ON A TRIGONOMETRIC SUM

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Introduction.

Let $d(n)$ denote the number of divisors of the positive integer n. The trigonometric sum*

$$D(x) = \sum_{n \leqslant x} d(n)\, e^{2n\pi\theta i},$$

where θ is real, has recently been considered by us†. In particular, Walfisz has proved that, for almost all θ (that is, for all θ with the exception of a set of Lebesgue measure zero),

$$D(x) = O(x^{\frac{1}{2}} \log^{2+\epsilon} x),$$

$$D(x) = \Omega(x^{\frac{1}{2}} \log^{\frac{1}{2}} x \log\log x). \tag{1}$$

In proving (1) use was made of the following deep result due to T. Estermann‡:

I. *Let*

$$h \text{ and } k \text{ be integers, } k > 0,\ (h, k) = 1;$$

$$\Re(y) > 0, \quad k|y| < 8\sqrt{\{\Re(y)\}};$$

$$z = e^{2\pi i(h/k) - y}, \quad f(s) = \sum_{r=1}^{\infty} d(r)\, s^r \quad \textit{for} \quad |s| < 1;$$

* When the lower limit of summation is not explicitly stated it is always 1.

† See S. D. Chowla, "Some problems of diophantine approximation (I)", *Math. Zeitschrift*, 33 (1931), 544–563, and A. Walfisz, "Über einige trigonometrische Summen", *Math. Zeitschrift*, 33 (1931), 564–601.

‡ T. Estermann, "On the representations of a number as the sum of three products", *Proc. London Math. Soc.* (2), 29 (1929), 453–478, Lemma 11.

then, $\log y$ *denoting the principal branch, we have**

$$f(z) = \frac{1}{ky}(\gamma - \log y - 2\log k) + \frac{B|y|k\log 2k}{\Re(y)},$$

where γ *is Euler's constant.*

We are now able, with the simplest apparatus, to prove the following result, which is indeed weaker than I, but is sufficient to prove (1).

II. *Let*

(2) h *and* k *be integers,* $k > 1$, $(h, k) = 1$;

(3) $y = \sigma + ti$, σ *and* t *real,* $0 < \sigma \leqslant 1/k^2$, $|t| \leqslant \sigma$;

(4) $z = e^{2\pi i(h/k) - y}$, $f(s) = \sum_{r=1}^{\infty} d(r)s^r$ *for* $|s| < 1$.

Then we have

(5) $$f(z) = \frac{1}{ky}(\gamma - \log y - 2\log k) + B(\sigma^{-\frac{1}{2}} + k\log k).$$

With the help of II we shall prove the following results which go beyond (1): for almost all θ,

(6) $$\Re\{D(x)\} = \Omega_R\{x^{\frac{1}{4}}\log^{\frac{1}{4}}x(\log\log x)^{\frac{3}{4}}\};$$

(7) $$\Im\{D(x)\} = \Omega_{\substack{R\\L}}\{x^{\frac{1}{4}}\log^{\frac{1}{4}}x(\log\log x)^{\frac{3}{4}}\}.$$

Our method, which leads to II, can be applied to related questions; these applications will be given elsewhere.

1. *Proof of* II.

Let $x \geqslant 3$ throughout. We put†

(8) $$S(x) = \sum_{m \leqslant x} d(m)\, e\left(\frac{mh}{k}\right), \quad T(x) = \sum_{n \leqslant x} S(n).$$

We shall show that, under the conditions (2), (3), and (4),

(9) $$T(x) = \frac{1}{2k}x^2\log x - \frac{1}{k}(\log k - \gamma + \tfrac{3}{4})x^2 + B(x^{\frac{3}{2}} + xk\log k + x^{\frac{1}{2}}k^2).$$

(5) will then be deduced without difficulty from (9).

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* We denote by A positive absolute constants, by B complex numbers for which $|B| < A$, by c positive constants which may depend only on θ, by (u, v) the greatest common divisor of u and v; $a|b$ means that a divides b, $a \nmid b$ that a does not divide b.

† We write, for brevity, $e(u)$ in place of $e^{2\pi iu}$.

From (8) we obtain

$$S(x) = \sum_{ab \leqslant x} e\left(\frac{abh}{k}\right) = 2 \sum_{a \leqslant \sqrt{x}} \sum_{a \leqslant b \leqslant x/a} e\left(\frac{abh}{k}\right) - \sum_{a \leqslant \sqrt{x}} e\left(\frac{a^2 h}{k}\right)$$

$$(10) \qquad = 2 \sum_{a \leqslant \sqrt{x}} \sum_{a \leqslant b \leqslant x/a} e\left(\frac{abh}{k}\right) + B\sqrt{x}.$$

