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JHEP11(2007)077

(http://iopscience.iop.org/1126-6708/2007/11/077)

Published by Institute of Physics Publishing for SISSA

Received: *October 3, 2007*
Accepted: *October 25, 2007*
Published: *November 26, 2007*

Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift

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Abstract: The degeneracies of supersymmetric quarter BPS dyons in four dimensions and of spinning black holes in five dimensions in a CHL compactification are computed exactly using Borcherds lift. The Hodge anomaly in the construction has a physical interpretation as the contribution of a single M-theory Kaluza-Klein 6-brane in the 4d-5d lift. Using factorization, it is shown that the resulting formula has a natural interpretation as a two-loop partition function of left-moving heterotic string, consistent with the heuristic picture of dyons in the M-theory lift of string webs.

Keywords: Superstrings and Heterotic Strings, Black Holes in String Theory.

Contents

1. Introduction

For heterotic string compactified on $\mathbf{T^6}$, there exists a remarkable formula that gives the exact degeneracies of the dyonic quarter BPS states in the four-dimensional theory [1 – 6]. A similar formula has been proposed also for Type-II string compactified on $\mathbf{T^6}$ [7, 8].

The spectrum of dyons encapsulates valuable information about the nonperturbative structure of the theory. Moreover, when the charge of the dyon is large, it gravitates to form a black hole. The logarithm of the degeneracies can then be compared with the Bekenstein-Hawking-Wald entropy. In favorable situations, one can hope to make exact comparisons between macroscopic and microscopic degeneracies beyond leading order as was done for the electric states [9 – 13] in an appropriate ensemble [14].

It is therefore of considerable interest to see if one can obtain a similar exact formula for counting dyons in more general compactifications. An interesting class of models where the computations are tractable are the CHL orbifolds [15, 16] of the heterotic string that result in $\mathcal{N} = 4$ models in four dimensions with gauge groups of reduced rank. Toroidally compactified heterotic string results in a gauge group of rank 28. A CHL compactification is obtained by orbifolding the heterotic string compactified on $\mathbf{T^4} \times \mathbf{\tilde{S}^1} \times \mathbf{S^1}$ by a $\mathbb{Z}_N$ symmetry generated by $\alpha T_{1/N}$, where α is an order N symmetry of the internal CFT of the heterotic string compactified on $\mathbf{T^4}$, and $T_{1/N}$ is an order N translation along the circle $\mathbf{\tilde{S}^1}$. The internal symmetry α has a nontrivial action on the gauge bosons and hence some combinations of bosons are projected out in the orbifold theory. Because of the order N shift that accompanies α, the twisted sectors are massive and no additional gauge bosons arise in the twisted sector. The resulting theory then has gauge group with a rank smaller than 28. Using string-string duality, this heterotic compactification is dual to an orbifold of Type-II on $\mathbf{K3} \times \tilde{\mathbf{S}}^{\mathbf{1}} \times \mathbf{S}^{\mathbf{1}}$ by a $\mathbb{Z}_N$ symmetry generated by $\tilde{\alpha} T_{1/N}$, where $\tilde{\alpha}$ is an order N symmetry in the internal CFT of the $\mathbf{K3}$.

The S-duality group of a CHL orbifold is a congruence subgroup $\Gamma_1(N)$ of the $\mathrm{SL}(2, \mathbb{Z})$ S-duality symmetry of the original toroidally compactified heterotic string theory. As a result, the dyonic degeneracies are expected to display interesting differences for various orbifolds. Recently a formula for the exact dyonic degeneracies for CHL orbifold was proposed [17] for the cases $N = 2, 3, 5, 7$ generalizing the work of [1]. Let us summarize this proposal. For a $\mathbb{Z}_N$ orbifold, the dyonic degeneracies are encapsulated by a Siegel modular forms $\Phi_k(\Omega)$ of level N and index k as a function of period matrices Ω of a genus two Riemann surface.[1] We denote each case by the pair (N, k). The index k is related to the level N by the relation

$$k = \frac{24}{N+1} - 2. \tag{1.1}$$

The original example of toroidally compactified heterotic string is included in this list as a special case $(1, 10)$. To obtain the dyonic degeneracy, Jatkar and Sen define $\tilde{\Phi}_k(\tilde{\Omega})$ related to $\Phi_k(\Omega)$ by an $SP(2, \mathbb{Z})$ transformations with a Fourier expansion

$$\frac{1}{\tilde{\Phi}_k(\tilde{\Omega})} = \frac{1}{N\,K} \sum_{\substack{m,n,p \\ m \geq -1, n \geq -1/N}} e^{2\pi i (m\tilde{\rho} + n\tilde{\sigma} + p\tilde{\nu})} g(m, n, p)\,, \tag{1.2}$$

where K is an appropriate constant and $\tilde{\Omega} = \begin{pmatrix} \tilde{\rho} & \tilde{\nu} \\ \tilde{\nu} & \tilde{\sigma} \end{pmatrix}$. Consider now a dyonic state with a charge vector $Q = (Q_e, Q_m)$ which is a doublet of the $\mathrm{SL}(2, \mathbb{R})$. Here Q_e and Q_m are the electric and magnetic charges that transform as vectors of the T-duality symmetry $O(r-6, 6; \mathbb{Z})$ for a rank r CHL compactification with $r = 2k+4$. The Q_e^2, Q_m^2, and $Q_e \cdot Q_m$ be the T-duality invariant combinations.[2] The degeneracy of dyons $d(Q)$ is then given in terms of the Fourier coefficients by

$$d(Q) = g\left(\frac{1}{2}Q_m^2, \frac{1}{2}Q_e^2, Q_e \cdot Q_m\right). \tag{1.3}$$

The degeneracy $d(Q)$ obtained this way satisfies three nontrivial physical consistency checks [17]. It is manifestly invariant under the S-duality group $\Gamma_1(N)$, it agrees with the Bekenstein-Hawking-Wald entropy of the corresponding black holes to leading and the first subleading order [1, 2], and finally it is integral as expected for an object that counts the number of states.

[1] Definitions of various quantities mentioned here in the introduction are given in section 2 and section 3.

[2] The T-duality group of the CHL model is actually a subgroup of $O(r-6, 6; \mathbb{R})$ symmetry that we have used here. The degeneracy formula proposed in [17] therefore is likely to be valid for a restricted class of dyons. In general, the degeneracy can also depend on more subtle invariants of the arithmetic subgroup that cannot be written as invariants of the continuous Lie group as is in the case of electric states.

