes the number of particles greater than the current threshold, a fixed number of particles, as specified by the matrix Δ, are transferred to other sites. A new random number is drawn from the PRNG, and used to choose a new threshold. It is easily shown that in any configuration with more than one unstable sites, the final state of this system does not depend on the order in which these sites are relaxed, for any particular sequence of thresholds generated by the PRNG. This establishes the abelian property for the rice-pile model.

4   Operator Algebra

Let us denote by $\mathcal{V}$ a linear vector space spanned by its basis vectors $|\mathcal{C}\rangle$, where $|\mathcal{C}\rangle$ are the stable configurations of the rice-pile. We shall characterize the configurations $\mathcal{C}$ by the slope variables $\{\sigma_i\}$. There are 3^L such configurations for a chain of length L. Note that the values of the critical slopes σ_i^c are not specified in $\mathcal{C}$.

The state of the rice-pile at time t is characterized by a probability vector $|Prob(t)\rangle$, which is an element of $\mathcal{V}$. If the probability that the rice-pile occurs

in configuration $\mathcal{C}$ at time t is given by $p(\mathcal{C}, t)$, we write

$$|Prob(t)\rangle = \sum_{\mathcal{C}} p(\mathcal{C}, t)|\mathcal{C}\rangle. \tag{1}$$

We now define linear operators a_i ($i = 1$ to L) acting on $\mathcal{V}$, by their actions on the basis vectors $|\mathcal{C}\rangle$. Starting from the stable configuration $\mathcal{C}$, we increase the slope variable σ_i by 1, and allow the resulting configuration to relax if necessary. If the resulting configuration is $\mathcal{C}'$ with probability $w_i(\mathcal{C}'|\mathcal{C})$, we define

$$a_i|\mathcal{C}\rangle = \sum_{\mathcal{C}'} w_i(\mathcal{C}'|\mathcal{C})|\mathcal{C}'\rangle. \tag{2}$$

We then have

$$[a_i, a_j] = 0, \quad \text{for all} \quad i, j. \tag{3}$$

The time-evolution of the system is Markovian. Let $|Prob(t)\rangle$ be the probability vector of the state of the system after t paticles have been added, and the system is allowed to relax. We then have the master equation

$$|Prob(t+1)\rangle = \mathcal{W} \, |Prob(t)\rangle, \tag{4}$$

which defines the time evolution operator $\mathcal{W}$. For the rice-pile model, with particles added only at the left end, clearly, we have $\mathcal{W} = a_1$.

Let us now give a more explicit representation of the operators $\{a_i\}$. If a_i acts on a configuration where $\sigma_i = 0$, we get a state with σ_i increased to value 1, and all other σ's are unchanged.

$$a_i|\ldots, \sigma_i = 0, \ldots\rangle = |\ldots, \sigma_i = 1, \ldots\rangle. \tag{5}$$

If a_i acts on a configuration with $\sigma_i = 1$, then with probability p, it increases its value to 2. With probability $(1-p)$, it causes a toppling there, which would change σ_i to 0, and add a particle at sites $(i \pm 1)$.

$$a_i|\ldots, \sigma_i = 1, \ldots\rangle = p|\ldots, \sigma_i = 2, \ldots\rangle + qa_{i-1}a_{i+1}|\ldots, \sigma_i = 0, \ldots\rangle. \tag{6}$$

Acting on a configuration with $\sigma_i = 2$, a_i will always cause a toppling, and we have

$$a_i|\ldots, \sigma_i = 2, \ldots\rangle = a_{i-1}a_{i+1}|\ldots, \sigma_i = 1, \ldots\rangle. \tag{7}$$

These equations fully define the action of the operators a_i. They also hold for the boundary sites $i = 1$, and $i = L$, if we assume the conventions

$$a_0 = 1, \quad \text{and} \quad a_{L+1} = a_L. \tag{8}$$

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Applying these rules repeatedly, we can determine the effect of any of the the operators $\{a_i\}$ on any stable configuration. For example, consider the case $L = 2$. Let us determine the result of a_1 acting on $|2, 2\rangle$. We get,

$$a_1|2,2\rangle \;=\; a_2|1,2\rangle \;=\; a_1a_2|1,1\rangle \;=\; a_1(\,p|1,2\rangle + qa_1|1,1\rangle). \tag{9}$$

