UK-05-04
hep-th/0507080

Branes wrapping Black Holes

Sumit R. Das[a], Stefano Giusto[b], Samir D. Mathur[b],
Yogesh Srivastava[b], Xinkai Wu[a] and Chengang Zhou[a]

[a]*Department of Physics and Astronomy*
University of Kentucky, Lexington, KY 40506, U.S.A.

[b]*Department of Physics*
Ohio State University, Columbus, OH 43210, U.S.A.

das@pa.uky.edu,giusto@pacific.mps.ohio-state.edu
mathur@pacific.mps.ohio-state.edu, srivastava.21@osu.edu
xinkaiwu@pa.uky.edu,czhou@pa.uky.edu

Abstract

We examine the dynamics of extended branes, carrying lower dimensional brane charges, wrapping black holes and black hole microstates in M and Type II string theory. We show that they have a universal dispersion relation typical of threshold bound states with a total energy equal to the sum of the contributions from the charges. In near-horizon geometries of black holes, these are BPS states, and the dispersion relation follows from supersymmetry as well as properties of the conformal algebra. However they break all supersymmetries of the full asymptotic geometries of black holes and microstates. We comment on a recent proposal which uses these states to explain black hole entropy.

1 Introduction

The dynamics of stable extended branes in backgrounds containing fluxes have played an important role in exploring non-perturbative aspects of string theory. A particularly important class of such objects are dielectric branes which are extended objects formed by a collection of lower dimensional extended objects moving in a transverse dimension via Myers' effect [1]. Dielectric branes wrap *contractible* cycles in the space-time and therefore do not carry any net charge appropriate to its dimensionality, but has nonvanishing higher multipole moments. In a class of backgrounds (e.g. AdS space-times or their plane wave limits and certain D-brane backgrounds) the energy due to the tension of the dielectric brane is completely cancelled by the effect of the background flux, so that its dispersion relation is that of a massless particle, which is why they are called giant gravitons [2]-[5] . The energetics of such branes are usually determined by a classical analysis : however these branes are BPS states which renders the classical results exact.

Recently a different class of extended brane configurations have been found in the near-horizon geometry of four dimensional extremal black holes [6]-[8] constructed e.g. from intersecting $D4$ branes and some additional $D0$ charge. The near-horizon geometry is $AdS_2 \times S^2 \times K$ where K is a suitable six dimensional internal space (e.g. Calabi-Yau). These are branes of various dimensionalities wrapping *non-contractible* cycles of the compact directions. The branes which are wrapped on cycles in K have a net charge in the full geometry and are similar to giant gravitons - the tension of the brane is cancelled and one is left with the dynamics of gravitons. More interestingly, there are BPS $D2$ branes wrapped on the S^2 with a worldvolume flux providing a $D0$ brane charge, and posessing momentum along K. These branes do not have net $D2$ charges in the full geometry - they only contribute a net $D0$ charge. The ground state is static in global time, located at a radial coordinate determined by the $D0$ charge. These configurations preserve half of the enhanced supersymmetries of the near-horizon geometry, but *do not preserve any supersymmetry of the full geometry*.

In [8] it has been argued that such brane configurations provide a natural understanding of the entropy of the black hole background. The presence of a magnetic type flux in the compact direction means that such a static brane carries a nonzero momentum and is in fact in the lowest Landau level. This means that the ground state is degenerate. It turns out that this degeneracy is *independent of the $D0$ charge q_0 of the background*. The idea then is to "construct" a black hole by starting from the set of $D4$ branes and then add $D0$ charges. However the $D0$ charge appear as these $D2$ branes which wrap the S^2, and each such $D2$ has a ground state degeneracy. The problem then reduces to a partitioning problem of distributing a given $D0$ charge N among $D2$ branes - the various possible ways of doing this give rise to the entropy of the final black hole. This argument has been extended to "small" black holes in [9].

In this paper, we show that such brane configurations are quite generic not only in near-horizon geometries of black holes, but in the *full* asymptotically flat geometries of certain black hole microstates. While these are supersymmetric states in near-horizon regions of black holes and near-cap regions of microstates, they break all the supersymmetries of the asymptotically flat backgrounds. We find that in all cases they have a universal dispersion relation characteristic of threshold bound states : the total energy is just the sum of the energies due to various brane charges. In near-horizon regions this simple dispersion relation follows from supersymmetry and conformal algebras. However, we have not been able to find a good reason why the same dispersion relation holds in the full microstate geometries.

One key feature of the examples which we provide is that the background does not have to posess the same kind of charge as the brane itself. This feature could be relevant for the proposal of [8], though we have reservations about this proposal as it stands.

In section (2) we consider generic $AdS_m \times S^n \times \mathcal{M}$ space-times with a brane wrapped around S^n and moving along a AdS direction with momentum P and derive the universal dispersion relation

$$E = P + M_n \tag{1}$$

where M_n denotes the mass of the brane.

Specific examples of solutions of M theory and Type IIA string theory which lead to $AdS_m \times S^n \times \mathcal{M}$ spacetimes are described in section (3). Our main example involves five dimensional black strings in M theory compactified on T^6 (in section (3.1)) and their dimensional reduction to four dimensional black holes in IIA theory (in section (3.4)). In section (3.2) it is argued that the dispersion relation (1) follows from the underlying conformal algebra. This is explicitly shown for AdS_3, but the considerations should generalize to other AdS_m. These branes are static in global time. In Poincare time, they correspond to branes coming out of the horizon upto a maximum distance and eventually returning back to the horizon. However, we find that for AdS_3 (section (3.3)) and for AdS_2 (section (3.4.1), the relation (1) is valid *both in global and Poincare coordinates*. Furthermore in AdS_3 the *Poincare momentum is equal to the global momentum* P. We argue that the equality of global and Poincare energies and momenta signifies that the brane is in a highest weight state of the conformal algebra.

The second class of backgrounds where we find such brane configurations with identical dispersion relations are geometries which represent microstate of 2-charge and 3-charge systems. In the examples of section (3) the existence of these brane configurations appears to be special to near-horizon limits. This is because they are states of lowest value of the global AdS energy and not of the Poincare energy and it is the latter which coincides with the energy defined in the full asymptotically flat geometry. In contrast, the microstate geometries are asymptotically flat and go over to a *global patch* of AdS in the interior. The time in the asymptotic region continues to the *global time* of the interior AdS. Consequently, the notion of energy is unambigious.

In sections (4) and (5) we find that the lowest energy states of such branes are indeed static configurations with dispersion relations given by (1). Section (4) deals with a T-dualized version of the 2-charge microstate geometry with D3 branes wrapping the S^3. We show, in section (4.3) that the energy has an interesting implication for the conformal field theory dual. In section (4.4) we determine the spectrum of vibrations of the brane and find a remarkably simple equispaced excitation spectrum with spacing determined only by the AdS scale - reminiscent of the spectrum of giant gravitons found in [4]. Section (5) deals with analogous treatments of a special 3-charge microstate geometry.

In section (6) we calculate the field produced by such a probe brane in the 2-charge microstate geometry and show that this leads to a *constant* field strength in the asymptotic region, pretty much like a domain wall.

In section (7) we examine the supersymmetry properties of these brane configurations. Section (7.1) deals with the case of D2 branes in the background of 4d black holes, which is the background of section (3.4). We show that in the near horizon limit this D2 brane preserves half of the supersymmetries. We calculate the topological charge on the brane and show that the supersymmetry algebra leads to our simple dispersion relation. It is then explicitly shown that the brane does not preserve any supersymmetry of the full black hole geometry. In section (7.2) we investigate the question in the 2 charge microstate geometry and show that while the near-cap limit (which is again $AdS_3 \times S^3$) the brane preserves supersymmetry, it breaks all the supersymmetries of the full background.

In an appendix we examine the validity of the near-horizon approximation our brane trajectories for the case of 4D black holes and show that the approximation is indeed valid when the energy due to D0 charge of the D2 brane is smaller than the D2 brane mass.

2 Spherical branes $AdS \times S \times M$ space-times

The simplest space-times in which these brane configurations occur are of the form $AdS_m \times S^n \times \mathcal{M}$, where $\mathcal{M}$ is some internal manifold.

Let us first consider branes in M-theory backgrounds. The metric is given by

$$ds^2 = R^2[-\cosh^2\chi\ d\tau^2 + d\chi^2 + \sinh^2\chi\ d\Omega_{m-2}^2] + \tilde{R}^2 d\Omega_n^2 + g_{ij}dy^i dy^j \tag{2}$$

where $R, \tilde{R}$ are length scales, g_{ij} is the metric on $\mathcal{M}$ and $d\Omega_p^2$ denotes the line element on a unit S^p. We will choose coordinates (θ_k, φ) on S^{m-2} leading to a metric

$$d\Omega_{m-2}^2 = d\theta_1^2 + \sin^2\theta_1\ d\theta_2^2 + \sin^2\theta_2 \sin^2\theta_1\ d\theta_3^2 + \cdots + \sin^2\theta_{m-3} \cdots \sin^2\theta_1\ d\varphi^2 \tag{3}$$

The background could have m-form and n form gauge field stengths which will not be relevant for our purposes.

In addition, the background contains $(n+1)$-form gauge potentials ($n=2$ or $n=5$) of the form

$$A^{(n+1)} = A_i(y^i)\; d\omega_n \wedge dy^i \tag{4}$$

where $d\omega_n$ denotes the volume form on the sphere. We will see explicit examples of these geometries later.

Consider the motion of a n-brane which is wrapped on the S^n, rotating in the S^{m-2} contained in the AdS_m and in general moving along both χ and y^i. The bosonic part of the brane action is of the form

$$S = -\mu_n \int d^{n+1}\xi\; \sqrt{\det\, G} + \mu_n \int P[A^{(n+1)}] \tag{5}$$

where G denotes the induced metric, the symbol P stands for pullback to the worldvolume and μ_n is the tension of the n-brane.

Let us fix a static gauge where the worldvolume time is chosen to be the target space time and the worldvolume angles are chosen to the angles on S^n. The remaining worldvolume fields are $\chi, y^i, \theta_k, \varphi$. When these fields are independent of the angles on the worldvolume, the dynamics is that of a point particle. The Hamiltonian can be easily seen to be

$$H = \cosh\chi \sqrt{M_n^2 + \frac{P_\chi^2}{R^2} + \frac{\Lambda^2}{R^2\sinh^2\chi} + g^{ij}(P_i - M_n A_i)(P_j - M_n A_j)} \tag{6}$$

where M_n is the mass of the brane

$$M_n = \mu_n \tilde{R}^n \Omega_n \tag{7}$$

Ω_n being the volume of unit S^n. Λ denotes the conserved angular momentum on S^{m-2}

$$\Lambda^2 = p_{\theta_1}^2 + \frac{p_{\theta_2}^2}{\sin^2\theta_1} + \frac{p_{\theta_3}^2}{\sin^2\theta_1\sin^2\theta_2} + \cdots \frac{p_\varphi^2}{\sin^2\theta_1\cdots\sin^2\theta_{m-3}} \tag{8}$$

Consider the lowest energy state for some given $|\Lambda|$. In the internal space this means that $P_i = M_n A_i$. (This can be considered to be the description of the lowest Landau level in the classical limit). In AdS this has a fixed value of the global coordinate $\chi = \chi_0$ determined by minimizing the hamiltonian :

$$\sinh^2\chi_0 = \frac{|\Lambda|}{R\; M_n} \tag{9}$$

The motion on the S^{m-1} contained in AdS_{m+1} is along an orbit with

$$p_{\theta_k} = 0 \quad \theta_k = \frac{\pi}{2} \qquad k = 1\cdots(m-3) \tag{10}$$

The ground state energy is

$$E_{global} = \frac{|\Lambda|}{R} + M_n \tag{11}$$

Finally it is easy to check that in this state

$$\dot{\varphi} = 1 \tag{12}$$

While the above formulae have been given for M-branes, they apply equally well for $D5$ branes in $AdS_5 \times S^5$ backgrounds of Type IIB string theory. This in fact provides the simplest example of such configurations. We will give a general explanation below for the simple form (11) of the energy E.

3 Extremal Black Strings in M theory and Black Holes in String Theory

In this section we will provide some concrete examples where branes in $AdS \times S \times \mathcal{M}$ appear.

3.1 5D Black Strings and 4D Black Holes

A specific example of interest is the geometry of an extremal black string in M-theory copmpactified on T^6 whose coordinates are denoted by $y^1 \cdots y^6$. The background is produced by three sets of M5 branes which are wrapped on the directions $y\,y^3\,y^4\,y^5\,y^6$, $y\,y^1\,y^2\,y^5\,y^6$ and $y\,y^1\,y^2\,y^3\,y^4$ and carrying momentum q_0 along y. The numbers n_i and charges p_i of the M5 branes are related as

$$p_i = \frac{2\pi^2\, n_i}{M_{11}^3\, T^{(i)}}\,,\ i = 1, 2, 3 \tag{13}$$

where $T^{(1)}$, $T^{(2)}$, $T^{(3)}$ are the volumes of the 2-tori $(1, 2)$, $(3, 4)$ and $(5, 6)$.

The metric and gauge fields produced by this system of branes is

$$\begin{aligned} ds^2 &= h^{-1/3}\left[-dt^2 + dy^2 + \frac{q_0}{r}\,(dt - dy)^2\right] + h^{2/3}\,[dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)] \\ &+ h^{-1/3} \sum_{i=1,2,3} H_i\, ds^2_{T^{(i)}} \end{aligned} \tag{14}$$

$$A^{(3)} = \sin\theta\ d\theta d\phi\,[p_3\ \frac{y^5 dy^6 - y^6 dy^5}{2} + p_2\ \frac{y^3 dy^4 - y^4 dy^3}{2} + p_1\ \frac{y^1 dy^2 - y^2 dy^1}{2}] \tag{15}$$

where we have defined

$$h = H_1\, H_2\, H_3\,, \quad H_i = 1 + \frac{p_i}{r}\,, \, i = 1, 2, 3\,, \quad H_0 = 1 + \frac{q_0}{r} \tag{16}$$

$ds^2_{T^{(i)}}$ is the flat metric on the 2-torus of volume $T^{(i)}$.

3.1.1 Near-horizon limit with $q_0 = 0$

When $q_0 = 0$ the near-horizon limit is given by $AdS_3 \times S^2 \times T^6$. This may be seen by re-defining coordinates

$$y = \lambda x \qquad t = \lambda T \qquad r = 4\lambda\, u^2 \tag{17}$$

where we have defined

$$\lambda \equiv (p_1\, p_2\, p_3)^{1/3} \tag{18}$$

Then for $r \ll p_i$ and $q_0 = 0$ the metric (14) becomes

$$ds^2 = (2\lambda)^2\Big[\frac{du^2}{u^2} + u^2\,(-dT^2 + dx^2) + \frac{1}{4}\,(d\theta^2 + \sin^2\theta\, d\phi^2)\Big] + \frac{1}{\lambda}\sum_{i=1,2,3} p_i\, ds^2_{T^{(i)}} \tag{19}$$

which is $AdS_3 \times S^2 \times T^6$ in Poincare coordinates.

One can further continue the metric to global AdS_3 using the transformations

$$T = \frac{\cosh\chi\, \sin\tau}{\cosh\chi\, \cos\tau - \sinh\chi\, \sin\varphi}\,, \qquad x = \frac{\sinh\chi\, \cos\varphi}{\cosh\chi\, \cos\tau - \sinh\chi\, \sin\varphi}$$
$$u = \cosh\chi\, \cos\tau - \sinh\chi\, \sin\varphi \tag{20}$$

The resulting metric is

$$ds^2 = (2\lambda)^2\Big[d\chi^2 - \sinh^2\chi\, d\tau^2 + \cosh\chi^2\, d\varphi^2 + \frac{1}{4}\,(d\theta^2 + \sin^2\theta\, d\phi^2)\Big] + \frac{1}{\lambda}\sum_{i=1,2,3} p_i\, ds^2_{T^{(i)}} \tag{21}$$

We can now consider a M2 brane wrapped around the S^2 and apply the general results in equations (5)-(11). For some given momentum P_φ in the φ direction, the lowest value of the global energy is given by

$$E_{gs} = P_\varphi + 8\pi\mu_2\lambda^3 \tag{22}$$

which corresponds to a brane which is static in global time.

