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*Inventiones mathematicae*


# Values of quadratic forms at primitive integral points

S.G. Dani[1] and G.A. Margulis[2]

[1] School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

[2] Institute for Problems of Information Transmission, ul. Ermolovoi 19, Moscow 101447, USSR

## Introduction

Let $B$ be a real nondegenerate indefinite quadratic form on $\mathbb{R}^n$, where $n \geqq 3$, which is not a multiple of a rational form (that is, $B$ is not of the form $tB_0$ where $t \in \mathbb{R}$ and $B_0$ a quadratic form whose coefficients with respect to the standard basis are rational). It was shown in [13], (see also [11] and [12]) that for any $\varepsilon > 0$ there exists $x \in \mathbb{Z}^n$, namely an integral vector, such that

$$0 < |B(x)| < \varepsilon. \tag{1}$$

The result was conjectured by Oppenheim (around 1930 in a somewhat weaker form and in the 1950's in the present form) and has been a subject of considerable work by Davenport, Oppenheim and various other mathematicians. We refer the reader to [9] and [13] for details on the developments and other references. It is well known that the above result also implies that $B(\mathbb{Z}^n)$, namely the set of values at integer points is dense in $\mathbb{R}$(cf. [9, 13]). We now consider the set of values of $B$ at primitive elements in $\mathbb{Z}^n$, namely on $\mathfrak{P}(\mathbb{Z}^n) = \{x \in \mathbb{Z}^n \mid x \neq ky$ for any $y \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$ with $|k| \geqq 2\}$. While it is evident that the element $x$ in (1) above can be chosen to be primitive, it no longer follows from this that $B(\mathfrak{P}(\mathbb{Z}^n))$ is dense in $\mathbb{R}$. In fact hardly anything seems to be known about the set. We now prove the following

**1. Theorem.** *Let $B$ be a real nondegenerate indefinite quadratic form on $\mathbb{R}^n$, where $n \geqq 3$, which is not a multiple of a rational form. Let $B_2$ be the corresponding bilinear form, defined by $B_2(v, w) = \frac{1}{4}\{B(v+w) - B(v-w)\}$ for all $v, w \in \mathbb{R}^n$. Let $a, b, c \in \mathbb{R}$ be such that there exist $v, w \in \mathbb{R}^n$ for which $B(v) = a$, $B(w) = b$, $B_2(v, w) = c$. Then for any $\varepsilon > 0$ there exist $x, y \in \mathfrak{P}(\mathbb{Z}^n)$ such that*

$$|B(x) - a| < \varepsilon, \; |B(y) - b| < \varepsilon \text{ and } |B_2(x, y) - c| < \varepsilon.$$

*In particular $\{B(x) \mid x \in \mathfrak{P}(\mathbb{Z}^n)\}$ is a dense subset of $\mathbb{R}$.*

It is enough to prove the theorem for the case $n = 3$ (see §5). As in the case of the above mentioned result from [13], Theorem 1 is deduced from a result on flows on the homogeneous space $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$. Let $H$ be the connected component of the identity in the subgroup of $G = SL(3, \mathbb{R})$ consisting of all elements leaving invariant the quadratic form $2x_1x_3 - x_2^2$, $x_1$, $x_2$, $x_3$ being the coordinates of $x$. Oppenheim's conjecture described above was deduced in [13] by proving that every *relatively compact* $H$-orbit on $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$ is compact (and hence closed). We now prove the following strengthening of this result and deduce Theorem 1.

**2. Theorem.** *Any $H$-orbit in* $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$ *is either closed or dense.*

The deduction of Theorem 1 from Theorem 2 is similar to (in fact simpler than) the deduction of Oppenheim's conjecture in [13]; the details are given in §5. The homogeneous space approach for studying values of quadratic forms was noted by M.S. Raghunathan who conjectured in this connection that if $G$ is a semisimple Lie group, $\Gamma$ is a lattice in $G$ and $U$ is a connected unipotent subgroup of $G$ then for any $x \in G/\Gamma$ there exists a closed subgroup $F$ such that the closure of $Ux$ in $G/\Gamma$ is $Fx$. The reader is referred to [13] for some details regarding the status of the conjecture. The results involved in the proof of Theorem 2 go some way towards verification of Raghunathan's conjecture for a unipotent one-parameter subgroup contained in $H$. In a forthcoming paper (to appear in Math. Ann.) we use the result to verify the conjecture for this one-parameter subgroup of the group $SL(3, \mathbb{R})$. The technique here involves, as in [13], finding orbits of larger subgroups inside closed invariant sets of unipotent subgroups; however unlike in [13] we now deal with noncompact closed invariant sets as well.

The paper is organized as follows. The first two sections contain various general results on orbits, closed invariant sets, minimal sets etc. for flows on homogeneous spaces of Lie groups; these would be of independent interest. In §3 we collect some further preliminaries and complete the proof of Theorem 2 in §4. The deduction of Theorem 1 from Theorem 2 is indicated in §5, where we also make some more observations.

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## §1. Minimal closed invariant sets

This section is devoted to various general results on minimal closed invariant subsets of actions on homogeneous spaces $G/\Gamma$, where $G$ is a Lie group and $\Gamma$ is a lattice in $G$, of subgroups of $G$ (acting on the left). For the most part these are deduced from results in [5] after recalling them suitably.

Let $\mathbb{R}^n$ be the $n$-dimensional euclidean space equipped with the usual inner product $\langle , \rangle$ and the corresponding norm $\|\cdot\|$. For any discrete subgroup $\Delta$ of $\mathbb{R}^n$ we denote by $\Delta_{\mathbb{R}}$ the subspace of $\mathbb{R}^n$ spanned by $\Delta$ and by $d(\Delta)$ the volume of the torus $\Delta_{\mathbb{R}}/\Delta$ (or equivalently that of any fundamental domain for $\Delta$ in $\Delta_{\mathbb{R}}$) with respect to the inner product induced by $\langle , \rangle$ on $\Delta_{\mathbb{R}}$. If $\Lambda$ is a lattice in $\mathbb{R}^n$ then a subgroup $\Delta$ is said to be complete if $\Delta_{\mathbb{R}} \cap \Lambda = \Delta$. We denote by $l$ the Lebesgue measure on $\mathbb{R}$. We have the following

**1.1. Theorem.** *Given $\varepsilon > 0$ and $\theta > 0$ there exists $\delta > 0$ such that for any lattice $\Lambda$ of $\mathbb{R}^n$, any unipotent one-parameter subgroup $\{u_t\}$ of $\mathrm{SL}(n, \mathbb{R})$ and $T \geqq 0$ at least one of the following conditions holds:*

i) $l(\{t \in [0, T] \mid \|u_t z\| \geqq \delta$ *for all* $z \in \Lambda\text{-}(0)\}) \geqq (1-\varepsilon)T$

ii) *there exists a complete subgroup $\Delta$ of $\Lambda$ such that $d^2(u_t \Delta) < \theta$ for all $t \in [0, T]$.*

*Proof.* Follows directly from Proposition 2.7 of [5] if we choose (in the notation of [5]) $S$ to be empty, $h$ and $k$ satisfying condition (2.3) formulated there, $a = 0$ and $b = Tk^{-1}$.

We deduce the following

**1.2. Theorem.** *Let $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. Let $\theta > 0$ be given. Then there exists a compact subset $C$ of $G/\Gamma$ such that for any connected unipotent subgroup $U$ of $G$ and any $g \in G$ at least one of the following conditions holds:*

i) *there exists a one-parameter subgroup $\{u_t\}$ of $U$ such that $\{t \geqq 0 \mid u_t g\Gamma \in C\}$ and $\{t \leqq 0 \mid u_t g\Gamma \in C\}$ are both unbounded subsets of $\mathbb{R}$.*

ii) *there exists a proper nonzero $U$-invariant subspace $W$ such that $W \cap g\Lambda_0$ is a lattice in $W$ and $d^2(W \cap g\Lambda_0) < \theta$, where $\Lambda_0 = \mathbb{Z}^n$ is the standard lattice in $\mathbb{R}^n$. In particular, $g^{-1}Ug$ is contained in a parabolic subgroup of $G$ defined over $\mathbb{Q}$.*

*Proof.* Let $\delta > 0$ be as in Theorem 1.1 corresponding to the given $\theta$ and some $0 < \varepsilon < 1$. Let $C$ be the subset of $G/\Gamma$ consisting of all $g\Gamma$ such that $g\Lambda_0$ contains no nonzero element in the $\delta$ neighbourhood of 0. Then by Mahler criterion (cf. [16]) $C$ is a compact subset and Theorem 1.1 implies (by a simple argument as in the proof of Theorem 2.1 of [5]) that if for a $g \in G$, assertion i) of the present theorem fails to hold for a (any) one-parameter subgroup $\{u_t\}$ of $U$ then there exists a $\{u_t\}$-invariant proper nonzero subspace $W$ such that $W \cap g\Lambda_0$ is a lattice in $W$ and $d^2(W \cap g\Lambda_0) < \theta$. An argument as in the proof of Theorem 3.8 of [5], using the countability of the set of rational subspaces, shows that $W$ can be chosen independent of the one-parameter subgroup $\{u_t\}$, and hence $U$-invariant. This shows that the first part of condition ii) holds. Observe that $g^{-1}Ug$ leaves invariant the subspace $g^{-1}W$ and that $g^{-1}W \cap \Lambda_0$ is a lattice in $g^{-1}W$. The latter condition implies that $g^{-1}W$ is a rational subspace. Hence $\{x \in G \mid xg^{-1}W = g^{-1}W\}$ is a parabolic subgroup defined over $\mathbb{Q}$ containing $g^{-1}Ug$.

**1.3. Corollary.** *Let $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ and let $C$ be a compact subset of $G/\Gamma$ as in Theorem 1.2. Let $U$ be a connected unipotent subgroup of $G$. Let $N(U)$ be the normaliser of $U$ in $G$ and let $\{f(t)\}_{t \geqq 0}$ be a curve in $N(U)$ such that if $L$ is a proper nonzero $U$-invariant subspace then $L$ is invariant under $f(t)$ for all $t$ and $|\det f(t)|_L| \to \infty$ as $t \to \infty$ ($\det f(t)|_L$ denotes the determinant of the restriction of $f(t)$ to $L$). Then for all $g \in G$, $C \cap Uf(t)g\Gamma/\Gamma$ is nonempty for all large $t$. If $F$ is the subgroup generated by $U$ and $\{f(t) \mid t \geqq 0\}$, then every nonempty closed $F$-invariant subset contains a minimal closed $F$-invariant subset.*

*Proof.* We note that if $L$ is a $U$-invariant subspace and $g\in G$ is such that $L\cap g\Lambda_0$ is a lattice in $L$ then for $\{f(t)\}_{t\geqq 0}$ as above $d^2(L\cap f(t)g\Lambda_0)=|\det f(t)|_L|^2 d^2(L\cap g\Lambda_0)$ and hence for all large $t$ condition ii) in Theorem 1.2 is violated. Therefore condition i) must hold, which proves the first part of the corollary. Let $F$ be the subgroup generated by $U$ and $\{f(t)\}_{t\geqq 0}$. Then the first part implies that every $F$-orbit intersects $C$, that is $C\cap Fg\Gamma/\Gamma$ is nonempty for all $g\in G$. Now if $Y$ is a nonempty closed $F$-invariant subset then any totally ordered family $\{Y_\alpha\}_{\alpha\in A}$ (where $A$ is an indexing set and the ordering is by inclusion) of nonempty closed $F$-invariant subsets has a common point in $C$ and therefore by Zorn's lemma $Y$ contains a minimal closed $F$-invariant subset.