Let $[x]$ denote the greatest integer contained in x. For $a \not\equiv 0 \pmod{k}$ we have

$$\sum_{a \leqslant b \leqslant x/a} e\left(\frac{abh}{k}\right) = \frac{e\left\{\frac{ah}{k}\left(\left[\frac{x}{a}\right]+1\right)\right\} - e\left(\frac{a^2 h}{k}\right)}{e\left(\frac{ah}{k}\right) - 1}$$

$$= \frac{1}{2i} \frac{2i}{e\left(\frac{ah}{2k}\right) - e\left(\frac{-ah}{2k}\right)} \left\{ e\left(\frac{ah}{k}\left[\frac{x}{a}\right]\right) e\left(\frac{ah}{2k}\right) - e\left(\frac{a^2 h}{k}\right) e\left(\frac{-ah}{2k}\right)\right\}$$

$$(11) \qquad = \frac{1}{2i} \cot\frac{ah\pi}{k}\, e\left(\frac{ah}{k}\left[\frac{x}{a}\right]\right) + \tfrac{1}{2} e\left(\frac{ah}{k}\left[\frac{x}{a}\right]\right)$$

$$- \frac{1}{2i} \cot\frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right) + \tfrac{1}{2} e\left(\frac{a^2 h}{k}\right).$$

From (10) and (11) it follows that

$$(12) \qquad S(x) = -i \sum_{\substack{a \leqslant \sqrt{x} \\ k \nmid a}} \cot\frac{ah\pi}{k}\, e\left(\frac{ah}{k}\left[\frac{x}{a}\right]\right) + i \sum_{\substack{a \leqslant \sqrt{x} \\ k \nmid a}} \cot\frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right)$$

$$+ 2 \sum_{\substack{a \leqslant \sqrt{x} \\ k \mid a}} \sum_{a \leqslant b \leqslant x/a} 1 + B\sqrt{x}$$

$$(13) \qquad = -iS_1(x) + iS_2(x) + 2S_3(x) + B\sqrt{x}.$$

We split $S_1(x)$ into the two sums $S_{11}(x)$ and $S_{12}(x)$, where

$$(14) \qquad ah \equiv r \pmod{k}, \quad 1 \leqslant r \leqslant \tfrac{1}{2}k \quad \text{in } S_{11}(x),$$

$$(15) \qquad ah \equiv r \pmod{k}, \quad \tfrac{1}{2}k < r \leqslant k-1 \quad \text{in } S_{12}(x).$$

From the equations (12) to (15) it follows that

$$(16) \qquad S_1(x) = S_{11}(x) + S_{12}(x),$$

$$(17) \qquad S_{11}(x) = \sum_{r \leqslant \frac{1}{2}k} \cot\frac{r\pi}{k} \sum_{\substack{a \leqslant \sqrt{x} \\ ah \equiv r \pmod{k}}} e\left(\frac{r}{k}\left[\frac{x}{a}\right]\right),$$

$$(18) \qquad S_{12}(x) = \sum_{\frac{1}{2}k < r \leqslant k-1} \cot\frac{r\pi}{k} \sum_{\substack{a \leqslant \sqrt{x} \\ ah \equiv r \pmod{k}}} e\left(\frac{r}{k}\left[\frac{x}{a}\right]\right).$$

For real u_1 and u_2 we have

(19) $$|e(u_1)-e(u_2)| = \left|2\pi i\int_{u_2}^{u_1} e(u)\,du\right| \leqslant 2\pi|u_1-u_2|.$$

From (19) we obtain

$$e\left(\frac{r}{k}\left[\frac{x}{a}\right]\right)-e\left(\frac{r}{k}\frac{x}{a}\right) = B\frac{r}{k}.$$

Applied to (17) this gives

(20) $$S_{11}(x) = \sum_{r\leqslant \frac{1}{2}k} \cot\frac{r\pi}{k} \sum_{\substack{a\leqslant \sqrt{x}\\ ah\equiv r \pmod{k}}} e\left(\frac{r}{k}\frac{x}{a}\right) + B\sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant \sqrt{x}\\ ah\equiv r \pmod{k}}}\frac{r}{k}.$$

In the latter sum

$$\sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant \sqrt{x}\\ ah\equiv r \pmod{k}}}\frac{r}{k} = B\sum_{r\leqslant\frac{1}{2}k}\frac{r}{k}\operatorname{cosec}\frac{r\pi}{k}\sum_{\substack{a\leqslant \sqrt{x}\\ ah\equiv r \pmod{k}}}1$$

(21) $$= B\sum_{r\leqslant\frac{1}{2}k}\sum_{\substack{a\leqslant \sqrt{x}\\ ah\equiv r \pmod{k}}}1 = B\sum_{a\leqslant\sqrt{x}}1 = B\sqrt{x}.$$

From (20) and (21) we obtain

(22) $$S_{11}(x) = \sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant \sqrt{x}\\ ah\equiv r \pmod{k}}} e\left(\frac{r}{k}\frac{x}{a}\right)+B\sqrt{x}.$$

