SOME SPECIAL TRIGONOMETRICAL SERIES RELATED TO THE DISTRIBUTION OF PRIME NUMBERS

P. T. Bateman *and* S. Chowla

1. In response to a query of N. J. Fine, Besicovitch [1] has constructed an example of a non-trivial real-valued continuous function f on the real line which has period unity, is not odd, and has the property

$$\sum_{h=1}^{k} f(h/k) = 0 \tag{1}$$

for every positive integer k. It is the purpose of this note to remark that the functions given by

$$f_1(\theta) = \Sigma\, \lambda(n)\, n^{-1} \cos 2\pi n\theta = \operatorname{Re} \Sigma\, \lambda(n)\, n^{-1} \exp 2\pi i n\theta, \tag{2}$$

where λ denotes the Liouville function, and

$$f_2(\theta) = \Sigma\, \mu(n)\, n^{-1} \cos 2\pi n\theta = \operatorname{Re} \Sigma\, \mu(n)\, n^{-1} \exp 2\pi i n\theta, \tag{3}$$

where μ denotes the Möbius function, also have these properties. The uniform convergence of the series in (2) and (3) (and consequently the continuity of f_1 and f_2) follows from a deep theorem of Davenport [2]. It is then comparatively trivial to show that f_1 and f_2 satisfy the condition (1).

We recall that the Liouville function is defined as follows: $\lambda(1) = 1$, $\lambda(n) = (-1)^j$ if n is the product of j (not necessarily distinct) prime numbers. The Möbius function is defined as follows: $\mu(n) = \lambda(n)$ if n is not divisible by the square of any prime, $\mu(n) = 0$ if n is divisible by the square of some prime. Throughout the paper all summations on n are to be taken over the positive integers and all products on p are to be taken over the primes, subject, in either case, to whatever further conditions are specified.

2. Davenport's theorem is embodied in the first assertion of the following lemma.

Lemma 1. *If A is a given positive constant, then for real x greater than unity we have*

$$T(x, \theta) = \sum_{n \leqslant x} \mu(n) \exp 2\pi i n\theta = O\left(x \log^{-A}(x+1)\right)$$

uniformly in θ, and also

$$S(x, \theta) = \sum_{n \leqslant x} \lambda(n) \exp 2\pi i n\theta = O\left(x \log^{-A}(x+1)\right)$$

uniformly in θ.

Received 27 July, 1962. This paper was written at the Institute of Number Theory sponsored during the year 1961–1962 by the National Science Foundation at the University of Pennsylvania, Philadelphia, Pa. Support was also received from the U.S. Office of Naval Research.

[Journal London Math. Soc. 38 (1963), 372–374]

The second assertion of Lemma 1 follows immediately from the first. In fact from the definitions of λ and μ we have

$$S(x, \theta) = \sum_{n \leqslant x} \sum_{d^2|n} \mu(nd^{-2}) \exp 2\pi i n\theta = \sum_{d \leqslant x^{\frac{1}{2}}} T(d^{-2}x, d^2\theta).$$

Now trivially

$$\left| \sum_{x^{\frac{1}{4}} < d \leqslant x^{\frac{1}{2}}} T(d^{-2}x, d^2\theta) \right| \leqslant \sum_{d > x^{\frac{1}{4}}} d^{-2}x \leqslant x(x^{\frac{1}{4}} - \tfrac{1}{2})^{-1}$$

$$= O\left(x \log^{-A}(x+1)\right).$$

On the other hand the first assertion of the lemma gives

$$\sum_{d \leqslant x^{\frac{1}{4}}} T(d^{-2}x, d^2\theta) = \sum_{d \leqslant x^{\frac{1}{4}}} O\left(d^{-2}x \log^{-A}(x^{\frac{1}{2}}+1)\right)$$

$$= O\left(x \log^{-A}(x+1)\right).$$

Thus the second assertion of Lemma 1 is established.

3. We now show that the power series $\Sigma \lambda(n) n^{-1} z^n$ and $\Sigma \mu(n) n^{-1} z^n$ converge uniformly on the unit circle, even though they do not converge absolutely there. (An example of a power series with this property was first given by Hardy [**3**]. However, the first of the two power series here fulfils the further requirement that the coefficient of z^n is $\pm n^{-1}$ for every positive integer n. *Cf.* [**5**].)