3.1.2 Near-horizon limit with $q_0 \neq 0$

The near-horizon geometry for $q_0 \neq 0$ is again $AdS_3 \times S^2 \times T^6$. For $r \ll q_0, p_i$ we have, from (14)

$$\begin{aligned} ds^2 &= \lambda^2[\rho'(-dT'^2 + dx'^2) + (dT' - dx')^2 + \frac{du'^2}{u'^2}] \\ &+ \lambda^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{\lambda}\sum_{i=1,2,3} p_i\, ds^2_{T^{(i)}} \end{aligned} \tag{23}$$

where we have defined

$$y = (\frac{\lambda^3}{q_0})^{1/2}\, x' \qquad t = (\frac{\lambda^3}{q_0})^{1/2}\, T' \qquad r = q_0 u' \tag{24}$$

With a further change of coordinates ([10])

$$\bar{T} - \bar{x} = e^{T'-x'} \qquad \bar{T} + \bar{x} = T' + x' + \frac{2}{u'} \qquad \bar{u} = \frac{\sqrt{u'}}{2}\, e^{-(T'-x')/2} \tag{25}$$

the metric reduces to the standard form of the Poincare metric on $AdS_3 \times S^2 \times T^6$

$$ds^2 = (2\lambda)^2 \Big[\frac{d\bar{u}^2}{\bar{u}^2} + \bar{u}^2\,(-d\bar{T}^2 + d\bar{x}^2) + \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)\Big] + \frac{1}{\lambda} \sum_{i=1,2,3} p_i\, ds^2_{T^{(i)}} \tag{26}$$

which is identical to the metric (19). As before, one can pass to the global AdS_3 using the formulae above.

Thus we see that whether the background has momentum in the y direction or not the near-horizon geometry has the local form $AdS_3 \times S^2 \times \mathcal{M}$, so the dynamics of the M2 brane will be similar in the two cases.

3.2 An explanation of the dispersion relation

For branes moving in flat space, we expect that the total energy E arises from the 'rest energy' M and the momentum P by a relation of the type $E = \sqrt{M^2 + P^2}$. But for the branes studied here we get a linear relation of the type $E = P + M$. The momentum P causes a shift in radial position of the brane, where the redshift factor is different, and in the end we end up with this simple energy law.

As we will see in a later section the brane configuration considered above is a BPS state which preserves half of the supersymmetries of the background. The dispersion relation then follows from the supersymmetry algebra.

It turns out that there is a simple derivation of this linear relation for branes in AdS spacetime based on the bosonic part of the conformal algebra. We will present this for the case of $AdS_3 \times S^n$. We suspect that similar considerations would hold for arbitrary AdS_m.

A n-brane wrapped on S^n becomes a point massive particle in AdS_3. Its lagrangian

$$L = -m[-\frac{\partial X^\mu}{\partial \tilde{\tau}} \frac{\partial X_\mu}{\partial \tilde{\tau}}]^{\frac{1}{2}} \tag{27}$$

where m is the mass of the brane and $\tilde{\tau}$ denotes the worldline parameter. The lagrangian is invariant under the $SL(2,R) \times SL(2,R)$ isometries of the background. Denoting the global AdS_3 coordinates by τ, χ, φ and defining $z = \tau + \varphi,\ \bar{z} = \tau - \varphi$ the generators are

$$\begin{aligned} L_0 &= i\,\partial_z \\ L_{-1} &= i\, e^{-iz}\, [\frac{\cosh 2\chi}{\sinh 2\chi}\partial_z - \frac{1}{\sinh 2\chi}\partial_{\bar{z}} + \frac{i}{2}\partial_\chi] \\ L_1 &= i\, e^{iz}\, [\frac{\cosh 2\chi}{\sinh 2\chi}\partial_z - \frac{1}{\sinh 2\chi}\partial_{\bar{z}} - \frac{i}{2}\partial_\chi] \end{aligned} \tag{28}$$

and

$$
\begin{aligned}
\bar{L}_0 &= i\,\partial_{\bar{z}} \\
\bar{L}_{-1} &= i\,e^{-i\bar{z}}\,[\frac{\cosh 2\chi}{\sinh 2\chi}\partial_{\bar{z}} - \frac{1}{\sinh 2\chi}\partial_z + \frac{i}{2}\partial_\chi] \\
\bar{L}_1 &= i\,e^{i\bar{z}}\,[\frac{\cosh 2\chi}{\sinh 2\chi}\partial_{\bar{z}} - \frac{1}{\sinh 2\chi}\partial_z - \frac{i}{2}\partial_\chi]
\end{aligned} \tag{29}
$$

We have the algebra

$$
[L_0, L_{-1}] = L_{-1}, \quad [L_0, L_1] = -L_1, \quad [L_1, L_{-1}] = 2L_0 \tag{30}
$$

$$
[\bar{L}_0, \bar{L}_{-1}] = \bar{L}_{-1}, \quad [\bar{L}_0, \bar{L}_1] = -\bar{L}_1, \quad [\bar{L}_1, \bar{L}_{-1}] = 2\bar{L}_0 \tag{31}
$$

The conserved quantities corresponding to these isometries are given by the replacement

$$
-i\partial_\mu \;\to\; P_\mu \tag{32}
$$

in (28),(29). The global coordinate energy E_{global} and momentum P_φ of the brane are related to the conserved charges under translations of t, φ

$$
E_{global} = -P_\tau \quad P = P_\varphi \tag{33}
$$

Denote the parameter on the worldline of the particle by $\tilde{\tau}$. The kind of solution we have been considering is of the form

$$
\chi = \chi_0, \quad t = \tilde{\tau}, \quad \varphi = \tilde{\tau} \tag{34}
$$

This is a geodesic in AdS_3. The isometries of AdS_3 will move this to other geodesics. The key property of our solution is that

$$
\bar{z} = \tau - \varphi = \text{constant} \tag{35}
$$

By a choice of the zero of τ we can choose this trajectory to be along $\bar{z} = 0$. On this trajectory the isometry $\bar{L}_1 - \bar{L}_{-1} = -\partial_\chi$ leads to a shift of the radial coordinate χ. Therefore applying this isometry transformation we will get a new solution to the equations of motion of the form

$$
\chi = \chi_0 + \epsilon \quad \tau = \tilde{\tau}, \quad \varphi = \tilde{\tau} \tag{36}
$$

the meomenta conjugate to $z, \bar{z}$ are

$$
P_z = \frac{1}{2}(P_\tau + P_\varphi) = \frac{1}{2}(P - E_{global}), \quad P_{\bar{z}} = \frac{1}{2}(P_\tau - P_\varphi) = -\frac{1}{2}(E_{global} + P) \tag{37}
$$

while the isometry $Q \equiv \bar{L}_1 - \bar{L}_{-1}$ is given by

$$
Q = -e^{-i\bar{z}}\,[\frac{\cosh 2\chi}{\sinh 2\chi}P_{\bar{z}} - \frac{1}{\sinh 2\chi}P_z + \frac{i}{2}P_\chi] + \;e^{i\bar{z}}\,[\frac{\cosh 2\chi}{\sinh 2\chi}P_{\bar{z}} - \frac{1}{\sinh 2\chi}P_z - \frac{i}{2}P_\chi] \tag{38}
$$

where we have used (32).

We now observe that

$$\{P_z, Q\} = 0 \tag{39}$$

so $P - E_{global}$ does not change under the shift. We thus see that for our family of solutions given by (36) we will have

$$E_{global} = P + \text{constant} \tag{40}$$

To fix the constant we can go to the geodesic at $\chi = 0$ which has $P = 0$. Then we just get the energy of the brane wrapped on the S^n, sitting at the center of AdS_3, Calling this energy M_n, we get

$$E_{global} = P + M_n \tag{41}$$

giving the simple additive relation between the mass and momentum contributions to the energy.

3.3 Poincare coordinate energies and momenta

The brane discussed above is static in global coordinates and would therefore correspond to a moving brane in Poincare time. In this subsection we discuss some properties of dynamical quantities in Poincare coordinates for branes in $AdS_3 \times S^n$. The coordinate transformations are given in equations (20).

A trajectory $\chi = \chi_0,\ \varphi = \tau$ becomes the following trajectory in Poincare coordinates

$$\begin{aligned} x &= \tanh \chi_0 \ (1 + T \tanh \chi_0) \\ u &= \frac{\cosh \chi_0}{\sqrt{T^2(1 + \tanh^2 \chi_0) + 2T \ \tanh \chi_0 + 1}} \end{aligned} \tag{42}$$

Thus the brane pops out of the horizon $u = 0$ at $T = -\infty$, goes out to a maximum distance u_{max} and returns back to the horizon at $T = -\infty$. At the same time the coordinate x increases monotonically with T. The total elapsed proper time is *finite*. The value of u_{max} can be calculated from the above trajectory and one gets

$$u_{max} = \sqrt{\cosh \ 2\chi_0} \tag{43}$$

The Poincare energy is given by

$$E_{Poincare} = \frac{M_n |g_{TT}|}{\sqrt{|g_{TT}| - g_{xx}\dot{x}^2 - g_{uu}\dot{u}^2}} = \frac{M_n u^2}{\sqrt{u^2(1 - \dot{x}^2) - \frac{1}{u^2}\dot{u}^2}} \tag{44}$$

Using the value of χ_0 in (9) one finds that

$$E_{Poincare} = M_n \ \cosh^2 \chi_0 = \frac{|L|}{R} + M_n = E_{global} \tag{45}$$

In an analogous way one can verify that the momentum in global coordinates, P_φ, equals the momentum in Poincarè coordinates, P_x:

$$P_x = M_n \, \frac{u^2 \, \dot{x}}{\sqrt{u^2(1-\dot{x}^2) - \frac{1}{u^2}\dot{u}^2}} = M_n \, \cosh^2 \chi_0 \, \tanh^2 \chi_0 = P_\varphi \tag{46}$$

where we have used the fact that, for the above trajectory $\dot{x} = \tanh^2 \chi_0$.

The trajectory $\chi = \chi_0, \varphi = \tau$ clearly does not have the smallest possible value of $E_{Poincare}$. The lowest value of $E_{Poincare}$ is in fact zero and corresponds to the brane being pushed to the horizon $u = 0$.

The equality of global and Poincare energies can be understood from the symmetries of AdS. The generators of the $SL(2,R) \times SL(2,R)$ isometries of the background have been given in global coordinates in equation (28) and (29). The generators in Poincare coordinates are given in terms of $w = T + x$ and $\bar{w} = T - x$ by

$$\begin{aligned} H_{-1} &= i\,\partial_w \\ H_0 &= i\left[w\,\partial_w - \frac{u}{2}\,\partial_u\right] \\ H_1 &= i\left[w^2\,\partial_w - w\,u\,\partial_u - \frac{1}{u^2}\,\partial_{\bar{w}}\right] \end{aligned} \tag{47}$$

and analogous ones with $H_i \to \bar{H}_i$ and $w \to \bar{w}$.

The relation between the two sets of generators is

$$H_0 = \frac{L_1 + L_{-1}}{2}\,, \quad H_{\pm 1} = L_0 \mp i\,\frac{L_1 - L_{-1}}{2} \tag{48}$$

Since the global energy E_{global} and the global momentum P_φ are equal to the Poincare energy $E_{Poincare}$ and the Poincare momentum P_x we must have

$$E_{global} = L_0 + \bar{L}_0 = E_{Poincare} = H_{-1} + \bar{H}_{-1}\,, \quad P_\varphi = -L_0 + \bar{L}_0 = P_x = -H_{-1} + \bar{H}_{-1} \tag{49}$$

which implies

$$L_1 - L_{-1} = \bar{L}_1 - \bar{L}_{-1} = 0 \tag{50}$$

The relations (50) may be readily verified for the trajectory under question by calculating the corresponding Noether charges. Computation of these charges require some care : since the transformations involve time, we cannot compute the charges starting from the static gauge lagrangian. Rather we should compute this *before* we choose the worldvolume time equal to the target space time. However we can choose the worldvolume angles equal to the target space angles as before. This partially gauge fixed lagrangian is given by

$$L = -\frac{M_n}{2}\sqrt{\dot{z}^2 + \dot{\bar{z}}^2 + 2\cosh 2\chi\,\dot{z}\,\dot{\bar{z}} - (2\dot{\chi})^2} \tag{51}$$

where the dot denotes derivative with respect to the worldvolume time $\tilde{\tau}$. The Noether charges corresponding to the $SL(2,R) \times SL(2,R)$ generators (28) and (29) are obtained by the substitutions

$$-i\partial_z = P_z \,, \quad -i\partial_{\bar{z}} = P_{\bar{z}} \,, \quad -i\partial_\chi = P_\chi \tag{52}$$

The momenta P_z, $P_{\bar{z}}$ and P_χ for a given configuration are given by

$$\begin{aligned}
P_z &= -\frac{M_n}{2} \frac{\dot{z} + \cosh 2\chi\, \dot{\bar{z}}}{\sqrt{\dot{z}^2 + \dot{\bar{z}}^2 + 2\cosh 2\chi\, \dot{z}\, \dot{\bar{z}} - (2\dot{\chi})^2}} \\
P_{\bar{z}} &= -\frac{M_n}{2} \frac{\dot{\bar{z}} + \cosh 2\chi\, \dot{z}}{\sqrt{\dot{z}^2 + \dot{\bar{z}}^2 + 2\cosh 2\chi\, \dot{z}\, \dot{\bar{z}} - (2\dot{\chi})^2}} \\
P_\chi &= M_n \frac{2\,\dot{\chi}}{\sqrt{\dot{z}^2 + \dot{\bar{z}}^2 + 2\cosh 2\chi\, \dot{z}\, \dot{\bar{z}} - (2\dot{\chi})^2}}
\end{aligned} \tag{53}$$

For our configuration with $\chi = \chi_0$, $z = 2\,\tilde{\tau}$, $\bar{z} = 0$ we find

$$P_z = -\frac{M_n}{2} \quad P_{\bar{z}} = -\frac{M_n}{2} \cosh 2\chi_0 \,, \quad P_\chi = 0 \tag{54}$$

and thus the Noether charges evaluate to[1]

$$\begin{aligned}
L_0 &= -P_z = \frac{M_n}{2} \\
L_{-1} &= -e^{-iz} \left[\frac{\cosh 2\chi}{\sinh 2\chi} P_z - \frac{1}{\sinh 2\chi} P_{\bar{z}} + \frac{i}{2} P_\chi\right] = 0 \\
L_1 &= -e^{iz} \left[\frac{\cosh 2\chi}{\sinh 2\chi} P_z - \frac{1}{\sinh 2\chi} P_{\bar{z}} - \frac{i}{2} P_\chi\right] = 0 \\
\bar{L}_0 &= -P_{\bar{z}} = \frac{M_n}{2} \cosh 2\chi_0 \\
\bar{L}_{-1} &= -e^{-i\bar{z}} \left[\frac{\cosh 2\chi}{\sinh 2\chi} P_{\bar{z}} - \frac{1}{\sinh 2\chi} P_z + \frac{i}{2} P_\chi\right] = \frac{M_n}{2} \sinh 2\chi_0 \\
\bar{L}_1 &= -e^{i\bar{z}} \left[\frac{\cosh 2\chi}{\sinh 2\chi} P_{\bar{z}} - \frac{1}{\sinh 2\chi} P_z - \frac{i}{2} P_\chi\right] = \frac{M_n}{2} \sinh 2\chi_0
\end{aligned} \tag{55}$$

From the expressions above we verify that $L_1 - L_{-1} = 0$ and $\bar{L}_1 - \bar{L}_{-1} = 0$, which explains the equality of E, P between the global and Poincare systems. We also note that the charges satisfy the constraints

$$L_0^2 - L_1\, L_{-1} = \bar{L}_0^2 - \bar{L}_1\, \bar{L}_{-1} = \frac{M_n^2}{4} \tag{56}$$

Further, note that $L_1 = L_{-1} = 0$, so the configuration is a highest weight state of one of the $SL(2,R)$ aglebras. This gives $L_0 = \frac{M_n}{2}$, which yields $E = P + M_n$, the linear relation observed for the energy of the brane.

