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W-INFINITY WARD IDENTITIES AND CORRELATION FUNCTIONS IN THE $C = 1$ MATRIX MODEL

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ABSTRACT

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We explore consequences of W-infinity symmetry in the fermionic field theory of the $c = 1$ matrix model. We derive exact Ward identities relating correlation functions of the bilocal operator. These identities can be expressed as equations satisfied by the effective action of a *three* dimensional theory and contain non-perturbative information about the model. We use these identities to calculate the two point function of the bilocal operator in the double scaling limit. We extract the operator whose two point correlator has a *single* pole at an (imaginary) integer value of the energy. We then rewrite the W-infinity charges in terms of operators in the matrix model and use this derive constraints satisfied by the partition function of the matrix model with a general time dependent potential.

1. Introduction.

In recent papers [1,2] we have developed a formulation of the $c = 1$ matrix model based on a gauge theory of fermions interacting with a background gauge field. This formulation was used to understand the symmetries of the theory as well as the general Ward identities relating correlation functions. The natural variables in the fermion theory are *gauge covariant* biolocal operators, $\Phi(x, y, t) \equiv \psi(x, t)\psi^\dagger(y, t)$, involving fermions at separated points in eigenvalue space. The gauge theory formulation led to a natural phase space bosonization of the model in terms of variables belonging to a coset space, $W^+_{1+\infty}/H$, where H denotes the group of transformations which commute with the matrix of eigenvalues of $\Phi(x, y, t)$.

In this paper we explore in greater detail the Ward identities and their consequences. The main result is an iteration formula for the n-point correlation function of the operator $W(p, q, t) = 1/2 \int dx\ e^{ipx}\ \psi^\dagger(x + q/2, t)\ \psi(x - q/2, t)$ which determines it in terms of all the lower order correlation function. The one point function cannot be determined in this fashion and needs to be calculated from the underlying theory, as an input to this system of equations. As a consequence we also present an exact equation for the effective action of this theory in terms of a three dimensional field. Our equations are completely non-perturbative. As an explicit example we calculate the two point correlation function of the bilocal operator exactly in the double scaling limit. From this general result we also extract the two point correlator of the operators $W_{rs} \equiv \int dx\ \psi^\dagger(x, t)\{(x + i\partial_x)^r, (x - i\partial_x)^s\}\psi$ and show that this two point function has a single pole at energy $E = i(r - s)$. Poles at discrete imaginary values of energy have been observed in the two point function of matrix model operators [3,4,5] and interpreted as signatures of the discrete states of the continuum $c = 1$ theory coupled to gravity. However, the correlation functions calculated in these works come as a *sum* of poles, so that the corresponding operators couple to a linear combination of these discrete states. Our result thus offers a way of isolating the *specific* operator which couples to a single such state.

We then proceed to show how the general bilocal operator is related to operators in the original matrix model by using the operator equations of motion. This allows us to write down the Ward identities as differential equations satisfied by the partition function considered as a functional of a general matrix model potential, i.e. Schwinger -Dyson equations of the matrix model itself. We expect that the corresponding differential operators will satisfy a $W^{+}_{1+\infty}$ algebra. While we have not proved this in general, we have checked by explicit computation that this is indeed true for the first four equations. These are the "string equations" for this model and may be regarded as generalizations of the W_p constraints of the p-matrix models. These string equations should enable us to understand renormalization group flows between various multicritical points.

2. Gauge Theory formulation of the $c = 1$ matrix model.

The double scaled fermion field theory is described by the action

$$S = \int dt\ dx\ dy\ \psi^{\dagger}(x,t)[i\partial_t \delta(x-y) + \bar{A}(x,y,t)]\psi(y,t) \tag{2.1}$$

where $\bar{A}(x,y,t) = \frac{1}{2}(\partial_x^2 + x^2)\ \delta(x-y)$ for the critical $c = 1$ matrix model which describes the noncritical string moving in two dimensions Let us introduce the operator $|\Psi(t)>= \sum_n c_n|n>$, where c_n are operators and $\{|n>\}$ denotes a complete basis in the single particle Hilbert space. The fermion field $\psi(x,t)$ is identified with the components of this ket in the x-basis, $<x|\Psi(t)>$. Introducing the background gauge field $\bar{A}(t)$ with the components $\bar{A}(x,y,t) \equiv <x|\bar{A}(t)|y>$ we can write (2.1) in the compact notation

$$S = \int <\Psi(t)|i\partial_t + \bar{A}(t)|\Psi(t)> \tag{2.2}$$

This action has the background gauge symmetry

$$\begin{aligned} &|\Psi(t)> \rightarrow \mathcal{U}(t)|\Psi(t)> \\ &\bar{A} \rightarrow \mathcal{U}(t)\bar{A}\mathcal{U}^{\dagger}(t) + i\mathcal{U}(t)\partial_t\mathcal{U}^{\dagger}(t) \end{aligned} \tag{2.3}$$

where $\mathcal{U}(t)$ is a unitary operator in the single particle Hilbert space.