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The task of understanding the dyon spectrum is then reduced to understanding the physics contained in the Siegel modular forms $\Phi_k(\Omega)$. In the best understood case $(1, 10)$, the modular form $\Phi_{10}(\Omega)$ is the well-known Igusa cusp form which is the unique cusp form of weight 10. In the original proposal of Dijkgraaf, Verlinde, Verlinde [1], the relation of this modular form to the counting of dyons was conjectured based on various consistency checks and a heuristic derivation of the properties of NS5-brane worldvolume theory [1, 18]. This conjecture used in an essential way the perturbative string computations in [19] of threshold corrections of heterotic string on $\mathbf{K3} \times \mathbf{T^2}$ where the Igusa cusp form naturally appears. In [20], the Igusa form made its appearance in an apparently unrelated context in connection with the elliptic genus of the symmetric product of **K3**, which counts the bound states comprising the D1-D5-P black hole in five dimensions [21]. A new perspective on the dyon counting formula and a definitive connection between the 4d and 5d formulae was provided in [3] using the 4d-5d lift of [22]. It related the 4d dyonic degeneracies to 5d degeneracies of the D1-D5-P black holes computed by the elliptic genus of the symmetric product of **K3** providing a physical re-derivation of the dyon counting formula.

Our objective is to obtain a similar physical derivation of the spectrum of dyons in CHL compactifications. Towards this end, we outline in section 2 a general procedure for deriving the modular forms Φ_k using a generalization of Borcherds lift which we call 'Multiplicative Lift'. This lift results in a a Siegel modular form of level N and weight k of $G_0(N)$ in a product representation, starting with a weak Jacobi form of weight zero and index one of $\Gamma_0(N)$. The resulting formula suggests a physical interpretation using the idea of 4d-5d lift proposed in [22, 3]. With this interpretation, the modular form Φ_k that counts dyons in four dimensions is naturally related to a quantity that counts the bound states of D1-D5-P in five dimensions. In section 4 we illustrate this general procedure for the special case $(2, 6)$ and the Siegel modular form Φ_6. We discuss various aspects of our physical interpretation using 4d-5d lift in section 5 and obtain as a byproduct the exact degeneracies of spinning black holes in CHL compactifications to five dimensions.

One of the mysterious aspects of the counting formula is the appearance of the genus two modular group. A physical explanation for this phenomenon was proposed in [4] by relating the dyon degeneracies to string webs. Lifting the string webs to Euclidean M-theory 5-branes wrapping on **K3** naturally results in a genus-two Riemann surface using the fact that M5-brane on **K3** is the fundamental heterotic string. We offer a similar interpretation of our results in section 6. We show using factorization that the product formula for Φ_6 has a natural representation as a chiral, genus-two partition function of the left-moving heterotic string.

Details of the construction of the modular forms Φ_k for the remaining pairs (N, k) using multiplicative lift will be presented in a forthcoming publication along with a more complete discussion of the physical interpretation outlined in this note [23].

2. Siegel modular forms of level N

Let us recall some relevant facts about Siegel modular forms. Let Ω be the period matrix of a genus two Riemann surface. It is given by a (2×2) symmetric matrix with complex entries

$$\Omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix} \tag{2.1}$$

satisfying

$$\mathrm{Im}(\rho) > 0, \quad \mathrm{Im}(\sigma) > 0, \quad \mathrm{Im}(\rho)\,\mathrm{Im}(\sigma) > \mathrm{Im}(\nu)^2, \tag{2.2}$$

and parametrizes the 'Siegel upper half plane' in the the space of (ρ, ν, σ). There is a natural symplectic action on the period matrix by the group $\mathrm{Sp}(2, \mathbb{Z})$ as follows. We write an element g of $\mathrm{Sp}(2, \mathbf{Z})$ as a (4×4) matrix in a block-diagonal form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.3}$$

where A, B, C, D are all (2×2) matrices with integer entries. They satisfy

$$AB^T = BA^T, \qquad CD^T = DC^T, \qquad AD^T - BC^T = I\,, \tag{2.4}$$

so that $g^t J g = J$ where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the symplectic form. The action of g on the period matrix is then given by

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}. \tag{2.5}$$

The $\mathrm{Sp}(2, \mathbb{Z})$ group is generated by the following three types of (4×4) matrices with integer entries

$$\begin{aligned} g_1(a, b, c, d) &\equiv \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad ad - bc = 1, \\ g_2 &\equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\ g_3(\lambda, \mu) &\equiv \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.6}$$

We are interested in a subgroup by $G_1(N)$ of $\mathrm{Sp}(2, \mathbb{Z})$ generated by the matrices in (2.6) with the additional restriction

$$c = 0 \bmod N, \quad a, d = 1 \bmod N. \tag{2.7}$$

Note that with the restriction (2.7), the elements $g_1(a, b, c, d)$ generate the congruence subgroup $\Gamma_1(N)$ of $\mathrm{SL}(2, \mathbb{Z})$ which is the reason for choosing the name $G_1(N)$ for the subgroup of $\mathrm{Sp}(2, \mathbb{Z})$ in this case. From the definition of $G_1(N)$ it follows that if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_1(N), \tag{2.8}$$

then

$$C = \mathbf{0} \bmod N, \quad \det A = 1 \bmod N, \quad \det D = 1 \bmod N \ . \tag{2.9}$$

One can similarly define $G_0(N)$ corresponding to $\Gamma_0(N)$ by relaxing the condition $a, d = 1$ mod N in (2.7).

We are interested in a modular form $\Phi_k(\Omega)$ which transforms as

$$\Phi_k[(A\Omega + B)(C\Omega + D)^{-1}] = \{\det (C\Omega + D)\}^k \Phi_k(\Omega), \tag{2.10}$$

for matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belonging to $G_1(N)$. We will actually construct modular forms of the bigger group $G_0(N)$. Such a modular form is called a Siegel modular form of level N and weight k. From the definition (2.10) it is clear that a product of two Siegel modular forms Φ_{k_1} and Φ_{k_2} gives a Siegel modular form $\Phi_{k_1+k_2}$. The space of modular forms is therefore a ring, graded by the integer k. The graded ring of Siegel Modular forms for $N = 1, 2, 3, 4$ is determined in a number of papers in the mathematics literature [24–28]. The special cases of our interest for the pairs (N, k) listed in the introduction were constructed explicitly in [17].