[1]Note that, for $\chi_0 \neq 0$, our configuartion is not symmetric under exchange of z and $\bar{z}$: this is obviously because we have chosen $\dot{\varphi} = 1$. Another solution can be obtained with the choice $\dot{\varphi} = -1$.

3.4 Reduction to IIA Black Holes

The geometry (14)-(16) can be reduced to IIA theory by a Kaluza Klein reduction along the y direction. Using the standard relation

$$ds_{11}^2 = e^{\frac{-2\Phi}{3}} ds_{10}^2 + e^{\frac{4\Phi}{3}} [dy - A_\mu dx^\mu]^2 \tag{57}$$

where ds_{10}^2 is the string metric, Φ is the dilaton and A_μ is the RR 1-form gauge field, it is straightforward to see that we get a 4-charge extremal black hole in four dimensions

$$\begin{aligned}
ds^2 &= -(H_0 h)^{-1/2} dt^2 + (H_0 h)^{1/2} [dr^2 + r^2 d\Omega_2^2] + (\frac{H_0}{h})^{1/2} \sum_i H_i ds_{T_i}^2 \\
A^{(1)} &= (1 - \frac{1}{H_0})\ dt \\
A^{(3)} &= \sin\theta\ d\theta d\phi\ [p_3\ \frac{y^5 dy^6 - y^6 dy^5}{2} + p_2\ \frac{y^3 dy^4 - y^4 dy^3}{2} + p_1\ \frac{y^1 dy^2 - y^2 dy^1}{2}] \\
e^\Phi &= \frac{H_0^3}{h}
\end{aligned} \tag{58}$$

The near-horizon limit of this IIA metric depends on whether or not q_0 is non-vanishing. For $q_0 = 0$ this has a null singularity at $r = 0$. Note that this limiting metric is *not* the dimensional reduction of the metric (19).

For $q_0 \neq 0$ the geometry is $AdS_2 \times S^2 \times T^6$. This may be seen by looking at the above formulae for $r \ll q_0, p_i$. The resulting metric, 1-form potential and dilaton are given by

$$\begin{aligned}
ds^2 &= -\frac{r^2}{R_{IIA}^2} dt^2 + \frac{R_{IIA}^2}{r^2} dr^2 + R_{IIA}^2 (d\theta^2 + \sin^2\theta d\phi^2) \\
&+ \sqrt{\frac{q_0 p_1}{p_2 p_3}} ((dy^1)^2 + (dy^2)^2) + \sqrt{\frac{q_0 p_2}{p_3 p_1}} ((dy^3)^2 + (dy^4)^2) \\
&+ \sqrt{\frac{q_0 p_3}{p_1 p_2}} ((dy^5)^2 + (dy^6)^2) \\
A^{(1)} &= [1 - \frac{r}{q_0}]\ dt \\
A^{(3)} &= \sin\theta\ d\theta d\phi\ [p_3\ \frac{y^5 dy^6 - y^6 dy^5}{2} + p_2\ \frac{y^3 dy^4 - y^4 dy^3}{2} + p_1\ \frac{y^1 dy^2 - y^2 dy^1}{2}] \\
e^\Phi &= \frac{q_0}{R_{IIA}}
\end{aligned} \tag{59}$$

where

$$R_{IIA} = (q_0 p_1 p_2 p_3 p_4)^{1/4} \tag{60}$$

If we replace the internal torus with a Calabi-Yau manifold, this is the background which is used in [6]- [8].

Equation (59) is the metric in Poincare coordinates. The coordinate transformations

$$\begin{aligned}
\frac{R_{IIA}}{r} &= \frac{1}{\cosh\chi\cos\tau + \sinh\chi} \\
t &= \frac{R_{IIA}\cosh\chi\sin\tau}{\cosh\chi\cos\tau + \sinh\chi}
\end{aligned} \tag{61}$$

can be used to continue this metric to global coordinates

$$\begin{aligned}
ds^2 &= R_{IIA}^2(-\cosh^2\chi\, d\tau^2 + d\chi^2) + R_{IIA}^2(d\theta^2 + \sin^2\theta d\phi^2) \\
&+ \sqrt{\frac{q_0 p_1}{p_2 p_3}}((dy^1)^2 + (dy^2)^2) + \sqrt{\frac{q_0 p_2}{p_3 p_1}}((dy^3)^2 + (dy^4)^2) \\
+ & \quad \sqrt{\frac{q_0 p_3}{p_1 p_2}}((dy^5)^2 + (dy^6)^2)
\end{aligned} \tag{62}$$

and one can choose a gauge in which the 1-form potential becomes

$$A^{(1)} = -\frac{R_{IIA}}{q_0}[1 - \sinh\chi]d\tau \tag{63}$$

In the IIA language the M2 brane becomes a D2 brane and the momentum along the y direction becomes a D0 charge because of the presence of a worldvolume gauge field

$$F = \frac{f}{2\pi\alpha'}\sin\theta d\theta \wedge d\phi \tag{64}$$

The contribution to the D0 brane charge to the mass of this brane in string metric is

$$M_0 = 4\pi\mu_2 f \tag{65}$$

where μ_2 is the D2 brane tension. The global hamiltonian may be written down using standard methods

$$H = \cosh\chi[(M_2^2 + M_0)^2\ e^{-2\Phi} + P_\chi^2 + \frac{(P_i - 4\pi\mu_2 A_i)^2}{g_{ii}}]^{\frac{1}{2}} + M_0\ e^{-\Phi}[1 - \sinh\chi] \tag{66}$$

where in writing down the last term we have used the explicit form of the dilaton in (59). (Here $A^{(3)} \equiv A_i\ \cos\theta\ d\theta d\phi dy^i$.) We have also denoted the mass of the D2 brane by M_2

$$M_2 = 4\pi R_{IIA}^2 \mu_2 \tag{67}$$

A static solution is obtained at a value of $\chi = \chi_0$ given by

$$\tanh\chi_0 = \frac{M_0}{\sqrt{M_2^2 + M_0^2}} \tag{68}$$

and the value of the energy is

$$E = (M_0 + M_2)e^{-\Phi} \tag{69}$$

which is what we expect from the dimensional reduction of the M theory result.

Note that the magnitude of the energy depends on the gauge choice for $A^{(1)}$. We have intentionally chosen a gauge which leads to an energy which is identical to the M-theory result. A gauge transformation on $A^{(1)}$ translates to a *coordinate* transformation in the M theory which redefines the coorinate y and therefore changes the Killing vector along which dimensional reduction is performed to obtain the IIA theory. For example instead of the choice in (59) we could have chosen

$$A^{(1)\prime} = -\frac{r}{q_0}\ dt \tag{70}$$

which is related to the original potential by a gauge transformation. From (57) it is easy to see that this corresponds to a coordinate transformation on y, $y \to y+t$. Thus this gauge potential would arise from a KK reduction of the 11 dimensional metric along $y+t$ rather than y. In this situation we do not of course expect the energy as calculated in IIA to agree with the energy as calculated in M theory.

The expression for the hamiltonian, (66) is not a sum of positive terms and it is not evident that the static solution has the lowest energy. However it is not hard to see that this is indeed the ground state, using the trick of [5]. It is convenient to use coordinates $\rho = \sinh\chi$ so that the metric of the AdS part becomes

$$ds^2 = -(1+\rho^2)\ d\tau^2 + \frac{d\rho^2}{1+\rho^2} \tag{71}$$

The expression for the energy is

$$E = \frac{\sqrt{M_2^2+M_0^2}|g_{\tau\tau}|e^{-\Phi}}{\sqrt{|g_{\tau\tau}| - g_{\rho\rho}(\partial_\tau\rho)^2}} + M_0 e^{-\Phi}(1-\rho) \tag{72}$$

This equation may be now re-written as

$$(\partial_\tau\rho)^2 + 2U(\rho) = 0 \tag{73}$$

where

$$2U(\rho) = \frac{(M_0^2+M_2^2)(1+\rho^2)^3}{((Ee^{\Phi}-M_0)+M_0\rho)^2} - (1+\rho^2)^2 \tag{74}$$

The relativistic dynamics of the $D2$ brane is thus identical to the *non-relativistic* dynamics of a particle of unit mass moving in a potential $U(\rho)$. The energy of this analog non-reltivistic problem is zero.

A solution to this non-relativistic problem will exist only if $U(\rho) = 0$ for some real ρ. From (74) we see that this happens when

$$M_2^2\rho^2 - 2M_0(Ee^{\Phi}-M_0)\ \rho - (Ee^{\Phi}-M_0)^2 + (M_2^2+M_0^2) = 0 \tag{75}$$

This has a real solution only if

$$E \geq (M_2 + M_0)e^{-\Phi} \tag{76}$$

which establishes the lower bound on the energy. When the energy saturates this bound the solution is static.

3.4.1 Poincare energies

The Poincare energies and momenta for this $D2$ brane are again equal to the global energies and momenta. The transformations are given in (61). The trajectory is then given by

$$\sinh \chi_0 = \frac{u}{2R_{IIA}}[1 - (\frac{R_{IIA}^2}{u^2} - \frac{t^2}{R_{IIA}^2})] \tag{77}$$

This is again a trajectory which comes out of the horizon and returns to it in finite proper time. The maximum value of u now turns out to be

$$u_{max} = R_{IIA}\ e^{\chi_0} \tag{78}$$

The value of the Poincare energy for this trajectory is

$$E_{Poincare} = \frac{M g_{tt} e^{-\Phi}}{\sqrt{g_{tt} - g_{rr}(\partial_\tau r)^2}} + M_0 e^{-\Phi}[1 - \frac{r}{R_{IIA}}] = (M_0 + M_2)\ e^{-\Phi} \tag{79}$$

which is again *exactly* equal to the global energy E_{global}.

Just as in the subsection (3.3), the equality of Poincare and global energies has a group theoretic significance. In terms of light cone coordinates $t_\pm = t \pm \frac{R_{IIA}^2}{r}$ the generators of the $SL(2,R)$ conformal isometries of AdS_2 are

$$h = L_{-1} = \frac{\partial}{\partial t_+} + \frac{\partial}{\partial t_-}, \qquad d = L_0 = t_+ \frac{\partial}{\partial t_+} + t_- \frac{\partial}{\partial t_-}, \quad k = L_1 = t_+^2 \frac{\partial}{\partial t_+} + t_-^2 \frac{\partial}{\partial t_-} \tag{80}$$

and the transformation to global coordinates is given by

$$t_\pm = \tan\ [\frac{1}{2}(\tau \pm \frac{1}{\cosh \chi})] \tag{81}$$

The global hamiltonian H is then

$$H_{global} = \frac{\partial}{\partial \tau} = h + k \tag{82}$$

Since the configurations we discussed have $H_{global} = h$ these must have $k = 0$. k is the generator of conformal boosts and the standard $SL(2,R)$ algebra obeyed by $L_\pm, L_0$ then implies that this state is a highest weight state.

The computations of these conserved charges follow the procedure of subsection (3.3). The partially gauge fixed action (for lowest Landau level orbits on the T^6)

$$S = -\frac{4\pi\mu_2 R_{IIA}^2}{q_0}\int \frac{d\tau}{v(\tau)}[\sqrt{R_{IIA}^4 + f^2}\ \sqrt{(\partial_\tau t)^2 - (\partial_\tau v)^2} - f(\partial_\tau t)] \tag{83}$$

The conserved charges in the static gauge are [2].

$$\begin{aligned} h &= \frac{4\pi\mu_2 R_{IIA}^2}{q_0 v}[\frac{A}{\sqrt{1-\dot{v}^2}} - f] \\ d &= \frac{4\pi\mu_2 R_{IIA}^2}{q_0 v}[-\frac{At}{\sqrt{1-\dot{v}^2}} + \frac{Av\dot{v}}{\sqrt{1-\dot{v}^2}} + ft] \\ k &= \frac{4\pi\mu_2 R_{IIA}^2}{q_0 v}[\frac{A}{\sqrt{1-\dot{v}^2}}(tv\dot{v} - \frac{1}{2}(t^2+v^2)) - \frac{f}{2}(v^2 - t^2)] \end{aligned} \tag{84}$$

Substituting the trajectory (77) we find that k evaluates to zero.

3.4.2 Validity of the near-horizon approximation

The branes we discussed so far were shown to be stable and static in global time in the near horizon geometry of the 4d extremal black hole. From the point of view of black hole physics these would be of interest only if they exist in the full asymptotically flat geometry. In the full geometry, the near-horizon region is a Poincare patch of AdS and we have seen that in Poincare coordinates the brane comes out of the horizon and goes back into it. This is what one would expect in the full geometry as well. However we have to check whether the approximation of restriction to the near-horizon limit is self-consistent. In the Appendix, this is done for four dimensional black hole geometry of section (3.4). We find that the brane remains in the near-horizon region so long as $M_0 \ll M_2$, but goes out of this region otherwise

3.5 Examples in Type IIB String Theory

Another example is provided by extremal black strings in Type IIB string theory compactified on T^4 formed by two sets of D3 branes intersecting along a line together with some momentum along the intersection, and its dimensional reduction to five dimensional black holes. The physics is identical to black strings in M theory and their reduction to four dimensional black holes considered above. The calculations are identical and will not be repeated here.

4 $D3$ branes in 2-charge microstate geometries

We have examined branes in $AdS_m \times S^n$ spaces, and computed their energy. But we can make a symmetry transformation in the AdS, and change what we call E. If on the other

[2]Note that the lagrangian is not invariant under special conformal transformations, though the action is - this results in an additional contribution to the Noether charge

hand we had an asymptotically flat spacetime then we might get a physically unique definition of energy. Note also that the goal of [8] was to study black hole states. Black holes have asymptotically flat geometries, and we measure the energy of different excitations using the time at infinity. So it would be helpful if we could study branes in spacetimes which have the *global* $AdS_m \times S^n$ structure in some region (we will wrap the test branes on the S^n) but which go over to asymptotically flat spacetime at large r.

Interestingly, such geometries are given by microstates of the 2-charge system. In [11, 12] it was found that metrics carrying D1 and D5 charges and a certain amount of rotation had the above mentioned property: they were asymptotically flat at large r but were $AdS_3 \times S^3 \times T^4$ in the small r region. The point to note is that the geometries were not just *locally* $AdS_3 \times S^3$ in the small r region; rather the small r region had the shape of a 'cap' which looked like the region $r < r_0$ of *global* $AdS_3 \times S^3$.

In detail, we take type IIB string theory, compactified on $T^4 \times S^1$. We wrap D1 branes on the S^1 and we wrap D5 branes on $S^1 \times T^4$. Let the S^1 be parametrized by y, with $0 < y < 2\pi R$, and the T^4 be parametrized by coordinates y^1, y^2, y^3, y^4 with a overall volume V. For our present purposes we will do two T-dualities, in the directions y^1, y^2, so that the system is composed of two sets of D3 branes. These branes extend along y^1, y^2, y and along y^3, y^4, y respectively. This does not change the nature of the geometry that we have described above.

We can now consider a D3 brane wrapped over the S^3, and let it move in the direction y. This situation with the D3 brane is very similar to the case of the M2 brane that we had studied above, and we expect to get similar results on the energy . But now we can extend our analysis to a spacetime which is asymptotically flat, so we can identify the charges which correspond to the energy E (conjugate to time t at infinity) and the momentum P (conjugate to the variable y).

We can extend the analysis to a class of geometries that carry *three charges*: the two D3 brane charges as above as well as momentum P along S^1. The geometries for specific microstates of this system were constructed in [13], and these again have an AdS type region at small r and go over to flat space at infinity.

