

Kinetic exchange models for income and wealth distributions

A. Chatterjee^{1,a} and B.K. Chakrabarti^{1,2,b}

¹ Theoretical Condensed Matter Physics Division and Centre for Applied Mathematics and Computational Science, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata 700064, India

² Economic Research Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108, India

Received 11 September 2007

Published online 8 December 2007 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2007

Abstract. Increasingly, a huge amount of statistics have been gathered which clearly indicates that income and wealth distributions in various countries or societies follow a robust pattern, close to the Gibbs distribution of energy in an ideal gas in equilibrium. However, it also deviates in the low income and more significantly for the high income ranges. Application of physics models provides illuminating ideas and understanding, complementing the observations.

PACS. 89.20.Hh World Wide Web, Internet – 89.75.Hc Networks and genealogical trees – 89.75.Da Systems obeying scaling laws – 43.38.Si Telephones, earphones, sound power telephones, and intercommunication systems

1 Introduction

In any society or country, if one can isolate its count on people and their money or wealth, one finds that while the total money or wealth remains fairly constant on a relatively longer time scale, its movement from individual to individual is not so due to its dynamics at shorter time scales (daily or weekly). Eventually, on overall average for the society or country, there appears very robust money or wealth distributions. Empirical data for society show a small variation in the value of the power-law exponent that characterises the ‘tail’ of the distribution, while it equals to unity for firms. Locally, of course, there appear many ‘obvious reasons’ for such uneven distribution of wealth or income within the societies. However, such ‘reasons’ seem to be very ineffective if the global robust structure of the income and wealth distribution in various societies is considered. Statistical physics based models for such distributions seem to succeed in capturing the essential ‘reasons’ for such universal aspects of the distributions.

Here, we review the empirical basis for considering kinetic exchange models for income and wealth distributions in Section 2. We then discuss the gas like models in Section 3 and give the details of their numerical analyses in Section 4, while in Section 5 we review the analytical studies done so far on these models. Section 6 discusses other

model studies, including an annealed savings model and a model with a non-consumable commodity. Finally, we end with discussions in Section 7.

2 Empirical studies of income and wealth distributions

The distribution of wealth among individuals in an economy has been an important area of research in economics, for more than a hundred years [1–4]. The same is true for income distribution in any society. Detailed analysis of the income distribution [3,4] so far indicate

$$P(m) \sim \begin{cases} m^\alpha \exp(-m/T) & \text{for } m < m_c, \\ m^{-(1+\nu)} & \text{for } m \geq m_c, \end{cases} \quad (1)$$

where P denotes the number density of people with income or wealth m and α , ν denote exponents and T denotes a scaling factor. The power law in income and wealth distribution (for $m \geq m_c$) is named after Pareto and the exponent ν is called the Pareto exponent. A historical account of Pareto’s data and that from recent sources can be found in reference [5]. The crossover point (m_c) is extracted from the crossover from the Gamma distribution form to the power law tail. One often fits the region below m_c to a log-normal form $\log P(m) \propto -(\log m)^2$. Although this form is often preferred by economists, we think that the other Gamma distribution form equation (1) fits better with the data, because of the remarkable fit with the Gibbs distribution in reference [6]. We consider that in the

^a *Present Address:* Condensed Matter and Statistical Physics Section, The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, Trieste 34014, Italy.

^b e-mail: bikask.chakrabarti@saha.ac.in

following discussion. This robust feature of $P(m)$ seems to be very well established for the analysis of the enormous amount of data available today (See Fig. 1). We consider this distribution (in view of its stability and universality) to be an ‘equilibrium’ (in the thermodynamic sense) distribution in a many-body (interacting, statistical) system like a gas, where the Gibbs distribution are established for more than 100 years. This paper reviews the various attempts in this direction.