These two terms can be evaluated further as

$$a_1|1,2\rangle = p|2,2\rangle + qa_2|0,2\rangle, \tag{10}$$

with

$$a_2|0,2\rangle = a_1a_2|0,1\rangle = a_2|1,1\rangle = p|1,2\rangle + qa_1|1,1\rangle, \tag{11}$$

and

$$a_1|1,1\rangle = p|2,1\rangle + qa_2|0,1\rangle = p|2,1\rangle + pq|0,2\rangle + q^2|1,1\rangle. \tag{12}$$

Putting all these together, we get

$$a_1|2,2\rangle = p^2|2,2\rangle + (1-p^2)\,[\,p|1,2\rangle + qp|2,1\rangle + q^2p|0,2\rangle + q^3|1,1\rangle] \tag{13}$$

We also note that these operators satisfy the equations

$$a_i^3 = a_{i-1}a_i a_{i+1}, \quad \text{for all}\ \ i. \tag{14}$$

This is equivalent to the observation that on adding three particles at site i, a toppling there must occur at site i, so that it is same as adding a particle each at the sites $i - 1, i$ and $i + 1$.

Given any product of the form $a_2a_1a_7a_2a_2\ldots$, we first use the commutativity property to bring it to the form $a_1^{n_1}a_2^{n_2}\ldots$, where $n_1, n_2, \ldots$ are non-negative integers. Then, we use the reduction rules Eq. (14) to express it as a sum of lower powers such that $n_i \leq 2$ for all i. Consider evaluating a_1^r for $r = 1, 2, \ldots$. As the number of possible answers is finite, we must have a minimum value of r, such that a_1^r is equal to a lower power of a_1. Normally, one would guess that r is of order $\exp(L)$, the number of different terms allowed. Interestingly, it turns out to be much smaller, and we now show that $r = L(L + 1) + 1$ for all L.

Using Eq.(14), it is straightforward to show that

$$a_1^{L(L+1)} = a_1^2 a_2^2 a_3^2 \ldots a_L^2. \tag{15}$$

Multiplying both sides of this equation by a_1, and again using the reduction rules Eq.(14), we get

$$a_1^{L(L+1)+1} = a_1^2 a_2^2 a_3^2 \ldots a_L^2. \tag{16}$$

As $a_1 = \mathcal{W}$, we get

$$\mathcal{W}^{L(L+1)+1} = \mathcal{W}^{L(L+1)}. \tag{17}$$

This equation holds as an operator equation, over the 3^L dimensional vector space $\mathcal{V}$, the space of all stable configurations. This has the remarkable consequence that eigenvalues of $\mathcal{W}$ are either 0 or 1. $\mathcal{W}$ is a stochastic matrix. As there is a finite probability that the maximal state $|2, 2, \ldots, 2\rangle$ can be reached from any stable configuration, all recurrent configurations are reachable from any other, and there is a unique steady state for this Markov process. This implies that there is only one eigenvector of $\mathcal{W}$ with eigenvalue 1. Let us call it $|\Psi_{st}\rangle$ Thus all the other $3^L - 1$ eigenvalues of $\mathcal{W}$ are 0.

From Eq.(17), it follows that for any vector $|\phi\rangle$ in $\mathcal{V}$, $\mathcal{W}^{L(L+1)}|\phi\rangle$ is proportional to $|\Psi_{st}\rangle$. Also, as $\mathcal{W}$ preserves the probability sum, for any initial basis vector $|\mathcal{C}\rangle$, we get

$$\mathcal{W}^{L(L+1)} |\mathcal{C}\rangle = |\Psi_{st}\rangle. \tag{18}$$

A simpler way to determine $|\Psi_{st}\rangle$ is to use the equation

$$|\Psi_{st}\rangle = a_1 |2, 2, 2 \ldots 2\rangle. \tag{19}$$

This equation is interesting, as it says that if we take the initial configuration as the one in which each σ_i has the highest allowed value, and just add one more grain, the stochastic evolution would result in different final stable configurations with probabilities exactly equal to their values in the steady state. To prove this result, we need only note that for any stable configuration,

$$a_i |\ldots, \sigma_i = 2, \ldots\rangle = a_i^3 |\ldots, \sigma_i = 0, \ldots\rangle. \tag{20}$$