We now introduce the operator $W(p,q,t)$ in a "classical" phase space (p,q) by the formula

$$W(p,q,t) = 1/2 \int dx\ e^{ipx}\ \psi^\dagger(x+q/2,t)\ \psi(x-q/2,t) \tag{2.4}$$

This operator may be also regarded as a generating function for the generators $W^{(r,s)}$ of the $W^+_{1+\infty}$symmetry algebra of the critical $c=1$ problem [1]. The $W^{(r,s)}$ can be obtained from $W(p,q,t)$ by the formula

$$W^{(r,s)} = e^{-(r-s)t} \int dp\ dq\ F_{rs}(\frac{p-q}{\sqrt{2}}, \frac{p+q}{\sqrt{2}})W(p,q,t) \tag{2.5}$$

where

$$F_{rs}(a,b) = 2\cos(\frac{1}{2}ab)\ (i\partial_a)^r(i\partial_b)^s\delta^{(2)}(a,b) \tag{2.6}$$

For example

$$W^{(1,1)} = (\partial_q^2 - \partial_p)^2 W(p,q,t)|_{p=q=0} \tag{2.7}$$

The operators $W(p,q,t)$ satisfy a simple algebra

$$[W(p,q,t), W(p',q',t)] = i\ \sin(\frac{\hbar}{2}(pq'-qp'))\ W(p+p',q+q',t) \tag{2.8}$$

In (2.8) we have explicitly exhibited the string coupling constant $g_{str} \sim \hbar$, so that in the limit $\hbar \to 0$ we regain the structure constants of the algebra of area-preserving diffeomorphisms on the plane. From (2.8) we can deduce the algebra of the operators $W^{(r,s)}$ defined in (2.5). The symmetries generated by these operators have been previously discussed in [1] and [6] in the matrix model and by [7] and [8] in the continuum formulation. The quantum deformation of the classical algebra of area-preserving diffeomorphisms has been discussed in [9]. Henceforth we shall work with $\hbar$ set to unity unless necessary for clarity.

One can derive the equation of motion for the operator $W(p,q,t)$ using the equation of motion for the fermion field. This equation is

$$\mathrm{D}_t W(p,q,t) \equiv \partial_t W(p,q,t) + \int dp' dq' \sin\frac{1}{2}(pq' - qp')\, \bar{A}(p',q',t)\, W(p+p', q+q', t) = 0 \tag{2.9}$$

where we have defined

$$\bar{A}(p,q,t) \equiv \frac{1}{\pi} \int dx\ e^{-ipx}\ \bar{A}(x+q/2, x-q/2, t) \tag{2.10}$$

The second term in (2.9) can be written as

$$\int dp' dq' \int dp'' dq''\ C^{p'',q''}_{p,q;p',q'} \bar{A}(p',q',t)\ W(p'',q'',t) \tag{2.11}$$

where

$$C^{p'',q''}_{p,q;p',q'} = sin\frac{1}{2}(pq' - p'q)\ \delta(p+p'-p'')\delta(q+q'-q'') \tag{2.12}$$

is the structure constant of the algebra (2.8). Thus D_t is really the W-infinity covariant derivative with $\bar{A}(p,q,t)$ as the (background) gauge field. For the critical $c=1$ theory

$$\bar{A}(p,q,t) = (\partial_q^2 - \partial_p^2)\delta(p)\ \delta(q) \tag{2.13}$$

In this case (2.9) becomes

$$[\partial_t + p\partial_q + q\partial_p] W(p,q,t) = 0 \tag{2.14}$$

Equations (2.4) and (2.8) can be understood in a framework that formulates quantum mechanics in terms of a classical phase space which is due to Wigner [10].

To explain this, let us introduce the Heisenberg-Weyl group with elements

$$g_{p,q} = e^{-i(q\hat{P} - p\hat{X})} \tag{2.15}$$

where (p, q) is a point in the plane $\mathbf{R}^2$ and $\hat{P}$ and $\hat{X}$ satisfy the commutation relation $[\hat{P}, \hat{X}] = -i\hbar$. The group multiplication law is given by

$$g_{p,q} g_{p',q'} = e^{i\hbar(pq' - qp')/2} g_{p+p',q+q'} \tag{2.16}$$

Now noting that $g_{p,q} = e^{-iq\hat{P}/2} e^{ip\hat{X}} e^{-iq\hat{P}/2}$, we see that

$$W(p, q, t) = 1/2 \int dx \; \psi^\dagger(x, t) \; g_{p,q} \; \psi(x, t) \tag{2.17}$$

One can now immediately see that the algebra (2.8) is an immediate consequence of the group law (2.16).

3. Symmetries

Among the background gauge transformations of the gauge field $\bar{A}$ (2.3) there are special ones which leave it invariant. These are symmetries of the action (2.2) generated by $W^{(r,s)}$. Here we ould like to discuss these in terms of the operators $W(p, q, t)$ introduced in (2.4). Let us consider the transformations

$$\begin{aligned} \delta\psi(x, t) &= -\frac{1}{2} i \int dp \; dq \; \epsilon(p, q, t) \; e^{ip(x - \frac{1}{2}q)} \psi(x - q, t) \\ \delta\psi^\dagger(x, t) &= \frac{1}{2} i \int dp \; dq \; \epsilon(p, q, t) \; e^{ip(x + \frac{1}{2}q)} \psi^\dagger(x + q, t) \end{aligned} \tag{3.1}$$

generated on the fermion field by $W(p, q, t)$. Equations (3.1) express the action of the Heisenberg-Weyl group on the fermion field. Under these transformations the action (2.2) changes by