In the theory of Siegel modular forms, the weak Jacobi forms of genus one play a fundamental role. A weak Jacobi form $\phi_{k,m}(\tau, z)$ of $\Gamma_0(N)$ transforms under modular transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ as

$$\phi_{k,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \exp\left[\frac{2\pi i m c z^2}{c\tau + d}\right] \phi_{k,m}(\tau, z). \tag{2.11}$$

and under lattice shifts as

$$\phi_{k,m}(\tau, z + \lambda\tau + \mu) = \exp\left[-2\pi i m(\lambda^2\tau + 2\lambda z)\right] \phi_{k,m}(\tau, z)\,, \quad \lambda, \mu \in \mathbb{Z}\,. \tag{2.12}$$

Furthermore, it has a Fourier expansion

$$\phi_{k,m}(\tau, z) = \sum_{n\geq 0,\, r\in\mathbb{Z}} c(4nm - r^2) q^n y^r. \tag{2.13}$$

The significance of weak Jacobi forms in this context stems from the fact that, with the transformation properties (2.11) and (2.12), the combination $\phi_{k,m}(\rho, \nu) \cdot \exp(2\pi i m\sigma)$ transforms with weight k under the group elements $g_1(a, b, c, d)$ and $g_3(\lambda, \mu)$ in (2.6). Then, with some additional ingredients using the property (2.13), one can also ensure the required transformation properties under g_2 to obtain a Siegel modular form.

There are two methods for constructing a Siegel modular form starting with a weak Jacobi form which we summarize below.

- Additive Lift

  This procedure generalizes the Maaß-Saito-Kurokawa lift explained in detail for example in [29]. We refer to it as the 'additive' lift because it naturally gives the sum representation of the modular form in terms of its Fourier expansion. The starting 'seed' for the additive lift is in general a weak Jacobi form $\phi_{k,1}(\rho,\nu)$ of weight k and index 1. Let us denote the operation of additive lift by the symbol $\mathcal{A}[.]$. If a given weak Jacobi form $\phi_{k,1}$ results in a Siegel modular form Φ_k after the additive lift, then we can write

$$\Phi_k(\Omega) = \mathcal{A}[\phi_{k,1}(\rho,\nu)]. \tag{2.14}$$

In the cases of our interest for the pairs (N,k) above, this procedure was used in [17] to obtain the modular forms Φ_k listed there. The seed in these cases can be expressed in terms of the unique cusp forms $f_k(\rho)$ of $\Gamma_1(N)$ of weight $(k+2)$,

$$f_k(\rho) = \eta^{k+2}(\rho)\eta^{k+2}(N\rho), \tag{2.15}$$

where $\eta(\rho)$ is the Dedekind eta function. The seed for the additive lift is then given by

$$\phi_{k,1}(\rho,\nu) = f_k(\rho)\,\frac{\theta_1^2(\rho,\nu)}{\eta^6(\rho)}, \tag{2.16}$$

where $\theta_1(\rho,\nu)$ is the usual Jacobi theta function.

- Multiplicative Lift

  This procedure is in a sense a logarithmic version of the Maaß-Saito-Kurokawa lift. We call it 'multiplicative' because it naturally results in the Borcherds product representation of the modular form. The starting 'seed' for this lift is a weak Jacobi form $\phi^k_{0,1}$ of weight zero and index one and the superscript k is added to denote the fact after multiplicative lift it gives a weight k form Φ_k. Let us denote the operation of multiplicative lift by the symbol $\mathcal{M}[.]$. If a given weak Jacobi form $\phi^k_{0,1}$ results in a Siegel modular form Φ_k after the multiplicative lift, then we can write

$$\Phi_k(\Omega) = \mathcal{M}[\phi^k_{0,1}(\rho,\nu)]. \tag{2.17}$$

Given the specific Siegel modular forms $\Phi_k(\Omega)$ obtained by additively lifting the seeds $\phi_{k,1}$ in (2.16) for the pairs $(N,k) = (1,10),(2,6),(3,4),(7,1)$ as in [17], we would like to know if the same Siegel forms can be obtained as multiplicative lifts of some weak Jacobi forms $\phi^k_{0,1}$. Such a relation between the additive and the multiplicative lift is very interesting mathematically for if it exists, it gives a Borcherds product representation of a given modular form. However, to our knowledge, at present there are no general theorems relating the two. Fortunately, as we describe next, in the examples of interest to us, it seems possible to determine the seed for the multiplicative lift from the seed for the additive lift quite easily and explicitly. Finding such a multiplicative seed to start with is a nontrivial step and is not guaranteed to work in general. But if one succeeds in finding the multiplicative seed $\phi^k_{1,0}$ given a Φ_k obtained from the additive seeds $\phi_{k,1}$ in (2.16) then one can write

$$\Phi_k(\Omega) = \mathcal{A}[f_k(\rho)\,\frac{\theta_1^2(\rho,\nu)}{\eta^6(\rho)}] = \mathcal{M}[\phi^k_{0,1}(\rho,\nu)]. \tag{2.18}$$

3. Multiplicative lift

We now outline the general procedure for constructing modular forms $\Phi_k(\Omega)$ as a Borcherds product [30] by a multiplicative lift following closely the treatment in [26 − 28]

For the special pair $(1, 10)$, which results in the Igusa cusp form Φ_{10}, the product representation was obtained by Gritsenko and Nikulin [31, 32]. The starting seed for this lift is a weak Jacobi form $\phi_{0,1}^{10}$ of weight zero and index one

$$\phi_{0,1}^{10} = 8\left[\frac{\theta_2(\rho,\nu)^2}{\theta_2(\rho)^2} + \frac{\theta_3(\rho,\nu)^2}{\theta_3(\rho)^2} + \frac{\theta_4(\rho,\nu)^2}{\theta_4(\rho)^2}\right], \tag{3.1}$$

where $\theta_i(\rho, \nu)$ are the usual Jacobi theta functions. We therefore have in this case the desired result

$$\Phi_{10}(\Omega) = \mathcal{A}(\phi_{10,1}) = \mathcal{M}(\phi_{0,1}^{10}). \tag{3.2}$$

This weak Jacobi form happens to also equal the elliptic genus of **K3**. As a result, the multiplicative lift is closely related to the elliptic genus of the symmetric product of **K3** [20] which counts the bound states of the D1-D5-P system in five dimensions. This coincidence, which at first sight is purely accidental, turns out to have a deeper significance based on the 4d-5d lift [22].