**1.4. Theorem.** *Let $G$ be a connected Lie group and $\Gamma$ be a lattice in $G$. Then there exists a compact subset $C$ of $G/\Gamma$ such that for any connected Ad-unipotent subgroup $U$ of $G$ (namely Ad $u$ is unipotent for all $u\in U$) and $g\in G$ at least one of the following conditions holds:*

i) *there exists a one-parameter subgroup $\{u_t\}$ of $U$ such that $\{t\geqq 0\,|\,u_tg\Gamma\in C\}$ and $\{t\leqq 0\,|\,u_tg\Gamma\in C\}$ are unbounded subsets of $\mathbb{R}$,*
ii) *there exists a proper closed subgroup $L$ of $G$ containing $g^{-1}Ug$ such that $L\Gamma$ is closed and $L\cap\Gamma$ is a lattice in $L$.*

This is only a slight variation of Theorem 3.8 of [5] whose validity is clear from the proof of the latter.

**1.5. Corollary.** *Let $G$ be a connected Lie group and $\Gamma$ be a lattice in $G$. Let $U$ be a connected Ad-unipotent subgroup of $G$. Then any nonempty closed $U$-invariant subset of $G/\Gamma$ contains a minimal closed $U$-invariant subset.*

*Proof.* We proceed by induction on the dimension of $G$. For low dimensions the assertion is obvious. Now suppose it to be true for dimensions less than $n$ and let $G$ be a $n$-dimensional Lie group. Let $\Gamma$ and $U$ be as in the hypothesis. Let $C$ be the compact subset of $G/\Gamma$ as in Theorem 1.4. Let $Y$ be a given nonempty closed $U$-invariant subset of $G/\Gamma$. First suppose that $C\cap Uy$ is nonempty for all $y\in Y$. Then any totally ordered family $\{Y_\alpha\}_{\alpha\in A}$, where $A$ is an indexing set, of nonempty closed $U$-invariant subsets of $Y$ (ordering by inclusion) has a common element in $C$. Hence by Zorn's lemma there exists a minimal (nonempty) closed $U$-invariant subset contained in $Y$. Now suppose that there exists $y\in Y$ such that $C\cap Uy=\emptyset$. Let $g\in G$ be such that $y=g\Gamma$. Then by Theorem 1.4 there exists a proper closed subgroup $L$ such that $g^{-1}Ug$ is contained in $L$, $L\Gamma$ is closed and $L\cap\Gamma$ is a lattice in $L$. Replacing $L$ by its connected component of the identity, we may assume $L$ to be connected. Now $g^{-1}Y\cap L\Gamma/\Gamma$ is a closed nonempty $g^{-1}Ug$-invariant subset of $L\Gamma/\Gamma$. Since $L\Gamma/\Gamma$ is canonically equivalent to $L/L\cap\Gamma$ and $L$ is of dimension $\leqq n-1$, by the induction hypothesis $g^{-1}Y\cap L\Gamma/\Gamma$ contains a minimal closed $g^{-1}Ug$-invariant subset. Hence $Y\cap gL\Gamma/\Gamma$ contains a minimal closed $U$-invariant subset. This proves the corollary.

We next show that the minimal sets of *Ad*-unipotent *one-parameter* subgroups are compact. For this we need the following lemma inspired by Lemma $K$ in [8].

**1.6. Lemma.** *Let $T$ be a homeomorphism of a locally compact space $Z$. Suppose that there exists a compact subset $K$ of $Z$ such that for each $z \in Z$ the sets $\{j \in \mathbb{N} \mid T^j z \in K\}$ and $\{j \in \mathbb{N} \mid T^{-j} z \in K\}$ of natural numbers, are both unbounded. Then $Z$ is compact.*

*Proof.* Let $K_1$ be a compact neighbourhood of $K$ in $Z$ and let $\Omega = Z - K_1$. Let $B = \bigcap_{j=0}^{\infty} T^j \bar{\Omega}$. Then $T^{-j} B \subseteq B \subseteq \bar{\Omega}$ for all $j \geqq 0$. Since $K \cap \bar{\Omega} = \varnothing$ the condition in the hypothesis implies that $B$ must be empty. Hence $TB$ is also empty. Since $Z - \Omega = K_1$ is compact it follows that there exists $m \geqq 1$ such that $\bigcap_1^m T^j \bar{\Omega}$ is contained in $\Omega$. Then $\bigcap_0^m T^j \Omega = \bigcap_1^m T^j \Omega$; call the set $D$. Then $TD \subset D$ and hence $T^j D \subset \Omega \subset Z - K$ for all $j$. Hence by hypothesis $D$ must be empty. This implies that $Z = \bigcup_{j=0}^m T^j K_1$ and hence it is compact.

**1.7. Corollary.** *Let $G$ be a connected Lie group and $\Gamma$ be a lattice in $G$. Let $\{u_t\}$ be a Ad-unipotent one-parameter subgroup of $G$. Let $X$ be a closed $\{u_t\}$-invariant subset. Suppose that for any $x = g\Gamma \in X$ the one-parameter subgroup $\{g^{-1} u_t g\}$ is not contained in any proper closed subgroup $L$ such that $L\Gamma$ is closed and $L \cap \Gamma$ is a lattice in $L$. Then $X$ is compact. In particular every minimal closed $\{u_t\}$-invariant set is compact.*

*Proof.* Let $C$ be a compact subset of $G/\Gamma$ as in Theorem 1.4. Let $u = u_1$ and $D = \{u_t x \mid 0 \leqq t \leqq 1 \text{ and } x \in C\}$. By Theorem 1.4 the condition in the hypothesis implies that for any $x \in X$ the subsets $\{t \geqq 0 \mid u_t x \in C\}$ and $\{t \leqq 0 \mid u_t x \in C\}$ are unbounded. Hence the subsets $\{j \in \mathbb{N} \mid u^j x \in D\}$ and $\{j \in \mathbb{N} \mid u^{-j} x \in D\}$ are unbounded for all $x \in X$. Thus the condition in Lemma 1.6 is satisfied for the action of $u$ on $X$. Hence the Lemma implies that $X$ is compact.

Now suppose that $X$ is a minimal closed $\{u_t\}$-invariant subset. If the condition in the first part is satisfied then we are through. Otherwise there exists $x = g\Gamma$ such that $\{g^{-1} u_t g\}$ is contained in a proper closed connected subgroup $L$ such that $L\Gamma$ is closed and $L \cap \Gamma$ is a lattice in $L$. Then $X \cap gL\Gamma/\Gamma$ is a nonempty closed $\{u_t\}$-invariant subset and hence, by minimality, $X$ is contained in $gL\Gamma/\Gamma$. The latter is canonically equivalent to $gLg^{-1}/g(L \cap \Gamma)g^{-1}$ and therefore we can conclude compactness of $X$ by repeating the above argument or equivalently by an obvious inductive procedure.

1.8 *Remark.* Theorem 1.2 shows that in the case when $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ the subgroup $L$ as in condition ii) of Theorem 1.4 can be chosen to be the subgroup generated by all unipotent elements in a parabolic subgroup; given $W$ as in condition ii) of Theorem 1.2 the subgroup $L = \{x \in G \mid xg^{-1} W = g^{-1} W \text{ and } \det x|_W = 1\}$ has these properties and satisfies condition ii) of Theorem 1.4 (cf. [2] §2, for instance). It turns out that in the general case also $L$ can be chosen such that $LR/R$, where $R$ is the radical of $G$, is the subgroup generated by all unipotent elements in a parabolic subgroup of $G/R$. Similarly it is also possible to generalize Corollary 1.3. The results are achieved by recasting the proofs in [5] in terms of general reduction theory. However a substantial amount of work is involved in supplying the details. As our main interest here lies in the lattice $\mathrm{SL}(3, \mathbb{Z})$ in $\mathrm{SL}(3, \mathbb{R})$ we shall not go into the details of the general case. However, it would be worthwhile to note for future reference that the arguments in the following sections including the proof of Theorem 2 apply to any lattice $\Gamma$ in $G = \mathrm{SL}(3, \mathbb{R})$ which is known to satisfy the following condition for the action of the one-parameter subgroup $V_1 = \{v_1(t)\}$ consisting of all upper triangular unipotent matrices contained in the subgroup $H$ as in the introduction (see also §3).

**Condition** (*) There exists a compact subset $C$ of $G/\Gamma$ such that for any $g \in G$, (a) the sets $\{t \geqq 0 | v_1(t) g\Gamma \in C\}$ and $\{t \leqq 0 | v_1(t) g\Gamma \in C\}$ are both unbounded unless there exists a proper parabolic subgroup $P$ such that if $L$ is the closed subgroup generated by all unipotent elements in $P$ then $g^{-1} V_1 g \subset L$, $L\Gamma$ is closed and $L \cap \Gamma$ is a lattice in $L$ and (b) if $\{f(t)\}_{t \geqq 0}$ is a curve in $N(V_1)$ (the normaliser of $V_1$) such that $|\det f(t)|_W| \to \infty$ as $t \to \infty$, for every proper nonzero $N(V_1)$-invariant subspace $W$ of $\mathbb{R}^3$ then $C \cap V_1 f(t) g\Gamma/\Gamma$ is nonempty for all large $t$.

In view of Theorem 1.2 and Corollary 1.3 (together with the fact that any $V_1$-invariant subspace is $N(V_1)$-invariant) Condition (*) holds for the lattice SL(3, $\mathbb{Z}$). Elsewhere we shall show that it in fact holds for all lattices.

## §2. Topological limits and inclusion of orbits

As noted in the introduction the proof of Theorem 2 depends on the technique of finding orbits of larger subgroups inside a given closed subset invariant under a (unipotent) subgroup. In this section we collect the details in this regard. We begin by recalling the following result from [13] (see also [11] and [12]).

**2.1. Lemma.** *Let $G$ be a locally compact (second countable) group and let $Z$ be a locally compact space with a given $G$-action. Let $F$, $P$ and $Q$ be closed subgroups of $G$ such that $F \subset P \cap Q$. Let $X$ and $Y$ be closed subsets of $Z$ invariant under the actions of $P$ and $Q$ respectively. Suppose also that $X$ is compact. Let $M$ be a subset of $G$ such that $gX \cap Y$ is nonempty for all $g \in M$. Then $hX \cap Y$ is nonempty for all $h \in \overline{QMP}$. If $X$ is a minimal $F$-invariant subset then $hX \subset Y$ for all $h \in \overline{QMP} \cap N(F)$, where $N(F)$ is the normaliser of $F$ in $G$. If further $X = Y$ and $P = Q$ then $Y$ is invariant under the closed subgroup generated by $\overline{QMP} \cap N(F)$.* □

In applying this in the present instance we use the following variation of Lemma 13 of [13] and Lemma 1 of [1]. (Though an appropriate analogue of the following lemma holds for any connected unipotent subgroup, for simplicity we restrict to one-parameter subgroups, which is the case needed in the sequel).