$$\delta S = \int dtdpdq\ W(p,q,t)\mathrm{D}_t^\dagger \epsilon(p,q,t) \tag{3.2}$$

where

$$\mathrm{D}_t^\dagger \epsilon(p,q,t) \equiv \partial_t \epsilon(p,q,t) + \int dp'dq'\ \sin\frac{1}{2}(pq'-qp')\ \bar{A}^*(p',q',t)\ \epsilon(p+p',q+q',t) \tag{3.3}$$

and we have used the hermiticity of $A(x,y,t)$, i.e. $A^*(x,y,t) = A(y,x,t)$. Thus the symmetries of the action are the set of $\epsilon(p,q,t)$'s which satisfy

$$\mathrm{D}_t^\dagger \epsilon(p,q,t) = 0 \tag{3.4}$$

For the critical $c = 1$ theory $\bar{A}^*(p,q,t) = \bar{A}(-p,-q,t)$ is given by (2.13). In this case (3.4) reads

$$[\partial_t + p\partial_q + q\partial_p]\epsilon(p,q,t) = 0 \tag{3.5}$$

The corresponding charges are

$$W[\epsilon] = \int dpdq\ \epsilon_0(p,q,t)W(p,q,t) \tag{3.6}$$

where $\epsilon_0(p,q,t)$ is a solution of (3.4). It is straightforward to check that (3.4) is satisfied by $\epsilon(p,q,t) = e^{-(r-s)t}F_{rs}(\frac{p-q}{\sqrt{2}}, \frac{p+q}{\sqrt{2}})$, thus verifying that the $W^{(r,s)}$ generate the symmetry algebra in the critical $c = 1$ model.

4. Ward Identities and the three dimensional effective action.

For more general $\epsilon(p,q,t)$, not necessarily satisfying (3.4), the variation of the action in (3.2) leads to Ward identities for the correlation functions of $W(p,q,t)$. We now turn to a discussion of these.

Consider a general n-point function of the $W(p,q,t)$'s

$$G_n(p_i,q_i,t_i) = <\mu|T(W(p_1,q_1,t_1),W(p_2,q_2,t_2)\cdots W(p_n,q_n,t_n))|\mu> \tag{4.1}$$

where by $|\mu>$ we have denoted the filled fermi vacuum. In the operator formalism the Ward identities can be derived by using the equation of motion for $W(p,q,t)$, (2.9), together with the symmetry algebra (2.8). For example, for the two point function we have

$$\begin{aligned}\mathrm{D}_t G_2(p,q,t;p',q',t') = \mathrm{D}_t\theta(t-t') <\mu|W(p,q,t)W(p',q',t')|\mu> +\\ \mathrm{D}_t\theta(t'-t) <\mu|W(p',q',t')W(p,q,t)|\mu>\end{aligned}$$

Using the equation of motion (2.9) and the symmetry algebra (setting $\hbar = 1$) we get

$$\begin{aligned}\mathrm{D}_t G_2(p,q,t;p',q',t') &= \delta(t-t') <\mu|[W(p,q,t),W(p',q',t')]|\mu> \\ &= i\delta(t-t')\sin\frac{1}{2}(pq'-p'q)\ G_1(p+p',q+q',t)\end{aligned} \tag{4.2}$$

Alternatively one may use the functional integral formalism to derive the Ward identities. In the functional formalism the n-point function (4.1) is given by the expression

$$G_n(p_i,q_i,t_i) = \frac{1}{\mathcal{Z}}\int \mathcal{D}\psi^\dagger\mathcal{D}\psi e^{iS}\prod_{i=1}^{n} W(p_i,q_i,t_i) \tag{4.3}$$

where $\mathcal{Z}$ is the partition function. To derive the Ward identites we make the change of variables $\psi \to \psi + \delta\psi$, $\psi^\dagger \to \psi^\dagger + \delta\psi^\dagger$ where the $\delta\psi$ and $\delta\psi^\dagger$ are given in (3.1).

Under this change of variables, the action S changes as in (3.2) while the change in $W(p,q,t)$ is

$$\delta W(p,q,t) = \int dp'dq' \, \sin\frac{1}{2}(pq' - qp') \; \epsilon(p',q',t')W(p+p',q+q',t) \tag{4.4}$$

Thus we get

$$\begin{aligned} 0 = \frac{1}{\mathcal{Z}} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; e^{iS}[i\delta S \; & \prod_{i=1}^{n} W(p_i,q_i,t_i) + \\ & \sum_{i=1}^{n} W(p_1,q_1,t_1)\cdots\delta W(p_i,q_i,t_i)\cdots W(p_n,q_n,t_n)] \end{aligned} \tag{4.5}$$

Substituting (3.2) and (4.4) in (4.5) we get the Ward identity

$$\begin{aligned} &\mathrm{D}_t G_{n+1}(p,q,t;p_i,q_i,t_i) \\ &= i\sum_{j=1}^{n} \delta(t-t_j) \, \sin\frac{1}{2}(pq_j - qp_j) \; G_n(p_1,q_1,t_1;\cdots;p_j+p,q_j+q,t_j;\cdots;p_n,q_n,t_n) \end{aligned} \tag{4.6}$$

Equation (4.6) gives a closed set of first order differential equations for the multipoint correlation function of the $W(p,q,t)$'s. It is appropriate to point out that this has happened because not only is the change in $W(p,q,t)$ under (3.1) proportional to another W operator, but more importantly the variation of the action also involves a $W(p,q,t)$ (equation (3.2)). The situation may be compared with that for the correlation functions of the Wilson loop operators in large-N QCD where a closed set of Dyson-Schwinger equations can be derived precisely because the variation of both the Wilson loop operator as well as the action involves another Wilson loop operator. Starting with the one point function which can be calculated in the fermion field theory (2.1), equation (4.6) can be used repeatedly to solve for the n-point function.