We would now like find a similar product representation for the remaining pairs of (N, k) using the multiplicative lift so that we can then try to find a similar physical interpretation using 4d-5d lift. We first describe the general procedure of the multiplicative lift for the group $G_0(N)$ and then specialize to the illustrative case $(2, 6)$ of our interest, to obtain the product representation of Φ_6 using these methods.

As we have defined in section 2, the group $G_0(N)$ consists of matrices with integer entries of the block-diagonal form

$$\left\{ \begin{pmatrix} A & B \\ NC & D \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{Z}) \right\} \tag{3.3}$$

which contains the subgroup $\Gamma_0(N)$. A basic ingredient in the construction of Siegel modular forms is the Hecke operator T_t of $\Gamma_0(N)$ where t is an integer. The main property of our interest is that acting on a weak Jacobi form $\phi_{k,m}$ of weight k and index m, the Hecke operator T_t generates a weak Jacobi form $\phi_{k,mt} = T_t(\phi_{k,m})$ of weight k and index mt. Thus, on a modular form $\phi_{k,1}$, the Hecke operator T_t acts like a raising operator that raises the index by $(t - 1)$ units. One subtlety that needs to be taken into account in the case of $\Gamma_0(N)$ that does not arise for $\mathrm{SL}(2, \mathbb{Z})$ is the fact that $\Gamma_0(N)$ has multiple cusps in its fundamental domain whereas $\mathrm{SL}(2, \mathbb{Z})$ has a unique cusp at $i\infty$. As a result, the Hecke operators that appear in the construction in this case are a little more involved as we review in appendix A.

Let us now explain the basic idea behind the lift. Given a seed weak Jacobi form $\phi_{0,1}^k(\rho, \nu)$ for the multiplicative lift, we define

$$(L\phi_{0,1}^k)(\rho, \nu, \sigma) = \sum_{t=1}^{\infty} T_t(\phi_{0,1}^k)(\rho, \nu) \exp(2\pi i \sigma t). \tag{3.4}$$

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Now, $T_t(\phi^k_{0,1})$ is a weak Jacobi form of weight 0 and index t. It then follows as explained in section 2, with the transformation properties (2.11) and (2.12), the combination $T_t(\phi^k_{0,m})(\rho,\nu)\cdot\exp(2\pi i t\sigma)$ is invariant under the group elements $g_1(a,b,c,d)$ and $g_3(\lambda,\mu)$ in (2.6). Thus, each term in the sum in (3.4) and therefor $L\phi$ is also invariant under these two elements.

If $L\phi$ were invariant also under the exchange of p and q then it would be invariant under the element g_2 defined in (2.6) and one would obtain a Siegel modular form of weight zero. This is almost true. To see this, we note that $\exp(L\phi^k_{0,1})$ can be written as an infinite product using the explicit representation of Hecke operators given in appendix (A):

$$\prod_{\substack{l,m,n\in\mathbb{Z}\\ m>0}} (1-(q^n y^l p^m)^{n_s})^{h_s n_s^{-1} c_{s,l}(4mn-l^2)}, \tag{3.5}$$

where $q \equiv \exp(2\pi i\rho), y \equiv \exp(2\pi i\nu), p \equiv \exp(2\pi i\sigma)$ (A.15). In the product presentation (3.5), the coefficients $c_{s,l}(4mn-l^2)$ are manifestly invariant under the exchange of m and n. The product, however, is not quite symmetric because the range of the products in (3.5) is not quite symmetric: m is strictly positive whereas n can be zero. This asymmetry can be remedied by multiplying the product (3.5) by an appropriate function as in [33, 28]. The required function can essentially be determined by inspection to render the final product symmetric in p and q. Following this procedure one then obtains a Siegel modular form as the multiplicative lift of the weak Jacobi form $\phi^k_{0,1}(\rho,\nu)$,

$$\Phi_k(\Omega) = \mathcal{M}[\phi^k_{0,1}] = q^a y^b p^c \prod_{(n,l,m)>0} (1-(q^n y^l p^m)^{n_s})^{h_s n_s^{-1} c_{s,l}(4mn-l^2)}, \tag{3.6}$$

for some integer b and positive integers a, c. Here the notation $(n,l,m)>0$ means that if (i) $m>0, n,l\in\mathbb{Z}$, or (ii) $m=0, n>0, l\in\mathbb{Z}$, or (iii) $m=n=0, l<0$.

It is useful to write the final answer for $\Phi_k(\Omega)$ as follows

$$\Phi_k(\Omega) = p^c H(\rho,\nu)\exp[L\phi^k_{0,1}(\rho,\nu,\sigma)], \tag{3.7}$$

$$\begin{aligned} H(\rho,\nu) = q^a y^b \prod_s \prod_{l,n\geq 1} &(1-(q^n y^l)^{n_s})(1-(q^n y^l)^{n_s})^{n_s^{-1} h_s c_{s,l}(-l^2)} \\ &\times \prod_{n=1}^{\infty}(1-q^{n n_s})^{n_s^{-1} h_s c_{s,l}(0)} \prod_{l<0}^{\infty}(1-y^{l n_s})^{n_s^{-1} h_s c_{s,l}(-l^2)}, \end{aligned} \tag{3.8}$$

in terms of the separate ingredients that go into the construction. This rewriting is more suggestive for the physical interpretation, as we discuss in the next section. Following Gritsenko [34], we refer to the function $H(\rho,\nu)$ as the 'Hodge Anomaly'. The construction thus far is general and applies to the construction of modular forms of weight k which may or may not be obtainable by an additive lift. In many cases however, as in the cases of our interest, it might be be possible to obtain the same modular form by using the two different lifts. To see the relation between the two lifts in such a situation and to illustrate the significance of the Hodge anomaly for our purpose, we next specialize to the case $(2,6)$. We show how to determine the multiplicative seed and the Borcherds product given the specific Φ_6 obtained from the additive lift.