**2.2. Lemma.** *Let $\{u_t\}$ be a unipotent one-parameter subgroup of* SL($n$, $\mathbb{R}$) *and consider the natural action of $\{u_t\}$ on $\mathbb{R}^n$. Let $L = \{x \in \mathbb{R}^n | u_t x = x \text{ for all } t \in \mathbb{R}\}$. Let $M_0$ be a subset of $\mathbb{R}^n$-$L$ and suppose that $p \in \overline{M}_0 \cap L$. Then there exists a nonconstant polynomial function $\phi$: $\mathbb{R} \to L$ such that $\phi(0) = p$ and the following condition holds: there exist sequences $\{x_i\}$ in $M_0$ and $\{t_i\}$ in $\mathbb{R}$, such that $t_i \to \infty$ and for any convergent sequence $\{\alpha_i\}$ in $\mathbb{R}$, say $\alpha_i \to \alpha$, $u_{\alpha_i t_i} x_i \to \phi(\alpha)$.*

*Proof.* By Jordan canonical form there exists a basis $\{e(j, k)\}$, where $1 \leqq k \leqq l$, for some $l$, and $1 \leqq j \leqq m_k$ for suitable $m_1, \ldots, m_l$ such that

$$u_t e(j, k) = e(j, k) + t e(j-1, k) + \tfrac{1}{2} t^2 e(j-2, k) + \ldots + \frac{1}{(j-1)!} t^{j-1} e(1, k)$$

for all $j$, $k$. For $x \in M_0$ let $x(j, k)$ denote the $e(j, k)$-component with respect to the basis and put

$$\theta(x) = \min\{|x(j,k)|^{-1/(j-1)} \mid 1 \leqq k \leqq l,\ 2 \leqq j \leqq m_k\}.$$

Then $|x(j,k)\theta(x)^{j-1}| \leqq 1$ whenever $j \geqq 2$ and the equality holds for at least one $(j, k)$ such that $j \geqq 2$. Now let $\{x_i\}$ be a sequence in $M_0$ converging to $p$. By passing to a subsequence we can arrange so that there exists a fixed $(j_0, k_0)$ with $j_0 \geqq 2$ such that $|x_i(j_0, k_0)\theta^{j_0-1}(x_i)| = 1$ for all $i$. Passing to subsequence once again we may further assume that $x_i(j, k)\theta^{j-1}(x_i)$ converges for all $(j, k)$; let $\lambda(j, k)$ denote the limit corresponding to $(j, k)$. Now choose

$$\phi(s) = \sum_{(j,k)} \frac{1}{(j-1)!} \lambda(j,k) s^{j-1} e(1,k).$$

Then it is straightforward to verify that $\phi$ has the required properties; for this, one chooses $x_i$ as above and $t_i = \theta(x_i)$.

We note that if $P$ is the isotropy subgroup of $p$ as in the Lemma, $M = \{g \mid gp \in M_0\}$ and $Q = \{u_t\}$ then the Lemma implies that $\overline{QMP}$ contains all $h$ such that $hp$ lies in the image of $\phi$. Thus the Lemma enables us to conclude existence of certain *subsets* of elements $h$ such that $hX \subset Y$, under appropriate conditions. We complement this by the following lemma to get orbits of subgroups.

**2.3. Lemma.** *Let $G$ be a connected Lie group and $\Gamma$ be a lattice in $G$. Let $U$ be a connected Ad-unipotent subgroup of $G$. Let $U_1$ be a subset of $U$ such that the following condition is satisfied: there exists a one-parameter subgroup $\{u_t\}$ of $U$ such that for any $u \in U$, $uu_t \in U_1$ for all large $t$ (say $t \geqq t_u$, $t_u$ depending on $u$). Let $Y$ be a closed subset of $G/\Gamma$ containing an element $x$ such that $u'x \in Y$ for all $u' \in U_1$. Then $Y$ contains a $U$-orbit.*

*Proof.* Let $u \in U$ be given and let $\{u_t\}$ be a one-parameter subgroup of $U$ as above. There exists a compact subset $C$ of $G/\Gamma$ such that $\{t \geqq 0 \mid u_t x \in C\}$ is unbounded (cf. [4], Theorem 4.1; the result may also be deduced from Theorem 1.4). Therefore there exists a sequence $\{t_i\}$ in $\mathbb{R}$, $t_i \to \infty$, such that $u_{t_i}x$ converges, say $u_{t_i}x \to y$. Then $uu_{t_i}x \to uy$. Since $t_i \to \infty$, for all large $i$, $uu_{t_i} \in U_1$ and hence $uu_{t_i}x \in Y$. Since $Y$ is closed, it follows that $uy \in Y$. As $u \in U$ is arbitrary this means that $Y$ contains the $U$-orbit of $y$.

Now let $A$ be a semidirect product of the groups $\mathbb{R}^*$ and $\mathbb{R}$. We take the underlying set of $A$ as $\{(\sigma, v) \mid \sigma \in \mathbb{R}^*, v \in \mathbb{R}\}$ and the product as given by $(\sigma', v')(\sigma, v) = (\sigma'\sigma, \sigma^{-d}v' + v)$, where $d \in \mathbb{R}$ is fixed. By a rational function on $\mathbb{R}$ we mean the quotient $\alpha/\beta$ of two polynomials with real coefficients, its value being defined at points where $\beta$ does not vanish. The number (degree $\alpha$) $-$ (degree $\beta$) is called the degree of $\alpha/\beta$.

**2.4. Proposition.** *Let $\sigma$ and $v$ be two rational functions on $\mathbb{R}$ and suppose that the degree of $\sigma$ is nonzero. Let $\phi$: $\mathbb{R} \to A \cup \{\infty\}$ be the function defined by $\phi(t) = (\sigma(t), v(t))$ if $\sigma(t) \in \mathbb{R}^*$ and $v(t) \in \mathbb{R}$, and $\infty$ otherwise. Then there exist a nontrivial one-parameter subgroup $\{\rho_s\}_{s \in \mathbb{R}}$ of $A$ and a family of functions $a_s$: $\mathbb{R}^+ \to \mathbb{R}$, $s \in \mathbb{R}$, such that as $t \to \infty$,*

$$t + a_s(t) \to \infty \text{ and } \phi(t + a_s(t))\phi(t)^{-1} \to \rho_s$$

*for all s in a subset E of* $\mathbb{R}$, *where* $E=\mathbb{R}$ *unless* $\{\rho_s | s\in\mathbb{R}\} = \{(1, s) | s\in\mathbb{R}\}$ *and in the latter case* $E=[0, \infty)$.

*Proof.* Let $d\in\mathbb{R}$ be as above. For any $t$ and $\xi\in\mathbb{R}$ we have

$$\begin{aligned}\phi(t+\xi)\phi(t)^{-1} &= (\sigma(t+\xi), v(t+\xi))(\sigma(t), v(t))^{-1}\\ &= (\sigma(t+\xi)\sigma(t)^{-1}, \sigma(t)^d(v(t+\xi)-v(t)))\end{aligned}$$

whenever it is finite. Let $v(t)=\alpha(t)/\beta(t)$ where $\alpha$ and $\beta$ are coprime polynomials. Let $k$ and $l$ be the degrees of $\alpha$ and $\beta$ respectively. The function $\xi \to \alpha(t+\xi)\beta(t) - \alpha(t)\beta(t+\xi)$ is a polynomial in $\xi$ whose constant coefficient is zero and other coefficients are polynomials in $t$ of degree $\leqq k+l-1$. For $i\geqq 1$ let $m_i$ be the degree in $t$ of the coefficient of $\xi^i$; $m_i = -\infty$ by convention, if the coefficient polynomial is 0.

Let $p$ be the degree of $\sigma$(as a rational function) and for each $i\geqq 1$ let $q_i = pd+m_i-2l$. Let $q=\max_{i\geqq 1}\{q_i/i\}$. We now consider two cases: First suppose that $q>-1$. Then we put $a_s(t)=st^{-q}$ for all $s$, $t\in\mathbb{R}$, $t>0$. The condition $q>-1$ ensures that $\beta(t+a_s(t))/\beta(t)$ converges to 1 as $t\to\infty$. On the other hand using the fact that $q_i\leqq iq$ it can be readily verified, substituting for $a_s(t)$, that

$$\sigma(t)^d\beta(t)^{-2}\{\alpha(t+a_s(t))\beta(t)-\alpha(t)\beta(t+a_s(t))\}$$

converges to a polynomial $\sum_{i=1}^m c_i s^i$ in $s$, where $m=\max\{k, l\}$ and $c_i$ is nonzero if and only if $q_i = iq$; in particular at least one $c_i$ is nonzero. Therefore $\sigma(t)^d\{v(t+a_s(t)) - v(t)\}$ which is the same as

$$\sigma(t)^d\beta(t+a_s(t))^{-1}\beta(t)^{-1}\{\alpha(t+a_s(t))\beta(t)-\alpha(t)\beta(t+a_s(t))\}$$

converges to a nonconstant polynomial in $s$. Also as $a_s(t)/t\to 0$ as $t\to\infty$, we have $\sigma(t+a_s(t))\sigma(t)^{-1}\to 1$. Combining, we find that $\phi(t+a_s(t))\phi(t)^{-1}$ converges for each $s$ to an element of $\{(1, v) | v\in\mathbb{R}\}$ and $(1, v)$ is such a limit if $v=\Sigma c_i s^i$ for some $s$. By altering the parameterisation of the family of functions $a_s$ we can adjust so that $\phi(t+a_s(t))\phi(t)^{-1}$ converges to $\rho_s$ for all $s\geqq 0$, where $\rho_s=(1, \pm s)$, the sign being the same throughout. Observe also that for any $s$, $t+a_s(t)\to\infty$ as $t\to\infty$. This proves the proposition for the case at hand.

Next suppose that $q\leqq -1$. We put $a_s(t)=(e^s-1)t$. Computations as before readily show that in this case $\sigma(t)^d\{v(t+a_s(t))-v(t)\}$ converges to a polynomial in $e^s$; this could however be zero. On the other hand $\sigma(t+a_s(t))\sigma(t)^{-1} = \sigma(e^s t)\sigma(t)^{-1} \to e^{sp}$. Thus $\phi(t+a_s(t))\,\phi(t)^{-1}$ converges in $A$ for all $s\in\mathbb{R}$. Let $\rho_s$ denote the limit of $\phi(t+a_s(t))\phi(t)^{-1}$ as $t\to\infty$. Then clearly $\rho_{s_1+s_2}=\rho_{s_1}\rho_{s_2}$ and $\rho_{-s}=(\rho_s)^{-1}$ for all $s_1$, $s_2$ and $s\in\mathbb{R}$. Also $s\to\rho_s$ is continuous. Since $p\neq 0$ the one-parameter subgroup is nontrivial. Also $t+a_s(t)=e^s t\to\infty$ as $t\to\infty$. This proves the proposition.

In the proof of Theorem 2 we also need to deal with situations where $X$ as in Lemma 2.1 is a noncompact closed subset. For this purpose we need the following notion.

**2.5. Definition.** Let $\{\psi_t\}$ be a flow on a locally compact space $Z$. An element $z\in Z$ is said to be a *point of uniform recurrence in linear time* if for any neighbourhood $\Omega$ of $z$ and $\alpha>0$, there exists a neighbourhood $\Omega'$ of $z$ and $T_0\in\mathbb{R}$ such that for any $y\in\Omega'$ and $T\geqq T_0$ the set $\Omega\cap\{\psi_t(y) | T\leqq t\leqq(1+\alpha)T\}$ is nonempty.

How this together with the later part of Lemma 2.2 substitutes for the compactness condition in Lemma 2.1 may be seen in the proof of Proposition 4.3. We shall not abstract it since it would seem too cumbersome and artificial that way. For the argument there we need the following instance of uniform recurrence in linear time.