Writing $k-r$ for r in the sum (18) we obtain

$$S_{12}(x) = -\sum_{r<\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant \sqrt{x}\\ ah\equiv -r \pmod{k}}} e\left(-\frac{r}{k}\left[\frac{x}{a}\right]\right).$$

From this, with the help of (19) and an equation similar to (20), we obtain

(23) $$S_{12}(x) = -\sum_{r<\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant \sqrt{x}\\ ah\equiv -r \pmod{k}}} e\left(-\frac{r}{k}\frac{x}{a}\right)+B\sqrt{x}.$$

We now make an evaluation of the sum

(24) $$T_{11}(x) = \sum_{n\leqslant x} S_{11}(n).$$

From (22) and (24) it follows that

(25) $$T_{11}(x) = \sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{n\leqslant x}\sum_{\substack{a\leqslant \sqrt{n}\\ ah\equiv r \pmod{k}}} e\left(\frac{r}{k}\frac{n}{a}\right)+Bx^{\frac{3}{2}},$$

and here we have

$$\sum_{n\leqslant x}\ \sum_{\substack{a\leqslant\sqrt{n}\\ ah\equiv r \pmod k}} e\left(\frac{r}{k}\frac{n}{a}\right)=\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r \pmod k}}\ \sum_{a^2\leqslant n\leqslant x} e\left(\frac{r}{k}\frac{n}{a}\right)$$

$$=B\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r \pmod k}} \operatorname{cosec}\frac{r\pi}{ka}=B\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r \pmod k}}\frac{ka}{r}$$

(26) $$=\frac{Bk}{r}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r \pmod k}} a=B\sqrt{x}\,\frac{k}{r}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r \pmod k}} 1.$$

From (25) and (26) we obtain

$$T_{11}(x)=B\sqrt{x}\sum_{r\leqslant\frac{1}{2}k}\frac{k}{r}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r \pmod k}} 1+Bx^{\frac{3}{4}}$$

(27) $$=B\sqrt{x}\,k^2\sum_{r\leqslant\frac{1}{2}k}\frac{1}{r^2}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r \pmod k}} 1+Bx^{\frac{3}{4}}.$$

In the inner sum in (27) we have

$$a\equiv\rho \pmod k,\quad 1\leqslant\rho\leqslant k-1.$$

Hence

$$\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r \pmod k}} 1=\sum_{\substack{a\leqslant\sqrt{x}\\ a\equiv\rho \pmod k}} 1=\sum_{mk+\rho\leqslant\sqrt{x}} 1=\sum_{0\leqslant m\leqslant(\sqrt{x}-\rho)/k} 1$$

(28) $$=\left[\frac{\sqrt{x}-\rho}{k}\right]+1\leqslant\frac{\sqrt{x}}{k}+1.$$

(27) and (28) give

(29) $$T_{11}(x)=B\sqrt{x}k^2\sum_{r\leqslant\frac{1}{2}k}\frac{1}{r^2}\left(\frac{\sqrt{x}}{k}+1\right)+Bx^{\frac{3}{4}}$$

$$=B(xk+\sqrt{x}k^2+x^{\frac{3}{4}}).$$

Similarly, starting from (23), we find exactly as above that, if

(30) $$T_{12}(x)=\sum_{n\leqslant x}S_{12}(n),$$

then

(31) $$T_{12}(x)=B(xk+\sqrt{x}k^2+x^{\frac{3}{4}}).$$

We now put

(32) $$T_1(x)=\sum_{n\leqslant x}S_1(n).$$

From (32), (16), (24), (30), (29), and (31) it follows that

(33) $$T_1(x)=B(x^{\frac{3}{4}}+xk+x^{\frac{1}{2}}k^2).$$

We consider now the sum

$$S_2(x) = \sum_{\substack{a \leqslant \sqrt{x} \\ k \nmid a}} \cot \frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right). \tag{34}$$

If a takes the values 1, 2, ..., $k-1$ (mod k), we have

$$\sum_a \cot \frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right) = \sum_{a=1}^{k-1} \cot \frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right)$$

$$= \sum_{a<\frac{1}{2}k} \left(\cot \frac{ah\pi}{k} + \cot \frac{(k-a)h\pi}{k}\right) e\left(\frac{a^2 h}{k}\right) = 0.$$