These Ward identities can be summarized in terms of a differential equation satisfied by the generating functional

$$Z[J(p,q,t);\bar{A}] \equiv \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \ \exp\ [iS + \int dpdqdt\ J(p,q,t)W(p,q,t)] \qquad (4.7)$$

The equation is

$$[i\mathrm{D}_t \frac{\delta}{\delta J(p,q,t)} + \int dp'dq'\ \sin\frac{1}{2}(p'q-q'p)\ J(p'-p,q'-q,t)\ \frac{\delta}{\delta J(p',q',t)}]Z[J;\bar{A}] = 0 \qquad (4.8)$$

The above equations lead to an equation satisfied by the effective action defined by a Legendre transformation

$$\Gamma[\Phi(p,q,t);\bar{A}] \equiv -\log\ Z[J;\bar{A}] + \int dp\ dq\ dt\ J(p,q,t)\Phi(p,q,t) \qquad (4.9)$$

Since

$$\frac{\delta \log\ Z}{\delta J} = \Phi \qquad \frac{\delta\Gamma}{\delta\Phi} = J \qquad (4.10)$$

we have

$$i\mathrm{D}_t\Phi(p,q,t) + \int dp'dq'\ \sin\frac{1}{2}(p'q-q'p)\ \Phi(p+p',q+q',t)\ \frac{\delta}{\delta\Phi(p',q',t)}\ \Gamma[\Phi;\bar{A}] = 0 \qquad (4.11)$$

Equation (4.11) gives an exact formulation of the $c = 1$ matrix model in terms of a three dimensional field $\Phi(p,q,t)^\star$.

For the critical $c = 1$ model (4.6) reduces to

$$\begin{aligned}&[\partial_t + p\partial_q + q\partial_p]G_{n+1}(p_1,q_1,t;p_i,q_i,t_i) = \\ &i\sum_{j=1}^{n}\delta(t-t_j)\ \sin\frac{1}{2}(pq_j - qp_j)\ G_n(p_1,q_1,t_1;\cdots;p_j+p,q_j+q,t_j;\cdots;p_n,q_n,t_n)\end{aligned} \qquad (4.12)$$

Because of time translation invariance, in this case the one point function G_1 is

$\star$ A three dimensional action for the discrete states of the $c = 1$ Liouville theory has been proposed in [8].

independent of time, so that we have the equation for G_1

$$(p\partial_q + q\partial_p)G_1(p,q,t) = 0 \tag{4.13}$$

This implies that $G_1(p,q,t)$ is a function of $(p^2 - q^2)$ only, as may be explicitly verified by a computation of G_1 in the critical fermion field theory.

Before ending this general discussion on Ward identities we would like to mention that we have so far obtained identities satisfied by correlation functions of the generators of W-infinity transformations of the theory. There are other operators in the fermion field theory besides these, viz. operators which transform homogeneously under W-infinity transformations. Examples of such operators are the group elements of W-infinity. There would be Ward identities satisfied by correlation functions of such operators as well. The situation is analogous to WZNW models where apart from the currents there are the primary fields $g(z,\bar{z})$ which are group elements and there are the Knizhnik-Zamolodchikov equations involving the correlators of these primary fields†.

5. The two-point function

In this section we focus our attention on the Ward identities for the $c = 1$ critical theory. Of particular interest is the two point function $G_2(p,q,t;p',q',t')$ of the $W(p,q,t)$'s since this contains information about the spectrum of the theory. In this case, equation (4.6) gives

$$[\partial_t + p\partial_q + q\partial_p]G_2(p,q,t;p',q',t') = i\delta(t-t')\sin\frac{1}{2}(pq' - qp')\;G_1(p+p',q+q') \tag{5.1}$$

where we have used the fact that G_1 is independent of time. In terms of the fourier

† We would like to thank S. Shatashvili and E. Verlinde for a discussion of this point.

transforms

$$W(p,q,E) \equiv \int dt\ e^{-iEt}\ W(p,q,t) \tag{5.2}$$

equation (5.1) can be written as

$$[iE + p\partial_q + q\partial_p]\tilde{G}_2(E;p,q;p',q') = 2\pi i\ \sin\frac{1}{2}(pq' - qp')\ G_1(p+p',q+q') \tag{5.3}$$

where we have defined $\tilde{G}_2$ by

$$\int dt\ dt'\ e^{-iEt}\ e^{-iE't'}\ G_2(p,q,t;p'q',t') = \delta(E+E')\ \tilde{G}_2(E;p,q,;p',q')$$

Note that since G_2 is symmetric in the interchange of the arguments $(p,q,t) \rightarrow (p',q',t')$, $\tilde{G}_2$ has the symmetry $\tilde{G}_2(E;p,q,;p',q') = \tilde{G}_2(-E;p'q';p,q)$.