**2.6. Theorem.** *Let $G=\mathrm{SL}(2,\mathbb{R})$ and $\Gamma$ be a lattice in $G$. Let $u_t=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ for all $t\in\mathbb{R}$. Let $x\in G/\Gamma$ be such that $u_t x \neq x$ for every $t\neq 0$. Then $x$ is a point of uniform recurrence in linear time for the flow induced by the action of $\{u_t\}$ on $G/\Gamma$.*

*Proof.* Suppose this is not true. Then there exist an element $x\in G/\Gamma$ such that $u_t x\neq x$ for every $t\neq 0$, a neighbourhood $\Omega$ of $x$, an $\alpha>0$ and sequences $\{x_j\}$ in $\Omega$ and $\{T_j\}$ in $\mathbb{R}$ such that $x_j\to x$, $T_j\to\infty$ and $\Omega\cap\{u_t x_j \mid T_j\leqq t\leqq(1+\alpha)T_j\}=\varnothing$ for all $j=1, 2, \ldots$. Without loss of generality we may assume $\Omega$ to be relatively compact. Let $X=G/\Gamma\cup\{\infty\}$ be the one-point compactification of $G/\Gamma$ and set $u_t\infty=\infty$ for all $t\in\mathbb{R}$. For $j=1, 2, \ldots$ let $\pi_j$ be the probability measure on $X$ defined by $\pi_j(E)=l(\{t\mid T_j\leqq t\leqq(1+\alpha)T_j,\ u_t x_j\in E\})/\alpha T_j$ for any Borel subset $E$ of $X$, where $l$ is the Lebesgue measure. Since $X$ is a compact second countable space, the space of probability measures on $X$ is compact and second countable when equipped with the usual weak topology. Hence the sequence $\{\pi_j\}$ has a convergent subsequence and by replacing the sequence $\{x_j\}$ by a suitable subsequence we may assume $\{\pi_j\}$ to be convergent, say $\pi_j\to\pi$. It is straightforward to verify that $\pi$ is invariant under the action of $\{u_t\}$ and that $\pi(\Omega)=0$.

The proof of Proposition 1.2 in [4] shows that given $\varepsilon>0$ and a relatively compact subset, say $\Omega$ as above, there exists a compact subset $C$ of $G/\Gamma$ and a constant $b_0$ such that $l(\{t\in[0, T]\mid u_t y\notin C\})\leqq\varepsilon T+b_0$ for all $y\in\Omega$ and $T>b_0$; in the set up as on page 31 in [4] choose $S$ to be such that $\Omega$ is disjoint from $X(\sigma, S)$ for all $\sigma\in\Sigma$ and having chosen $s$ as required there, depending on $S$, choose $C$ to be a compact subset of $G/\Gamma$ whose complement is contained in $\bigcup_\sigma X(\sigma, s)$. (We note that the boundedness of the $\{u_t\}$-orbit assumed in [4] is involved only to ensure that for any $a>0$, $\{u_t y\mid t>a\}$ is not contained in $X(\sigma, S)$ for any $\sigma$, which is automatic in the above variation). Since $x_j\in\Omega$ for all $j$ we get that $\pi_j(X-C)\leqq\alpha^{-1}(1+\alpha)\varepsilon+b_0/T_j\alpha$ and hence $\pi(X-C)\leqq\alpha^{-1}(1+\alpha)\varepsilon$. In particular this implies that $\pi(\{\infty\})=0$.

Thus $\pi$ is a $\{u_t\}$-invariant measure on $X$ such that $\pi(\Omega)=0$ and $\pi(\{\infty\})=0$. By the classification theorem for $\{u_t\}$-invariant measures (cf. [2]) this implies that $\pi$ is supported on the set of periodic orbits, namely $\pi(P)=1$, where $P=\{y\mid u_t y=y$ for some $t>0\}$. We claim that actually $\pi(P)=0$, the contradiction showing that the theorem is true.

The proof of $\pi(P)$ being 0 is very similar to that of Corollary 3.8 in [7]. Therefore, rather than going through the whole argument, we only give a sketch indicating the comparable steps in [7]. For this purpose we also follow the notation as in [7], without further mention. As in [7] it is enough to prove that $\pi(P_i)=0$ for all $i=1, 2, \ldots, r$. Again for each $i$, considered fixed as in [7], it is enough to show that $\pi(J^0(0,\rho))=0$ for all $\rho>0$. Let $\rho>0$ and $\varepsilon>0$ be arbitrary and let $M$ and $\sigma$ be as in the proof of Proposition 3.2 in [7]. Then the argument as in that Proposition shows that $l(\{t\in[0, T]\mid u_t y\in J^\delta(0,\rho)\})\leqq\varepsilon T$ for all $T\geqq 0$, $0<\delta\leqq\sigma$ and $y$ not belonging to $J^\delta(0, M)$. Since $u_t x\neq x$ for any $t\neq 0$ it follows that

$x \notin J^0(0, M)$). Hence there exists a $0<\delta<\sigma$ and a neighbourhood $\Omega'$ of $x$ such that $J^\delta(0, M) \cap \Omega' = \emptyset$. For all large $j$, $x_j \in \Omega'$ and hence $l(\{t \in [0,(1+\alpha)T_j] \mid u_t x_j \in j^\delta(0, \rho)\}) < \varepsilon(1+\alpha)T_j$. Hence $\pi_j(J^\delta(0,\rho)) \leqq \alpha^{-1}(1+\alpha)\varepsilon$. This implies that $\pi(J^\delta(0, \rho)) \leqq \alpha^{-1}(1+\alpha)\varepsilon$ for all $\rho$ such that $\pi(J^0(\rho, \rho))=0$ (cf. [3] Lemma 3.2). As $\varepsilon > 0$ is arbitrary we get that $\pi(J^0(0, \rho))=0$ for all $\rho$ such that $\pi(J^0(\rho, \rho))=0$. Since $\pi$ is a probability measure the latter condition holds for all but countably many $\rho$ and hence, by monotonicity, $\pi(J^0(0, \rho))=0$ for all $\rho$. This completes the proof.

## §3. Some more preliminary results

We now set up notation and note some more (specialized) results which will be used in the next section in proving Theorem 2.

Let $G=\mathrm{SL}(3, \mathbb{R})$ and $\Gamma$ be a lattice in $G$ satisfying Condition (*) formulated in Remark 1.8; we draw the readers' attention to the comments following the statement of the condition. We denote by $e$ the identity element in $G$. Let, for $t \in \mathbb{R}$

$$v_1(t) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v_2(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$V_1 = \{v_1(t) \mid t \in \mathbb{R}\}, \quad V_2 = \{v_2(t) \mid t \in \mathbb{R}\}, \; V = V_1 V_2,$$

$$V_2^+ = \{v_2(t) \mid t \geqq 0\} \quad V_2^- = \{v_2(t) \mid t \leqq 0\}$$

$$W = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} \quad \text{and} \quad D = \left\{ \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d^{-1} \end{pmatrix} \middle| \, d > 0 \right\}$$

Let $H$ be the subgroup as in the introduction; namely $H$ is the connected component of $e$ in the subgroup of $G$ of elements leaving invariant the quadratic form $2x_1x_3 - x_2^2$. We note that $D$ normalises $V_1$, $V_2$, $V$ and $W$, the subgroup $DV_1$ is contained in $H$ and $H/DV_1$ is compact.

**3.1. Proposition.** *Let $X$ be a closed $H$-orbit in $G/\Gamma$. Then the following conditions are satisfied:*

i) *$X$ admits a finite $H$-invariant measure $\mu$.*

ii) *For all $x \in X$, $DV_1 x$ is dense in $X$; that is, the $DV_1$-action on $X$ is minimal.*

iii) *The $V_1$-action on $X$ is ergodic with respect to $\mu$. For $x \in X$ the $V_1$-orbit of $X$ is either periodic (that is, $v_1(t)x = x$ for some $t > 0$) or dense and uniformly distributed with respect to $\mu$.*

iv) *If $x \in X$ is such that the $V_1$-orbit is not periodic then $x$ is a point of uniform recurrence in linear time for the $V_1$-action.*

*Proof.* It is well-known and easy to see that $H$ is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$. Therefore Theorem 3.11 of [14] implies assertion i) as above. In view of this, $X$ can be realized as $\mathrm{SL}(2, \mathbb{R})/\Delta$ for some lattice $\Delta$ in $\mathrm{SL}(2, \mathbb{R})$, with the $DV_1$ and $V_1$ actions on $X$ corresponding to the actions on $\mathrm{SL}(2, \mathbb{R})/\Delta$ of $\{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a, b \in \mathbb{R}, a > 0\}$ and $\{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mid t \in \mathbb{R}\}$ respectively. Assertions ii) and iii) therefore follows from the corresponding well-known results for the latter actions (cf. [6], for instance). Assertion iv) follows in the same way from Theorem 2.6.

**3.2. Proposition.** *Consider the action of $DV_1$ on $G/\Gamma$. Then we have the following:*
i) *every nonempty closed invariant set contains a minimal closed invariant set.*
ii) *there are no closed orbits.*

*Proof.* Let $f(t)=\mathrm{diag}(e^t, 1, e^{-t})$ for all $t\in\mathbb{R}$. Then $\det f(t)|_S\to\infty$ as $t\to\infty$, $S$ being any of the two proper nonzero $V_1$-invariant subspaces. Hence, by Corollary 1.3 in the case of $\mathrm{SL}(3,\mathbb{Z})$ and by Condition (*) in the general case, there exists a compact subset $C$ such that for each $g\in G$, $C\cap V_1f(t)g\Gamma/\Gamma$ is nonempty for all large $t$. Assertion i) now follows from a simple argument as in Corollary 1.3. Now suppose, if possible, that there exists $g\in G$ such that the $DV_1$-orbit of $g\Gamma$ in $G/\Gamma$, namely the set $Y=DV_1g\Gamma/\Gamma$ is closed. We note that since $DV_1$ is not unimodular $Y$ cannot be compact. Let $X$ be a minimal $V_1$-invariant subset of $Y$ (cf. Corollary 1.5). For each $d\in D$, $dX$ is $V_1$-invariant. By minimality of $X$ this implies that $dX\cap X=\emptyset$ unless $dX=X$. Since $X$ is compact (cf. Corollary 1.7), $Y=DX$ and $Y$ is not compact, there does not exist any nontrivial $d$ such that $dX=X$. Hence $dX\cap X=\emptyset$ for all $d\neq e$. Since $Y=DX$ this implies that $X$ is a $V_1$-orbit. As $X$ is compact this means that the isotropy subgroup of $g\Gamma/\Gamma$ for the $DV_1$-action is a nontrivial cyclic subgroup of $V_1$. Let $\Delta$ be the isotropy subgroup. Since $Y$ is a closed $DV_1$-orbit, the canonical orbit map $\eta$: $DV_1/\Delta\to G/\Gamma$ must be a proper map. However as $\Delta$ is contained in $V_1$ this contradicts the fact that $C\cap V_1f(t)g\Gamma/\Gamma$ is nonempty for all large $t$. Hence there are no closed $DV_1$-orbits, which proves ii).