4. Multiplicative lift for Φ_6

We want to determine the seed $\phi^6_{0,1}$ whose multiplicative lift equals Φ_6 constructed from the additive lift of (2.16). From the p expansion of the additive representation of Φ_6 we conclude that the integer c in (3.6) and (3.7) equals one. Then we see from (3.7) that if Φ_6 is to be a weight six Siegel modular form, $H(\rho, \nu)$ must be a weak Jacobi form of weight six and index one. Such a weak Jacobi form is in fact unique and hence must equal the seed $\phi_{6,1}$ that we used for the additive lift. In summary, the Hodge anomaly is given by

$$\begin{aligned} H(\rho,\nu) &= \phi_{6,1}(\rho,\nu) = \eta^2(\rho)\eta^8(2\rho)\theta_1^2(\rho,\nu) \\ &= qy(1-y^{-1})^2\prod_{n=1}^{\infty}(1-q^{2n})^8(1-q^n)^4(1-q^ny)^2(1-q^ny^{-1})^2. \end{aligned} \tag{4.1}$$

Comparing this product representation with (3.8), we determine that

$$c_1(0) = 4, \quad c_1(-1) = 2; \quad c_2(0) = 8, \quad c_2(-1) = 0; \tag{4.2}$$

and moreover $c_1(n) = c_2(n) = 0, \quad \forall\, n < -1$. This information about the leading coefficients $c_s(n)$ obtained from the Hodge anomaly is sufficient to determine completely the multiplicative seed $\phi^6_{0,1}$. Let us assume the seed to be a weak Jacobi form.[3] Now, proposition (6.1) in [28] states that the space of weak Jacobi forms of even weight is generated as linear combinations of two weak forms $\phi_{-2,1}$ and $\phi_{0,1}$ which in turn are given in terms of elementary theta functions by

$$\phi_{-2,1}(\rho,\nu) = \frac{\theta_1^2(\rho,\nu)}{\eta^6(\rho)} \tag{4.3}$$

$$\phi_{0,1}(\rho,\nu) = 4\left[\frac{\theta_2(\rho,\nu)^2}{\theta_2(\rho)^2} + \frac{\theta_3(\rho,\nu)^2}{\theta_3(\rho)^2} + \frac{\theta_4(\rho,\nu)^2}{\theta_4(\rho)^2}\right] \tag{4.4}$$

The coefficients for this linear combination can take values in the ring $A(\Gamma(N))$ of holomorphic modular forms of $\Gamma(N)$. Basically, the coefficients have to be chosen so as to get the correct weight. For our case, with $N = 2$, the relevant holomorphic modular form, is the one of weight two

$$\alpha(\rho) = \theta_3^4(2\rho) + \theta_2^4(2\rho). \tag{4.5}$$

By virtue of the above-mentioned proposition, and using the definitions in (4.3) and (4.5), we can then write our desired seed as the linear combination

$$\phi^6_{0,1}(\rho,\nu) = A\alpha(\rho)\phi_{-2,1}(\rho,\nu) + B\phi_{0,1}(\rho,\nu), \tag{4.6}$$

where A and B are constants. To determine the constants we investigate the behavior near the cusps. For $\Gamma_0(2)$, there are only two cusps, one at $i\infty$ and the other at 0 in the

[3] Strictly, it is enough that it is a 'very weak' Jacobi form as defined in [28] but from the physical interpretation that we give in the next section, we expect and hence assume it to be a weak Jacobi form to find a consistent solution.

fundamental domain which we label by $s = 1, 2$ respectively. Then the various relevant quantities required in the final expression (3.6) are given in our case by

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h_1 = 1, \quad z_1 = 0, \quad n_1 = 1 \tag{4.7}$$

$$g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad h_1 = 2, \quad z_2 = 1, \quad n_2 = 2. \tag{4.8}$$

The q expansion for $\phi_{-2,1}$ and $\phi_{0,1}$ at the cusp $q = 0$ is given by

$$\phi_{-2,1} = (-2 + y + y^{-1}) + q(-12 + 8y + 8y^{-1} - 2y^2 - 2y^{-2}) + \dots \tag{4.9}$$

$$\phi_{0,1} = (10 + y + y^{-1}) + \ \dots . \tag{4.10}$$

The Fourier expansion of $\alpha(\rho)$ at the cusps $i\infty$ and 0 is given by,

$$\alpha(\rho) \ = 1 + 24q + 24q^2 + \dots \tag{4.11}$$

near infinity and by

$$\rho^{-2}\alpha\Big(-1/\rho\Big) \ = -\frac{1}{2}\alpha\Big(\frac{\rho}{2}\Big) \tag{4.12}$$

$$= -\frac{1}{2} + \dots \tag{4.13}$$

near zero. Demanding that the leading terms in the Fourier expansion of the linear combination (4.6) match with those given by (4.2) determines the coefficients $A = 4/3$ and $B = 2/3$ in (4.6). The constraints are actually over-determined so the fact that a solution exists at all gives a check of the procedure. Our final answer for the multiplicative lift is then

$$\phi^6_{0,1}(\rho,\nu) = \frac{4}{3}\alpha(\rho)\phi_{-2,1}(\rho,\nu) + \frac{2}{3}\phi_{0,1}(\rho,\nu). \tag{4.14}$$

With this determination we can simply apply the formalism in the previous section to determine

$$\Phi_6(\Omega) = \mathcal{M}\left[\frac{4}{3}\alpha(\rho)\phi_{-2,1}(\rho,\nu) + \frac{2}{3}\phi_{0,1}(\rho,\nu)\right] \tag{4.15}$$

by using the formula (3.6).

5. Physical interpretation of the multiplicative lift

Both $\exp(-L\phi)$ and the inverse of the Hodge anomaly $H^{-1}(\rho,\nu)$ that appear in the multiplicative lift in (3) have a natural physical interpretation using the 4d-5d lift, which we discuss in this section and also in terms of M-theory lift of string webs which we discuss in the next section.

Let us recall the basic idea behind the 4d-5d lift [22]. Consider Type-IIA compactified on a five-dimensional space $\mathbf{M_5}$ to five dimensions. Given a BPS black hole in Type-IIA string theory in five dimensions, we can obtain a black hole in four dimensions as

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follows. A five-dimensional black hole situated in an asymptotically flat space $\mathbb{R}^4$ can be embedded into an asymptotically Taub-NUT geometry of unit charge. Intuitively, this is possible because near the origin, the Taub-NUT geometry reduces to $\mathbb{R}^4$, so when the Taub-NUT radius is much larger than the black hole radius, the black hole does not see the difference between $\mathbb{R}^4$ and Taub-NUT. Asymptotically, however, the Taub-NUT geometry is $\mathbb{R}^3 \times \mathbf{S^1_{\rm tn}}$. We can dimensionally reduce on the Taub-NUT circle to obtain a four-dimensional compactification. Now, Type-IIA is dual to M-theory compactified on the M-theory circle $\mathbf{S}^1_m$ so we can regard four-dimensional theory as an M-theory compactification on $\mathbf{M_5} \times \mathbf{S^1_{\rm tn}} \times \mathbf{S^1_m}$. Now flipping the two circles, we can choose to regard the Taub-NUT circle $\mathbf{S^1_{\rm tn}}$ as the new M-theory circle. This in turn is dual to a Type-IIA theory but in a different duality frame than the original one. In this duality frame, the Taub-NUT space is just the Kaluza-Klein 6-brane of M-theory dual to the D6-brane. Thus the Taub-NUT charge of the original Type-IIA frame is interpreted in as the D6 brane charge in the new Type-IIA frame and we obtain a BPS state in four dimensions with a D6-brane charge. Since we can go between the two descriptions by smoothly varying various moduli such as the Taub-NUT radius and choosing appropriate duality frames, the spectrum of BPS states is not expected to change. In this way, we relate the spectrum of four-dimensional BPS states with D6-brane charge to five-dimensional BPS states in Type-IIA.