The zero modes of the operator $[iE + p\partial_q + q\partial_p]$ are precisely the solutions of the equations of motion. They occur at $E_{rs} = i(r-s)$ and are given by the functions $F_{rs}(\frac{p-q}{\sqrt{2}}, \frac{p+q}{\sqrt{2}})$ which were introduced in (2.6).

In order to solve (5.3) we continue the energy E to complex values. Absorbing the i in the definition of E we have

$$\tilde{G}_2(E;p,q;p',q') = \frac{1}{E + p\partial_q + q\partial_p}[2\pi i\ \sin\frac{1}{2}(pq' - qp')\ G_1((p+p')^2 - (q+q')^2)] \tag{5.4}$$

Using the exponential representation of the inverse operator we get

$$\tilde{G}_2(E;p,q,;p',q') = \int_0^{(\text{sgn})\times\infty} ds\ e^{-sE}\ e^{-s(p\partial_q + q\partial_p)}\ [2\pi i\ (\sin\frac{Q}{2})\ G_1(R + R' + 2P)] \tag{5.5}$$

where we have defined

$$P \equiv pp' - qq' \qquad Q \equiv pq' - p'q \qquad R = p^2 - q^2 \qquad R' \equiv p'^2 - q'^2 \tag{5.6}$$

Now, noting that the operator $(p\partial_q + q\partial_p)$ is the generator of Lorentz rotations in the (p,q) space we can easily write (5.5) as

$$\tilde{G}_2(E;p,q;p',q') = 2\pi i \int\limits_0^{(sgnE)\times\infty} ds\ e^{-sE}\ \sin[\frac{1}{2}(Q\ cosh\ s - P\ sinh\ s)]$$
$$\times\ G_1(R + R' + 2P\ cosh\ s - 2Q\ sinh\ s) \tag{5.7}$$

The solution (5.7) satisfies the symmetry requirement $\tilde{G}(E;p,q,;p',q') = \tilde{G}(-E;p',q';p,q)$. This expression for $\tilde{G}_2$ exactly matches with the explicit calculation of the two point function for $q = q' = 0$ given in [4].

In a similar way one can obtain the three point function in terms of the two point function and the one point function and so on.

To proceed further we need to know the one point function G_1. This can be computed in the fermionic theory. Using the notations and conventions of [4]* the result may be written as

$$G_1(p,q) = \frac{1}{2}\int dx\ e^{ipx} \int\limits_\mu^\infty d\nu \sum_{\epsilon=\pm} \psi_\epsilon(\nu, x + \frac{1}{2}q)\psi_\epsilon(\nu, x - \frac{1}{2}q) \tag{5.8}$$

Taking a derivative of (5.8) with respect to μ we get

$$\partial_\mu G_1(p,q) = -\frac{1}{2}\int dx\ e^{ipx} \sum_{\epsilon=\pm} \psi_\epsilon(\mu, x + \frac{1}{2}q)\psi_\epsilon(\mu, x - \frac{1}{2}q) \tag{5.9}$$

Using the relation (A.12) in [4] one can show that

$$\partial_\mu G_1(p,q) = \frac{i}{4\sqrt{2}\pi}\mathrm{Re}\int\limits_0^\infty d\xi \frac{e^{i\mu\xi + \frac{i}{4}(p^2-q^2)\coth\frac{\xi}{2}}}{\sinh\frac{\xi}{2}} \tag{5.10}$$

The integral on the right hand side is well defined for complex $(p^2 - q^2)$ with a small positive imaginary part. The result is then analytically continued for real

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* Note that our x stands for the matrix eigenvalue while time is denoted by t. Also, the eigenvalue variable λ used in [4] is $\sqrt{2}x$.

$(p^2 - q^2)$. Now, taking a μ-derivative of (5.7) and using (5.10) one gets

$$\partial_\mu \tilde{G}_2(E;p,q;p',q') = -\frac{1}{2\sqrt{2}} \mathrm{Re} \int\limits_0^\infty \frac{d\xi}{\sinh\frac{\xi}{2}} \int\limits_0^{(\mathrm{sgn}\ E)\times\infty} ds\ e^{-sE} \sin\frac{1}{2}(-P\sinh\ s + Q\cosh\ s)$$
$$\times \exp\ [i\mu\xi + \frac{i}{4}(R + R' + 2P\cosh\ s - 2Q\sinh\ s)\coth\frac{\xi}{2}] \tag{5.11}$$

or,

$$\partial_\mu \tilde{G}_2(E;p,q,;p',q') = -\frac{1}{4\sqrt{2}} \mathrm{Im} \int\limits_0^\infty d\xi \frac{e^{i\mu\xi + \frac{i}{4}(R+R')\coth\frac{\xi}{2}}}{\sinh\frac{\xi}{2}}$$
$$\times \int\limits_0^{(\mathrm{sgn}E)\times\infty} ds\ e^{-sE} [\exp\ (\frac{iP\cosh(s-\frac{\xi}{2})}{2\sinh\frac{\xi}{2}} - \frac{iQ\sinh(s-\frac{\xi}{2})}{2\sinh\frac{\xi}{2}}) -$$
$$\exp\ (\frac{iP\cosh(s+\frac{\xi}{2})}{2\sinh\frac{\xi}{2}} - \frac{iQ\sinh(s+\frac{\xi}{2})}{2\sinh\frac{\xi}{2}})] \tag{5.12}$$