**3.3. Proposition.** *Let $X$ be a minimal closed $V$-invariant subset. Then $X$ is compact. Further, either it is a $V$-orbit or it contains a $W$-orbit.*

*Proof.* First suppose that there exists $g\in G$ such that $g\Gamma\in X$ and $g^{-1}V_1g$ is contained in a subgroup $L$ as in Condition (*), namely $L$ is generated by all unipotent elements in a proper parabolic subgroup $P$, $L\Gamma$ is closed and $L\cap\Gamma$ is a lattice in $L$. Then $gPg^{-1}$ is a parabolic subgroup containing $V_1$ and hence it contains $V$. Hence $g^{-1}Vg$ is contained in $P$. Since $L$ contains all unipotent elements in $P$ it follows that $g^{-1}Vg$ is contained in $L$. Then $Vg\Gamma/\Gamma$ is contained in $gL\Gamma/\Gamma$, which is a closed subset, and hence, by minimality of $X$ as a closed $V$-invariant subset, $X$ is contained in $gL\Gamma/\Gamma$. Hence to prove the compactness of $X$ it is enough to prove that any minimal closed $g^{-1}Vg$-invariant subset, say $Y$, of $L/L\cap\Gamma$ is compact. Since $L$ is the closed subgroup generated by all unipotent elements in $P$ we see that either $L=g^{-1}Wg$ or $L/N$, where $N$ is the radical of $L$, is topologically isomorphic to $\mathrm{SL}(2,\mathbb{R})$. Recall that $L\cap\Gamma$ is a lattice in $L$. Now if $L=g^{-1}Wg$ then $L/L\cap\Gamma$ is compact and hence so is $Y$. In the other case $N\cap\Gamma$ is a lattice in $N$ and consequently the canonical quotient map $\eta$: $L/L\cap\Gamma\to L/N(L\cap\Gamma)$ is a proper map. (cf. [16] for instance). Therefore to prove compactness of $Y$ we have only to show that $\eta(Y)$ is compact. But this follows from Corollary 1.7, since $\eta(Y)$ is a minimal closed $g^{-1}VgN/N$-invariant subset and $g^{-1}VgN/N$ is, as can be readily seen, a (unipotent) one-parameter subgroup of $L/N$. Therefore $Y$ is compact and hence so is $X$.

We now suppose that there does not exist any $g$ as above. Then, by Theorem 1.2 in the case of SL(3, $\mathbb{Z}$) and by Condition (*) in the general case, there exists a compact subset $C$ of $G/\Gamma$ such that for all $x = g\Gamma \in G/\Gamma$ the sets $\{t \geqq 0 | v_1(t)x \in C\}$ and $\{t \leqq 0 | v_1(t)x \in C\}$ are both unbounded. Then the argument as in Corollary 1.7, using Lemma 1.6, implies that $X$ is compact.

Lemma 4 of [13] now implies that either $X$ is a (compact) $V$-orbit or there exists a subset $M$ of $G$-$V$ such that $e \in \bar{M}$ and $gX \cap X \neq \emptyset$ for all $g \in M$. In the latter case, by Lemma 2.1, $X$ is invariant under the closed subgroup, say $F$, generated by $\overline{VMV} \cap N(V)$, where $N(V)$ is the normaliser of $V$. It is easy to see that $N(V) \supset DW$. By Lemma 8 i) of [13] $F$ contains a one-parameter subgroup of $DW$ not contained in $V$. It follows that $F$ contains either $W$ or $w(DV)w^{-1}$ for some $w \in W$. By Lemma 6 of [13], $W \subset F$ if $M \subset G - N(V)$. As $DV$ has no compact orbits on $G/\Gamma$ the two assertions together imply that $X$ contains a $W$-orbit, which proves the proposition.

Now let $\mathfrak{g}$ be the Lie algebra of $G$, realised as the Lie algebra of $3 \times 3$ matrices of trace 0. Let $\mathfrak{h}$ be the Lie subalgebra corresponding to $H$. Let $\mathfrak{p}$ be the orthocomplement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form (this can be expressed explicitly — see [13]). Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $\mathfrak{h}$ and $\mathfrak{p}$ are invariant under the restriction of the adjoint action (of $G$ on $\mathfrak{g}$) to $H$.

**3.4. Proposition.** *Let $M$ be a subset of* $\exp \mathfrak{p}$ *such that $M \cap V_2 = \emptyset$ and $e \in \bar{M}$. Then there exists a polynomial map $\phi$: $\mathbb{R} \to V_2$ such that $\phi(\mathbb{R})$ contains $V_2^+$ or $V_2^-$ and there exist sequences $\{g_i\}$ in $M$ and $\{t_i\}$ in $\mathbb{R}^+$ such that for any convergent sequence $\{\beta_i\}$ of real numbers, say $\beta_i \to \beta$, $v_1(\beta_i t_i) g_i v_1(-\beta_i t_i) \to \phi(\beta)$.*

*Proof.* Let $M_0 = \{\xi \in \mathfrak{p} | \exp \xi \in M\}$. Then the conditions of Lemma 2.2 satisfied for the action of the one-parameter subgroup $\{v_1(t)\}$ on $\mathfrak{p}$, obtained by restricting the adjoint action as above, with $p = 0$; we note that for $\xi \in \mathfrak{p}$, $\mathrm{Ad}\, v_1(t)\xi = \xi$ for all $t$ if and only if $\exp \xi \in V_2$. Hence by that Lemma there exists a non-constant polynomial map $\phi$: $\mathbb{R} \to V_2$ such that $\phi(0) = e$ and there exist sequences $\{\xi_i\}$ in $M_0$ and $\{t_i\}$ in $\mathbb{R}^+$ such that for any convergent sequence $\{\beta_i\}$ in $\mathbb{R}$, say $\beta_i \to \beta$, $\exp \mathrm{Ad}\, v_1(\beta_i t_i)(\xi_i) \to \phi(\beta)$; setting $g_i = \exp \xi_i$ yields the assertion as in the Proposition. Since $\phi$ is a nonconstant polynomial map and $\phi(0) = e$ it follows that $\phi(\mathbb{R})$ contains either $V_2^+$ or $V_2^-$.

**3.5. Corollary.** *For any subset $M \subset G - HV_2$ such that $e \in \bar{M}$, $\overline{DV_1 M V_1}$ contains either $V_2^+$ or $V_2^-$.*

*Proof.* Let $\{g_i\}$ be a sequence in $M$ such that $g_i \to e$. Since $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$, there exist neighbourhoods $A$ and $B$ of $e$ in $H$ and $\exp \mathfrak{p}$ respectively, such that the map $(a, b) \to ab$ is a homeomorphism of $A \times B$ onto a neighbourhood of $e$ in $G$ and hence for all large $i$, $g_i$ has the form $h_i q_i$, where $h_i \in H$, $q_i \in \exp \mathfrak{p}$, with $h_i \to e$ and $q_i \to e$ as $i \to \infty$. We also note that since $M \subset G - HV_2$, $q_i \notin V_2$. Let $\phi$ be the polynomial as in Proposition 3.4 for the subset $\{q_i | i = 1, 2, \ldots\}$ (in the place of $M$ there). Then by the Proposition there exist sequences $\{k_i\}$ in $\mathbb{N}$ and $\{t_i\}$ in $\mathbb{R}^+$ such that $v_1(t_i s)$ $q_{k_i} v_1(-t_i s) \to \phi(s)$ for all $s \in \mathbb{R}$. Clearly $t_i \to \infty$. Recall that $H$ is a quotient group of SL(2, $\mathbb{R}$) and $DV_1$ is a parabolic subgroup in $H$. A straightforward computation shows that if $P$ is a parabolic subgroup of $\mathrm{SL}(2, \mathbb{R})$ if $\{u_t\}$ is a unipotent one-parameter subgroup contained in $P$ and if $\{h_i\}$ is a sequence in $\mathrm{SL}(2, \mathbb{R})$ then given $\{t_i\}$ as above, the sequence of cosets $Ph_i u(-t_i s)$ converges to the coset $P$ (in the quotient space $P\backslash \mathrm{SL}(2, \mathbb{R})$), except possibly for one particular value of $s$. It follows that there exists $s_0 \in \mathbb{R}$ such that for every $s \neq s_0$, the sequence of cosets $DV_1 h_{k_i} v_1 (-t_i s)$ converges to the coset $DV_1$ in $DV_1 \backslash H$. It follows that for every $s \neq s_0$, $DV_1 g_{k_i} v_1(-t_i s) = DV_1 h_{k_i} v_1(-t_i s) \cdot v_1(t_i s) q_{k_i} v_1(-t_i s) \to DV_1 \phi(s)$. But then $\phi(s)$ must be contained in $\overline{DV_1 M V_1}$ for all $s \in \mathbb{R}$. Since the image of $\phi$ contains either $V_2^+$ or $V_2^-$ it follows that $\overline{DVMV_1}$ contains either $V_2^+$ or $V_2^-$.

**3.6. Proposition.** *Let $Y$ be the closure of a $V_1$-orbit in $G/\Gamma$. Then either $Y$ is a $V_1$-orbit or it contains an orbit of $V$ or $v(DV_1)v^{-1}$ for some $v \in V_2$.*

*Proof.* Let $X$ be a minimal closed $V_1$-invariant subset contained in $Y$; such a subset exists by Corollary 1.5 and is compact by Corollary 1.7. Let $x \in X$ be fixed. First suppose that there exists a neighbourhood $\Omega$ of $e$ in $G$ such that $\{g \in \Omega \mid gx \in Y\} \subset N(V_1)$, the normaliser of $V_1$ in $G$. Then $(\Omega \cap N(V_1))x \cap Y$ is open in $Y$ and hence contains an element $y$ such that $V_1 y$ is dense in $Y$. Let $y = px$ where $p \in N(V_1)$. Then $V_1 y = V_1 px = pV_1 x \subset pX$ and hence $Y \subset pX$. But since $p \in N(V_1)$ and $X$ is a minimal closed $V_1$-invariant subset, $pX$ and $X$ are disjoint unless $pX = X$. Since $X \subset Y \subset pX$ we get that $Y = X$. Thus $Y$ is a compact minimal $V_1$-invariant set. Then by Lemma 4 of [13] either $Y$ is a $V_1$-orbit or there exists a subset $M \subset G - V_1$ such that $e \in \overline{M}$ and $gx \in Y$ for all $g \in M$. If the second condition holds then by Lemma 2.1 $Y$ is invariant under the subgroup generated by $\overline{V_1 M V_1} \cap N(V_1)$ and by Lemma 8 of [13] the latter contains either $V$ or $v(DV_1)v^{-1}$ for some $v \in V_2$. Hence the proposition holds in this case.

Now suppose that there does not exist any $\Omega$ as above. Then there exists a subset $M \subset G - N(V_1)$ such that $e \in \overline{M}$ and $gx \in Y$ for all $g \in M$. Then by Lemma 2.1 $hX \subset Y$ for all $h \in \overline{V_1 M V_1} \cap N(V_1)$. For $\lambda \in \mathbb{R}^*$ let $d(\lambda)$ denote the diagonal matrix $\mathrm{diag}(\lambda, 1, \lambda^{-1})$. We shall show that there exist rational functions $\sigma$ and $\nu$ on $\mathbb{R}$ such that the following conditions are satisfied: i) $\sigma$ and $\sigma^2 \nu$ are polynomials, $\sigma(0) = 1$ and at least one of $\sigma$ and $\nu$ is nonconstant and ii) for any $t \in \mathbb{R}$ such that $\sigma(t) \neq 0$ the element $d(\sigma(t)) v_2(\nu(t))$ is contained in $\overline{V_1 M V_1}$.

Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{s}$ be the space of all symmetric $3 \times 3$ matrices. Let $\mathfrak{x} = \mathfrak{g} \oplus \mathfrak{s}$ and consider the linear action of $G$ on $\mathfrak{x}$ obtained as the direct sum of the adjoint action on $\mathfrak{g}$ and the action on $\mathfrak{s}$ defined by $(g, \theta) \to g\theta{}^t g$ (where ${}^t g$ is the transpose of $g$). Let $p = \xi_0 + \theta_0$, where $\xi_0$ is a nonzero element in the Lie subalgebra of $\mathfrak{g}$ corresponding to $V_1$ and $\theta_0$ is a nonzero element fixed by the action of $H$ (say the matrix corresponding to the quadratic form $2x_1 x_3 - x_2^2$). Then $V_1$ is the isotropy subgroup of $p$ under the $G$ action on $\mathfrak{x}$. Let $\mathfrak{v}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to $V$, viewed as a subspace of $\mathfrak{x}$ in a natural way. Let $\theta_1 = v_2(1)\,\theta_0 - \theta_0$ (action as above) and let $\theta_0$ and $\theta_1$ be viewed as elements of $\mathfrak{x}$. Let $L$ be the subspace of $\mathfrak{x}$ consisting of all elements fixed by $V_1$. It is easy to see that $L$ is precisely the subspace spanned by $\mathfrak{v} \cup \{\theta_0, \theta_1\}$. By Lemma 2.2 there exists a nonconstant polynomial map $\tilde{\phi}\colon \mathbb{R} \to L$ such that $\tilde{\phi}(0) = p$ and the image of $\tilde{\phi}$ is contained in $\overline{V_1 M p}$. Since the $G$-orbit of $p$ is locally closed, the contention of the

Lemma also implies that for all $t$ in a suitable neighbourhood of 0 in $\mathbb{R}$, $\tilde{\phi}(t)$ is contained in the closure of $Gp \cap L$. It is straightforward to verify that the latter set is precisely $E := \{\alpha\xi_0 + \theta_0 + \beta\theta_1 \mid \alpha, \beta \in \mathbb{R}\}$. Since $\tilde{\phi}$ is a polynomial map it follows that $\tilde{\phi}(t) \in E$ for all $t \in \mathbb{R}$. Hence we can write $\tilde{\phi}(t)$ as $\phi_0(t)\xi_0 + \theta_0 + \phi_1(t)\theta_1$ (uniquely), where $\phi_0$ and $\phi_1$ are polynomials on $\mathbb{R}$. We define $\sigma = \phi_0$ and $v = \phi_1/\phi_0^2$. We note that since $\tilde{\phi}(0) = p$, $\sigma(0) = \phi_0(0) = 1$ and hence in particular $v$ is a genuine rational function. Clearly $\sigma$ and $\sigma^2 v$ are polynomials. Also since $\tilde{\phi}$ is nonconstant, either $\phi_0$ or $\phi_1$ and hence $\sigma$ or $v$ is nonconstant. It is easy to verify that for any $\lambda \in \mathbb{R}^*$, $d(\lambda)\xi_0 = \lambda\xi_0$, $d(\lambda)\theta_0 = \theta_0$ and $d(\lambda)\theta_1 = \lambda^2\theta_1$ and for any $s \in \mathbb{R}$, $v_2(s)\xi_0 = \xi_0$ and $v_2(s)\theta_0 = \theta_0 + s\theta_1$. Hence for any $\lambda \in \mathbb{R}^*$ and $s \in \mathbb{R}$, $d(\lambda)v_2(s)p = \lambda\xi_0 + \theta_0 + \lambda^2 s\theta_1$. Substituting, we find that if $t \in \mathbb{R}$ is such that $\sigma(t) \neq 0$ then $d(\sigma(t))v_2(v(t))p = \sigma(t)\xi_0 + \theta_0 + \sigma^2(t)v(t)\theta_1 = \phi_0(t)\xi_0 + \theta_0 + \phi_1(t)\theta_1 = \tilde{\phi}(t) \in \overline{V_1 Mp}$ and consequently $d(\sigma(t))v_2(v(t)) \in \overline{V_1 M V_1}$. Thus $\sigma$ and $v$ have the desired properties.

Now first suppose that $\sigma$ is constant, namely $\sigma(t) = 1$ for all $t$. Then $v(t)$ is a nonconstant polynomial and $v_2(v(t)) \in \overline{V_1 M V_1}$ and hence $v_2(v(t))X \subset Y$ for all $t \in \mathbb{R}$. By Lemma 2.3 this implies that $Y$ contains a $V_2$-orbit. Being $V_1$-invariant it must in turn contain a $V$-orbit as desired. Next suppose that $\sigma$ is a non-constant polynomial. Let $\phi(t) = d(\sigma(t))v_2(v(t))$. Then by Proposition 2.4 there exist a nontrivial one-parameter subgroup $\{\rho_s\}$ of $DV_2$ and a family of functions $a_s\colon \mathbb{R}^+ \to \mathbb{R}$, $s \in \mathbb{R}$, such that as $t \to \infty$, $t + a_s(t) \to \infty$ and $\phi(t + a_s(t))\phi(t)^{-1} \to \rho_s$ for all $s$ in a subset $E$ of $\mathbb{R}$, where $E = \mathbb{R}$ unless $\{\rho_s \mid s \in \mathbb{R}\} = V_2$ and in the latter case $E = [0, \infty)$.

Observe that $|\det \phi(t)|_L|$ where $L$ is any of the two proper nonzero $V_1$-invariant subspaces, tends to $\infty$ as $t \to \infty$. Hence by Corollary 1.3 or Condition (*) there exists a compact subset $C$ of $G/\Gamma$ such that $\phi(t)V_1 x \cap C$ is nonempty for all large $t$. Hence there exist sequences $\{t_i\}$ in $\mathbb{R}$ and $\{v_i\}$ in $V_1$ such that $t_i \to \infty$ and $\phi(t_i)v_i x$ converges, to say an element $y$. Since $\phi(t)X \subset Y$ for all large $t$ (for which $\sigma(t) \neq 0$ as above) it follows that $y \in Y$. Now for any $s \in E$ we have

$$\phi(t_i + a_s(t_i))v_i x = \phi(t_i + a_s(t_i))\phi(t_i)^{-1}\phi(t_i)v_i x \to \rho_s y \ .$$

Since $t_i + a_s(t_i) \to \infty$ as $i \to \infty$ and $\phi(t)X \subset Y$ for all large $t$ it follows that $\rho_s y \in Y$ for all $s \in E$. Recall that if $E$ is a proper subset of $\mathbb{R}$ then we have $E = [0, \infty)$ and $\{\rho_s \mid s \in \mathbb{R}\} = V_2$. Hence using Lemma 2.3 we conclude that in any case $Y$ contains a $\{\rho_s\}$-orbit. Recall that $\{\rho_s\}$ is a nontrivial one-parameter subgroup of $DV_2$. Let $F$ be the (closed) subgroup generated by $\{\rho_s\}$ and $V_1$. Since $Y$ is a $V_1$-invariant subset and contains a $\{\rho_s\}$-orbit and $\{\rho_s\}$ normalises $V_1$ it follows that $Y$ contains a $F$-orbit. Clearly $F$ is a two dimensional connected Lie subgroup of $DV$. It is well known and easy to check that such a subgroup is either $V$ or $v(DV_1)v^{-1}$ for some $v \in V_2$. This proves the proposition.

3.7. *Remark*. Following the method of Proposition 3.6 one can prove that if $Y$ is the closure of a $U$-orbit where $U$ is one of the subgroups $V$ or $W$ then $Y$ is either a (compact) $U$-orbit or it contains an orbit of a closed connected subgroup $F$ properly containing $U$ and contained in the normaliser of $U$.

## §4. Proof of Theorem 2

In this section we complete the proof of Theorem 2. We follow the notation as in §3. We recall in particular, that $\Gamma$ is allowed to be any lattice in $G=\mathrm{SL}(3,\mathbb{R})$ satisfying Condition (*) formulated in Remark 1.8, which is seen to hold for $\mathrm{SL}(3,\mathbb{Z})$ and would be upheld for an arbitrary lattice elsewhere.

**4.1. Proposition.** *Let $Y$ be the closure of a $DV_1$-orbit. Then $Y$ contains either a $V$-orbit or a closed $H$-orbit.*

*Proof.* In view of Proposition 3.2 without loss of generality we may assume $Y$ to be a minimal closed $DV_1$-invariant subset. Let $X$ be a minimal closed $V_1$-invariant subset contained in $Y$; it may be recalled that such a subset exists by Corollary 1.5 and is compact by Corollary 1.7. Let $x\in X$ be fixed.

First suppose that there exists a subset $M$ of $G-HV_2$ such that $e\in\overline{M}$ and $gx\in Y$ for all $g\in M$. Since $X$ is a compact minimal $V_1$-invariant set and $Y$ is $DV_1$-invariant, Lemma 2.1 implies that $hX\subset Y$ for all $h\in\overline{DV_1MV_1}\cap N(V_1)$, where $N(V_1)$ is the normaliser of $V_1$ in $G$. Since $M\subset G-HV_2$ and $e\in\overline{M}$, by Corollary 3.5 $\overline{DV_1MV_1}$ contains either $V_2^+$ or $V_2^-$. Since $V_2\subset N(V_1)$ we get that either $V_2^+X$ or $V_2^-X$ is contained in $Y$. By Lemma 2.3 this implies that $Y$ contains a $V$-orbit.

Now suppose that there does not exist any subset $M$ as above. Then there exists an open neighbourhood $\Omega$ of $e$ in $G$ such that $\{g\in\Omega\mid gx\in Y\}\subset HV_2$. By replacing $\Omega$ by a smaller neighbourhood we may further assume that $g\to gx$, $(g\in\Omega)$, is a homeomorphism of $\Omega$ onto $\Omega x$ and that the map $(h,v)\to hv$, $(h\in\Omega\cap H$ and $v\in\Omega\cap V_2)$ is a homeomorphism of $(\Omega\cap H)\times(\Omega\cap V_2)$ onto a neighbourhood of $e$ in $HV_2$ contained in $\Omega$.

Let $y\in(\Omega\cap HV_2)x\cap Y$, say $y=hvx$ with $h\in\Omega\cap H$ and $v\in\Omega\cap V_2$, be arbitrary; we show that then $vx\in Y$. Since $X$ is a minimal closed $V_1$-invariant subset there exists a sequence $\{u_i\}$ in $V_1$ such that $u_i\to\infty$ (namely it has no limit point in $V_1$) and $u_ix\to x$. In the group $H$, which is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$, for $h$ and $\{u_i\}$ as above there exists a sequence $\{p_i\}$ in $DV_1$ such that $p_ihu_i^{-1}\to e$; one only has to note that $(DV_1)hu_i^{-1}\to DV_1$ in the quotient space $DV_1\backslash H$. Hence $p_iy=p_ihvx=(p_ihu_i^{-1})(u_ivx)=(p_ihu_i^{-1})v(u_ix)\to vx$. This shows that $vx\in Y$.

In view of the above observation either there exists a neighbourhood $\Omega'$ of $e$, contained in $\Omega$, such that $\{g\in\Omega'\mid gx\in Y\}\subset H$ or there exists a sequence $\{v_i\}$ in $V_2\{e\}$ such that $v_i\to e$ and $v_ix\in Y$ for all $i$. Suppose the first condition holds. Then $(\Omega'\cap H)x\cap Y$ is open in $Y$ and as the $DV_1$-action on $Y$ is minimal it follows that $Y$ is contained in $DV_1(\Omega'\cap H)x$ and hence in $Hx$. Since $H/DV_1$ is compact and $Y$ is $DV_1$-invariant, $HY$ is a closed subset. But then, since $Hx=HY$, $Hx$ is a closed $H$-orbit and hence, by Proposition 3.1, the $DV_1$-action on $Hx$ is minimal. As $Y$ is a closed $DV_1$-invariant subset of $Hx$ it follows that $Y=Hx$, namely a closed $H$-orbit. Thus we are through in this case.