In each of the above cases we find, somewhat surprisingly, that we again get a relation of the form $E = P + Constant$. This might suggest that there is again an underlying symmetry that rotates orbits of the wrapped brane, but we have not been able to identify such a symmetry.

4.1 The 2-charge microstate geometry

The string frame metric is given by

$$ds^2 \;=\; -h^{-1}\,(dt^2 - dy^2) + hf\,\Big(\frac{dr^2}{r^2 + a^2\,\gamma^2} + d\theta^2\Big)$$

$$
\begin{aligned}
&+ \quad h\Big(r^2+\frac{Q_1Q_2a^2\,\gamma^2\,\cos^2\theta}{h^2f^2}\Big)\,\cos^2\theta\,d\psi^2+h\Big(r^2+a^2\,\gamma^2-\frac{Q_1Q_2a^2\,\gamma^2\sin^2\theta}{h^2f^2}\Big)\,\sin^2\theta\,d\phi^2 \\
&- \quad 2\,\frac{a\,\gamma\,\sqrt{Q_1Q_2}}{hf}\,\cos^2\theta\,d\psi\,dy-2\,\frac{a\,\gamma\,\sqrt{Q_1Q_2}}{hf}\,\sin^2\theta\,d\phi\,dt \\
&+ \quad \sqrt{\frac{Q_2}{Q_1}}\,(dy_1^2+dy_2^2)+\sqrt{\frac{Q_1}{Q_2}}\,(dy_3^2+dy_4^2) \\
h \quad &= \quad \sqrt{\Big(1+\frac{Q_1}{f}\Big)\Big(1+\frac{Q_2}{f}\Big)}\,,\qquad f=r^2+a^2\,\gamma^2\cos^2\theta
\end{aligned}
\tag{85}
$$

while the dilaton field vanishes. There is a 4-form potential given by

$$
\begin{aligned}
A^{(4)} \quad &= \quad \Big[-\frac{Q_1}{f+Q_1}\,dt\wedge dy-\frac{Q_2\,(r^2+a^2\,\gamma^2+Q_1)}{f+Q_1}\,\cos^2\theta\,d\psi\wedge d\phi \\
&- \quad \frac{a\,\gamma\,\sqrt{Q_1Q_2}}{f+Q_1}\,\cos^2\theta\,dt\wedge d\psi-\frac{a\,\gamma\,\sqrt{Q_1Q_2}}{f+Q_1}\,\sin^2\theta\,dy\wedge d\phi\Big]\wedge dy^1\wedge dy^2
\end{aligned}
\tag{86}
$$

However the experience of the previous sections show that the only role of this is to put a probe $D3$ brane in a Lowest Landau level orbit on the T^4. We will therefore ignore this in the following discussion.

This geomtery reduces to the asymptotically flat space-time $M^{1,5}\times T^4$ in the large r limit. In the limit $r^2, a^2 \ll \sqrt{Q_1Q_2}$ the metric becomes

$$
\begin{aligned}
ds^2 \quad &= \quad \sqrt{Q_1Q_2}\Big(\frac{dr^2}{r^2+a^2\,\gamma^2}+\frac{r^2}{Q_1Q_2}\,dy^2-\frac{r^2+a^2\,\gamma^2}{Q_1Q_2}\,dt^2\Big) \\
&+ \quad \sqrt{Q_1Q_2}(d\theta^2+\cos^2\theta\,d\psi'^2+\sin^2\theta\,d\phi'^2)+\sqrt{\frac{Q_2}{Q_1}}\,(dy_1^2+dy_2^2)+\sqrt{\frac{Q_1}{Q_2}}\,(dy_3^2+dy_4^2)
\end{aligned}
\tag{87}
$$

where ψ' and ϕ' are "NS sector coordinates"

$$
\psi'=\psi-\frac{a\,\gamma}{\sqrt{Q_1Q_2}}\,y\,,\quad \phi'=\phi-\frac{a\,\gamma}{\sqrt{Q_1Q_2}}\,t \tag{88}
$$

For $\gamma = 1$, this is precisely global $Ad_3\times S^3\times T^4$ as may be seen by making the coordinate transformations to

$$
\tau=\frac{a\gamma t}{\sqrt{Q_1Q_2}}\qquad \varphi=\frac{a\;y\gamma}{\sqrt{Q_1Q_2}}\qquad r=a\gamma\;\sinh\chi \tag{89}
$$

For $\gamma = 1/k$, with k integer greater than 1, the "near horizon" geometry is an orbifod space of the type $(Ad_3\times S^3)/Z_k\times T^4$.

The geometry therefore smoothly interpolates between *global AdS* (or an orbifold of it) and flat space. The key fact about this geometry is that in the small r region t is the *global* time in AdS_3, while in the large r region the same t is the usual Minkowski time in the asymptoically flat space-time. This is in contrast to the geometry of three charge black holes in five dimensions where the Minkowski time of the asymptotic region becomes the *Poincare* time of the near-horizon region. Therefore we can address the question of wrapped $D3$ branes in the full geometry.

4.2 $D3$ branes in 2-charge microstate geometry

In the geometry described above, consider a D3 brane wrapping the angular S^3 and carrying momentum P along the circle y. This brane couples to the background $F^{(5)}$ flux, which extends in the S^3 directions as well as two of the directions of T^4, and hence behaves like a charged particle moving in a magnetic field on T^4. This system represents thus a five dimensional analogue of the S^2 wrapped D2 brane in a 4d black hole, studied in section 2.

Choosing t, θ, ψ and ϕ as worldvolume coordinates, the square root of the determinant of the metric induced on the D3 brane can be written in the form

$$\sqrt{-\det P(g)} = \sin\theta\,\cos\theta\,\sqrt{(r^2+a^2\,\gamma^2)\,F_1 - r^2\,F_2\,\dot{y}^2} \tag{90}$$

where we have defined

$$F_1 = r^2\,(f+Q_1+Q_2)+Q_1Q_2\,,\quad F_2 = (r^2+a^2\,\gamma^2)\,(f+Q_1+Q_2)+Q_1Q_2 \tag{91}$$

It is then straightforward to compute the D3 brane Lagrangian

$$L = -\mu_3\,(2\pi)^2\int d\theta\,\sin\theta\,\cos\theta\,\sqrt{(r^2+a^2\,\gamma^2)\,F_1 - r^2\,F_2\,\dot{y}^2} \tag{92}$$

the momentum conjugate to y

$$P = \mu_3\,(2\pi)^2\int d\theta\,\sin\theta\,\cos\theta\,\frac{r^2\,F_2\,\dot{y}}{\sqrt{(r^2+a^2\,\gamma^2)\,F_1 - r^2\,F_2\,\dot{y}^2}} \tag{93}$$

and the energy of the D3 brane

$$E = \mu_3\,(2\pi)^2\int d\theta\,\sin\theta\,\cos\theta\,\frac{(r^2+a^2\,\gamma^2)\,F_1}{\sqrt{(r^2+a^2\,\gamma^2)\,F_1 - r^2\,F_2\,\dot{y}^2}} \tag{94}$$

Though the θ integrals could be explicitly computed, we find it more convenient to perform integrations only after having minimized the energy.

The location at which the D3 brane stabilizes can be found by either minimizing E with respect to r^2 keeping P fixed or minimizing L with respect to r^2 keeping $\dot{y}$ fixed. The second way is the most convenient and yelds the following, surprinsingly simple, result:

$$\frac{\partial L}{\partial r^2} = 0 \;\Rightarrow\; \partial_{r^2}[(r^2+a^2\,\gamma^2)F_1] - \partial_{r^2}[r^2F_2]\,\dot{y}^2 = 0 \tag{95}$$

$$\Rightarrow \dot{y}^2 = \frac{r^2(f+Q_1+Q_2)+Q_1Q_2+(r^2+a^2\,\gamma^2)(f+r^2+Q_1+Q_2)}{(r^2+a^2\,\gamma^2)(f+Q_1+Q_2)+Q_1Q_2+r^2(f+r^2+a^2\,\gamma^2+Q_1+Q_2)} = 1$$

The location at which the D3 brane sits is then found by putting $\dot{y}=1$ in the expression (93) for P and solving with respect to r. Note that for $\dot{y}=1$ the square root which appears in the expression for P and E simplifies

$$\sqrt{(r^2+a^2\,\gamma^2)\,F_1 - r^2\,F_2} = a\,\gamma\,\sqrt{Q_1Q_2} \tag{96}$$

The expressions for the momentum and energy of the D3 at its stable point are then

$$\begin{aligned} P &= \frac{\mu_3\,(2\pi)^2}{a\,\gamma\,\sqrt{Q_1Q_2}}\int d\theta\,\sin\theta\,\cos\theta\,r^2 F_2 \\ &= \frac{\mu_3\,(2\pi)^2}{2\,a\,\gamma\,\sqrt{Q_1Q_2}}\,r^2\left[Q_1Q_2+(r^2+a^2\,\gamma^2)\left(r^2+\frac{a^2\,\gamma^2}{2}+Q_1+Q_2\right)\right] \\ E &= \frac{\mu_3\,(2\pi)^2}{a\,\gamma\,\sqrt{Q_1Q_2}}\int d\theta\,\sin\theta\,\cos\theta\,(r^2+a^2\,\gamma^2)F_1 \\ &= \frac{\mu_3\,(2\pi)^2}{2\,a\,\gamma\,\sqrt{Q_1Q_2}}\,(r^2+a^2\,\gamma^2)\left[Q_1Q_2+r^2\left(r^2+\frac{a^2\,\gamma^2}{2}+Q_1+Q_2\right)\right] \end{aligned} \tag{97}$$

From the expressions above we see that the dispersion relation of the D3 brane is

$$E = P + 2\pi^2\,\mu_3\,\sqrt{Q_1Q_2}\,a\,\gamma \tag{98}$$

Remarkably, this is *identical* to the formula we would have obtained if we performed the analysis in the AdS limit. This may be easily seen from the general formulae of section (2) and noting that the standard AdS coordinates are related to the coordinates r, t, y by the equations in (89) and that the AdS scale is given by $(Q_1Q_2)^{1/4}$.

We would like to emphasize that the definition of energy is completely unambigious in this geometry because of the presence of an asymptotically flat region. Furthermore from general grounds we know that if we simply added pure momentum to the 2-charge microstate geometry the additional ADM energy is simply equal to the momentum. This is what happens if we take the formal limit $\mu_3 = 0$ in (98) which shows we have taken the zero of the energy correctly.

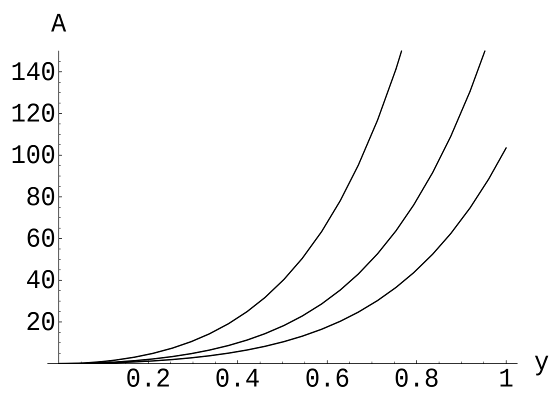

Figure 1: The ratio A plotted as a function of y. The curves have $b = 0.1, 0.15, 0.2$ starting from top to bottom

Even though the dispersion relation is the same as in the AdS limit, the location of the brane obtained by solving the first equation of (97) has a modified dependence on the momentum P. We would like to determine the range of parameters for which this location lies in the AdS region. We have not obtained the general solution of the equation. However to get an idea we examine the solution for $Q_1 = Q_2 = \lambda^2$. The quantity λ will become the scale of the AdS in the appropriate region. In this case it is useful to express this equation in terms of the following quantities

$$A \equiv \frac{P}{2\pi^2 a\gamma\lambda^2\mu_3} \qquad y \equiv \frac{r}{\lambda} \qquad b = \frac{a\gamma}{\lambda} \tag{99}$$

Note that A is the ratio of the contributions to the from the momentum and the $D3$ brane (as in (98). The AdS region of the solution corresponds to $y, b \ll 1$.

The first equation of (97) then becomes

$$A = (\frac{y}{b})^2\ [1 + (y^2 + b^2)(y^2 + \frac{1}{2}b^2 + 2)] \tag{100}$$

Figure (1) shows a plot of A versus y for various values of b. The brane location moves further away from the center of AdS as we increase the ratio A, and for a given value of A, the brane location $r = r_0$ is larger for larger values of a. This shows that for small values of b there is a large range of values of A for which the brane sits in the AdS region of small y.

4.3 CFT Duals

In order to gain some insight on the dual CFT significance of the D3 brane configuarion discussed here, let us rewrite the expression above in terms of microscopic quantities. If R is the radius of the y circle, $V = L_1\, L_2\, L_3\, L_4$ is the volume of T^4, g the string coupling and n_1 and n_2 are the numbers of D3 branes wrapped on y^1, y^2, y and y^3, y^4, y, one has

$$a = \frac{\sqrt{Q_1 Q_2}}{R}\,, \quad \mu_3 = \frac{1}{(2\pi)^3\,\alpha'^2\, g}\,, \quad Q_1 = \frac{(2\pi)^2\, g\,\alpha'^2}{L_3\, L_4}\, n_1\,, \quad Q_2 = \frac{(2\pi)^2\, g\,\alpha'^2}{L_1\, L_2}\, n_2 \tag{101}$$

and thus

$$E = P + 2\pi^2\,\mu_3\,\frac{Q_1 Q_2}{k\, R} = P + \frac{4\pi^3\,\alpha'^3\, g}{V}\,\frac{1}{R}\,\frac{n_1\, n_2}{k} \tag{102}$$

While the significance of this result is not clear to us, it s interesting that the powers of the charges are integral, so we get a quantity $\frac{n_1 n_2}{k}$ that counts the number of 'component strings' in the CFT microstate (see [14] for a discussion of the microstate in terms of component strings). Further the energy comes in units of $\frac{1}{R}$ which is the natural qantum of energy in the CFT which lives on a circle of radius R.

4.4 Vibration modes

Let us look at the D3 brane cosidered above, and restrict attention to the small r region where the geometry is $AdS_3 \times S^3$. We have found the energy E of the brane in a specific configuration (which minimised E for a given P), but we can now ask for the properties of small vibratons of the brane around this configuration. We will only consider oscillations in the AdS_3 directions, so that, in a static gauge, we can write

$$\chi = \chi_0 + \epsilon\, \delta\chi(\tau, \theta_i)\,, \quad y^5 = \dot{y}\,\tau + \epsilon\, \delta y^5(\tau, \theta_i)\,, \quad y^i = y_0^i\,, i = 1\ldots, 4 \tag{103}$$

where we have denoted coordinates on S^3 by θ_i, $i = 1, \ldots, 3$ and the metric on a S^3 of unit radius by g_3 .