It now follows from (34) that there is a positive integer $m \leqslant k-1$ such that

$$S_2(x) = \sum_{a \leqslant m} \cot \frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right),$$

whence

$$S_2(x) = B \sum_{a \leqslant k-1} \left|\operatorname{cosec} \frac{a\pi}{k}\right| = B \sum_{a \leqslant \frac{1}{2}k} \operatorname{cosec} \frac{a\pi}{k}$$

$$= Bk \sum_{a \leqslant \frac{1}{2}k} \frac{1}{a} = Bk \log k$$

and so

$$T_2(x) = \sum_{n \leqslant x} S_2(n) = Bx\, k \log k. \tag{35}$$

It remains to evaluate the sum $S_3(x)$ [given by (12) and (13)] and the iterated sum

$$T_3(x) = \sum_{n \leqslant x} S_3(n). \tag{36}$$

We have

$$S_3(x) = \sum_{\substack{a \leqslant \sqrt{x} \\ k \mid a}} \sum_{a \leqslant b \leqslant x/a} 1 = \sum_{a \leqslant \sqrt{x}/k} \sum_{ka \leqslant b \leqslant x/ka} 1$$

$$= \sum_{a \leqslant \sqrt{x}/k} \left(\frac{x}{ka} - ka\right) + B\sqrt{x} = \frac{x}{k} \sum_{a \leqslant \sqrt{x}/k} \frac{1}{a} - k \sum_{a \leqslant \sqrt{x}/k} a + B\sqrt{x}$$

$$= \frac{x}{k}\left(\log \frac{\sqrt{x}}{k} + \gamma + \frac{Bk}{\sqrt{x}}\right) - \tfrac{1}{2}k \left[\frac{\sqrt{x}}{k}\right]\left[\frac{\sqrt{x}}{k} + 1\right] + B\sqrt{x}$$

$$= \frac{x}{k}\left(\log \frac{\sqrt{x}}{k} + \gamma + \frac{Bk}{\sqrt{x}}\right) - \tfrac{1}{2}k \frac{\sqrt{x}}{k} \frac{\sqrt{x}}{k} + B\sqrt{x} + Bk$$

$$= \frac{1}{2k}\, x \log x - \frac{1}{k}\left(\log k - \gamma + \tfrac{1}{2}\right) x + B(\sqrt{x} + k). \tag{37}$$

From (36) and (37) it follows that

$$2T_3(x) = \frac{1}{k} \sum_{n \leqslant x} n \log n - \frac{2}{k} (\log k - \gamma + \tfrac{1}{2}) \sum_{n \leqslant x} n + B(x^{\frac{3}{2}} + xk)$$

$$= \frac{1}{k} \sum_{n \leqslant x} n \left(\sum_{m=1}^{n} \frac{1}{m} - \gamma + \frac{B}{n} \right)$$

$$- \frac{2}{k} (\log k - \gamma + \tfrac{1}{2}) \sum_{n \leqslant x} n + B(x^{\frac{3}{2}} + xk)$$

$$= \frac{1}{k} \sum_{n \leqslant x} n \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{k} (2 \log k - \gamma + 1) \sum_{n \leqslant x} n + B(x^{\frac{3}{2}} + xk)$$

$$= \frac{1}{k} \sum_{m \leqslant x} \frac{1}{m} \sum_{m \leqslant n \leqslant x} n - \frac{1}{2k} (2 \log k - \gamma + 1)[x][x+1] + B(x^{\frac{3}{2}} + xk)$$

$$= \frac{1}{2k} [x][x+1] \sum_{m \leqslant x} \frac{1}{m} - \frac{1}{2k} \sum_{m \leqslant x} (m-1)$$

$$- \frac{1}{2k} (2 \log k - \gamma + 1) x^2 + B(x^{\frac{3}{2}} + xk)$$

$$= \frac{1}{2k} x^2 \sum_{m \leqslant x} \frac{1}{m} - \frac{1}{2k} \sum_{m \leqslant x} m - \frac{1}{2k} (2 \log k - \gamma + 1) x^2 + B(x^{\frac{3}{2}} + xk)$$

$$= \frac{1}{2k} x^2 (\log x + \gamma) - \frac{1}{2k} \frac{x^2}{2} - \frac{1}{2k} (2 \log k - \gamma + 1) x^2 + B(x^{\frac{3}{2}} + xk)$$

(38) $$= \frac{1}{2k} x^2 \log x - \frac{1}{k} (\log k - \gamma + \tfrac{3}{4}) x^2 + B(x^{\frac{3}{2}} + xk).$$

In order to prove (9) we have only to combine (8), (13), (32), (33), (35), (36) and (38).

We shall now obtain (5) from (9), making frequent use of (2), (3), and (4). For brevity we write

(39) $$p = 2 \log k - 2\gamma + \tfrac{3}{2}, \quad q = 2 \log k - \gamma + \tfrac{3}{2}, \quad r = \sigma^{-\frac{1}{2}} + k \log k.$$

From (8), (9), and (39) we obtain

$$f(z) = (1 - e^{-y}) \sum_{n=1}^{\infty} S(n) e^{-ny} = (1 - e^{-y})^2 \sum_{n=1}^{\infty} T(n) e^{-ny}$$