Although Pareto [1] and Gini [10] had respectively identified the power-law tail and the log-normal bulk of the income distribution, the demonstration of both features in the same distribution was possibly first demonstrated by Montroll and Shlesinger [11] through an analysis of fine-scale income data obtained from the US Internal Revenue Service (IRS) for the year 1935–36. It was observed that while the top 2–3% of the population (in terms of income) followed a power law with Pareto exponent $\nu \simeq 1.63$; the rest followed a lognormal distribution. Later work on Japanese personal income data based on detailed records obtained from the Japanese National Tax Administration indicated that the tail of the distribution followed a power law value that fluctuated from year to year around the mean value of 2 [12]. Further work [13] showed that the power law region described the top 10% or less of the population (in terms of income), while the remaining income distribution was well-described by the log-normal form. While the value fluctuated significantly from year to year, it was observed that the parameter describing the log-normal bulk, the Gibrat index [14], remained relatively unchanged. The change of income from year to year, i.e., the growth rate as measured by the log ratio of the income tax paid in successive years, was observed by Fujiwara et al. [15] to be also a heavy tailed distribution, although skewed, and centered about zero. Later work on the US income distribution based on data from IRS for the years 1997–1998, while still indicating a power-law tail (with $\nu \simeq 1.7$), have suggested that the lower 95% of the population have income whose distribution may be better described by an exponential form [8,16]. The same observation has been made for income distribution in the UK for the years 1994–1999, where the value was found to vary between 2.0 and 2.3, but the bulk seemed to be well-described by an exponential decay.

It is interesting to note that, when one shifts attention from the income of individuals to the income of companies, one still observes the power law tail. A study of the income distribution of Japanese firms [17] concluded that it follows a power law with $\nu \simeq 1$, which is also often referred to as the Zipf’s law. Similar observation has been reported for the income distribution of US companies [18].

Compared to the empirical work done on income distribution, relatively few studies have looked at the distribution of wealth, which consist of the net value of assets (financial holdings and/or tangible items) owned at a given point in time. The lack of an easily available data source for measuring wealth, analogous to income tax returns for measuring income, means that one has to resort to indirect methods. Levy and Solomon [19] used a

published list of wealthiest people to generate a rank-order distribution, from which they inferred the Pareto exponent for wealth distribution in USA. References [16] and [20] used an alternative technique based on adjusted data reported for the purpose of inheritance tax to obtain the Pareto exponent for UK. Another study used tangible asset (namely house area) as a measure of wealth to obtain the wealth distribution exponent in ancient Egyptian society during the reign of Akhenaten (14th century BC) [21]. The wealth distribution in Hungarian medieval society has also been seen to follow a Pareto law [22]. More recently, the wealth distribution in India at present was also observed to follow a power law tail with the exponent varying around 0.9 [23]. The general feature observed in the limited empirical study of wealth distribution is that of a power law behavior for the wealthiest 5–10% of the population, and exponential or log-normal distribution for the rest of the population. The Pareto exponent as measured from the wealth distribution is found to be always lower than the exponent for the income distribution, which is consistent with the general observation that, in market economies, wealth is much more unequally distributed than income [24].

The striking regularities (see Fig. 1) observed in the income distribution for different countries, have led to several new attempts at explaining them on theoretical grounds. Much of the current impetus is from physicists’ modelling of economic behavior in analogy with large systems of interacting particles, as treated, e.g., in the kinetic theory of gases. According to physicists working on this problem, the regular patterns observed in the income (and wealth) distribution may be indicative of a natural law for the statistical properties of a many-body dynamical system representing the entire set of economic interactions in a society, analogous to those previously derived for gases and liquids. By viewing the economy as a thermodynamic system, one can identify the income distribution with the distribution of energy among the particles in a gas. In particular, a class of kinetic exchange models have provided a simple mechanism for understanding the unequal accumulation of assets. Many of these models, while simple from the perspective of economics, has the benefit of coming to grips with the key factor in socioeconomic interactions that results in very different societies converging to similar forms of unequal distribution of resources (see Refs. [3,4], which consists of a collection of large number of technical papers in this field; see also [25–30] for some popular discussions and criticisms as well as appreciations).

Considerable investigations with real data during the last ten years revealed that the tail of the income distribution indeed follows the above mentioned behavior and the value of the Pareto exponent ν is generally seen to vary between 1 and 3 [16,19,23,31–36]. It is also known that typically less than 10% of the population in any country possesses about 40% of the total wealth of that country and they follow the above law. The rest of the low income population, follow a different distribution which is debated to be either Gibbs [16,19,32,37–39] or log-normal [33–35].