Now suppose that there exists a sequence $\{v_i\}$ in $V_2$-$\{e\}$ such that $v_i\to e$ and $v_ix\in Y$ for all $i$. Let $p\in DV_1$ be any element such that $px\in\Omega x$. Then, by the choice of $\Omega$, we have $px\in(\Omega\cap HV_2)x$ and hence there exist $h'\in\Omega\cap H$ and $v'\in\Omega\cap V_2$ such that $px=h'v'x$. For each $i$ we have $pv_ix=(pv_ip^{-1})px=(pv_ip^{-1})h'v'x$

$=h'(h'^{-1}\,pv_ip^{-1}h')v'x$. Also, since $v_ix\in Y$, $pv_ix\in Y$ for all $i$ and since $v_i\to e$, $pv_ix\in\Omega x$ for all large $i$. Hence $pv_ix\in\Omega x\cap Y=(\Omega\cap HV_2)x\cap Y$ for all large $i$. Therefore $h'(h'^{-1}pv_ip^{-1}h')v'x\in(\Omega\cap HV_2)x$ for all large $i$. Since $v_i\to e$ and the map $g\to gx$ is injective on $\Omega$, this readily implies that $h'^{-1}pv_ip^{-1}h'\in HV_2$ for all large $i$. By considering the action of $H$ on the subspace $\mathfrak{p}$ as in §3, obtained by restriction of the adjoint action of $G$, it is easy to see that there exists a neighbourhood $N$ of $e$ in $V_2$ such that for any $v\in N-\{e\}$ and $h\in H$, $hvh^{-1}\in HV_2$ if and only if $h\in DV_1$. Therefore the preceding condition yields that $h'\in DV_1$. Hence $px=h'v'x\in(\Omega\cap DV)x$. Thus $\Omega x\cap DV_1x$ is contained in $(\Omega\cap DV)x$.

In view of the above conclusion and the $DV_1$-invariance of $Y$, either there exists a neighbourhood $\Omega''$ of $e$ contained in $\Omega$ such that $\Omega''x\cap DV_1x$ is contained in $(\Omega''\cap DV_1)x$ or there exists a sequence $\{v_i'\}$ in $V_2$-$\{e\}$ such that $v_i'\to e$ and $v_i'x\in DV_1x$ for all $i$. If the first condition holds then $DV_1x$ is open in $Y$ and hence by minimality of $Y$ it is the whole of $Y$; but that is a contradiction since by Proposition 3.1 $DV_1$ has no closed orbits on $G/\Gamma$. Hence the second condition must hold. Let $\{v_i'\}$ be a sequence as above and let $\Delta=\{g\in DV|gx=x\}$. Then $\Delta$ is a discrete subgroup of $DV$ containing for each $i$ an element of the form $d_iu_iv_i'$, where $d_i\in D$ and $u_i\in V_1$. An elementary argument shows that any discrete subgroup of $DV$ is either contained in $V$ or it is a cyclic subgroup generated by an element of the form $wdw^{-1}$ with $d\in D$ and $w\in V$. It is also easy to see that in the latter case the subgroup does not contain any sequence of elements of the form $\{d_iu_iv_i'\}$ with $d_i\in D$, $u_i\in V_1$ and $v_i'\in V_2$-$\{e\}$ with $v_i'\to e$. Hence $\Delta$ as above must be contained in $V$ and $d_i=e$ for all $i$. Thus $\Delta$ contains $u_iv_i'$ for all $i$, where $u_i\in V_1$, $v_i'\in V_2$-$\{e\}$ and $v_i'\to e$. In particular this forces that $u_i$ is nontrivial for all large $i$. Thus $x$ is fixed by an element $u_iv_i'$, where $u_i\in V_1$-$\{e\}$ and $v_i'\in V_2$-$\{e\}$. We deduce from this that $Wx$ is compact; this may be done either by applying Theorem 6.4 of [10] or, if we grant $\Gamma$ being arithmetic then, by a direct argument. Since $X=\overline{V_1x}$ it is contained in $Wx$ and it follows from the well-known results on flows on nilmanifolds (cf. [16], Chapter II) that $V_2x$ is compact and $X$ is an orbit of one of the subgroups $V_1$, $V$ and $W$. It is easy to see that since $V_2x$ is compact the subgroup $\Delta$ as above cannot contain a sequence of elements of the form $u_iv_i'$ with $u_i\in V_1$ and $v_i'\in V_2$-$\{e\}$ with $v_i'\to e$ if $V_1x$ is also compact. Hence $X$ must be an orbit of $V$ or $W$, in which case the contention of the proposition is satisfied.

**4.2. Proposition.** *Every $DV$-orbit is dense in $G/\Gamma$.*

*Proof.* Let $Y$ be the closure of a $DV$-orbit. If $Y$ contains a $W$-orbit then, since it is $D$-invariant and $D$ normalises $W$, it would contain a $DW$-orbit; since every $DW$-orbit is dense (cf. [6] Proposition 1.2) this implies that $Y=G/\Gamma$ as desired. Now suppose, if possible, $Y$ does not contain any $W$-orbit. Let $X$ be a minimal closed $V$-invariant subset of $Y$; such a subset exists by Corollary 1.5. Since $Y$ does not contain any $W$-orbit, by Proposition 3.3, $X$ must be a (compact) $V$-orbit. By an argument in the proof of Proposition 3.2 (ii) this implies that $DX$ is not closed. Hence in particular $DX\neq Y$. By replacing $X$ by another minimal subset (compact orbit) if necessary, we may also assume that $DX$ is not open in $Y$.

Let $x \in X$ and suppose that there exists an open neighbourhood $\Omega$ of $e$ in $G$ such that $\bar{\Omega}$ is compact and $\{g \in \Omega \mid gx \in Y\} \subset DW \subset N(V)$, the normaliser of $V$. Then $(\Omega \cap DW)x \cap Y$ is a neighbourhood of $x$ in $Y$ and hence it contains an element $y$ such that $DVy$ is dense in $Y$. Let $y = dwx$ where $d \in D$ and $w \in W$. Then $DVy = DVdwx = D(dw)Vx = DwVx = DwX$. Let $\Delta = \{g \in DW \mid gx = x\}$. Since $X = Vx$ is compact it follows as in the last Proposition that $\Delta$ is a lattice in $W$; thus the map $(d, z) \to dz$ of $D \times Wx$ into $G/\Gamma$ is injective and its restriction to any compact subset is a homeomorphism onto the image. In particular $(\Omega \cap DW)x \cap DwX$ is a closed subset of $(\Omega \cap DW)x$. As $DVy$ is dense in $Y$ it follows that $(\Omega \cap DW)x \cap Y$ is contained in $DwX$. Since $x \in Y$, this further implies that $wX = X$. Hence $\Omega x \cap Y = (\Omega \cap DW)x \cap Y$ is contained in $DX$. This implies that $DX$ is open in $Y$, contrary to our assumption. Hence there does not exist any $\Omega$ as above. Therefore there exists $M \subset G - DW$ such that $e \in \bar{M}$ and $gx \in Y$ for all $g \in M$. By Lemma 2.1 we have $hX \subset Y$ for all $h \in \overline{DVMV} \cap N(V)$. Since $M \subset G - DW$, $\overline{DVMV}$ contains either $W^+$ or $W^-$ (cf. [13] proof of Lemma 6), where $W^+ = \{g = (g_{kl}) \in W \mid g_{12} \geqq g_{23}\}$ and $W^- = \{g = (g_{kl}) \in W \mid g_{12} \leqq g_{23}\}$, $(g_{kl})$ being the matrix form of $g$ in $G$. Since $W \subset N(V)$ this implies that either $W^+X$ or $W^-X$ is contained in $Y$. By Lemma 2.3 this implies that $Y$ contains a $W$-orbit, contrary to our assumption. This shows that $Y = G/\Gamma$.

**4.3. Proposition.** *Let $Y$ be the closure of a $V_1$-orbit and suppose that it contains a closed orbit of $H$, say $X$. Then either $Y = X$ or $Y = G/\Gamma$.*

*Proof.* First suppose that there exist $x \in X$ and a neighbourhood $\Omega$ of $e$ in $G$ such that $\{g \in \Omega \mid gx \in Y\}$ is contained in $V_2H$. Then $(\Omega \cap V_2H)x \cap Y$ is open in $Y$. As $Y$ is the closure of a $V_1$-orbit there exists $y \in (\Omega \cap V_2H)x$ such that $V_1y$ is dense in $Y$. Let $y = v_2hx$, where $v_2 \in V_2$ and $h \in H$ such that $v_2h \in \Omega$. Then for any $v_1 \in V_1$, $v_1y = v_1v_2hx = v_2v_1hx \in v_2Hx = v_2X$. Hence $V_1y$ is contained in $v_2X$ and since the latter set is closed it also follows that $Y = \overline{V_1y} \subset v_2X$. Thus we have $X \subset Y \subset v_2X$. Since $X$ is a closed $H$-orbit this is impossible unless $v_2X = X$ and consequently $Y = X$.

Now suppose that the above condition is not satisfied and let $x \in X$ be such that the $V_1$-orbit of $x$ is not periodic. Then there exists $M \subset G - V_2H$ such that $e \in \bar{M}$ and $gx \in Y$ for all $g \in M$. Let $\Psi$ be any compact neighbourhood of $x$ and let

$$H(\Psi) = \{h \in H \mid x = hy \text{ for some } y \in \Psi\}$$

The set of $g$ such that $g\Psi \cap Y \neq \emptyset$ is closed and contains $V_1MH(\Psi)$; therefore it contains $\overline{V_1MH(\Psi)}$. We now claim that $\overline{V_1MH(\Psi)}$ contains either $V_2^+$ or $V_2^-$.