(40) $$= \frac{1}{2k} (1 - e^{-y})^2 \sum_{n=1}^{\infty} n^2 (\log n - p) e^{-ny}$$

$$+ B\sigma^2 \sum_{n=1}^{\infty} (n^{\frac{3}{2}} + nk \log k + n^{\frac{1}{2}} k^2) e^{-n\sigma}.$$

The error term here is, by (39),

$$B\sigma^2(\sigma^{-\frac{3}{2}}+\sigma^{-2}k\log k+\sigma^{-\frac{3}{2}}k^2)=B(\sigma^{-\frac{1}{2}}+k\log k+\sigma^{\frac{1}{2}}k^2)=Br.$$

If we substitute $\sum_{m=1}^{n}\frac{1}{m}-\gamma$ for $\log n$ in (40), we obtain an error

$$B\sigma^2\sum_{n=1}^{\infty}n^2\frac{1}{n}e^{-n\sigma}=B.$$

Hence

$$f(z)=\frac{1}{2k}(1-e^{-y})^2\sum_{n=1}^{\infty}n^2\left(\sum_{m=1}^{n}\frac{1}{m}-q\right)e^{-ny}+Br$$

$$(41)\qquad =\frac{1}{2k}(1-e^{-y})^2\left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m}^{\infty}n^2e^{-ny}-q\sum_{n=1}^{\infty}n^2e^{-ny}\right)+Br.$$

We write for brevity

$$(42)\qquad e^{-y}=u.$$

Since

$$\sum_{n=m}^{\infty}n^2u^n=\frac{m^2u^m}{1-u}+\frac{(2m+1)u^{m+1}}{(1-u)^2}+\frac{2u^{m+2}}{(1-u)^3},$$

we obtain from (41), noting (42) and (39),

$$f(z)=\frac{1}{2k}(1-u)^2\left\{\sum_{m=1}^{\infty}\frac{mu^m}{1-u}+2\sum_{m=1}^{\infty}\frac{u^{m+1}}{(1-u)^2}+\sum_{m=1}^{\infty}\frac{u^{m+1}}{m(1-u)^2}\right.$$

$$\left.+2\sum_{m=1}^{\infty}\frac{u^{m+2}}{m(1-u)^3}-q\left(\frac{u}{1-u}+\frac{3u^2}{(1-u)^2}+\frac{2u^3}{(1-u)^3}\right)\right\}+Br$$

$$=\frac{1}{2k}(1-u)^2\left(\frac{u}{(1-u)^2}+\frac{u^2}{(1-u)^3}+\frac{2u^2}{(1-u)^3}+\frac{u}{(1-u)^2}\log\frac{1}{1-u}\right.$$

$$\left.+\frac{2u^2}{(1-u)^3}\log\frac{1}{1-u}\right)-\frac{1}{k}q\frac{u^3}{1-u}+Br$$

$$=\frac{1}{2k}\left(\frac{3u^2}{1-u}+u\log\frac{1}{1-u}+\frac{2u^2}{1-u}\log\frac{1}{1-u}\right)-\frac{1}{k}q\frac{u^3}{1-u}+Br$$

$$(43)\qquad =\frac{1}{2k}\left(\frac{3}{1-u}-3\log\frac{1}{1-u}+\frac{2}{1-u}\log\frac{1}{1-u}\right)-\frac{1}{k}q\frac{1}{1-u}+Br.$$

In (43) we have, from (42),

$$\frac{1}{1-u}=\frac{1}{y}+B,\quad \log\frac{1}{1-u}=-\log y+B\sigma=B\log\frac{1}{\sigma}.$$

Making these substitutions and noting (39), we obtain

$$f(z) = \frac{1}{2k}\left(\frac{3}{y} + B\log\frac{1}{\sigma} - \frac{2}{y}\log y + B\log\frac{1}{\sigma}\right) - \frac{1}{k}\,q\,\frac{1}{y} + Br$$

$$= \frac{1}{ky}\left(\tfrac{3}{2} - \log y - q\right) + B(\sigma^{-\frac{1}{2}} + k\log k)$$

$$(5) \qquad = \frac{1}{ky}(\gamma - \log y - 2\log k) + B(\sigma^{-\frac{1}{2}} + k\log k).$$

Thus II is proved.

2. *Proof of* (6) *and* (7).