For $q = q' = 0$ this expression for $\partial_\mu \tilde{G}_2$ matches exactly with that given in [4]. As in that reference, the integration over s may be performed by changing variables from s to $y = e^s$ and expanding in inverse powers of y. The result is most naturally expressed in terms of the new variables $z_\pm, z'_\pm$ defined as

$$z_\pm = \frac{1}{\sqrt{2}}(p \pm q) \qquad z'_\pm = \frac{1}{\sqrt{2}}(p' \pm q') \tag{5.13}$$

In terms of these variables we finally get

$$\partial_\mu \tilde{G}_2(E;p,q,;p',q') = -\frac{1}{4\sqrt{2}} \mathrm{Im} \int\limits_0^\infty \frac{d\xi}{\sinh\frac{\xi}{2}}\ e^{i\mu\xi + \frac{i}{2}(z_+z_- + z'_+z'_-)\coth\frac{\xi}{2}}$$
$$\times [(\frac{z_+z'_-}{z_-z'_+})^{\frac{E}{2}}\ 2\pi e^{-i\frac{\pi|E|}{2}}\ \frac{\sinh\ |E|\frac{\xi}{2}}{\sin\ \pi|E|}\ J_{|E|}(\frac{\sqrt{z_+z_-z'_+z'_-}}{\sinh\ \frac{\xi}{2}}) +$$
$$2\sum_{n=1}^\infty i^n \{(\frac{z_+z'_-}{z_-z'_+})^{\frac{n}{2}}\ \frac{1}{(n-E)} + (\frac{z'_+z_-}{z'_-z_+})^{\frac{n}{2}}\ \frac{1}{(n+E)}\}\ \sinh\ n\frac{\xi}{2}\ J_n(\frac{\sqrt{z_+z_-z'_+z'_-}}{\sinh\ \frac{\xi}{2}})] \tag{5.14}$$

Not surprisingly, equation (5.14) shows more structure than the corresponding expression for $q = q' = 0$. In particular, we can now obtain the two point functions of the operators $\hat{W}_{rs}(E)$, defined in terms of $W(p, q, E)$ (eqn. (5.2)) by an expression similar to (2.5)

$$\hat{W}_{rs}(E) = 2(-i\partial_-)^r(-i\partial_+)^s[\cos(z_-z_+/2)W(p, q, E)]_{z_\pm=0} \tag{5.15}$$

where $\partial_\pm \equiv \frac{\partial}{\partial z_\pm}$. The limit $z_\pm \to 0$ in (5.15) has to be taken carefully. In fact, a sensible limit is defined by going to a region of the complex E plane in which the expressions are well-defined. To see how this comes about, let us compute the two point function $< \hat{W}_{rs}(E)W(p', q', -E) >$. We will assume that Re E is positive and that $r < s$. The other cases can be treated similarly. This two point function can be projected out from (5.14) by taking suitable derivatives with respect to $z_\pm$ as given in the definition of $\hat{W}_{rs}(E)$ in (5.15). We will calculate only the leading behavior for large μ, which corresponds to the planar limit. In this limit the terms in which the derivatives in (5.15) hit the cosine factor are lower order in μ. So we can set this factor to unity and directly take the derivatives $\partial_-^r\ \partial_+^s$ on $\tilde{G}_2$ in (5.14). To perform the limit $z_\pm \to 0$ we first work in the region Re $E > (s - r)$ where the limit exists and then analytically continue to other regions of the complex E plane*.

Let us now compute the two-point function $< \hat{W}_{rs}(E)\ \hat{W}_{r's'}(-E) >$. As before we will assume that E is positive and consider the case $r < s$. It turns out that the above two point function vanishes unless $(s - r) = (r' - s')$. Thus the region in complex E plane where the limit $z_\pm \to 0$ has to be performed in order to define $\hat{W}_{rs}(E)$ is precisely the region where $\hat{W}_{r's'}(-E)$ can be defined. We thus have a completely well defined two point function. In fact, we find the leading μ behavior

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* The limit $E \to Z$ is more tricky and one needs to be careful while taking this limit. In fact, just as for $q = q' = 0$, the poles in the expression in square brackets in (5.14) cancel in this limit.

to be

$$< \hat{W}_{rs}(E)\hat{W}_{r's'}(-E) >\sim \delta_{s-r,r'-s'}\mu^{\frac{1}{2}(r+s+r'+s')} \log \mu \, \frac{(s-r)}{E-(s-r)} \tag{5.16}$$

It is interesting to note that the above two point function contains a pole at a single integer value of the energy, which is actually a consequence of the fact that $\hat{W}_{rs}$ are eigenoperators of the hamiltonian. In fact a standard matrix model operator $O_m = \int dx\ x^m \psi^\dagger \psi$ may be written as binomial sum of $\hat{W}_{rs}$'s with $r + s = m$. This explains why the two point function $< O_m\ O_n >$ contains a sum over poles : the operators O_m are not energy eigenoperators, but linear combinations of energy eigenoperators.