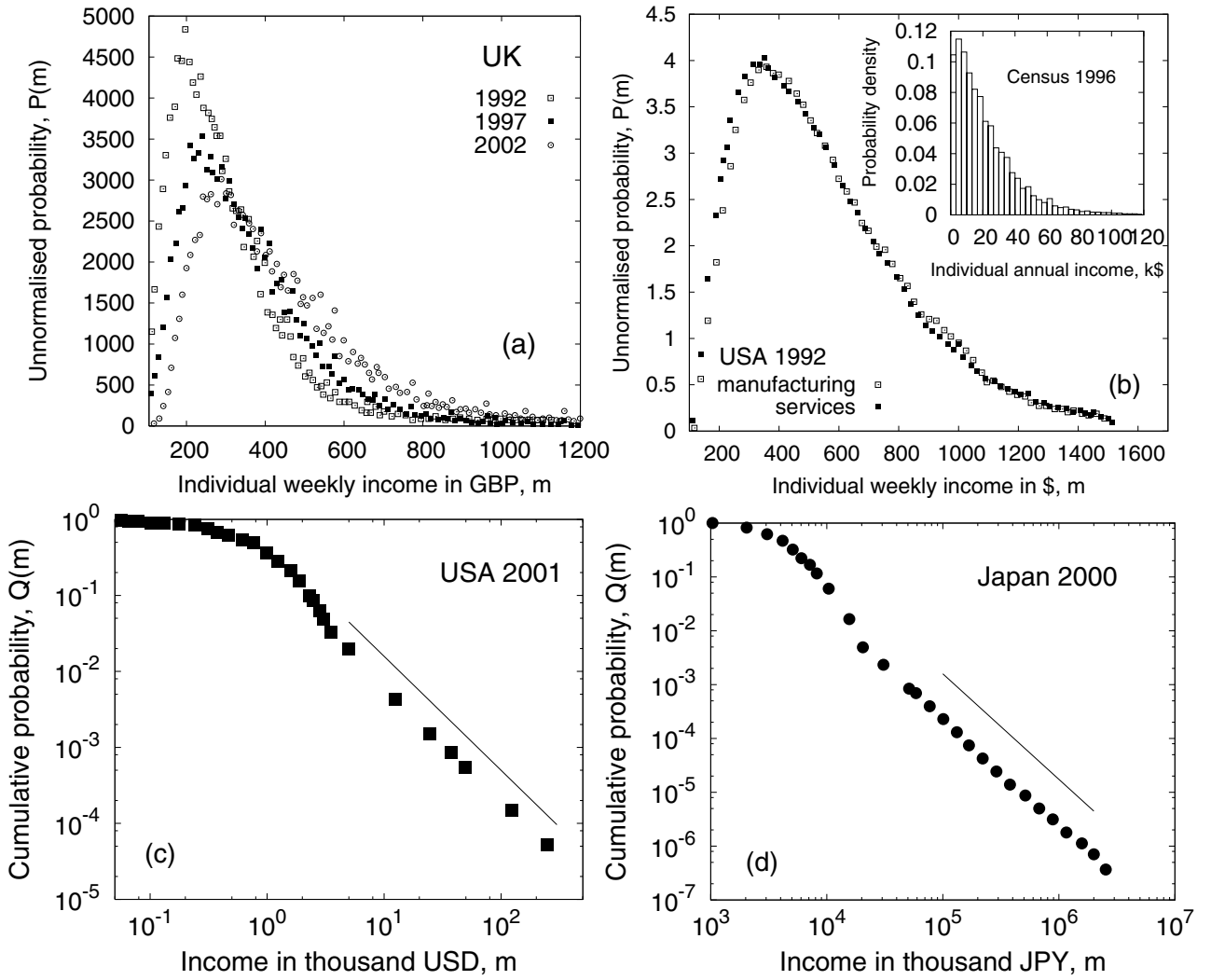


Fig. 1. (a) Distribution $P(m)$ of individual weekly income in UK for 1992, 1997 and 2002; data adapted from reference [7]. (b) Distribution $P(m)$ of individual weekly income for manufacturing and service sectors in USA for 1992; data for US Statistical survey, taken from reference [7]. The inset shows the probability distribution of individual annual income, from US census data of 1996. The data is adapted from reference [8]. (c) Cumulative probability $Q(m) = \int_m^\infty P(m)dm$ of rescaled adjusted gross personal annual income in US for IRS data from 2001 (adapted from Ref. [6]), with Pareto exponent $\nu \approx 1.5$ (given by the slope of the solid line). (d) Cumulative probability distribution of Japanese personal income in the year 2000 (data adapted from Ref. [9]). The power law (Pareto) region approximately fits to $\nu = 1.96$.

3 Gas-like models

In 1960, Mandelbrot wrote “There is a great temptation to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physical shocks between molecules. In the loosest possible terms, both kinds of interactions *should* lead to *similar* states of equilibrium. That is, one *should* be able to explain the law of income distribution by a model similar to that used in statistical thermodynamics: many authors have done so explicitly, and all the others of whom we know have done so implicitly.” [2]. However, Mandelbrot does not provide any references to this bulk of material! Here, we discuss the recent literature and the developments.

In analogy to two-particle collisions with a resulting change in their individual kinetic energy (or momenta), income exchange models may be based on two-agent interactions. Here two randomly selected agents exchange money by some pre-defined mechanism. Assuming the exchange process does not depend on previous exchanges, the dynamics follows a Markovian process:

$$\begin{pmatrix} m_i(t+1) \\ m_j(t+1) \end{pmatrix} = \mathcal{M} \begin{pmatrix} m_i(t) \\ m_j(t) \end{pmatrix} \quad (2)$$