Under the hypotheses of II we have $\sigma^{-\frac{1}{2}} \leqslant 1/(k\sigma)$. Hence it follows from (5) that

$$f(z) = \frac{1}{ky}(\gamma - \log y - 2\log k) + B\left(\frac{1}{k\sigma} + k\log k\right)$$

$$= \frac{\sigma - ti}{k|y|^2}\{\gamma - \log|y| - i\Im(\log y) - 2\log k\} + B\left(\frac{1}{k\sigma} + k\log k\right)$$

$$= \frac{\sigma - ti}{k|y|^2}\log\frac{1}{k^2|y|} + B\left(\frac{1}{k\sigma} + k\log k\right),$$

$$(44) \qquad \Re\{f(z)\} = \frac{\sigma}{k|y|^2}\log\frac{1}{k^2|y|} + B\left(\frac{1}{k\sigma} + k\log k\right),$$

$$(45) \qquad \Im\{f(z)\} = -\frac{t}{k|y|^2}\log\frac{1}{k^2|y|} + B\left(\frac{1}{k\sigma} + k\log k\right).$$

We shall now make use of the following theorem, due to Khintchine*:

Let $F(u)$ *be positive and continuous for* $u \geqslant 1$ *and let* $uF(u)$ *be steadily decreasing. When the integral*

$$(46) \qquad \int_1^\infty F(u)\,du$$

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* A. Khintchine, "Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen", *Math. Annalen*, 92 (1924), 115–125. Khintchine only considers the inequality $|\theta - (h/k)| < F(k)/k$. His proof leads, however, without further trouble to the theorem attributed to him in the text when we remember that (after Bernstein) each of the inequalities

$$a_{2n} > \frac{1}{G(2n-1)}, \quad a_{2n+1} > \frac{1}{G(2n)}$$

(for the elements in the simple continued fraction for θ) has infinitely many solutions, for almost all θ, whenever $G(n)$ is a positive, steadily decreasing function and $\sum_{n=1}^{\infty} G(n)$ diverges.

converges, each of the inequalities

$$0 \leqslant \pm\left(\theta - \frac{h}{k}\right) < \frac{F(k)}{k} \tag{47}$$

has only a finite number of integral solutions for almost all θ. *If, however, the integral* (46) *is divergent, then, for almost all* θ, *each of the inequalities* (47) *has infinitely many solutions with* $k > 0$, $(h, k) = 1$.

With the help of this result we shall deduce the following relations* from (44) and (45), for almost all θ:

$$\Re\{f(e^{-\rho+2\pi\theta i})\} = \Omega_R\left\{\left(\frac{1}{\rho}\,\mathrm{l}\frac{1}{\rho}\right)^{\frac{1}{2}}\left(\mathrm{ll}\frac{1}{\rho}\right)^{\frac{3}{2}}\left(\mathrm{lll}\frac{1}{\rho}\right)^{\frac{1}{2}}\right\} \quad (\rho > 0,\ \rho \to 0). \tag{48}$$

$$\Im\{f(e^{-\rho+2\pi\theta i})\} = \Omega_{\substack{R\\L}}\left\{\left(\frac{1}{\rho}\,\mathrm{l}\frac{1}{\rho}\right)^{\frac{1}{2}}\left(\mathrm{ll}\frac{1}{\rho}\right)^{\frac{3}{2}}\left(\mathrm{lll}\frac{1}{\rho}\right)^{\frac{1}{2}}\right\} \quad (\rho > 0,\ \rho \to 0). \tag{49}$$

To obtain (6) from (48) and (7) from (49) we make use of an artifice which has been used by Hardy and Littlewood† in a similar connection.

We apply Khintchine's theorem with

$$F(u) = (2\pi u\,\mathrm{l}\,2u\,\mathrm{ll}\,3u\,\mathrm{lll}\,27u)^{-1} \quad (u \geqslant 1),$$

so that the integral (46) is divergent. Hence, corresponding to almost every θ, there is an infinite sequence of positive integers $k_1, k_2, k_3, \ldots$ such that

$$3^{27} \leqslant k_1 < k_2 < k_3 < \ldots \tag{50}$$

and an infinite sequence of integers $h_1, h_2, h_3, \ldots$ such that, for all positive integral values of n,

$$(h_n, k_n) = 1, \tag{51}$$

$$(-1)^n\left(\theta - \frac{h_n}{k_n}\right) = \left|\theta - \frac{h_n}{k_n}\right| < (2\pi k_n^2\,\mathrm{l}\,k_n\,\mathrm{ll}\,k_n\,\mathrm{lll}\,k_n)^{-1}. \tag{52}$$

We put

$$\begin{cases} \sigma_n = (k_n^2\,\mathrm{l}\,k_n\,\mathrm{ll}\,k_n\,\mathrm{lll}\,k_n)^{-1}, \quad t_n = -2\pi\left(\theta - \dfrac{h_n}{k_n}\right), \quad y_n = \sigma_n + t_n i, \\ z_n = e^{(2h_n\pi i/k_n) - y_n}, \end{cases} \tag{53}$$

* For the sake of brevity we now write l, ll, lll for log, log log, log log log.