We will compute the action of the D3 brane up to quadratic order in ϵ. Having suppressed oscillations in the T^4 directions, only the DBI term contributes. The term of first order in ϵ is

$$S_1 = \epsilon\, \mu_3\, \lambda^4 \int d\tau\, d^3\theta_i\, \sqrt{g_3}\, \frac{\sinh\chi_0}{\sqrt{\cosh^2\chi_0 - \dot{y}^2\, \sinh^2\chi_0}}\, [(\dot{y}^2 - 1)\, \cosh\chi\, \delta\chi + \dot{y}\, \sinh\chi_0\, \partial_\tau \delta y^5] \tag{104}$$

The term proportional to $\delta\chi$ vanishes for $\dot{y}^2 = 1$, while the coefficient of $\partial_\tau \delta y^5$ is a constant and thus this term does not contribute to the equations of motion. Restricting to $\dot{y}^2 = 1$, and performing the change of coordinates $\rho = \sinh\chi$, the term of second order in ϵ is

$$\begin{aligned} S_2 \;\; = \;\; & -\epsilon^2\, \mu_3\, \lambda^4 \int d\tau\, d^3\theta_i\, \sqrt{g_3}\, \Big[\frac{g_3^{ij}}{2}\, \frac{\partial_i\delta\rho\, \partial_j\delta\rho}{\rho_0^2 + 1} - \frac{1}{2}\, \frac{(\partial_t\delta\rho)^2}{\rho_0^2 + 1} \\ & + \frac{g_3^{ij}}{2}\, \rho_0^2(\rho_0^2 + 1)\, \partial_i\delta y^5\, \partial_j\delta y^5 - \frac{1}{2}\, \rho_0^2(\rho_0^2 + 1)\, (\partial_t\delta y^5)^2 - 2\, \rho_0\, \delta\rho\, \partial_t\delta y^5\Big] \end{aligned} \tag{105}$$

If one expands the perturbations $\delta\rho$ and δy^5 as

$$\delta\rho(\tau, \theta_i) = \delta\tilde{\rho}\, e^{-i\omega\tau}\, Y_l(\theta_i)\,, \quad \delta y^5(\tau, \theta_i) = \delta\tilde{y}^5\, e^{-i\omega\tau}\, Y_l(\theta_i) \tag{106}$$

where Y_l are spherical harmonics on S^3

$$\frac{1}{\sqrt{g_3}}\, \partial_i\, (g_3^{ij} \partial_j Y_l(\theta_i)) = -Q_l\, Y_l(\theta_i)\,, \quad Q_l = l(l+2) \tag{107}$$

the equations of motion derived from the action (105) become

$$\begin{pmatrix} (\rho_0^2 + 1)^{-1}\, (-Q_l + \omega^2) & -2\, i\, \omega\, \rho_0 \\ 2\, i\, \omega\, \rho_0 & \rho_0^2\, (\rho_0^2 + 1)\, (-Q_l + \omega^2) \end{pmatrix} = \begin{pmatrix} \delta\tilde{\rho} \\ \delta\tilde{y}^5 \end{pmatrix} \tag{108}$$

The vibration frequencies are then

$$\omega^2 = Q_l + 2 \pm 2\sqrt{Q_l + 1} = l(l+2) + 2 \pm 2(l+1) \tag{109}$$

or equivalently

$$\omega = l + 2 \quad \text{and} \quad \omega = l \tag{110}$$

Note that ω denotes the conjugate of the dimensionless coordinate τ. This is related to the physical energies by a suitable factor of the AdS scale. We therefore see that *the frequencies are universal.* They depend only on the AdS scale of the background and not on the value of the momentum P of the brane. This is similar to what happens for giant gravitons [4].

5 D3 branes in 3-charge microstates

By applying a spectral flow to the two charge microstate of the previous subsections one obtains a geometry dual to a three charge microstate. This is described, in the string frame, by the following metric and dilaton [13]

$$
\begin{aligned}
ds^2 &= -\frac{1}{h}(dt^2-dy^2)+\frac{Q_p}{hf}\,(dt-dy)^2+hf\left(\frac{dr^2}{r^2+(\gamma_1+\gamma_2)^2\eta}+d\theta^2\right) \\
&+ h\Big(r^2+\gamma_1\,(\gamma_1+\gamma_2)\,\eta-\frac{Q_1Q_2\,(\gamma_1^2-\gamma_2^2)\,\eta\,\cos^2\theta}{h^2f^2}\Big)\cos^2\theta d\psi^2 \\
&+ h\Big(r^2+\gamma_2\,(\gamma_1+\gamma_2)\,\eta+\frac{Q_1Q_2\,(\gamma_1^2-\gamma_2^2)\,\eta\,\sin^2\theta}{h^2f^2}\Big)\sin^2\theta d\phi^2 \\
&+ \frac{Q_p\,(\gamma_1+\gamma_2)^2\,\eta^2}{hf}\Big(\cos^2\theta d\psi+\sin^2\theta d\phi\Big)^2 \\
&- \frac{2\sqrt{Q_1Q_2}}{hf}\Big(\gamma_1\cos^2\theta d\psi+\gamma_2\sin^2\theta d\phi\Big)\,(dt-dy) \\
&- \frac{2\sqrt{Q_1Q_2}\,(\gamma_1+\gamma_2)\,\eta}{hf}\Big(\cos^2\theta d\psi+\sin^2\theta d\phi\Big)\,dy+\sqrt{\frac{H_1}{H_2}}(dy_1^2+dy_2^2)+\sqrt{\frac{H_2}{H_1}}(dy_3^2+dy_4^2)
\end{aligned}
\tag{111}
$$

$$
e^{2\Phi}=\frac{H_1}{H_2}
\tag{112}
$$

where

$$
\begin{aligned}
\eta &= \frac{Q_1Q_2}{Q_1Q_2+Q_1Q_p+Q_2Q_p} \\
f &= r^2+(\gamma_1+\gamma_2)\,\eta\,(\gamma_1\,\sin^2\theta+\gamma_2\,\cos^2\theta) \\
H_1 &= 1+\frac{Q_1}{f}\,,\quad H_2=1+\frac{Q_2}{f}\,,\quad h=\sqrt{H_1H_2}
\end{aligned}
\tag{113}
$$

For the solution obtained by spectral flow from the 2-charge microstate geometry, the parameters γ_1 and γ_2 take the values

$$
\gamma_1=-a\,n\,,\quad \gamma_2=a\left(n+\frac{1}{k}\right),\quad a=\frac{\sqrt{Q_1Q_2}}{R}
\tag{114}
$$

where R is the y radius and n and k are integers. Geometries corresponding to other values of γ_1 and γ_2 can be obtained by S and T dualities.

In this geometry, consider a D3 brane wrapping the angular S^3 and rotating along y. The determinant of the induced metric in static gauge can be cast the the form

$$
\sqrt{-\det P(g)}=-\sin\theta\cos\theta\,\sqrt{c_0+\dot y\,c_1+\dot y^2\,c_2}
\tag{115}
$$

where c_0, c_1 and c_2 are functions of r and θ that can be computed using Mathematica. As we did not manage to bring these functions to reasonably simple form, we do not give their explicit expressions here. We can however proceed with the help of Mathematica and verify that the r-derivative of the Lagrangian

$$L = -\mu_3\,(2\pi)^2 \int d\theta\,\sin\theta\,\cos\theta\,\sqrt{c_0 + \dot{y}\,c_1 + \dot{y}^2\,c_2} \tag{116}$$

at fixed $\dot{y}$ vanishes for $\dot{y} = 1$ (note that in this case the invariance under $y \to -y$ is broken by the momentum carried by the background metric (111) and $\dot{y} = -1$ is not a local minimum). For this value of $\dot{y}$ the determinant of the induced metric simplifies to

$$\sqrt{-\mathrm{det}P(g)} = -\sin\theta\cos\theta\,(\gamma_1+\gamma_2)\,\eta\,\sqrt{Q_1 Q_2} \tag{117}$$

Following the same steps as in the previous subsection, one can compute the energy and momentum conjugate to y at the stable point $\dot{y} = 1$:

$$\begin{aligned} E &= \frac{\mu_3\,(2\pi)^2}{(\gamma_1+\gamma_2)\,\eta\,\sqrt{Q_1 Q_2}} \int d\theta\,\sin\theta\,\cos\theta\,\frac{2c_0+c_1}{2} \\ P &= -\frac{\mu_3\,(2\pi)^2}{(\gamma_1+\gamma_2)\,\eta\,\sqrt{Q_1 Q_2}} \int d\theta\,\sin\theta\,\cos\theta\,\frac{2c_2+c_1}{2} \end{aligned} \tag{118}$$

Neither E or P have particularly good looking expressions, but their difference is simply given by

$$E = P + 2\pi^2\,\mu_3\,\sqrt{Q_1 Q_2}\,(\gamma_1+\gamma_2)\,\eta = P + 2\pi^2\,\mu_3\,\frac{\sqrt{Q_1 Q_2}\,a}{k}\,\eta \tag{119}$$

where in the last equality we have used the values (114) for γ_1 and γ_2.

We thus conclude that the dispersion relation of the D3 brane in the three charge geometry differs from that in the two charge geometry only by a factor of η.

6 The field produced by the wrapped brane

In this section we look at the gauge field produced by the D3 brane that we wrap on the S^3, in the asymptotically flat 2-charge microstate geometry. If we think of the brane as a small perturbation of strength ϵ on the background, then the field strength produced by the brane is also of order ϵ, and the energy carried by this field is $O(\epsilon^2)$. But we find that the field strength goes to a constant at large r, so that its overall energy would diverge. The brane wrapped on the S^3 appears to behave like a domain wall in the spacetime, making the field nonzero on the outside everywhere.

The action for the 4-form RR field $A^{(4)}$ sourced by the D3 brane is

$$S = \frac{1}{2}\int F^{(5)} \wedge \star F^{(5)} + \mu_2 \int dr\,dy\,dt\,d\theta\,d\phi\,d\psi\,dV\,\delta(r-r_0)\,[A^{(4)}_{t\theta\phi\psi} + A^{(4)}_{y\theta\phi\psi}] \tag{120}$$

($dV = dy^1 \wedge \ldots \wedge dy^4$ is the volume form on T^4). We have assumed the brane to be smeared along y and the torus directions y^i and, in writing the source term, we have taken into account that the brane moves with velocity $\dot{y} = 1$ along y.

We will make the following ansatz for $A^{(4)}$

$$\begin{aligned} A^{(4)} &= A^{(4)}_{t\theta\phi\psi}\, dt\wedge d\theta\wedge d\phi\wedge d\psi + A^{(4)}_{t\theta\phi y}\, dt\wedge d\theta\wedge d\phi\wedge dy \\ &+ A^{(4)}_{y\theta\phi\psi}\, dy\wedge d\theta\wedge d\phi\wedge d\psi + A^{(4)}_{y\theta t\psi}\, dy\wedge d\theta\wedge dt\wedge d\psi \end{aligned} \quad (121)$$

(At the same order in μ_2, the gauge field also has components $A^{(4)}_{\mu\nu y^1 y^2}$, where $\mu,\nu = t, y, \psi, \phi$ and y^1, y^2 are directions in T^4: these components arise from the fact that the background metric is perturbed by the D3 brane together with the fact that the unperturbed background has non-zero values of $A^{(4)}_{\mu\nu y^1 y^2}$. Since the equations of motion do not mix the components $A^{(4)}_{\mu\nu y^1 y^2}$ with the ones contained in the ansatz (121), we can consistently ignore these extra components in the following).

Ona has

$$F^{(5)} = dr\wedge \partial_r A^{(4)} \quad (122)$$

The star operation in a geometry with $t\phi$ and $y\psi$ mixings is given by[3]

$$\begin{aligned} \star(dr\wedge dt\wedge d\theta\wedge d\phi\wedge d\psi) &= \sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{\psi\psi}\, dy - g^{\psi y}\, d\psi)\wedge dV \\ \star(dr\wedge dt\wedge d\theta\wedge d\phi\wedge dy) &= \sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{\psi y}\, dy - g^{yy}\, d\psi)\wedge dV \\ \star(dr\wedge dy\wedge d\theta\wedge d\phi\wedge d\psi) &= -\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{\phi\phi}\, dt - g^{\phi t}\, d\phi)\wedge dV \\ \star(dr\wedge dy\wedge d\theta\wedge dt\wedge d\psi) &= -\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{\phi t}\, dt - g^{tt}\, d\phi)\wedge dV \end{aligned} \quad (123)$$

The equations of motion are

$$d\star F^{(5)} + \mu_2\, \delta(r-r_0)\, dr\wedge(dy - dt)\wedge dV = 0 \quad (124)$$

which yeld

$$\begin{aligned} &\partial_r[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{\psi\psi}\partial_r A^{(4)}_{t\theta\phi\psi} + g^{\psi y}\partial_r A^{(4)}_{t\theta\phi y})] + \mu_2\, \delta(r-r_0) = 0 \\ &\partial_r[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{yy}\partial_r A^{(4)}_{t\theta\phi y} + g^{\psi y}\partial_r A^{(4)}_{t\theta\phi\psi})] = 0 \\ &\partial_r[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{\phi\phi}\partial_r A^{(4)}_{y\theta\phi\psi} + g^{\phi t}\partial_r A^{(4)}_{y\theta t\psi})] + \mu_2\, \delta(r-r_0) = 0 \\ &\partial_r[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{tt}\partial_r A^{(4)}_{y\theta t\psi} + g^{\phi t}\partial_r A^{(4)}_{y\theta\phi\psi})] = 0 \end{aligned} \quad (125)$$

$$\begin{aligned} &\partial_\theta[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{\psi\psi}\partial_r A^{(4)}_{t\theta\phi\psi} + g^{\psi y}\partial_r A^{(4)}_{t\theta\phi y})] = 0 \\ &\partial_\theta[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{yy}\partial_r A^{(4)}_{t\theta\phi y} + g^{\psi y}\partial_r A^{(4)}_{t\theta\phi\psi})] = 0 \\ &\partial_\theta[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{\phi\phi}\partial_r A^{(4)}_{y\theta\phi\psi} + g^{\phi t}\partial_r A^{(4)}_{y\theta t\psi})] = 0 \\ &\partial_\theta[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{tt}\partial_r A^{(4)}_{y\theta t\psi} + g^{\phi t}\partial_r A^{(4)}_{y\theta\phi\psi})] = 0 \end{aligned} \quad (126)$$

[3]We are using the orientation $\epsilon_{tyr\theta\phi\psi} = 1$.

Their solution is

$$\begin{aligned}
F^{(5)}_{rt\theta\phi\psi} &= \frac{a_\pm\, g_{\psi\psi} + b_\pm\, g_{\psi y}}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\phi\phi} - g^{t\phi} g^{t\phi})} \\
F^{(5)}_{rt\theta\phi y} &= \frac{a_\pm\, g_{\psi y} + b_\pm\, g_{yy}}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\phi\phi} - g^{t\phi} g^{t\phi})} \\
F^{(5)}_{ry\theta\phi\psi} &= \frac{c_\pm\, g_{\phi\phi} + d_\pm\, g_{\phi t}}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy} g^{\psi\psi} - g^{y\psi} g^{y\psi})} \\
F^{(5)}_{ry\theta t\psi} &= \frac{c_\pm\, g_{\phi t} + d_\pm\, g_{tt}}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy} g^{\psi\psi} - g^{y\psi} g^{y\psi})}
\end{aligned} \tag{127}$$

where $a_\pm$, $b_\pm$, $c_\pm$ and $d_\pm$ are r and θ independent constants: the subscript $+$ applies to the region $r > r_0$ while the subscript $-$ applies to $r < r_0$. Because of the delta function source we have $a_+ - a_- = -\mu_2$, $b_+ - b_- = 0$, $c_+ - c_- = -\mu_2$, $d_+ - d_- = 0$.

In order to fix the values of these constants let us impose regularity of the field strength. It will be convenient to work in "NS-sector coordinates"

$$\phi' = \phi - \frac{a}{\sqrt{Q_1 Q_2}}\, t\,, \quad \psi' = \psi - \frac{a}{\sqrt{Q_1 Q_2}}\, y \tag{128}$$

Consider first regularity at $\theta = 0, \pi/2$. One has

$$\begin{aligned}
\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\phi'\phi'} - g^{t\phi'} g^{t\phi'}) &= -\frac{r}{hf}\, \frac{\cos\theta}{\sin\theta} \\
\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy} g^{\psi'\psi'} - g^{y\psi'} g^{y\psi'}) &= \frac{r^2 + a^2}{r\, hf}\, \frac{\sin\theta}{\cos\theta}
\end{aligned} \tag{129}$$

Moreover $g_{\phi'\phi'} \sim \sin^2\theta$, $g_{t\phi'} \sim \sin^2\theta$, $g_{\psi'\psi'} \sim \cos^2\theta$, $g_{y\psi'} \sim \cos^2\theta$ while g_{tt} and g_{yy} go to some finite non-zero values as $\theta \to 0, \pi/2$. We thus see that the term proportional to $b_\pm$ in $F^{(5)}_{rt\theta\phi y}$ is singular for $\theta = \pi/2$ and the term proportional to $d_\pm$ in $F^{(5)}_{ry\theta t\psi}$ is singular at $\theta = 0$. Therefore we have to take $b_\pm = d_\pm = 0$.