where $m_i(t)$ is the income of agent i at time t and the collision matrix \mathcal{M} defines the exchange mechanism.

In this class of models, one considers a closed economic system where total money M and total number of agents N is fixed. This corresponds to a situation where no

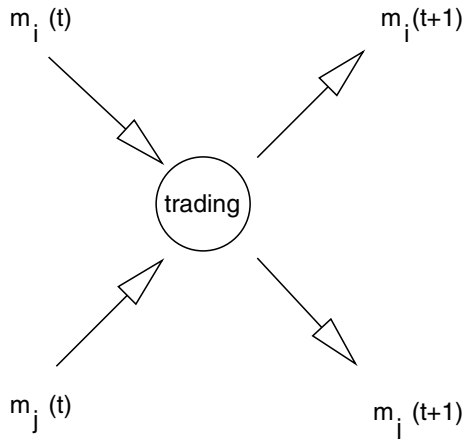


Fig. 2. Schematic diagram of the trading process. Agents i and j redistribute their money in the market: $m_i(t)$ and $m_j(t)$, their respective money before trading, changes over to $m_i(t+1)$ and $m_j(t+1)$ after trading.

production or migration occurs and the only economic activity is confined to trading. Each agent i , individual or corporate, possesses money $m_i(t)$ at time t . In any trading, a pair of traders i and j exchange their money [37–40], such that their total money is (locally) conserved (Fig. 2) and none end up with negative money ($m_i(t) \geq 0$, i.e., debt not allowed):

$$m_i(t+1) = m_i(t) + \Delta m; \quad m_j(t+1) = m_j(t) - \Delta m \quad (3)$$

following local conservation:

$$m_i(t) + m_j(t) = m_i(t+1) + m_j(t+1); \quad (4)$$

time (t) changes by one unit after each trading.

3.1 Model A: Without any savings

The simplest model considers a random fraction of total money to be shared [39]:

$$\Delta m = \epsilon_{ij}[m_i(t) + m_j(t)] - m_i(t), \quad (5)$$

where ϵ_{ij} is a random fraction ($0 \leq \epsilon_{ij} \leq 1$) changing with time or trading. The steady-state ($t \rightarrow \infty$) distribution of money is Gibbs one:

$$P(m) = (1/T) \exp(-m/T); \quad T = M/N. \quad (6)$$

Hence, no matter how uniform or justified the initial distribution is, the eventual steady state correspond to Gibbs a distribution where most of the people have got very little money. This follows from the conservation of money and additivity of entropy:

$$P(m_1)P(m_2) = P(m_1 + m_2). \quad (7)$$

This steady state result is quite robust and realistic too! In fact, several variations of the trading, and of the ‘lattice’ (on which the agents can be put and each agent trade with