It is also interesting to note that the above expression for the two point function is *exactly* reproduced in the analytic continuation of the theory described in [1]. In the phase space formulation one continues $t \to it$ as well as $p \to -ip$ (where p denotes the single particle momentum). Note that this analytic continuation is not the standard euclidean continuation. It leads to a hamiltonian $h = -\frac{1}{2}(p^2 + x^2)$. The fermi sea is filled from $-\infty$ to some level $-\mu$. The operators $\hat{W}_{rs}$ are well-defined in this framework and their two point function may be computed in a standard manner. In the planar limit the result is exactly (5.16). However, in this framework the "tachyon" pole corresponding to the first term in (5.14) is not obtained since the energy levels are all discrete. The equivalence of these two calculations is not entirely unexpected since the calculation of two point functions of operators like $\int dx\ x^m \psi^\dagger \psi$ using harmonic oscillator wavefunctions with imaginary frequencies [5] agrees with that of [4].

Finally we would like to emphasize that in the Minkowski section, E in (5.16) is purely imaginary so that in Minkowski space the poles are at purely imaginary values of the physical energy. Consequently they cannot be associated with states in the spectrum of the theory.

6. $c = 1$ Matrix Model with a general potential and String equations

At first sight the operators $W(p,q,t)$ for $q \neq 0$ do not have an obvious correspondence with operators of the original matrix model. This is because the most general invariant local operator in the matrix model is of the form Tr $\exp(ikM(t))$ whose expression in the fermion field theory is simply $W(k,0,t)$. In this section we will show that the $q \neq 0$ operators can be directly expressed as matrix model operators through the Heisenberg equations of motion.

Consider a matrix model with an arbitrary time dependent potential $U(M(t))$. The corresponding fermion field theory (2.1) has a background gauge field

$$\bar{A}(x,y,t) = \frac{1}{2}(\partial_x^2 - V(x,t)) \tag{6.1}$$

where $V(x,t)$ denotes the corresponding scaled potential. Corresponding to this gauge field the expression for $\bar{A}(p,q,t)$ defined in (2.10) is

$$\bar{A}(p,q,t) = (\partial_q^2 - V(i\partial_p,t))\delta(p)\ \delta(q) \tag{6.2}$$

Therefore, in this case the equation of motion (2.9) for $W(p,q,t)$ reads

$$[\partial_t + p\partial_q - \frac{i}{2}[V(-i\partial_p + \frac{1}{2}q,t) - V(-i\partial_p - \frac{1}{2}q,t)]]W(p,q,t) = 0 \tag{6.3}$$

This equation may be used to relate the operators

$$W_n(p,t) \equiv \partial_q^n W(p,q,t)|_{q=0} \tag{6.4}$$

to matrix model operators $O_n(t) \equiv \mathrm{Tr} M^n(t) = \int dx\ x^n\ \psi^\dagger(x,t)\psi(x,t)$. Using the above definition of $W_n(p,t)$ we can obtain, from (6.3)

$$\partial_t W_n(p,t) + pW_{n+1}(p,t) - i\sum_{m=0}^{n} \frac{\epsilon_{n-m}}{2^{n-m}} \binom{n}{m} V^{(n-m)}(-i\partial_p,t)W_m(p,t) = 0, \qquad n = 0,1,2,\cdots \tag{6.5}$$

where $\epsilon_k = 1$ for k odd and zero otherwise and we have used the notation $V^{(k)}(x,t)$ to denote $\partial_x^k V(x,t)$. Clearly we can use (6.5) repeatedly to relate $W_n(p,t)$ to

$W_0(p,t)$ by an equation which contains at most n derivatives with respect to time. The $W_0(p,t)$ may be used, in turn, to generate all the operators $O_n(t)$. In fact,

$$O_n(t) = 2(-i\partial_p)^n W_0(p,t)|_{p=0} \tag{6.6}$$

(The factor 2 comes from the definition of $W(p,q,t$ which has a factor of $\frac{1}{2}$ in it).

The above relations mean that the set of operators $W_n(p,t)$ which have no direct correspondence with the operators of the matrix model, is equivalent to the set of operators $\partial_t^n W_0(p,t)$ which are simply $\partial_t^n \mathrm{Tr} e^{ipM(t)}$. The role of q is replaced by time derivatives.

It is now clear that the equations (6.5) imply an infinite set of equations for the one point functions of the $O_n(t)$; the equation for $O_n(t)$ involving at most $(n+1)$ time derivatives. Equivalently, these equations may be expressed as an infinite set of constraints on the partition function, which we may write as

$$\mathcal{W}_n(t)\ \mathcal{Z} = 0, \qquad n = 0,1,2,\cdots \tag{6.7}$$

These may be written in terms of the time dependent couplings $V_k(t)$ defined as

$$V(x,t) = \sum_{k=1}^{\infty} V_k(t) x^k \tag{6.8}$$

Below we list the first few of these constraints

$$\mathcal{W}_0(t) = \partial_t \frac{\delta}{\delta V_0(t)} \tag{6.9}$$

$$\mathcal{W}_1(t) = \partial_t^2 \frac{\delta}{\delta V_1(t)} + \frac{1}{2}\sum_{k=1}^{\infty} k V_k(t) \frac{\delta}{\delta V_{k-1}(t)} \tag{6.10}$$

$$\mathcal{W}_2(t) = \partial_t^3 \frac{\delta}{\delta V_2(t)} + \sum_{k=1}^{\infty}[(k+2)V_k(t)\partial_t \frac{\delta}{\delta V_k(t)} + k\partial_t V_k(t) \frac{\delta}{\delta V_k(t)}] \tag{6.11}$$