Let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{p}$ be as in §3. Since $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ there exist neighbourhoods $A$ and $B$ of $e$ in $H$ and $\exp \mathfrak{p}$ respectively such that the map $(a, b) \to ba$ is a homeomorphism of $A \times B$ onto a neighbourhood of $e$ in $G$. Let $\{m_j\}$ be a sequence in $M$ converging to $e$ and $M_1 = \{g \in B \mid m_j \in gA \text{ for some } j\}$. Clearly $e \in \bar{M}_1$ and $M_1 \cap V_2 = \emptyset$. Then by Proposition 3.4 there exists a polynomial map $\phi$: $\mathbb{R} \to V_2$ such that $\phi(\mathbb{R})$ contains $V_2^+$ or $V_2^-$ and there exist sequences $\{t_i\}$ in $\mathbb{R}^+$ and $\{g_i\}$ in $M_1$ such that $g_i \to e$ and $v_1(\beta_i t_i) g_i v_1(-\beta_i t_i) \to \phi(\beta)$ for any sequence $\{\beta_i\}$ in $\mathbb{R}$ such that $\beta_i \to \beta$. Since $X$ is a closed $H$-orbit, by Proposition 3.1 the element $x$ as above is a point of uniform recurrence in linear time for the $V_1$-action. Thus given $\alpha > 0$ there exists a neighbourhood $\Psi'$ of $x$ in $X$ and $T_0 \in \mathbb{R}$ such that for all $y \in \Psi'$ and $T \geqq T_0$ there exists $t\in[T,(1+\alpha)T]$ such that $v_1(t)y\in\Psi$. Let $s\in\mathbb{R}$ be given. Observe that there exists a sequence $\{h_i\}$ in $H$ such that $h_i\to e$ and $g_ih_i\in M$ for all $i$. The elements $h_ix$ are contained in $\Psi'$ for all large $i$ and hence there exists a sequence $\{\beta_i\}$ in $[1,1+\alpha]$ such that $v_1(\beta_it_is)h_ix\in\Psi$. Hence for $i$, $h_i^{-1}v_1(-\beta_it_is)\in H(\Psi)$ and in turn $V_1g_iv_1(-\beta_it_is)=V_1(g_ih_i)h_i^{-1}v_1(-\beta_it_is)\in V_1MH(\Psi)$. As $1\leqq\beta_i\leqq 1+\alpha$ for all $i$, passing to a subsequence we may assume that $\{\beta_i\}$ converges, say $\beta_i\to\beta$, where $1\leqq\beta\leqq 1+\alpha$. Therefore $V_1g_iv_1(-\beta_it_is)\to V_1\phi(\beta s)$ and hence $\phi(\beta s)\in\overline{V_1MH(\Psi)}$. Thus we have shown that for any $s\in\mathbb{R}$ and $\alpha>0$ there exists $\beta$ such that $1\leqq\beta\leqq 1+\alpha$ and $\phi(\beta s)\in\overline{V_1MH(\Psi)}$. This implies that $\phi(s)\in\overline{V_1MH(\Psi)}$ for all $s\in\mathbb{R}$. As the image of $\phi$ contains $V_2^+$ or $V_2^-$, this proves the claim.

Thus $g\Psi\cap Y$ is nonempty for all $g$ in $V_2^+$ or $V_2^-$. Since $\Psi$ was an arbitrary compact neighbourhood of $x$ this implies that either $V_2^+x$ or $V_2^-x$ is contained in $Y$. Since $Y$ is a closed $V_1$-invariant set this means that $Y$ contains the closure of either $V_1V_2^+x$ or $V_1V_2^-x$. But $V_1V_2^+x=V_2^+V_1x$, $V_1V_2^-x=V_2^-V_1x$ and by Proposition 3.1 $V_1x$ is dense in $X$. Therefore $Y$ contains either $V_2^+X$ or $V_2^-X$. We complete the proof by showing that $V_2^+X$ and $V_2^-X$ are dense in $G/\Gamma$. Let $Y_1$ be the closure of say $V_2^+X$. Since $DV_1V_2^+=V_2^+D\cdot V_1$, it follows that $Y_1$ is $DV_1$-invariant. On the other hand, by Lemma 2.3, $Y_1$ contains an orbit of $V_2$. Together the two assertions imply that $Y_1$ contains an orbit of $DV$. Hence by Proposition 4.2, $Y_1=G/\Gamma$ which means that $V_2^+X$ is dense in $G/\Gamma$. A similar argument shows that $V_2^-X$ is also dense in $G/\Gamma$.

**4.4. Corollary.** *Let $Y$ be the closure of a $V_1$-orbit in $G/\Gamma$. Suppose that it contains an orbit of $DV_1$. Then either $Y$ is a $H$-orbit or $Y=G/\Gamma$.*

*Proof.* Let $Y_1$ be the closure of a $DV_1$-orbit in $Y$. By Proposition 4.1 $Y_1$ contains either a $V$-orbit or a closed $H$-orbit. If $Y_1$ contains a closed $H$-orbit then so does $Y$ and hence Proposition 4.3 implies the Corollary. Now suppose that $Y_1$ contains a $V$-orbit. Since $Y_1$ is $DV_1$-invariant and $D$ normalises $V$, this implies that $Y_1$ contains a $DV$-orbit. Hence by Proposition 4.2 $Y_1=G/\Gamma$ which proves the Corollary.

*Proof of Theorem 2.* Let $y=g\Gamma\in G/\Gamma$, where $g\in G$, and let $Y=\overline{Hy}$. For $h\in H$, $V_1hy$ is compact if and only if $g^{-1}h^{-1}V_1hg$ contains a nontrivial element of $\Gamma$. As $\{g^{-1}h^{-1}V_1hg\}_{h\in H}$ is an uncountable family of subgroups no two of which have any nontrivial element in common it follows that there exists $h\in H$ such that $V_1hy$ is not compact. Then by Proposition 3.6 $\overline{V_1hy}$ contains an orbit of either $V$ or $v(DV_1)v^{-1}$ for some $v\in V$. Suppose $\overline{V_1hy}$, and hence $Y$, contains a $V$-orbit. Since $Y$ is $D$-invariant and $D$ normalises $V$ this implies that $Y$ contains a $DV$-orbit and hence $\overline{Y}=G/\Gamma$ by Proposition 4.2. Now suppose that there exists $v\in V$ such that $\overline{V_1hy}$ contains a $v(DV_1)v^{-1}$ orbit, say of a point $z\in G/\Gamma$. First consider the case where $v$ is not contained in $V_1$. Then $DvD$ contains $\{dvd^{-1}|d\in D\}$ which is dense in either $V_2^+$ or $V_2^-$. As $\overline{V_1hy}$ contains $vDV_1v^{-1}z$, $Y$ contains $DvDV_1v^{-1}z$ and hence it contains either $V_2^+v^{-1}z$ or $V_2^-v^{-1}z$, by the preceding observation. Hence by Lemma 2.3 $Y$ contains a $V$ orbit and, as seen above, this implies that $Y=G/\Gamma$. Now suppose that $v\in V_1$, which means that $\overline{V_1hy}$ contains a $DV_1$-orbit. Hence by Corollary 4.4 $\overline{V_1hy}$

is either a closed $H$-orbit or $G/\Gamma$. Then clearly either $Y=Hy$, a closed $H$-orbit, or $Y=G/\Gamma$, which proves the theorem.

## §5. Proof of Theorem 1

We now deduce Theorem 1 from Theorem 2. We first note that it is enough to prove Theorem 1 for $n=3$. While an expert may readily recognize this, a proof is indicated for the convenience of the general reader. Let the notation be as in the statement of the theorem, with $n\geq 4$. We can find linearly independent rational vectors $v_0$ and $w_0$ arbitrarily close to $v$ and $w$ respectively such that the restriction of $B$ to the subspace spanned by $\{v_0, w_0\}$ is nondegenerate. Hence by modifying the data we may assume $v$ and $w$ linearly independent rational vectors such that the restriction of $B$ to the of span of $\{v, w\}$ is nondegenerate. As $B$ is nondegenerate and indefinite there exists a rational vector $w_1$ such that $v$, $w$ and $w_1$ are linearly independent and the restriction of $B$ to the subspace spanned by them is nondegenerate and indefinite; we first find a $w_1\in\mathbb{R}^n$, using orthogonal decomposition, and then observe that the required properties continue to hold if it is replaced by a rational vector close to it. By adjoining more rational vectors we find a rational hyperplane $L$ containing $v$ and $w$ such that the restriction of $B$ to $L$ is nondegenerate and indefinite. Let $\lambda$ be a nonzero rational linear form on $L$ such that $\lambda(v)=\lambda(w)=0$. Let $x$ be a rational vector in $\mathbb{R}^n-L$ and for $t\in\mathbb{R}$ let $L_t$ be the hyperplane $\{z+t\lambda(z)x\,|\,z\in L\}$. Then $L_t$ is a rational hyperplane for each rational $t$ and for all small $t$ the restriction of $B$ to $L_t$ is nondegenerate and indefinite. We claim that there exists such a $t$ for which the restriction of $B$ to $L_t$ is not a multiple of a rational form. Suppose this is not true. Since the restriction of $B$ to $L\cap L_t$ is nonzero we can conclude from this that there exists a common $\alpha\in\mathbb{R}^*$ such that the restrictions of $\alpha B$ to $L$ and $L_t$ are rational forms. Since this applies to any $L_t$ as above we get that $\alpha B$ is a rational form, contradicting the hypothesis. Therefore the claim holds. Now if $x$ and $y$ are primitive elements in $L\cap\mathbb{Z}^n$ or $L_t\cap\mathbb{Z}^n$ for which the conclusion of the theorem holds then the theorem will stand proved for $\mathbb{R}^n$ as well. This shows that for $n\geq 4$ the theorem holds for a value of $n$ whenever it holds for $n-1$. Hence it is enough to prove the theorem for $n=3$.

Now let $n=3$. Let the notation be as in the theorem and §3. Let $L$ be the identity component of the subgroup of $G$ leaving invariant the quadratic form $B$. Since for $n=3$ any nondegenerate indefinite quadratic form is equivalent over $\mathbb{R}$, to one of the forms $\pm(2x_1x_3-x_2^2)$ it follows that $L=gHg^{-1}$ for some $g\in G$. Now consider the function $f\colon\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ defined by $f(v, w)=(B(v), B(w), B_2(v, w))$ for all $v$, $w\in\mathbb{R}^3$. Let $\Gamma=\mathrm{SL}(3,\mathbb{Z})$ and $E\subset\mathbb{R}^3\times\mathbb{R}^3$ be a nonempty subset invariant under the (componentwise)$\Gamma$-action on $\mathbb{R}^3\times\mathbb{R}^3$. Since $B$ is invariant under elements of $L$ it follows that $f(E)=f(LE)=f(L\Gamma E)$. Since $f$ is continuous we get that $f(\overline{L\Gamma E})\subset\overline{f(E)}$. By Theorem 2, $\overline{Hg^{-1}\Gamma}$ is either $G$ or $Hg^{-1}\Gamma$. Hence $\overline{L\Gamma}$ which is the same as $\overline{gHg^{-1}\Gamma}$ coincides with either $G$ or $L\Gamma$. Therefore we get that $\overline{f(E)}$ contains $f(GE)$ unless $L\Gamma$ is closed. An argument as in [13] shows that as $B$ is not a multiple of a rational form $L\Gamma$ is not closed. Hence $\overline{f(E)}$ contains $f(GE)$ for any nonempty $\Gamma$-invariant subset of $\mathbb{R}^3\times\mathbb{R}^3$. Now choose $E=\mathfrak{P}(\mathbb{Z}^3)\times\mathfrak{P}(\mathbb{Z}^3)$. Then $GE=\mathbb{R}^3\times\mathbb{R}^3$

and hence $f(E)$ is dense in the image of $f$, which is precisely the assertion of the theorem.

*Remark.* Raghunathan's conjecture stated in the introduction entails that if $B$ is a nondegenerate indefinite quadratic form on $\mathbb{R}^n$ and $L$ is the subgroup of $\mathrm{SL}(n, \mathbb{R})$ leaving $B$ invariant then any $L$-orbit is either closed or dense. By the argument as in the deduction of Theorem 1 above this implies that if $B$ as above is not a multiple of a rational form and $B_2$ is the corresponding bilinear form then for any $\{a_{ij} | i, j=1, 2, \ldots, n-1\} \subset \mathbb{R}$ for which there exist $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ such that $B_2(v_i, v_j) = a_{ij}$ and $\varepsilon > 0$ there also exist $x_1, \ldots, x_{n-1} \in \mathfrak{P}(\mathbb{Z}^n)$ such that

$$|B_2(x_i, x_j) - a_{ij}| < \varepsilon$$

for all $i, j = 1, \ldots, n-1$.

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**Note added in proof**

A simpler proof is now obtained for the last part of Theorem 1, on values at primitive integral points, and will appear elsewhere.