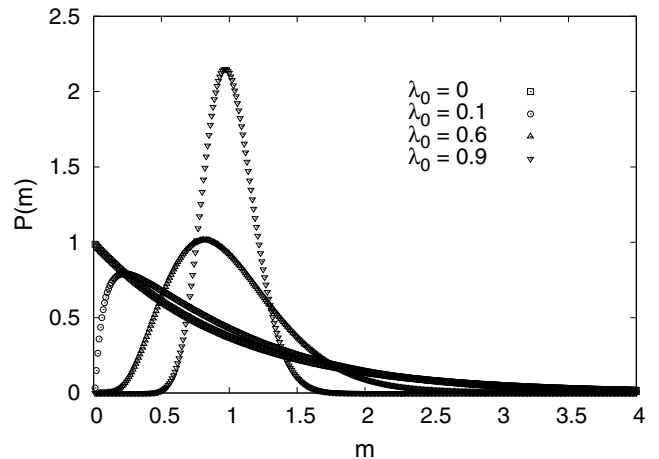


Fig. 3. Steady state money distribution $P(m)$ for the model with uniform savings. The data shown are for different values of λ : 0, 0.1, 0.6, 0.9 for a system size $N = 100$. All data sets shown are for average money per agent $M/N = 1$.

its ‘lattice neighbors’ only), whether compact, fractal or small-world like [31], leaves the distribution unchanged. Some other variations like random sharing of an amount $2m_2$ only (not of $m_1 + m_2$) when $m_1 > m_2$ (trading at the level of lower economic class in the trade), lead even to a drastic situation: all the money in the market drifts to one agent and the rest become truly pauper [41, 25].

3.2 Model B: With uniform savings

In any trading, savings come naturally [24]. A saving propensity factor λ was therefore introduced in the random exchange model [40] (see [39] for model without savings), where each trader at time t saves a fraction λ of its money $m_i(t)$ and trades randomly with the rest:

$$m_i(t+1) = \lambda m_i(t) + \epsilon_{ij} [(1-\lambda)(m_i(t) + m_j(t))], \quad (8)$$

$$m_j(t+1) = \lambda m_j(t) + (1-\epsilon_{ij}) [(1-\lambda)(m_i(t) + m_j(t))], \quad (9)$$

where

$$\Delta m = (1-\lambda)[\epsilon_{ij}\{m_i(t) + m_j(t)\} - m_i(t)], \quad (10)$$

ϵ_{ij} being a random fraction, coming from the stochastic nature of the trading.

The market (non-interacting at $\lambda = 0$ and 1) becomes ‘interacting’ for any non-vanishing λ (< 1): for fixed λ (same for all agents), the steady state distribution $P(m)$ of money is exponentially decaying on both sides with the most-probable money per agent shifting away from $m = 0$ (for $\lambda = 0$) to M/N as $\lambda \rightarrow 1$ (Fig. 3). This self-organizing feature of the market, induced by sheer self-interest of saving by each agent without any global perspective, is quite significant as the fraction of paupers decrease with saving fraction λ and most people end up with some finite fraction of the average money in the market (for $\lambda \rightarrow 1$, the socialists’ dream is achieved with just people’s self-interest

of saving!). Interestingly, self-organisation also occurs in such market models when there is restriction in the commodity market [42]. Although this fixed saving propensity does not give yet the Pareto-like power-law distribution, the Markovian nature of the scattering or trading processes (Eq. (7)) is effectively lost. Indirectly through λ , the agents get to know (start interacting with) each other and the system co-operatively self-organises towards a stable form with a non-vanishing most-probable income (see Fig. 3).