With this physical picture in mind, we now interpret the term $\exp(-L\phi)$ in (3.7) as counting the degeneracies of the five dimensional BPS states that correspond to the four-dimensional BPS states after the 4d-5d lift. For example, in the familiar case $(1, 10)$ of toroidally compactified heterotic string, the dual Type-II theory is compactified on $\mathbf{K3} \times \tilde{\mathbf{S}}^\mathbf{1} \times \mathbf{S^1}$. In the notation of the previous paragraph, we then have $\mathbf{M_5} = \mathbf{K3} \times \tilde{\mathbf{S}}^\mathbf{1}$. The five-dimensional BPS state is described by the D1-D5-P system. Its degeneracies are counted by the elliptic genus of the symmetric product of $\mathbf{K_3}$. In this case, indeed $\exp(-L\phi)$ above gives nothing but the symmetric product elliptic genus evaluated in [20].

In our case $(2, 6)$, D-brane configuration in five dimensions corresponding to our dyonic state in four dimensions is obtained simply by implementing the CHL orbifolding action in the open string sector on the D1-D5-P system in five dimensions. The term $\exp(-L\phi)$ in (3.7) then has a natural interpretation as a symmetric product elliptic genus. Because of the shift in the orbifolding action, the resulting orbifold is a little unusual and the elliptic genus is weak Jacobi form not of $\mathrm{SL}(2, \mathbb{Z})$ but of $\Gamma_0(2)$. The details of the orbifold interpretation will be presented in [23].

The Hodge anomaly plays a special role in the 4d-5d lift. It is naturally interpreted as the contribution of the bound states of momentum and the single Taub-NUT 5-brane in the Type-IIB description. A KK5-brane of IIB wrapping $\mathbf{K_3} \times \mathbf{S^1}$ carrying momentum along the $\mathbf{S^1}$ is T-dual to an NS5-brane of IIA wrapping $\mathbf{K_3} \times \mathbf{S^1}$ carrying momentum which in turn is dual to the heterotic fundamental string wrapping the circle with momentum.[4].

[4]In [7], the Hodge anomaly for the $(1, 10)$ example is interpreted as a single 5-brane contribution. This, however, is not dual to the heterotic F1-P system and would not give the desired form of the Hodge anomaly. For the purposes of 4d-5d lift, it is essential to introduce Taub-NUT geometry which appears like KK5-brane in IIB. In the 5d elliptic genus the bound states of this KK5-brane and momentum are not accounted for. Therefore, the Hodge anomaly is naturally identified as this additional contribution that must be taken

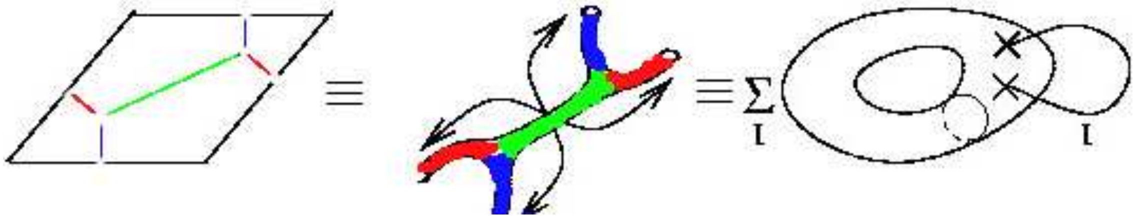

Figure 1: A dyon can be represented as a string web on a torus which in M-theory looks like a genus two Riemann surface. Factorization of the product representation of $1/\Phi_k(\Omega)$ reveals this Riemann surface.

These can be counted in perturbation theory [35, 36, 13, 12] in both cases $(1,10)$ and $(2,6)$. The $y(1-y^{-1})$ term in the Hodge anomaly in (4.1) is more subtle and would require a more detailed analysis.

6. M-theory lift of string webs

The appearance in the dyon counting formulae of objects related to a genus two Riemann surface such as the period matrix and the $G_0(N)$ subgroups of $\mathrm{Sp}(2,\mathbb{Z})$ is quite surprising and demands a deeper physical explanation. We now offer such an explanation combining earlier work of [37] and [22] in the toroidal $(1,10)$ case and generalizing it to CHL orbifolds.

To start with, let us reinterpret the Hodge anomaly following Kawai [37]. It can be written as

$$H(\rho,\nu) = \eta^8(\rho)\eta^8(2\rho)\frac{\theta_1^2(\rho,\nu)}{\eta^6(\rho)} = Z(\rho)K^2(\rho,\nu), \tag{6.1}$$

where $Z(\rho) \equiv \eta^8(\rho)\eta^8(2\rho)$ is the one-loop partition function of the left-moving chiral 24-dimensional bosonic string with the $\mathbb{Z}_2$ twist α of the CHL orbifold action, and $K(\rho,\tau)$ is the prime form on the torus. Let us also expand

$$\exp(-L\phi^6_{0,1}(\rho,\nu)) = \sum_{N=0}^{\infty} p^N \chi_N \tag{6.2}$$

We can then write from (3.7),

$$\frac{1}{\Phi_6(\Omega)} = \frac{1}{p}\frac{1}{H(\rho,\nu)}\exp(-L\phi^6_{0,1}(\rho,\nu)) \tag{6.3}$$

$$= \sum_{N=0}^{\infty} p^{N-1}\frac{1}{K(\rho,\nu)^2}\chi_N \tag{6.4}$$

$$\sim \frac{1}{p}\frac{1}{K(\rho,\nu)^2}\frac{1}{Z(\rho)} + \dots \tag{6.5}$$

into account.