Then we can write

$$a_1 |2, 2, 2 \ldots 2\rangle = a_1^3 |0, 2, 2 \ldots\rangle = a_1 a_2 |0, 2, 2 \ldots\rangle = a_1 a_2^3 |0, 0, 2 \ldots\rangle \tag{21}$$

$$= a_1^2 a_2 a_3 |0, 0, 2 \ldots\rangle = a_1^2 a_2 a_3^3 |0, 0, 0 \ldots\rangle = etc., \tag{22}$$

finally giving

$$a_1 |2, 2, 2 \ldots\rangle = a_1^2 a_2^2 a_3^2 \ldots a_L^2 |0, 0, 0 \ldots\rangle = |\Psi_{st}\rangle. \tag{23}$$

In fact, it is possible to get a result stronger than Eq.(18). We note that most of the stable configurations are transient, and do not occur in the steady

state. The different recurrent configurations of the Oslo rice-pile have been characterized [15]. For any recurrent configuration $\mathcal{C}$ with m grains, arguing as above, we can show that

$$a_1^{L(L+1)-m}|\mathcal{C}\rangle = a_1^2 a_2^2 a_3^2 \ldots a_L^2 |0,0,0\ldots0\rangle \tag{24}$$

Since the lowest allowed value of m in recurrent configurations is $L(L+1)/2$, corresponding to the configuration $|1,1,1,\ldots\rangle$, we get that for any recurrent configuration $\mathcal{R}$,

$$a_1^{L(L+1)/2}|\mathcal{R}\rangle = |\Psi_{st}\rangle \tag{25}$$

Let $X(t)$ be some scalar observable whose value depends on the stable configuration of the pile after t grains have been added. Then any particular evolution of the rice-pile will generate a stochastic time-series $\{X(j)\}, j = 1, 2, \ldots$. For example, $X(t)$ may be the total height of the pile h_1 at time t, or the number of grains in the pile. Clearly, this time-series has nontrivial correlations. For example, $h_1(t+1) \leq h_1(t)+1$. For any two such observables $X(t)$ and $Y(t)$, we define the connected time-dependent correlation function $C_{XY}(\tau)$ in the steady state as

$$C_{XY}(\tau) = \langle X(t)Y(t+\tau)\rangle - \langle X(t)\rangle\langle Y(t+\tau)\rangle \tag{26}$$

However, from Eq.(25), it follows that the conditional expectation value of $Y(t + \tau)$, given that the observable X has a particular value $X(t)$, must be equal to its unconditional expectation value, whenever $\tau \geq L(L + 1)/2$. Thus, $C_{XY}(\tau)$ is exactly equal to zero for $\tau \geq L(L+1)/2$, for all observables X and Y.

For $\tau < L(L + 1)/2$, the correlation function $C_{XY}(\tau)$ is not always zero. This correlation function can be nontrivial, even though $\mathcal{W}$ has only one nonzero eigenvalue, because $\mathcal{W}$ is non-hermitian. This is easily checked for the simple cases $L = 2, 3$.

5   Generalizations

It is straightforward to generalize the discussion given above to more general abelian sandpile model (ASM) [19]. Consider a stochastic sandpile model defined on a set on N vertices labelled by integers 1 to N. At each site is a height variable z_i, which takes non-negative integer values. There is an $N \times N$ integer matrix Δ, which specifies how particles are transferred

under topplings: If z_i is greater than or equal to a site-dependent threshold value $z_{i,c}$, there is a toppling at site i, the heights at all sites j are updated according to the rule

$$z_j \to z_j - \Delta_{i,j}. \tag{27}$$

The matrix Δ is assumed to satisfy the good-behavior conditions, same as for ASM [19]. After each toppling at a site i, the critical threshold at that site is randomly reset to a new value $z_{ic} = r$, with r lying in a finite range of values $z_{i,c}^{min}$ and $z_{i,c}^{max}$ from a known probability distribution $Prob_i(z_{ic} = r)$. This distribution functions can be different at different sites. Note that while the threshold is randomly reset, we are assuming that the matrix Δ specifying how particles are transferred under toppling does not change. Without any loss of generality, we may assume that $z_{i,c}^{min} = \Delta_{i,i}$. This corresponds to setting the minimum allowed value of z_i to be zero for all sites i.