Patriarca et al. [43] claimed through heuristic arguments (based on numerical results) that the distribution is a close approximate form of the Gamma distribution

$$P(m) = Cm^\alpha \exp[-m/T] \quad (11)$$

where $T = 1/(\alpha + 1)$ and $C = (\alpha + 1)^{\alpha+1}/\Gamma(\alpha + 1)$, Γ being the Gamma function whose argument α is related to the savings factor λ as:

$$\alpha = \frac{3\lambda}{1-\lambda}. \quad (12)$$

When compared with equation (6) for $\lambda = 0$ limit, it is to be noted that $M/N = 1$ here. Also, when compared with equation (1), $m_c \rightarrow \infty$. The qualitative argument forwarded here [43] is that, as λ increases, effectively the agents (particles) retain more of its money (energy) in any trading (scattering). This can be taken as implying that with increasing λ , the effective dimensionality increases and temperature of the scattering process changes [43].

This result has also been supported by numerical results in reference [44]. However, a later study [45,46] analyzed the moments, and found that moments up to the third order agree with those obtained from the form of the equation (12), and discrepancies start from fourth order onwards. Hence, the actual form of the distribution for this model still remains to be found out.

It seems that a very similar model was proposed by Angle [47–49] several years back in sociology journals. Angle’s ‘One Parameter Inequality Process’ model is described by the equations:

$$\begin{aligned} m_i(t+1) &= m_i(t) + D_t w m_j(t) - (1 - D_t) w m_i(t) \\ m_j(t+1) &= m_j(t) + (1 - D_t) w m_i(t) - D_t w m_j(t) \end{aligned} \quad (13)$$

where w is a fixed fraction and D_t takes value 0 or 1 randomly. The numerical simulation results of Angle’s model fit well to Gamma distributions.

In the gas like models with uniform savings, the distribution of wealth shows a self organizing feature. A peaked distribution with a most-probable value indicates an economic scale. Empirical observations in homogeneous groups of individuals as in waged income of factory laborers in UK and USA [7] and data from population survey in USA among students of different school and colleges produce similar distributions [49]. This is a simple case where a homogeneous population (say, characterised by a unique value of λ) has been identified.

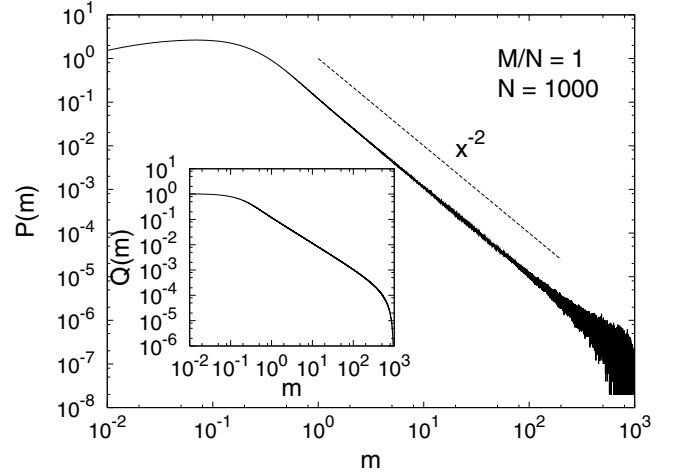


Fig. 4. Steady state money distribution $P(m)$ for the distributed λ model with $0 \leq \lambda < 1$ for a system of $N = 1000$ agents. The x^{-2} is a guide to the observed power-law, with $1 + \nu = 2$. Here, the average money per agent $M/N = 1$.

3.3 Model C: With distributed savings

In a real society or economy, the interest of saving varies from person to person, which implies that λ is a very inhomogeneous parameter. To imitate this situation, we move a step closer to the real situation where saving factor λ is widely distributed within the population [50–52]. The evolution of money in such a trading can be written as:

$$m_i(t+1) = \lambda_i m_i(t) + \epsilon_{ij} [(1 - \lambda_i) m_i(t) + (1 - \lambda_j) m_j(t)], \quad (14)$$

$$m_j(t+1) = \lambda_j m_j(t) + (1 - \epsilon_{ij}) [(1 - \lambda_i) m_i(t) + (1 - \lambda_j) m_j(t)]. \quad (15)$$