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In (6.5) above, we can identify $K^{-2}(\rho,\nu)$ as the on-shell (chiral) tachyon propagator, and $Z(\rho)$ as the one-loop left-moving partition function. If we denote by X the bosonic spacetime coordinate, then we have

$$< e^{ik\cdot X}(\nu)e^{-ik\cdot X}(0) >= K^{-2}(\rho,\nu), \tag{6.6}$$

where k is the momentum of an on-shell tachyon and the correlator is evaluated on a genus one Riemann surface with complex structure ρ. This is exactly the first term in the factorized expansion in figure (1). The subleading terms at higher N denoted by $\ldots$ in (6.5) come from contributions of string states at higher mass-level $N-1$. Summing over all states gives the genus two partition function.

This reinterpretation of $1/\Phi_6$ as the two-loop partition function of the chiral bosonic string explains at a mathematical level the appearance of genus two Riemann surface generalizing the results of Kawai to the $(2,6)$ case. Note that the partition function $Z(\rho)$ will be different in the two cases. In the (1, 10) case it equals $\eta^{-24}(\rho)$ and in the $(2,6)$ case it equals $\eta^{-8}(\rho)\eta^{-8}(2\rho)$. This precisely captures the effect of CHL orbifolding on the chiral left moving bosons of the heterotic string. To describe the $N=2$ orbifold action let us consider the $E_8\times E_8$ heterotic string. The orbifold twist α then simply flips the two E_8 factors. We can compute the partition function with a twist in the time direction $\mathrm{Tr}(\alpha q^H)$ where H is the left-moving bosonic Hamiltonian. Then, the eight light-cone bosons will contribute $\eta^{-8}(\rho)$ as usual to the trace, but the sixteen bosons of the internal $E_8\times E_8$ torus will contribute $\eta^{-8}(2\rho)$ instead of $\eta^{-16}(2\rho)$. The power changes from -16 to -8 because eight bosons of the first E_8 factor are mapped by α to the eight bosons of the second E_8. Thus only those states that have equal number of oscillators from the two E_8 factors contribute to the trace, thereby reducing effectively the number of oscillators to 8. The argument on the other hand is doubled to 2ρ because when equal number of oscillators from the two factors are present, the worldsheet energy is effectively doubled. The tachyon propagator in the two cases is unchanged because only light-cone bosons appear on shell which are not affected by the orbifolding.

This mathematical rewriting does not explain at a fundamental level why the chiral bosonic string has anything to do with dyon counting. This connection can be completed using with the heuristic picture suggested in [4].

Under string-string duality, the $\mathrm{SL}(2,\mathbb{Z})$ S-duality group of the heterotic string gets mapped to the geometric $\mathrm{SL}(2,\mathbb{Z})$ of the Type-IIB string [38, 39]. Thus, electric states correspond to branes wrapping the a cycle of the torus and magnetic states correspond to branes wrapping the b cycle of the torus. A general dyon with electric and magnetic charges (n_e,n_m) of a given $\mathrm{U}(1)$ symmetry is then represented as a brane wrapping (n_e,n_m) cycle of the torus. If a state is charged under more than one $\mathrm{U}(1)$ fields then one gets instead a (p,q) string web with different winding numbers along the a and the b cycles. The angles and lengths of the web are fixed by energetic considerations for a given charge assignment [40, 41]. For our purpose, we can consider D5 and NS5 branes wrapping the **K3** resulting in two different kinds of $(1,0)$ and $(0,1)$ strings. A dyon in a particular duality frame then looks like the string web made of these strings as in the first diagram figure (1). In the M-theory lift of this diagram, both D5 and NS5 branes correspond to M5 branes so the string in the web arises from M-theory brane wrapping **K3**. To count states, we require a partition function with Euclidean time. Adding the circle direction of time we can fatten the string web diagram which looks like a particle Feynman diagram into a genus-two Riemann surface representing a closed-string Feynman diagram as in the second diagram in figure (1). Now, **K3**-wrapped M5 brane is nothing but the heterotic string. Furthermore, since we are counting BPS states by an elliptic genus, the right-movers are in the ground state and we are left with the two-loop partition function of the bosonic string. This partition function is what we have constructed by in the third diagram in figure (1) as explained above using factorization.

7. Conclusions

The exact spectrum of dyons in four dimensions and of spinning black holes in five dimensions in CHL compactifications can be determined using a Borcherds product representation of level N Siegel modular forms of $\mathrm{Sp}(2, \mathbb{Z})$. Various elements in the Borcherds product have a natural interpretation from the perspective of 4d-5d lift. The Hodge anomaly is identified the contribution of bound states of Type-IIB KK5-brane with momentum. The remaining piece is interpreted as arising from the symmetric product of the orbifolded D1-D5-P system. The appearance of an underlying chiral bosonic string on a genus two Riemann surface in this construction has a natural interpretation as the Euclidean world-sheet of the **K3** wrapped M5 brane on a string web in orbifolded theory. By factorization, this connection with the Siegel modular form can be made precise.

We have seen that a very rich and interesting mathematical structure underlies the counting of BPS dyons and black holes. Given the relation of Siegel modular forms to Generalized Kac-Moody algebras [30, 42, 33, 43], their appearance in the counting is perhaps indicative of a larger underlying symmetry of string theory. If so, investigating this structure further might prove to be a fruitful avenue towards uncovering the full structure of M-theory.

Note: During the course of writing this paper, a related paper appeared [44] with some overlap with our work where the product representation is derived using yet another lift called the 'Theta Lift'.

Acknowledgments

We would like to thank Eknath Ghate, Dileep Jatkar, Arvind Nair, Dipendra Prasad, Ashoke Sen, and T. Venkataramana for useful discussions. We are especially grateful to Davide Gaiotto for explaining his paper and to Rainer Schulze-Pillot for explaining some elements of the Borcherds lift.