$$\mathcal{W}_3(t) = \partial_t^4 \frac{\delta}{\delta V_3(t)} + \frac{3}{2}\sum_{k=1}^{\infty}[\frac{(k+2)(k+3)}{(k+1)} V_k(t)\partial_t^2 \frac{\delta}{\delta V_{k+1}(t)} + 3\frac{k(k+3)}{(k+1)}\partial_t V_k(t)\frac{\delta}{\delta V_{k+1}(t)}$$
$$+ \frac{3}{2}k\partial_t^2 V_k(t)\frac{\delta}{\delta V_{k+1}(t)} - \frac{3}{4}k(k-1)(k-2)V_k(t)\frac{\delta}{\delta V_{k-3}(t)}]$$
$$+ \frac{9}{4}\sum_{k,l=1}^{\infty}(k+l)V_k(t)V_l(t)\frac{\delta}{\delta V_{k+l-1}(t)} \tag{6.12}$$

Alternatively, these equations may be derived by performing a set of specific W transformations with time dependent coefficients on the fermionic fields, in such a way that the transformed action is of the same form as the original action but with a modified potential. These transformations are generated by the charges

$$I_n = \sum_{k=0}^{n} \int dx \; \psi^\dagger(x,t)\epsilon_k(x,t)[-i\partial_x]^k \psi(x,t) \tag{6.13}$$

where the functions $\epsilon_k(t)$ satisfy the following set of equations

$$i\partial_t \epsilon_m + \frac{1}{2}\partial_x^2 \epsilon_m - \frac{1}{2}\sum_{j=1}^{n-m} \epsilon_{j+m}(-1)^{j+1} i^j \binom{j+m}{j} \partial_x^j V(x,t) + i\partial_x \epsilon_{m-1} = 0 \qquad m = 1, 2, \cdots (n-1) \tag{6.14}$$

$$\partial_x \epsilon_{n-1} + \partial_t \epsilon_n = 0, \qquad \epsilon_n = \epsilon_n(t)$$

Clearly these equations can be used to determine any ϵ_j in terms of ϵ_{j+1} and thus express all of them in terms of ϵ_n. Since $\epsilon_n(t)$ is a function of t alone we can choose a fourier basis to write the constraints on on the partition function as

$$\mathcal{W}_n(E)Z = [\int dxdt \; e^{-iEt}[i\partial_t \epsilon_0 + \frac{1}{2}\partial_x^2 \epsilon_0 - \frac{1}{2}\sum_{j=1}^{n} \epsilon_j(x,t)(-1)^{j+1} i^j \partial_x^j V(x,t)]\frac{\delta}{\delta V(x,t)}]\mathcal{Z} = 0 \tag{6.15}$$

The operators $\mathcal{W}$ clearly form a closed algebra. Since the equations (6.15) are rewritings of the original Ward identities, this algebra is expected to be isomorphic to the W - algebra*. The role of spin is played by n, while the "mode number" is E. Note that the mode number can be any real number and not necessarily an integer.

For time independent potential and for time independent ϵ_n in (6.13), I_n reduce to the well known transformations which generate the isospectral flows of the Schrodinger operator

$$I_n^{(0)} = (-\partial_x^2 + V(x))^{\frac{n}{2}}|_+ \tag{6.16}$$

where $|_+$ denotes the part of (6.16) which involves only positive powers of the differential operator $-i\partial_x$. and the corresponding string equations are simply given by

$$\int dx \partial_x R_n(V(x)) \frac{\delta}{\delta V(x)} \mathcal{Z} = 0 \tag{6.17}$$

where $R_n(V(x))$ are the Gelfand-Dickii coefficients [12]. For the special case of time independent quadratic critical potential $V(x) = -x^2$ these isospectral deformations have been discussed in [11]. Time dependent transformations I_n and the corresponding algebra for the special case of $V(x) = -x^2$ have been given in [6]. A different set of identities (once again for time independent potentials) on the puncture operator, rather than the partition function, have been derived in [5].

The $\mathcal{W}_n$ equations would be probably useful in studying the flows between various multicritical points. In understanding such flows one must remember that the potential $V(x,t)$ used above is a *scaled* potential. In general, starting from some matrix model potential $U(M)$ in which the N dependence of the various terms appear in such a way that the large-N expansion is a topological expansion, the critical potential in the double scaling limit will contain only a single monomial in x. To obtain a scaled potential $V(x)$ which has various powers of x the N dependence of the terms in the original matrix model potential have to be different from the standard N dependence. This does not, however, prevent us from discussing flows

$\star$ We have explicitly checked this for $n = 0, 1, 2, 3$

between various multicritical points. Rather, this means that when we consider, e.g. flows between a critical point of the form $V(x) = -x^2$ and another critical point $V(x) = -x^4$ by an interpolating potential $V(x) = ax^2 + bx^4$, only the end points $a = -1, b = 0$ and $a = 0, b = -1$ correspond to random surface theories. For generic (a, b) the matrix model $\frac{1}{N}$ expansion no longer correspond to a string perturbation theory.

Note added : After this work was complete we learnt that I. Klebanov and A. M. Polyakov [13], D. Kutasov, E. Matrinec and N. Seiberg [14] and G. Moore, M. Plesser and S. Ramgoolam [15] have also considered the problem of constraining correlation functions using different methods.

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