LEMMA 2. *The series* $\Sigma \lambda(n) n^{-1} \exp 2\pi i n\theta$ *and* $\Sigma \mu(n) n^{-1} \exp 2\pi i n\theta$ *converge uniformly in* θ, *for real* θ.

Proof. We apply Lemma 1 with some fixed A greater than 1. If a and b are positive integers with $a < b$, we have by partial summation

$$\sum_{a < n \leqslant b} \mu(n) n^{-1} \exp 2\pi i n\theta$$

$$= -(a+1)^{-1} T(a, \theta) + \sum_{a<n<b} n^{-1}(n+1)^{-1} T(n, \theta) + b^{-1} T(b, \theta)$$

$$= O\left(\log^{-A}(a+1)\right) + \sum_{a<n<b} O\left((n+1)^{-1} \log^{-A}(n+1)\right) + O\left(\log^{-A}(b+1)\right)$$

$$= O\left(\log^{-A+1}(a+1)\right).$$

Since $\log^{-A+1}(a+1)$ tends to zero with increasing a (in view of our choice of A), the second assertion of Lemma 2 follows. The first assertion is proved in the same way.

We remark that for any particular *rational* value of θ the estimates of Lemma 1 and the convergence assertions of Lemma 2 were proved by Landau [**4**]. However, the above uniformity results stemming from Davenport's work are somewhat more difficult to obtain. In fact, Davenport's proof of the first assertion of Lemma 1 is similar to Vinogradov's proof that every sufficiently large odd integer is expressible as a sum of three odd primes.

4. THEOREM. *The functions f_1 and f_2 defined by* (2) *and* (3) *are non-trivial even real-valued continuous functions of period unity such that*

$$\sum_{h=1}^{k} f_1(h/k) = 0, \quad \sum_{h=1}^{k} f_2(h/k) = 0, \tag{4}$$

for every positive integer k.

Proof. In view of Lemma 2 we need only prove (4). First

$$\begin{aligned}\sum_{h=1}^{k} f_1(h/k) &= \mathrm{Re}\,\Sigma\,\lambda(n)\,n^{-1}\sum_{h=1}^{k}\exp(2\pi ihn/k)\\ &= \sum_{n\equiv 0 (\mathrm{mod}\ k)}\lambda(n)\,n^{-1}k = \Sigma\,\lambda(kn)\,n^{-1}\\ &= \lambda(k)\,\Sigma\,\lambda(n)\,n^{-1},\end{aligned}$$

where all the infinite series occurring converge by Lemma 2. But by the continuity theorem for Dirichlet series (*cf.* [**6**], p. 291)

$$\Sigma\,\lambda(n)\,n^{-1} = \lim_{s\to 1+}\Sigma\,\lambda(n)\,n^{-s} = \lim_{s\to 1+}\{\zeta(2s)/\zeta(s)\} = 0.$$

Thus the first assertion of (4) is proved. In the same way

$$\sum_{h=1}^{k} f_2(h/k) = \Sigma\,\mu(kn)\,n^{-1} = \mu(k)\sum_{(n,k)=1}\mu(n)\,n^{-1}$$

and, since the infinite series here likewise converge by Lemma 2,

$$\begin{aligned}\sum_{(n,k)=1}\mu(n)\,n^{-1} &= \lim_{s\to 1+}\sum_{(n,k)=1}\mu(n)\,n^{-s} = \lim_{s\to 1+}\prod_{p\nmid k}(1-p^{-s})\\ &= \lim_{s\to 1+}\left\{\zeta(s)\prod_{p\mid k}(1-p^{-s})\right\}^{-1} = 0.\end{aligned}$$

Thus the second assertion of (4) is proved.

References.

1. A. S. Besicovitch, " Problem on continuity ", *Journal London Math. Soc.*, 36 (1961),. 388–392.
2. H. Davenport, " On some infinite series involving arithmetical functions (II) ", *Quart J. of Math.* (Oxford), 8 (1937), 313–320.
3. G. H. Hardy, " A theorem concerning Taylor's series ", *Quart. J. of Pure and Applied Math.*, 44 (1913), 147–160.
4. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig, 1909), particularly pp. 567–637.
5. D. J. Newman, Problem 4449, *American Math. Monthly*, 58 (1951), 495.
6. E. C. Titchmarsh, *The theory of functions* (Oxford, 1939).

University of Illinois,
Urbana, Illinois, U.S.A.

University of Colorado,
Boulder, Colorado, U.S.A.