A. Hecke operators and the multiplicative lift

In this section we summarise the construction of Hecke operators and the multiplicative

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lift, following [28]. Let us define $\Delta_N(t)$ as

$$\Delta_N(t) = \left\{ g = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{Z}, \quad \det(g) = t \right\}. \tag{A.1}$$

The action of the Hecke operator T_t on a weak Jacobi form $\phi_{k,m}$ is then given by

$$T_t(\phi_{k,m})(\tau, z) = t^{k-1} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\backslash\Delta_N(t)} (c\tau + d)^{-k} \exp\left(- \frac{2\pi i m c z^2}{c\tau + d} \right) \phi_{k,m}\left(\frac{a\tau + b}{d}, az \right). \tag{A.2}$$

To compute everything concretely, we need to define representatives of $\Gamma_0(N)\backslash\Delta_N(t)$. Choose the complete set of cusps $\{s\}$ of $\Gamma_0(N)$ represented by the set of representative matrices $\{g_s\}$. Let

$$g_s \,\epsilon\, \mathrm{SL}(2, \mathbb{Z}) = \begin{pmatrix} x_s & y_s \\ z_s & w_s \end{pmatrix} \tag{A.3}$$

Define a natural number h_s by

$$g_s^{-1} \Gamma_0(N) g_s \cap P(\mathbb{Z}) = \left\{ \pm \begin{pmatrix} 1 & h_s n \\ 0 & 1 \end{pmatrix}; n \in \mathbb{Z} \right\} \tag{A.4}$$

where $P(\mathbb{Z})$ is the set of all upper-triangular matrices over integers with unit determinant. We can then write

$$\Gamma_0(N)\backslash\Delta_N(t) = \cup_s \left\{ g_s \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; a, b, d \in \mathbb{Z}, ad = t, \, a z_s = 0 \, mod \, N, \, b = 0, \ldots, h_s d - 1 \right\}. \tag{A.5}$$

For each cusp we define $n_s = \frac{N}{\text{g.c.d}(z_s, N)}$. We define

$$\phi_s(\tau, z) = \phi\left(\frac{x_s \tau + y_s}{z_s \tau + \omega_s}, \frac{z}{z_s \tau + \omega_s} \right), \tag{A.6}$$

with Fourier expansion

$$\phi_s(\tau, z) = \sum_{n,l} c_s(n, l) \exp(2\pi i(n\tau + lz)). \tag{A.7}$$

As usual, one can show that $c_s(n, l)$ depends only on $4n - l^2$ and $l \, mod \, 2$ so we write $c_s(n, l) = c_{s,l}(4n - l^2)$ following the notation in [28]. In general $n \in h_s^{-1}\mathbb{Z}$ need not be an integer. If 4 does not divide h_s, which is true for all cases of our interest, then $l \, mod \, 2$ is determined only by $4n - l^2$ and in that case we can write simply $c_s(4n - l^2) = c_{s,l}(4n - l^2)$.

For $\Gamma_0(N)$ with N prime, there are only two cusps, one at $i\infty$ and the other at 0 in the fundamental domain. Hence the index s runs over 1 and 2. For this case, various objects with the subscript s defined in the formula for the lift above take the following values:

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h_1 = 1, \quad z_1 = 0, \quad n_1 = 1 \tag{A.8}$$

$$g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad h_2 = N, \quad z_2 = 1, \quad n_2 = N \tag{A.9}$$

In this case we can then write

$$\Gamma_0(N)\backslash\Delta_N(t) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL(2,\mathbb{Z}); ad = t, b = 0,\ldots,d-1 \right\} \tag{A.10}$$
$$\cup \left\{ g_2 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL(2,\mathbb{Z}); ad = t, a \equiv 0\, mod\, N, b = 0,\ldots,Nd-1 \right\}.$$

Given a weak Jacobi form ϕ of weight 0 and index 1, we can define

$$L\phi(\rho,\nu,\sigma) = \sum_{t=1}^{\infty} T_t(\phi)(\rho,\nu)\exp(2\pi i\sigma t). \tag{A.11}$$

Using the explicit representation of the Hecke operators, one can then show [28]

$$\begin{aligned} L\phi &= \sum_s \sum_{t=1}^{\infty} \sum_{\substack{ad=t \\ az_s=0\, mod\, N}} \sum_{b=0}^{h_s d-1} \phi_s\left(\frac{a\rho+b}{d}, a\nu\right)\exp(2\pi i t\sigma) & \text{(A.12)} \\ &= \sum_s \sum_{t=1}^{\infty} \sum_{\substack{ad=t \\ a\in n_s\mathbb{Z}}} (ad)^{-1} d h_s \sum_{n,l\in\mathbb{Z}} c_{s,l}(4nd-l^2)\exp(2\pi i(an\rho+al\nu+t\sigma)) & \text{(A.13)} \\ &= \sum_s h_s \sum_{a=1}^{\infty} \frac{1}{an_s} \sum_{m=1}^{\infty} \sum_{n,l\in\mathbb{Z}} c_{s,l}(4nd-l^2)\exp(2\pi i(an\rho+al\nu+m\sigma)) & \text{(A.14)} \\ &= \sum_s \frac{h_s}{n_s}\log\left(\prod_{\substack{l,m,n\,\in\mathbb{Z} \\ m\geq 1}} (1-e^{n_s(n\rho+l\nu+m\sigma)})^{c_{s,l}(4mn-l^2)}\right). & \text{(A.15)} \end{aligned}$$

B. Consistency check

As a consistency check we compare the coefficients of the leading powers of p,q,y in the multiplicative lift with the Fourier expansion of Φ_6 obtained using the additive lift in [17]. The leading terms, corresponding to a single power of p, in the expansion are

$$-pqy\prod_n(1-q^n)^{c_1(0)}(1-q^n y)^{c_1(-1)}(1-q^n y^{-1})^{c_1(-1)}(1-q^{2n})^{c_2(0)}. \tag{B.1}$$

Substituting the values of the c_1 and c_2 coefficients and collecting terms with the same powers in q and y together, we obtain

$$\Phi_6(\Omega) = \left[\left(2-y-\frac{1}{y}\right)q+\left(-4+\frac{2}{y^2}\right)q^2+\left(-16-\frac{1}{y^3}\right)-\frac{4}{y^2}+\frac{13}{y}+13y-4y^2-y^3\right)q^3\right]p+\ldots \tag{B.2}$$

To compare, we now read off the coefficients from its sum representation derived in [17] by the additive lift. The seed for the additive lift is

$$\phi_{6,1} = \eta^2(\tau)\eta^8(2\tau)\theta_1^2 = \sum_{l,n\geq 0} C(4n-l^2)q^n y^l \tag{B.3}$$

The lift is then given by

$$\Phi_6(\Omega) = \sum_{m\geq 1} T_m[\phi_{6,1}(\rho,\nu)]p^m, \tag{B.4}$$

with the Fourier expansion

$$\Phi_6(\Omega) = \sum_{\substack{m>0,\\ n\geq 0, r\in\mathbb{Z}}} a(n,m,r)q^n p^m y^r. \tag{B.5}$$

Given the action of the Hecke operators, $a(n,m,r)$ can be read off from this expansion knowing $C(N)$ as in (B.3). These are in precise agreement with the same coefficients in the expansion of the product representation given above in (B.2).

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