† G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation, II: The trigonometrical series associated with the elliptic ϑ-functions", *Acta Math.*, 37 (1914), 193–239 (228).

and we can apply (44) and (45) with

$$(54) \qquad h = h_n, \quad k = k_n, \quad \sigma = \sigma_n, \quad t = t_n, \quad y = y_n, \quad z = z_n,$$

since it follows from the equations (50) to (54) that the conditions (2), (3), (4) are satisfied. We notice, further, that, in consequence of (50),

$$(55) \qquad \mathrm{l}\, k_n > 27, \quad \mathrm{ll}\, k_n > 3, \quad \mathrm{lll}\, k_n > 1.$$

It follows now that

$$(56) \qquad \Re\{f(z_n)\} = \frac{\sigma_n}{k_n |y_n|^2} \mathrm{l} \frac{1}{k_n^2 |y_n|} + B\left(\frac{1}{k_n \sigma_n} + k_n \mathrm{l}\, k_n\right),$$

$$(57) \qquad \Im\{f(z_n)\} = -\frac{t_n}{k_n |y_n|^2} \mathrm{l} \frac{1}{k_n^2 |y_n|} + B\left(\frac{1}{k_n \sigma_n} + k_n \mathrm{l}\, k_n\right).$$

We have here, on account of (53) and (55),

$$\frac{1}{|y_n|^2} = \frac{1}{\sigma_n^2 + t_n^2} \geqslant \frac{1}{2\sigma_n^2} = \tfrac{1}{2}(k_n^2 \mathrm{l}\, k_n \mathrm{ll}\, k_n \mathrm{lll}\, k_n)^2,$$

$$(58) \qquad \frac{1}{k_n |y_n|^2} \geqslant \tfrac{1}{2} k_n^3 (\mathrm{l}\, k_n \mathrm{ll}\, k_n \mathrm{lll}\, k_n)^2,$$

$$(59) \qquad \frac{1}{k_n^2 |y_n|} \geqslant \tfrac{1}{2} \mathrm{l}\, k_n \mathrm{ll}\, k_n \mathrm{lll}\, k_n \geqslant \mathrm{l}\, k_n,$$

$$(60) \qquad \frac{1}{k_n \sigma_n} = k_n \mathrm{l}\, k_n \mathrm{ll}\, k_n \mathrm{lll}\, k_n.$$

From (56), (58), (53), (59), and (60) we obtain*

$$\Re\{f(z_n)\} \geqslant \tfrac{1}{2} k_n^3 (\mathrm{l}\, k_n \mathrm{ll}\, k_n \mathrm{lll}\, k_n)^2 (k_n^2 \mathrm{l}\, k_n \mathrm{ll}\, k_n \mathrm{lll}\, k_n)^{-1} \mathrm{ll}\, k_n - A k_n \mathrm{l}\, k_n \mathrm{ll}\, k_n \mathrm{lll}\, k_n$$

$$= \tfrac{1}{2} k_n \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2 \mathrm{lll}\, k_n - A k_n \mathrm{l}\, k_n \mathrm{ll}\, k_n \mathrm{lll}\, k_n,$$

whence

$$(61) \qquad \Re\{f(z_n)\} \geqslant \tfrac{1}{3} k_n \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2 \mathrm{lll}\, k_n \quad \text{for } n > A.$$

We now need a lower limit for $|t_n|$ and to this end we apply Khintchine's theorem with

$$F(u) = \{u \,\mathrm{l}\, 2u \,\mathrm{ll}\, 3u\, (\mathrm{lll}\, 27u)^{\frac{4}{3}}\}^{-1} \quad (u \geqslant 1).$$

The integral (46) converges and hence, for almost all θ, (47) can have only a finite number of integral solutions. We may suppose here that the set of values of θ specified is exactly the same as that for which (57) was

* See foot-note *, p. 402.

proved (for the common part of two sets of " almost all " θ is a set of almost all θ, as is seen immediately by considering the complementary sets). Hence, on account of (52) and (53), we have*

(62) $$(-1)^{n+1} t_n = |t_n| \geqslant c\{k_n^2 \, \mathrm{l}\, k_n \, \mathrm{ll}\, k_n \, (\mathrm{lll}\, k_n)^{\frac{5}{4}}\}^{-1}.$$

From (57), (58), (62), (59), and (60) it follows that

$$(-1)^n \Im\{f(z_n)\} \geqslant c\, k_n^3 (\mathrm{l}\, k_n \, \mathrm{ll}\, k_n \, \mathrm{lll}\, k_n)^2 \{k_n^2 \, \mathrm{l}\, k_n \, \mathrm{ll}\, k_n (\mathrm{lll}\, k_n)^{\frac{5}{4}}\}^{-1} \, \mathrm{ll}\, k_n$$
$$- A k_n \, \mathrm{l}\, k_n \, \mathrm{ll}\, k_n \, \mathrm{lll}\, k_n,$$

so that

(63) $$(-1)^n \Im\{f(z_n)\} \geqslant c\, k_n \, \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2 (\mathrm{lll}\, k_n)^{\frac{3}{4}} \quad \text{for} \quad n > c.$$

We now put

(64) $$k_n^2 \, \mathrm{l}\, k_n \, \mathrm{ll}\, k_n \, \mathrm{lll}\, k_n = \frac{1}{\rho_n}.$$