Consider now the behaviour around $f = 0$ (i.e. $r = 0$ and $\theta = \pi/2$), where the metric goes to

$$\begin{aligned}
\frac{ds^2}{\sqrt{Q_1 Q_2}} \;\approx\;& \frac{dr^2}{r^2 + a^2} + \frac{r^2}{Q_1 Q_2}\, dy^2 - \frac{r^2 + a^2}{Q_1 Q_2} \Big(1 - 2\frac{a^2}{\sqrt{Q_1 Q_2}}\Big) + d\theta^2 + \cos^2\theta\, d\psi'^2 \\
&+\; \sin^2\theta\, d\phi'^2 \Big(1 + 2\frac{a^2}{\sqrt{Q_1 Q_2}}\Big) + 4\, \frac{a\, r^2}{\sqrt{Q_1 Q_2}}\, \cos^2\theta\, dy\, d\psi' + 4\, \frac{a\,(r^2 + a^2)}{\sqrt{Q_1 Q_2}}\, \sin^2\theta\, dt\, d\phi'
\end{aligned} \tag{130}$$

Then we have

$$\begin{aligned}
F^{(5)}_{rt\theta\phi'\psi'} &\approx -a_-\, Q_1 Q_2\, \frac{\sin\theta\cos\theta}{r} \\
F^{(5)}_{rt\theta\phi' y} &\approx -2a_-\, a\, r\, \sin\theta\, \cos\theta \\
F^{(5)}_{ry\theta\phi'\psi'} &\approx c_-\, Q_1 Q_2 \Big(1 + 2\frac{a^2}{\sqrt{Q_1 Q_2}}\Big)\, \frac{\sin\theta\cos\theta\, r}{r^2 + a^2} \\
F^{(5)}_{ry\theta t\psi'} &\approx 2c_-\, a\, r\, \sin\theta\, \cos\theta
\end{aligned} \tag{131}$$

Regularity at $f = 0$ then requires $a_- = 0$ (and thus $c_+ = -\mu_2$), while c_- is left arbitrary.

Let us now consider the behaviour of the field strength at asymptotic infinity:

$$
\begin{aligned}
F^{(5)} &= a_+ \, dr \wedge dt \wedge d\theta \wedge d\phi' \wedge \frac{g_{\psi'\psi'}\, d\psi' + g_{\psi' y}\, dy}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\phi\phi} - g^{t\phi} g^{t\phi})} \\
&+ c_+ \, dr \wedge dy \wedge d\theta \wedge \frac{g_{\phi'\phi'}\, d\phi' + g_{\phi' t}\, dt}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy} g^{\psi\psi} - g^{y\psi} g^{y\psi})} \wedge d\psi' \\
&\approx -a_+ \, r^3 \sin\theta \, \cos\theta \, dr \wedge dt \wedge d\theta \wedge d\phi \wedge d\psi \\
&+ c_+ \, r^3 \sin\theta \, \cos\theta \, dr \wedge dy \wedge d\theta \wedge d\phi \wedge d\psi
\end{aligned}
\tag{132}
$$

The formula above shows that, asymptotically, the field strength is constant in local orthonomal coordinates.

We have the freedom to choose the constant c_- to have any value that we want; this freedom corresponds to adding a smooth magnetic field everywhere to the background. A simple choice of c_- would be the one that makes $c_+ = 0$, so that this magnetic field vanishes at infinity. Then we get

$$
\begin{aligned}
F^{(5)} &= -\mu_2 \, dr \wedge dt \wedge d\theta \wedge d\phi' \wedge \frac{g_{\psi'\psi'}\, d\psi' + g_{\psi' y}\, dy}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\phi'\phi'} - g^{t\phi'} g^{t\phi'})} \\
&= \mu_2 \, \frac{\sin\theta \, \cos\theta}{r} \, dr \wedge dt \wedge d\theta \wedge d\phi' \wedge \Big[h^2 \, f \Big(r^2 + \frac{Q_1 Q_2 \, a^2 \, \cos^2\theta}{h^2 f^2} \Big) d\psi' \\
&\qquad + \frac{a}{\sqrt{Q_1 Q_2}} \, r^2 \, (f + Q_1 + Q_2) \, dy \Big] \, , \text{ for } r > r_0 \\
F^{(5)} &= -\mu_2 \, dr \wedge dy \wedge d\theta \wedge d\psi' \wedge \frac{g_{\phi'\phi'}\, d\phi' + g_{\phi' t}\, dt}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\psi'\psi'} - g^{y\psi'} g^{y\psi'})} \\
&= -\mu_2 \, \frac{r \, \sin\theta \, \cos\theta}{r^2 + a^2} \, dr \wedge dy \wedge d\theta \wedge d\psi' \wedge \Big[h^2 \, f \Big(r^2 + a^2 - \frac{Q_1 Q_2 \, a^2 \, \sin^2\theta}{h^2 f^2} \Big) d\phi' \\
&\qquad + \frac{a}{\sqrt{Q_1 Q_2}} \, (r^2 + a^2) \, (f + Q_1 + Q_2) \, dt \Big] \, , \text{ for } r < r_0
\end{aligned}
\tag{133}
$$

For any choice of c_- we find the the stress tensor of the field goes to a constant rather than vanish at infinity. We can thus generate a uniform cosmological type contribution in the spacetime dimensionally reduced on the y direction. The only way to cancel this contribution would be to have a $\overline{D3}$ brane in the 'throat' of the microstate geometry, or in the throat of a different microstate geometry located at some other spacetime point. We have to be aware that the energy computed from the DBI action in the above sections does not iclude this (possibly divergent) field contribution.

7 Supersymmetry properties of the branes

The simple expressions for the energies as a sum of the contribution from individual charges signifies a threshold bound state. As is usual in such situations, this usually follows from supersymmetry and BPS bounds. In this section we will examine the supersymmetry properties of these brane configurations.

7.1 Supersymmetry of the $D2$ brane

In this section we will examine the supersymmetry properties for the case of $D2$ branes in IIA theory. The considerations can be easily generalized to the M-branes.

7.1.1 Killing spinors of the near-horizon background

We work in global coordinates. The metric, dilaton, RR 1-form, and RR 3-forms of the near-horizon background were given in (58). We use $m, n... = \tau, \chi, \theta, \phi, 1, ..., 6$ as the ten-dimensional curved space indices, $a, b... = \hat{\tau}, \hat{\chi}, \hat{\theta}, \hat{\phi}, \hat{1}, ..., \hat{6}$ (or sometimes, equivalently, $a, b... = \underline{0}, ..., \underline{9}$) as the tangent space indices. The Clifford algebra is

$$\left\{\Gamma^a, \Gamma^b\right\} = 2\eta^{ab} \tag{134}$$

with η^{ab} having signature $(-,+,...,+)$, and the gamma matrices Γ^a's are 32 by 32 real matrices ($\Gamma^{\hat{\tau}}$ being antisymmetric, and $\Gamma^{\hat{\chi}}, ..., \Gamma^{\hat{6}}$ being symmetric). $\Gamma^{\underline{10}} \equiv \Gamma^{\underline{0}\cdots\underline{9}}$ and $(\Gamma^{\underline{10}})^2 = 1$. We use 32-component real spinors y, and define $\bar{y} \equiv y^T \Gamma^{\underline{0}}$.

The local supersymmetry variation of the dilatino, parameterized by a 32-component real spinor ϵ, is

$$\delta\lambda = \frac{1}{8} e^{\Phi} \left(\frac{3}{2!} F^{(2)}_{ab} \Gamma^{ab} \Gamma^{\underline{\varphi}} + \frac{1}{4!} F^{(4)}_{abcd} \Gamma^{abcd} \right) \epsilon \tag{135}$$

and the gravitino variation is

$$\delta\psi_m = \left[\partial_m + \frac{1}{4}\omega_{mab}\Gamma^{ab} + \frac{1}{8} e^{\Phi} \left(\frac{1}{2!} F^{(2)}_{ab} \Gamma^{ab} \Gamma_m \Gamma^{\underline{\varphi}} + \frac{1}{4!} F^{(4)}_{abcd} \Gamma^{abcd} \Gamma_m \right) \right] \epsilon \tag{136}$$

where $\Gamma^{\underline{\varphi}} \equiv -\Gamma^{\underline{10}} = -\Gamma^{\hat{\tau}\hat{\chi}\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}}$. Plugging in the expressions of the RR field strength, we get

$$\delta\lambda = \frac{1}{R} N \epsilon, \quad \delta\psi_m = \left[\partial_m + \frac{1}{4}\omega_{mab}\Gamma^{ab} + \frac{1}{R} M \Gamma_m \right] \epsilon \tag{137}$$

where the matrices N and M are given by

$$\begin{aligned} N &= \frac{1}{8} \left[3\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} + \Gamma^{\hat{\theta}\hat{\phi}} \left(\Gamma^{\hat{1}\hat{2}} + \Gamma^{\hat{3}\hat{4}} + \Gamma^{\hat{5}\hat{6}} \right) \right] \\ M &= \frac{1}{8} \left[-\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} + \Gamma^{\hat{\theta}\hat{\phi}} \left(\Gamma^{\hat{1}\hat{2}} + \Gamma^{\hat{3}\hat{4}} + \Gamma^{\hat{5}\hat{6}} \right) \right] \end{aligned} \tag{138}$$

(note the only nonvanishing $\frac{1}{4}\omega_{mab}\Gamma^{ab}$'s are $\frac{1}{4}\omega_{\tau ab}\Gamma^{ab} = -\frac{\sinh\chi}{2}\Gamma^{\hat{\tau}\hat{\chi}}$ and $\frac{1}{4}\omega_{\phi ab}\Gamma^{ab} = \frac{\cos\theta}{2}\Gamma^{\hat{\phi}\hat{\theta}}$; also note that $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} = -\left(\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}\right)\left(\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}\right)\left(\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}\right)$.) Next we solve $\delta\lambda = 0$ and $\delta\psi_m = 0$ to find the Killing spinors.

Let's divide the 32-dimensional vector space of ϵ into eight subspaces of simultaneous eigenvectors of $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}$, and $\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}$, labeled as $(\pm\pm\pm)$ (with the $\pm$'s denotes the ± 1 eigenvalues of these three matrices, respectively). Each of these subspaces is four-dimensional by itself. It is easy to see that, $\delta\lambda = 0$ if and only if

$$\epsilon = \epsilon_+ + \epsilon_- \tag{139}$$

with $\epsilon_+ \in (+++)$ and $\epsilon_- \in (---)$.

Plugging the above expression for ϵ into $\delta\psi_m$ and integrating, we then get the explicit expression of the eight Killing spinors of $AdS_2 \times S^2 \times T^6$, four of them being

$$\epsilon_1 = \left[e^{-\frac{1}{2}\chi\Gamma^{\hat{\chi}}} e^{\frac{1}{2}\tau\Gamma^{\hat{\tau}}} \sin\frac{\theta}{2} e^{\frac{1}{2}\phi\Gamma^{\hat{\phi}\hat{\theta}}} + e^{\frac{1}{2}\chi\Gamma^{\hat{\chi}}} e^{-\frac{1}{2}\tau\Gamma^{\hat{\tau}}} \left(-\cos\frac{\theta}{2}\right)\Gamma^{\hat{\theta}} e^{\frac{1}{2}\phi\Gamma^{\hat{\phi}\hat{\theta}}}\right]\Phi_0 \tag{140}$$

with Φ_0 being an arbitrary constant 32-component real spinor in the four-dimensional $(+++)$ subspace, i.e. $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}\Phi_0 = \Phi_0$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}\Phi_0 = \Phi_0$, and $\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}\Phi_0 = \Phi_0$; and the other four being

$$\epsilon_2 = \left[e^{-\frac{1}{2}\chi\Gamma^{\hat{\chi}}} e^{\frac{1}{2}\tau\Gamma^{\hat{\tau}}} \cos\frac{\theta}{2} e^{-\frac{1}{2}\phi\Gamma^{\hat{\phi}\hat{\theta}}} + e^{\frac{1}{2}\chi\Gamma^{\hat{\chi}}} e^{-\frac{1}{2}\tau\Gamma^{\hat{\tau}}} \sin\frac{\theta}{2}\Gamma^{\hat{\theta}} e^{-\frac{1}{2}\phi\Gamma^{\hat{\phi}\hat{\theta}}}\right]\Phi_0' \tag{141}$$

with Φ_0' being another arbitrary constant 32-component real spinor in the four-dimensional $(+++)$ subspace. A general Killing spinor is given by $\epsilon = \epsilon_1 + \epsilon_2$.

7.1.2 Supersymmetric D2 configuration

Next we show that the D2 trajectory considered in Subsection 3.4 preserves half of the background supersymmetries. Recall that the trajectory is

$$\tau = \chi^0,\ \theta = \chi^1,\ \phi = \chi^2,\ \chi = \chi_0,\ y_1 = 0, ..., y_6 = 0 \tag{142}$$

for which the κ projection matrix as given in [15] evaluates to

$$\Gamma = \frac{-1}{\cosh\chi_0}\left(1 + \sinh\chi_0 \Gamma^{\hat{\theta}\hat{\phi}}\Gamma^{\underline{10}}\right)\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \tag{143}$$

The supersymmetries preserved by the D2 brane are the Killing spinors ϵ that satisfy

$$(1 - \Gamma)\epsilon = 0 \tag{144}$$

After some manipulation, one finds that there are four supersymmetries preserved, with two of the corresponding Killing spinors given by eqn. (140) constrained by $(1 + \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}})\Phi_0 = 0$, and the other two given by eqn. (141) constrained by $(1 + \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}})\Phi_0' = 0$. Note that these projection conditions turn out to be independent of the D0 charge (i.e., independent of the value of χ_0).

7.1.3 Topological charge of the brane

In [16] p-forms constructed from background Killing spinors are integrated over probe branes' spatial worldvolumes to give topological charges in M-theory. [17] generalize this to IIA theory, whose approach we shall now apply to the above D2 brane. We shall find a central charge $C_{D2} = M_2 e^{-\Phi} + M_0 e^{-\Phi}$ in the superalgebra, which equals the D2's global energy and shows that the D2 indeed saturates a BPS bound.

After being sandwiched between ϵ^T and ϵ (where ϵ is a Killing spinor, and is treated as a commuting rather than anti-commuting variable), the superalgebra with the probe brane can be written as

$$(Q\epsilon)^2 = \int d^2\chi K_\mu p^\mu \pm \int \omega_{D2} \tag{145}$$

where the integrals are over the spatial worldvolume of the brane, K is a one-form defined as a bilinear of ϵ

$$K = \bar{\epsilon}\Gamma_a \epsilon \; e^a \tag{146}$$

(e^a being the vielbein one-form) and ω_{D2} is a closed two-form also constructed from bilinears of ϵ. The choice of ω_{D2} is background-specific[4], and we shall take the one used in [17] to consider supertubes

$$\omega_{D2} = \mu_2 \left(e^{-\Phi}\Omega + K \cdot A^{(3)} + \tilde{K} \wedge A^{(1)} - 2\pi\alpha' F \right) \tag{147}$$

with the $\cdot$ denoting the inner product of q-forms with p-forms ($p < q$) $(\alpha_p \cdot \beta_q)_{a_1 \ldots a_{q-p}} = (1/p!)\, \alpha^{b_1 \ldots b_p} \beta_{b_1 \ldots b_p a_1 \ldots a_{q-p}}$, and

$$\Omega = \frac{1}{2}\bar{\epsilon}\Gamma_{ab}\epsilon \; e^{ab}, \quad \tilde{K} = \bar{\epsilon}\Gamma_a \Gamma^{\underline{10}} \epsilon \; e^a \tag{148}$$

Note that our choice for the contribution of the worldvolume field strength to ω_{D2} differs from that of [17] by a minus sign. Due to ϵ's being a Killing spinor, K turns out to be a Killing vector, and $K, \Omega, \tilde{K}$ satisfy the following differential relations (which are obtained by plugging our background into equations (3.18) and (3.19) of [17])

$$d\tilde{K} = 0, \quad d\left(e^{-\Phi}\Omega\right) = \tilde{K} \wedge F^{(2)} + K \cdot F^{(4)} \tag{149}$$

Using these relations one finds

$$d\omega_{D2} = K \cdot F^{(4)} + d\left(K \cdot A^{(3)}\right) = \mathcal{L}_K A^{(3)} \tag{150}$$

[4] For a string probe, there is a general expression for the closed one-form ω_{string}, see [17] for details.