It is easy to see that this model is still abelian, and that just like the ASM with deterministic rules, there are forbidden subconfigurations in this model. In fact the forbidden subconfigurations are exactly the same as in the deterministic ASM specified by the toppling martix Δ [19]. The number of allowed configurations in the stochastic model is larger, as the maximum allowed value of z_i is larger. We define a new matrix $\tilde{\Delta}$ such that

$$\tilde{\Delta}_{i,j} = \Delta_{i,j}, \quad for \quad i \neq j; \tag{28}$$

and

$$\tilde{\Delta}_{i,i} = z_{i,c}^{max}. \tag{29}$$

Then, it is easy to see that any allowed configuration for a deterministic ASM with toppling matrix $\tilde{\Delta}$ is also a recurrent configuration of the stochastic model, and vice versa. The number of recurrent configurations is given by $det(\tilde{\Delta})$. However, all configurations are not equally likely to occur in the steady state.

For the 1-dimensional model studied in this paper, the only non-zero entries of $\tilde{\Delta}$ are $\tilde{\Delta}_{i,i} = 3$, for $i < L$, and $\tilde{\Delta}_{L,L} = 2$, and $\tilde{\Delta}_{i,i\pm1} = -1$. The determinant is easy to calculate, recovering the results of [15].

Again, for this stochastic model, we define operators $\{a_i\}$ corresponding to adding a particle at site i, and relaxing. Then these operators commute with each other. In addition, they satisfy the equations

$$a_i^{z_{i,c}^{max}} = a_i^{(z_{i,c}^{max} - z_{i,c}^{min})} \prod_{j \neq i} a_j^{-\Delta_{i,j}}. \tag{30}$$

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This equation is a generalization of the Eq.(14) in the previous section. As before, we find that the eigenvalues of the operators $\{a_i\}$ are either all zero, or same as that for the deterministic ASM with toppling matrix Δ. The steady state $|\Psi_{st}\rangle$ of the stochastic model is given by

$$|\Psi_{st}\rangle = \prod_i a_i^{(z_{i,c}^{max} - z_{i,c}^{min})}|\Phi_{st}\rangle, \tag{31}$$

where $|\Phi_{st}\rangle$ is the steady state of the deterministic ASM with toppling matrix Δ. To prove this statement, we note that if for the deterministic model $a_i|\mathcal{C}\rangle = |\mathcal{C}'\rangle$, then for the stochastic model we must have

$$a_i \prod_i a_i^{(z_{i,c}^{max} - z_{i,c}^{min})}|\mathcal{C}\rangle = \prod_i a_i^{(z_{i,c}^{max} - z_{i,c}^{min})}|\mathcal{C}'\rangle, \tag{32}$$

as each toppling which is required to relax from $\mathcal{C}$ to $\mathcal{C}'$ in the deterministic case, will also be allowed in the stochastic case with the larger threshold values, so long as there are enough extra untoppled particles present[18]. Then, if $|\Phi_{st}\rangle$ is an eigenvector of all a_i with eigenvalue 1 in the deterministic model, $|\Psi_{st}\rangle$ would be so for the stochastic one.

It may be noted that this operator algebra does not depend on the probability distribution of the random thresholds, and depends only on the values of $z_{i,c}^{max}$ and $z_{i,c}^{min}$, and the toppling matrix Δ. All nontrivial eigenvalues of $\{a_i\}$ are solutions of algebraic equations (30), and are the same as for the corresponding ASM with toppling matrix Δ.

The Oslo rice-pile model is special in that in this case, the deterministic ASM with same matrix Δ has only one recurrent configuration, and the $|\Phi_{st}\rangle = |1, 1, 1, \ldots\rangle$. This explains why $\mathcal{W}$ has only one non-trivial eigenvalue.

The distribution of avalanche sizes, and the weights of different configurations in the steady state certainly do depend on the probability distribution of thresholds. These are rather difficult to calculate explicitly even for the original one-dimensional model, except for very small values of L. Exact calculation of these for general L seems more difficult. This seems to be an interesting open problem.

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