We then have the following asymptotic relations as $n \to \infty$:

$$\mathrm{l}\, k_n \sim \tfrac{1}{2} \, \mathrm{l} \frac{1}{\rho_n}, \quad \mathrm{ll}\, k_n \sim \mathrm{ll} \frac{1}{\rho_n}, \quad \mathrm{lll}\, k_n \sim \mathrm{lll} \frac{1}{\rho_n},$$

$$k_n^2 \sim 2 \left(\rho_n \, \mathrm{l} \frac{1}{\rho_n} \, \mathrm{ll} \frac{1}{\rho_n} \, \mathrm{lll} \frac{1}{\rho_n}\right)^{-1}, \quad k_n \sim \sqrt{2} \left(\rho_n \, \mathrm{l} \frac{1}{\rho_n} \, \mathrm{ll} \frac{1}{\rho_n} \, \mathrm{lll} \frac{1}{\rho_n}\right)^{-\frac{1}{2}},$$

whence

(65) $$k_n \, \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2 \, \mathrm{lll}\, k_n \sim \frac{1}{\sqrt{2}} \left(\frac{1}{\rho_n} \, \mathrm{l} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \left(\mathrm{ll} \frac{1}{\rho_n}\right)^{\frac{3}{2}} \left(\mathrm{lll} \frac{1}{\rho_n}\right)^{\frac{1}{2}},$$

(66) $$k_n \, \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2 (\mathrm{lll}\, k_n)^{\frac{3}{4}} \sim \frac{1}{\sqrt{2}} \left(\frac{1}{\rho_n} \, \mathrm{l} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \left(\mathrm{ll} \frac{1}{\rho_n}\right)^{\frac{3}{2}} \left(\mathrm{lll} \frac{1}{\rho_n}\right)^{\frac{1}{4}}.$$

From (53) and (64) it follows that

$$z_n = e^{-\rho_n + 2\pi\theta i}.$$

Hence, from (61), (65), and (63), (66), we obtain

$$\Re\{f(e^{-\rho_n + 2\pi\theta i})\} \geqslant \left(\frac{1}{\rho_n} \, \mathrm{l} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \left(\mathrm{ll} \frac{1}{\rho_n}\right)^{\frac{3}{2}} \left(\mathrm{lll} \frac{1}{\rho_n}\right)^{\frac{1}{8}} \quad (n > A),$$

$$(-1)^n \Im\{f(e^{-\rho_n + 2\pi\theta i})\} \geqslant \left(\frac{1}{\rho_n} \, \mathrm{l} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \left(\mathrm{ll} \frac{1}{\rho_n}\right)^{\frac{3}{2}} \left(\mathrm{lll} \frac{1}{\rho_n}\right)^{\frac{1}{8}} \quad (n > c).$$

Since $\rho > 0$, $\rho \to 0$, these two inequalities show the truth of (48) and (49).

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* See foot-note *, p. 402.

Let us suppose that

(67) $$\mathfrak{R}\{D(n)\} \leqslant c(n\,\mathrm{l}\,n)^{\frac{1}{2}}(\mathrm{ll}\,n)^{\frac{3}{2}} \quad (n \geqslant 27).$$

It would then follow that, for $\rho > 0$, $\rho \to 0$,

$$\mathfrak{R}\{f(e^{-\rho+2\pi\theta i})\} = \sum_{n=1}^{\infty} d(n) \cos 2n\pi\theta\, e^{-n\rho}$$

$$= (1-e^{-\rho}) \sum_{n=1}^{\infty} \mathfrak{R}\{D(n)\} e^{-n\rho} \leqslant c + c\rho \sum_{n=27}^{\infty} (n\,\mathrm{l}\,n)^{\frac{1}{2}}(\mathrm{ll}\,n)^{\frac{3}{2}} e^{-n\rho}$$

$$\leqslant c + c\rho \frac{1}{\rho} \left(\frac{1}{\rho}\,\mathrm{l}\,\frac{1}{\rho}\right)^{\frac{1}{2}} \left(\mathrm{ll}\,\frac{1}{\rho}\right)^{\frac{3}{2}} \leqslant c \left(\frac{1}{\rho}\,\mathrm{l}\,\frac{1}{\rho}\right)^{\frac{1}{2}} \left(\mathrm{ll}\,\frac{1}{\rho}\right)^{\frac{3}{2}}.$$

Since this contradicts (48), (67) cannot be true for any value of c. This proves (6).

If we now suppose that either of the inequalities

$$\pm\,\mathfrak{I}\{D(n)\} \leqslant c(n\,\mathrm{l}\,n)^{\frac{1}{2}}(\mathrm{ll}\,n)^{\frac{3}{2}} \quad (n \geqslant 27)$$

is true, we are led, by exactly similar reasoning, to a contradiction of (49), and thus (7) is proved.