Hence ω_{D2} will be closed if $A^{(3)}$ is invariant under the Lie derivative $\mathcal{L}_K$, and now we turn our attention to K.

One readily sees that $K_{\hat{1}} = 0$, since ϵ only has components in $(+++)$ and $(---)$ while $\Gamma^{\tau\hat{1}}$ takes $(+++)$ to $(-++)$ and $(---)$ to $(+--)$, and orthogonality of the subspaces then gives $\epsilon^T\Gamma^{\hat{\tau}\hat{1}}\epsilon = 0$. Similiarly, $K_{\hat{2}}, ..., K_{\hat{6}}$ all vanish.

After some algebra, one finds

$$K_{\hat{\chi}} = \epsilon^T\Gamma^{\hat{\tau}\hat{\chi}}\epsilon = \cos\tau\left(\Phi_0^T\Gamma^{\hat{\tau}\hat{\chi}}\Phi_0 + \Phi_0'^T\Gamma^{\hat{\tau}\hat{\chi}}\Phi_0'\right) + \sin\tau\left(\Phi_0^T\Gamma^{\hat{\chi}}\Phi_0 + \Phi_0'^T\Gamma^{\hat{\chi}}\Phi_0'\right) \tag{151}$$

$$K_{\hat{\theta}} = \epsilon^T\Gamma^{\hat{\tau}\hat{\theta}}\epsilon = 2\Phi_0^T\Gamma^{\hat{\tau}}\exp\left(-\phi\Gamma^{\hat{\phi}\hat{\theta}}\right)\Phi_0' \tag{152}$$

$$K_{\hat{\phi}} = \epsilon^T\Gamma^{\hat{\tau}\hat{\phi}}\epsilon = 2\cos\theta\Phi_0'^T\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\exp\left(\phi\Gamma^{\hat{\phi}\hat{\theta}}\right)\Phi_0 + \sin\theta\left(\Phi_0^T\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\Phi_0 - \Phi_0'^T\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\Phi_0'\right) \tag{153}$$

Now let's pick out a unique Killing spinor by further imposing the projection and normalization conditions

$$\begin{aligned}
&\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\Phi_0 = -\Phi_0, \quad \Gamma^{\hat{\tau}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}}\Phi_0 = \Phi_0\\
&\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\Phi_0' = -\Phi_0', \quad \Gamma^{\hat{\tau}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}}\Phi_0' = -\Phi_0'\\
&\Phi_0^T\Phi_0 = \frac{\Delta}{2}, \quad \Phi_0'^T\Phi_0' = \frac{\Delta}{2}
\end{aligned} \tag{154}$$

where Δ is some positive normalization number whose value shall be determined soon. Note that this Killing spinor is preserved by the D2 (see subsection 7.1.2). For this Killing spinor, one immediately finds

$$K_{\hat{\chi}} = 0,\ K_{\hat{\theta}} = 0,\ K_{\hat{\phi}} = 0, and,\ K_{\hat{\tau}} = \epsilon^T\epsilon = \Delta\cosh\chi \tag{155}$$

i.e. $K = \frac{-\Delta}{R}\frac{\partial}{\partial\tau}$, which is the Killing vector generating global time translation. For this K $\mathcal{L}_K A^{(3)}$ vanishes, and we then find ω_{D2} is indeed closed. (Actually, the story here is quite trivial: since $A^{(3)}, F^{(4)}$ don't have any τ-component, $K\cdot A^{(3)}, K\cdot F^{(4)}$ both vanish.)

Having established the closedness of ω_{D2}, we now integrate it over the spatial worldvolume of the D2. Since $A^{(1)} \sim d\tau$ and $K\cdot A^{(3)}$ vanishes, only the $e^{-\Phi}\Omega$ term and the worldvolume flux term contributes to the integral

$$\begin{aligned}
\int_{S^2}\omega_{D2} &= \mu_2\int_{S^2}\frac{R}{q_0}\left(\epsilon^T\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\epsilon\right)R^2\sin\theta d\theta\wedge d\phi - \mu_2 2\pi\alpha'\int_{S^2}F\\
&= -4\pi\mu_2\Delta\frac{R^3}{q_0} - M_0 = -\Delta M_2 e^{-\Phi} - M_0
\end{aligned} \tag{156}$$

where in the second line we've used the fact that $\epsilon^T\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\epsilon$ evaluates to $-\Delta$ for the particular Killing spinor we've chosen. Since $\int d^2\chi K_\mu p^\mu = K^\tau P_\tau = -\frac{\Delta}{R}P_\tau = -\Delta E$ (recall that the physical energy is $E = P_\tau/R$), and the particular ϵ is perserved by the D2, the supersymmetry algegra (145) becomes

$$-\Delta E = \mp\left(-\Delta M_2 e^{-\Phi} - M_0\right) \tag{157}$$

which upon taking the lower sign on the r.h.s. gives

$$E = C_{D2} = M_2 e^{-\Phi} + \frac{M_0}{\Delta} \tag{158}$$

From this we see that we should take the normalization number Δ to be $e^\Phi = \frac{q_0}{R}$, which results in $C_{D2} = M_2 e^{-\Phi} + M_0 e^{-\Phi}$. This is the same as the global energy we computed earlier for this D2 trajectory and shows this D2 saturates the BPS bound.

7.1.4 Supersymmetry of D2 in the full black hole geometry

In this subsection, we show that the D2 considered above does not preserve any of the supersymmetries of the full black hole geometry (except in the $\chi_0 \to \infty$ limit where it is effectively a bunch of D0 branes), and is thus not really a stable configuration in the full geometry. First let's work out the Killing spinors of the full geometry.

Recall that the metric of the full geometry is given by

$$\begin{aligned} ds^2 = \quad & \frac{-1}{\sqrt{H_0H_1H_2H_3}}dt^2 + \sqrt{H_0H_1H_2H_3}\left(dr^2 + r^2 d\Omega_2^2\right) + \sqrt{\frac{H_0H_1}{H_2H_3}}\left(dy_1^2 + dy_2^2\right) \\ & + \sqrt{\frac{H_0H_2}{H_1H_3}}\left(dy_3^2 + dy_4^2\right) + \sqrt{\frac{H_0H_3}{H_1H_2}}\left(dy_5^2 + dy_6^2\right) \end{aligned} \tag{159}$$

where $H_0 = 1 + \frac{q_0}{r}$, $H_i = 1 + \frac{p_i}{r}, i = 1, 2, 3$. The nonvanishing components of the RR four-form and two-form field strengths are given by

$$\begin{aligned} F^{(4)}_{\theta\phi 12} &= -\frac{dH_1}{dr} r^2 \sin\theta, \;\; F^{(4)}_{\theta\phi 34} = -\frac{dH_2}{dr} r^2 \sin\theta, \quad F^{(4)}_{\theta\phi 56} = -\frac{dH_3}{dr} r^2 \sin\theta \\ F^{(2)}_{rt} &= -\frac{1}{(H_0)^2}\frac{dH_0}{dr} \end{aligned} \tag{160}$$

And the dilaton is

$$e^\Phi = \left(\frac{H_1H_2H_3}{(H_0)^3}\right)^{-1/4} \tag{161}$$

Note that the dilaton is no longer constant once we go beyond the near-horizon region. Now the local supersymmetry variation of the dilatino is given by

$$\delta\lambda = \left[\frac{1}{2}\Gamma^m \partial_m \Phi + \frac{1}{8} e^\Phi \left(\frac{3}{2!} F^{(2)}_{ab} \Gamma^{ab} \Gamma^{\underline{\varphi}} + \frac{1}{4!} F^{(4)}_{abcd} \Gamma^{abcd}\right)\right] \epsilon \tag{162}$$

which after plugging in the expression of the RR fields becomes

$$\delta\lambda = \frac{1}{8}\left(H_0 H_1 H_2 H_3\right)^{-1/4} \left\{ -\left(\sum_{i=1}^{3}\frac{1}{H_i}\frac{dH_i}{dr} - \frac{3}{H_0}\frac{dH_0}{dr}\right)\Gamma^{\hat{r}} + \left[\frac{-3}{H_0}\frac{dH_0}{dr}\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} \right.\right.$$
$$\left.\left. + \left(-\frac{1}{H_1}\frac{dH_1}{dr}\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}} - \frac{1}{H_2}\frac{dH_2}{dr}\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}} - \frac{1}{H_3}\frac{dH_3}{dr}\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}\right)\right]\right\}\epsilon \quad (163)$$

Now we divide the 32-component spinor ϵ into sixteen subspaces labeled by $(s_1 s_2 s_3 w)$ with $s_1, s_2, s_3 = \pm 1$ being eigenvalues of $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}$, and $w = \pm 1$ being eigenvalue of $\Gamma^{\hat{r}}$. It is then easy to see that, $\delta\lambda = 0$ if and only if

$$\epsilon = \epsilon_{+++-} + \epsilon_{---+} \quad (164)$$

where the subscripts denote the subspace the spinors belong to. This gives us the four Killing spinors of the full black hole geometry, and we shall denote them as ϵ_{full}. The concrete coordinate-dependence of ϵ_{full} can be worked out by requiring the vanishing of the gravitino variation, however we don't need this detailed knowledge for the analysis below.

Now let's look at the kappa-projection matrix Γ given in eqn. (143) in the near-horizon region. Note that

$$\Gamma^{\hat{\theta}\hat{\phi}}\Gamma^{\underline{10}}\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} = \Gamma^{\hat{\chi}}\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}} \quad (165)$$

and that $\Gamma^{\hat{\chi}}$ is the same as $\Gamma^{\hat{r}}$ because both are tangent-indiced gamma matrices. We see that Γ commutes with $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}$. Hence requiring the supersymmetry of the full geometry to be preserved by D2, i.e.,

$$\Gamma\epsilon_{full} = \epsilon_{full} \quad (166)$$

is equivalent to requiring

$$\Gamma\epsilon_{+++-} = \epsilon_{+++-}, \; and \; \; \Gamma\epsilon_{---+} = \epsilon_{---+} \quad (167)$$

which is immediately seen to be impossible to satisfy for any finite value of χ_0, because

$$\begin{aligned}\Gamma\epsilon_{+++-} &= \frac{-1}{\cosh\chi_0}\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\epsilon_{+++-} + \tanh\chi_0\ \epsilon_{+++-} \\ \Gamma\epsilon_{---+} &= \frac{-1}{\cosh\chi_0}\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}\epsilon_{---+} + \tanh\chi_0\ \epsilon_{---+}\end{aligned} \quad (168)$$

(where the identity (165) has been used) and we see that the first terms on the right hand sides have the wrong eigenvalue under $\Gamma^{\hat{r}}$ (because $\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}}$ anticommutes with $\Gamma^{\hat{r}}$). This proves our claim that, for finite χ_0 the D2 brane doesn't preserve any of the four usual supersymmetries of the full geometry (the four supersymmetries preserved by the D2 as shown in subsection 7.1.2

have to be formed out of linear combinations of the usual supersymmetries of the full geometry and the conformal supersymmetries that are present only in the near-horizon region). What about the case $\chi_0 \to \infty$? In this case, the first terms on the right hand sides of eqn. (168) vanish, and the second terms become ϵ_{+++-} and ϵ_{---+} respectively, giving exactly what is needed for $\Gamma\epsilon_{full} = \epsilon_{full}$. This comes as no surprise since in the infinite χ_0 limit the D2 has an infinite D0 charge and is effectively just a bunch of D0 branes, which is known to preserve all the four usual supersymmetries of the full black hole geometry.

7.2 Supersymmetry of D3 branes in Microstate geometry

In this subsection we examine supersymmetry properties of D3 brane in the 2 charge microstate geometry discussed in section (4). Analogously to the D2 case considered above, we shall find that the D3 brane preserves half of the supersymmetries of the near-hoziron geometry, but doesn't preserve any of the supersymmetries of the full asymptotically flat geometry.

As in the above IIA case, we use $m, n... = t, y, r, \theta, \phi, \psi, y^1, y^2, y^3, y^4$ to denote curved space indices, and $a, b... = \hat{0}, \hat{1}, ..., \hat{9}$ to denote tangent space indices. $\hat{\Gamma}_a$ are ten dimensional Gamma matrices, which we will decompose into direct products of 6-d Gamma matrices denoted as $\tilde{\Gamma}_a$ and 4-d Gamma matrices denoted as Γ_a. The analysis in the near-horizon region $AdS_3 \times S^3 \times T^4$ is similar to the D2 case, hence instead of giving all the details here we will simply quote the near-horizon results when needed without proof.

Let us consider the D3 brane at its stable point $\dot{y} = -1$. (In Section 4 the choice of $\dot{y} = +1$ was made. This difference in choices does not affect the conclusion of the analysis below, because they just correspond to conjugate Killing spinors preserved by the D3 brane). As shown in Section 4, for $\dot{y} = -1$ the determinant of the metric induced on the brane simplifies to $\sqrt{-g} = a\sqrt{Q_1 Q_2}\sin\theta\cos\theta$. Then the kappa symmetry condition (after getting rid of antisymmetrization and combinatorial factors) becomes

$$\gamma_t\gamma_\theta\gamma_\psi\gamma_\phi\xi = -ia\sqrt{Q_1 Q_2}\sin\theta\cos\theta\xi \tag{169}$$

where γ_i are the pull backs on the brane worldvolume of the space time Gamma matrices. Using the vielbeins for the six dimensional 2-charge microstate metric

$$e^{\hat{0}} = \frac{1}{\sqrt{h}}\Big(dt + \frac{a\sqrt{Q_1 Q_2}}{f}\sin^2\theta d\phi\Big)\ ,\ e^{\hat{1}} = \frac{1}{\sqrt{h}}\Big(dy - \frac{a\sqrt{Q_1 Q_2}}{f}\cos^2\theta d\psi\Big) \tag{170}$$

$$e^{\hat{2}} = \sqrt{\frac{hf}{r^2+a^2}}dr\ ,\ e^{\hat{3}} = \sqrt{hf}d\theta\ ,\ e^{\hat{4}} = \sqrt{h}r\cos\theta d\psi\ ,\ e^{\hat{5}} = \sqrt{h(r^2+a^2)}\sin\theta d\phi \tag{171}$$

the induced gamma matices are found to be

$$\gamma_\theta = e^{\hat{3}}_\theta\tilde{\Gamma}_{\hat{3}}\ ,\ \gamma_\phi = e^{\hat{5}}_\phi\tilde{\Gamma}_{\hat{5}} + e^{\hat{0}}_\phi\tilde{\Gamma}_{\hat{0}}\ ,\ \gamma_t = e^{\hat{0}}_t\tilde{\Gamma}_{\hat{0}} + \dot{y}e^{\hat{1}}_y\tilde{\Gamma}_{\hat{1}}\ ,\ \gamma_\psi = e^{\hat{4}}_\psi\tilde{\Gamma}_{\hat{4}} + e^{\hat{1}}_\psi\tilde{\Gamma}_{\hat{1}} \tag{172}$$

Setting $\dot{y} = -1$ in the expression for γ_t and using using $e_t^{\hat{0}} = e_y^{\hat{1}}$ we can then rewrite the kappa symmetry matrix in terms of constant Gamma matrices and vielbeins as

$$\gamma_{t\theta\psi\phi}\,\xi = e_t^{\hat{0}} e_\theta^{\hat{3}} \left[(e_\psi^{\hat{1}} e_\phi^{\hat{5}} \tilde{\Gamma}_{\hat{3}} \tilde{\Gamma}_{\hat{5}} - e_\psi^{\hat{4}} e_\phi^{\hat{0}} \tilde{\Gamma}_{\hat{3}} \tilde{\Gamma}_{\hat{4}})(1 - \tilde{\Gamma}_{\hat{0}\hat{1}}) + (e_\psi^{\hat{4}} e_\phi^{\hat{5}} \tilde{\Gamma}_{\hat{0}} \tilde{\Gamma}_{\hat{3}} \tilde{\Gamma}_{\hat{4}} \tilde{\Gamma}_{\hat{5}} - e_\psi^{\hat{1}} e_\phi^{\hat{0}} \tilde{\Gamma}_{\hat{0}} \tilde{\Gamma}_{\hat{3}})(1 + \tilde{\Gamma}_{\hat{0}\hat{1}}) \right] \xi \tag{173}$$

Now let's look at a Killing spinor for the asymptotically flat metric generated by the background $D3$ branes. We know that it will be of the form $\xi = g(x)\xi_0$ (see, e.g., [18]). Here $g(x)$ is some spacetime dependent part which will cancel from both sides of kappa symmetry matrix as it doesn't depend on gamma matrices (unlike the near horizon case). The constant part ξ_0 satisfies projection conditions corresponding to two orthogonal sets of $D3$ branes. Our $D3$ branes are along directions $y67$ and $y89$, hence

$$\xi_0 + i\hat{\Gamma}_{\hat{0}\hat{1}\hat{6}\hat{7}}\xi_0 = 0 \;,\; \xi_0 + i\hat{\Gamma}_{\hat{0}\hat{1}\hat{8}\hat{9}}\xi_0 = 0 \tag{174}$$

We decompose the constant spinor ξ_0 as $\xi_{M6}^{(0)} \otimes \xi_{T4}^{(0)}$. This gives three constraints

$$\xi_{M6}^{(0)} + \tilde{\Gamma}_{\hat{0}\hat{1}}\xi_{M6}^{(0)} = 0 \;,\; \xi_{T4}^{(0)} + i\Gamma_{\hat{6}\hat{7}}\xi_{T4}^{(0)} = 0 \;,\; \xi_{T4}^{(0)} + i\Gamma_{\hat{8}\hat{9}}\xi_{T4}^{(0)} = 0 \tag{175}$$

The second and third constraints can be seen to be satisfied as in the near horizon case by using an explicit representation of gamma matrices. For now we concentrate on the $M6$ part. Using the first constraint in (175), we see that the term containg $(I + \tilde{\Gamma}_{\hat{0}\hat{1}})\xi$ in the kappa symmetry matrix gives zero. Plugging in the values of vielbeins, we get, from the remaining term,

$$\frac{1}{\sqrt{f}}(\sqrt{r^2+a^2}\cos\theta\tilde{\Gamma}_{\hat{3}\hat{5}} + r\sin\theta\tilde{\Gamma}_{\hat{3}\hat{4}})(1 - \tilde{\Gamma}_{\hat{0}\hat{1}})\xi_{M6}^{(0)} = -\xi_{M6}^{(0)} \tag{176}$$

It is apparent that the kappa symmetry condition cannot be satisfied for $r \neq 0$. For $r = 0$ we get a projection condition on $\xi_{M6}^{(0)}$ that can be easily seen to be inconsistent with the first of the constraints in (175). We conclude that the D3 brane is not supersymmetric in the full asymptotically flat geometry for any value of r.

Let's ask why the supersymmetry of the D3 brane is broken in full 2-charge microstate geometry. We have seen that the Killing spinors of the full six dimensional background geometry of $D3-D3$ system satisfy the projection condition $(I+\tilde{\Gamma}_{\hat{0}\hat{1}})\xi_{M6}^{(0)} = 0$. In the near horizon region, the geometry neatly separates into AdS and sphere parts, hence we can write gamma matrices for AdS part and they act on the AdS part of Killing spinor. So we have

$$\tilde{\Gamma}_{\hat{0}\hat{1}}\xi_{ads}^{(0)} = -\xi_{ads}^{(0)} \tag{177}$$

In the near horizon region we have two types of supersymmetries. In addition to ordinary supersymmetries, there are also the superconformal supersymmetries. Only ordinary supersymmetries continue to the far, i.e., asymptotically flat region. Now we want to see if the projection condition (177) is compatible with the kappa symmetry condition for $D3$ brane wrapping the sphere in the near horizon region. The condition one finds in the near-horizon region is, with $\xi^{(0)} = \epsilon_0$,

$$\tilde{\Gamma}_{\hat{0}}\epsilon_0 = -\epsilon_0 \tag{178}$$

The condition to be continuable to the far region is that it be in an eigenvector of $\tilde{\Gamma}_{\hat{0}\hat{1}} = \tilde{\Gamma}_{\hat{2}}$ i.e

$$\tilde{\Gamma}_{\hat{2}}\epsilon_0 = -\epsilon_0 \tag{179}$$

In three dimensions, $\tilde{\Gamma}_{\hat{0},\hat{1},\hat{2}}$ are just Pauli matrices which don't commute. Hence it is not possible for them to have simultaneous eigenvectors. As a result, the two conditions (178) and (179) are not compatible and hence Killing spinors in the far region that are preserved by this $D3$ brane do not exist.

8 Discussion

In [8] the 4-charge black hole was considered. The charges were D4-D4-D4-D0. It was argued that the D0 branes swell up into D2 branes which wrap the horizon, and which occupies a Landau level on the torus. The different ways to partition the D0 branes into such groups gave the entropy of the hole.

We must however ask if the energy of the D0 branes remains the same when we try to make them form a D2 brane; since we are looking at the states of an extremal hole we do not have any 'extra' energy to make the D2 brane. It is not clear to us how this would work in general, since in the limit where we have a very small D0 charge the mass of the D2 brane would seem to be just the area of the horizon times the tension, and this is much more than the mass of the D0 branes attached to it. In fact in the work of [8] the global energy which follows from the supersymmetry algebra turned out to be equal to the mass of the D2 brane *with no contribution to the D0 charge*! This is what would follow in our treatment if we chose a gauge for the 1-form potential of the background to be $A^{(1)} = \frac{R_{IIA}}{q_0} \sinh\chi \, d\tau$ rather than (63). As we have noted, there is always an ambiguity in calculating energies from brane actions.

The situation would be clearer if we had an asymptotically flat space-time. With this in mind we have looked at 2-charge microstates which have a similar structure to the system of [8], but where the AdS space inside goes over to asymptotically flat space at large r. We find that the mass of the D3+P system (which is analogous to the D2+D0 system) is given by the *sum* of two contributions: the energy carried by P and an energy coming from the tension of the D3. It is interesting that the energy is given by such a simple relation, because this configuration is not supersymmetric in the full asymptotically flat geometry. This suggests that there is some hidden symmetry in this 2-charge background, but we do not have any clear understanding of this as yet. But this also raises a puzzle about the relation of this computation with that of [8], since the mass of the D3+P system is more than the mass of the P charge alone.

We also computed the gauge field produced by the D3 brane wrapped on the S^3 in the full asymptotically flat geometry, and found that the field strength went to a nonzero constant at infinity. This suggests a divergent total energy for the field produced by the brane, or alternatively, that the D3 branes and anti-branes wrapped in this way are 'confined' and cannot be separated to large distances without generating a uniform energy density in the intervening spacetime. Note that the energy $E = P + M$ computed using the DBI action ignores this field energy. The field energy is quadratic in the test brane charge, and would be ignored in a linear analysis if it were finite.

9 Acknowledgements

S.R.D. would like to thank the Tata Institute of Fundamental Research for hospitality during the completion of this paper. The work of S.R.D., X.W and C.Z was partially supported by the National Science Foundation Grant No. PHY-0244811 and by the Department of Energy Contract No. DE-FG-01-00ER45832. S.G. was supported by an I.N.F.N. fellowship. We are grateful to B. de Wit, T. McLoughlin, and J. Michelson for discussions.

10 Appendix : Trajectories in the full Black Hole geometry

Consider the motion of a brane in the full four dimensional black hole geometry which has an energy (as measured in terms of the time in the asymptotically flat region) which is given by $E = (M_2 + M_0)\frac{R}{q_0}$, i.e. the same energy which we found in the near-horizon approximation. We will verify that this brane comes out of the horizon and goes back and examine the parameter space for which the brane remains in the near-hroizon region. In this analysis we will set the motion along the T^6 to zero from the beginning, so that we will deal with the four dimensional part of the geometry.

The black hole solution is described in terms of harmonic functions

$$H_0(r) = 1 + \frac{q_0}{r} \qquad H_i(r) = 1 + \frac{p_i}{r} \quad (i = 1, \cdots 3) \tag{180}$$

The (four dimensional part) string metric, dilaton and the 1-form RR fields are given by

$$\begin{aligned} ds^2 &= -\frac{dt^2}{[H(r)]^2} + [H(r)]^2 \; [dr^2 + r^2 d\Omega_2^2] \\ A_t &= 1 - \frac{1}{H_0(r)} \\ e^{\Phi} &= \frac{H_0(r)}{H(r)} \end{aligned} \tag{181}$$

where we have defined

$$H(r) = (H_0 H_1 H_2 H_3)^{\frac{1}{4}} \tag{182}$$

The lagrangian for a $D2$ brane which is wrapped on the S^2 at some value of r then becomes

$$S = -\mu(r)\sqrt{[H(r)]^{-2} - [H(r)]^2(\dot{r})^2} + \frac{M_0}{H_0(r)} \tag{183}$$

where we have defined

$$\mu(r) = 4\pi\mu_2 \frac{H(r)}{H_0(r)}\sqrt{(H(r))^4 r^4 + f^2} \tag{184}$$

and the other quantities have been defined above.

The expression for the energy is

$$E = \frac{\mu(r)[H(r)]^{-2}}{\sqrt{[H(r)]^{-2} - [H(r)]^2(\dot{r})^2}} - \frac{M_0}{H_0(r)} \tag{185}$$

Following the strategy of section (2.2) we will cast the problem as that of a non-relativistic particle in some potential with the non-relativistic energy equal to zero. The equation of motion may be written using (185) as

$$\frac{1}{2}(\dot{r})^2 + W(r) = 0 \tag{186}$$

where

$$W(r) = -\frac{1}{2H^2(r)}\,[\frac{1}{H^2(r)} - \frac{\mu^2(r)}{H^4(r)(E + \frac{M_0}{H_0(r)})^2}] \tag{187}$$

The potential $W(r)$ behaves as $-r^4$ for small r and $+r^4$ for large r and has a single minimum. For any E the brane therefore starts from the horizon, goes upto a maximum distance $r = r_0$ given by the point $W(r_0) = 0$ and turns back to the horizon.

The near-horizon region has $r \ll q_0, p_i$ and we want to examine whether r_0 lies in this region. The general problem is difficult to analyze. However we get some indication by looking at the simpler case where

$$q_0 = p_1 = p_2 = p_3 \equiv q \tag{188}$$

so that $H_0(r) = H_1(r) = H_2(r) = H_3(r) = H(r)$. In this case

$$\mu^2(r) = M_2^2(1 + \frac{r}{q})^4 + M_0^2 \tag{189}$$

where M_2 is the $D2$ mass of the previous subsections.

In terms of the dimensionless distance

$$y \equiv \frac{r}{q} \tag{190}$$

the potential $W(r)$ becomes

$$W(y) = \frac{y^4}{2(1+y)^3}\frac{y^2(1+y)^3 - \epsilon^2(1+y) - 2\alpha\epsilon y}{(\epsilon(1+y) + \alpha y)^2} \tag{191}$$

where we have defined

$$\epsilon \equiv \frac{E}{M_2} \qquad \alpha = \frac{M_0}{M_2} \tag{192}$$

We want to examine only the special trajectory with $E = M_2$. The function $W(y)$ for $E = M_2$ is shown in Figure (2) for various values of the ratio $\alpha = M_0/M_2$

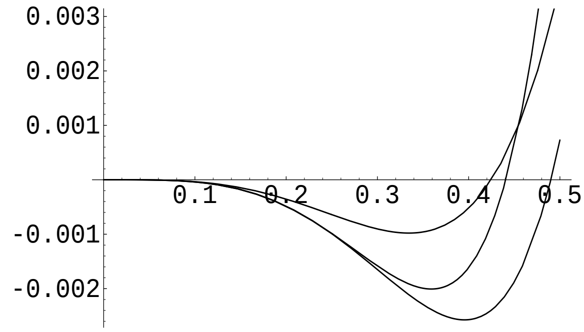

Figure 2: The potential $W(y)$ as a function of y for $E = M_2$. The curves have $\frac{M_0}{M_2} = 0, 1, 6$ starting from the top

The trajectory will proceed to the zero of $W(y)$ at $y = y_0(\alpha) \neq 0$. The function $W(y)$ is plotted against y for various values of α in Figure (2). It is clear that the value of y_0 increases as α increases and becomes *greater than unity* for sufficiently large α. Thus the $D2$ brane goes beyond the near-horizon region for large enough α and strictly speaking the near-horizon approxiomation can be trusted only when $M_0 \ll M_2$.

References

[1] R. C. Myers, JHEP **9912**, 022 (1999) [arXiv:hep-th/9910053].

[2] J. McGreevy, L. Susskind and N. Toumbas, JHEP **0006**, 008 (2000) [arXiv:hep-th/0003075].

[3] M. T. Grisaru, R. C. Myers and O. Tafjord, JHEP **0008**, 040 (2000) [arXiv:hep-th/0008015]; A. Hashimoto, S. Hirano and N. Itzhaki, JHEP **0008**, 051 (2000) [arXiv:hep-th/0008016].

[4] S. R. Das, A. Jevicki and S. D. Mathur, Phys. Rev. D **63**, 024013 (2001) [arXiv:hep-th/0009019]; S. R. Das, A. Jevicki and S. D. Mathur, Phys. Rev. D **63**, 044001 (2001) [arXiv:hep-th/0008088].

[5] S. R. Das, S. P. Trivedi and S. Vaidya, JHEP **0010**, 037 (2000) [arXiv:hep-th/0008203].

[6] A. Simons, A. Strominger, D. M. Thompson and X. Yin, Phys. Rev. D **71**, 066008 (2005) [arXiv:hep-th/0406121].

[7] D. Gaiotto, A. Simons, A. Strominger and X. Yin, arXiv:hep-th/0412179.

[8] D. Gaiotto, A. Strominger and X. Yin, arXiv:hep-th/0412322.

[9] S. Kim and J. Raeymaekers, arXiv:hep-th/0505176.

[10] J. M. Maldacena and A. Strominger, JHEP **9812**, 005 (1998) [arXiv:hep-th/9804085].

[11] V. Balasubramanian, J. de Boer, E. Keski-Vakkuri and S. F. Ross, Phys. Rev. D **64**, 064011 (2001) [arXiv:hep-th/0011217].

[12] J. M. Maldacena and L. Maoz, JHEP **0212**, 055 (2002) [arXiv:hep-th/0012025].

[13] S. Giusto, S. D. Mathur and A. Saxena, Nucl. Phys. B **701**, 357 (2004) [arXiv:hep-th/0405017]; S. Giusto, S. D. Mathur and A. Saxena, Nucl. Phys. B **710**, 425 (2005) [arXiv:hep-th/0406103]; S. Giusto and S. D. Mathur, arXiv:hep-th/0409067.

[14] O. Lunin and S. D. Mathur, Nucl. Phys. B **623**, 342 (2002) [arXiv:hep-th/0109154].

[15] E. Bergshoeff and P. K. Townsend, Nucl. Phys. B **490**, 145 (1997) [arXiv:hep-th/9611173].

[16] E. J. Hackett-Jones, D. C. Page and D. J. Smith, JHEP **0310**, 005 (2003) [arXiv:hep-th/0306267].

[17] P. M. Saffin, Phys. Rev. D **71**, 025018 (2005) [arXiv:hep-th/0407156].

[18] R. Emparan, D. Mateos and P. K. Townsend, JHEP **0107**, 011 (2001) [arXiv:hep-th/0106012].