The trading rules are same as before, except that

$$\Delta m = \epsilon_{ij} (1 - \lambda_j) m_j(t) - (1 - \lambda_i) (1 - \epsilon_{ij}) m_i(t) \quad (16)$$

here; where λ_i and λ_j are the saving propensities of agents i and j . The agents have fixed (over time) saving propensities, distributed independently, randomly and uniformly (white) within an interval 0 to 1: agent i saves a random fraction λ_i ($0 \leq \lambda_i < 1$) and this λ_i value is quenched for each agent (λ_i are independent of trading or t). Studies show that for uniformly distributed saving propensities, $\rho(\lambda) = 1$ for $0 \leq \lambda < 1$, one gets eventually $P(m) \sim m^{(1+\nu)}$, with $\nu = 1$ (see Fig. 4). The eventual deviation from the power law in $Q(m)$ in the inset of Figure 4 is due to the exponential cutoff contributed by the rare statistics for high m value.

4 Numerical analysis of models A, B and C

Starting with an arbitrary initial (uniform or random) distribution of money among the agents, the market evolves with the trading. At each time, two agents are randomly

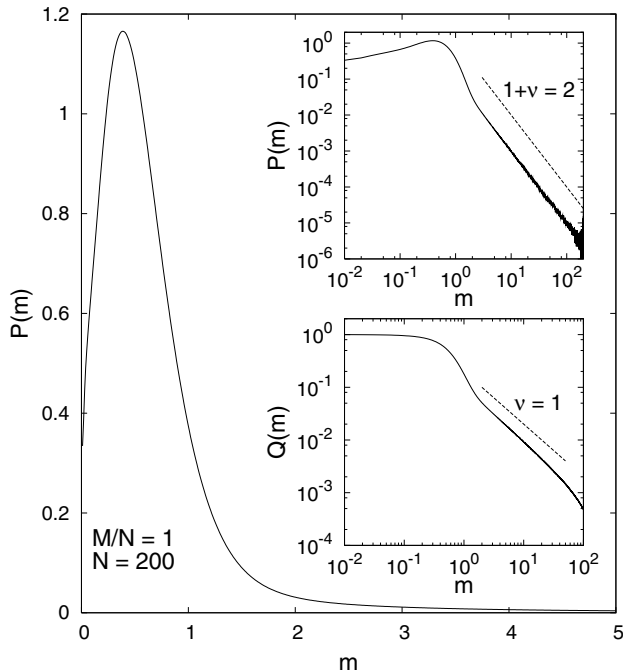


Fig. 5. Steady state money distribution $P(m)$ for a model with $f = 0.6$ fraction of agents with a uniform saving propensity $\lambda_1 = 0.6$ and the rest $1 - f$ fraction having random uniformly distributed (quenched) savings, in $0 \leq \lambda < 1$ for a system of $N = 200$ agents. Here, the average money per agent $M/N = 1$. The top inset shows $P(m)$ in log-log scale for the full range, while the bottom inset shows the cumulative distribution $Q(m)$. In addition to the power law tail in $P(m)$ and $Q(m)$ (as in the basic, distributed savings model), $Q(m)$ resembles a behavior similar to observed in empirical data (see Fig. 1).

selected and the money exchange among them occurs, following the above mentioned scheme. We check for the steady state, by looking at the stability of the money distribution in successive Monte Carlo steps t (we define one Monte Carlo time step as N pairwise exchanges). Eventually, after a typical relaxation time the money distribution becomes stationary. This relaxation time is dependent on system size N and the distribution of λ (e.g., $\sim 10^6$ for $N = 1000$ and uniformly distributed λ). After this, we average the money distribution over $\sim 10^3$ time steps. Finally we take configurational average over $\sim 10^5$ realizations of the λ distribution to get the money distribution $P(m)$. It is found to follow a power law for the wealthiest population ($\sim 10\%$). This decay fits to Pareto law equation (1) with $\nu \simeq 1$ (Fig. 4). We also checked that for a mixed population where a fraction f has fixed saving propensity $\lambda = \lambda_1$ and for the rest $(1 - f)$ fraction, λ is distributed uniformly within $0 \leq \lambda < 1$, we find a money distribution resembling very much the observed empirical distributions (see Fig. 5), as shown in Figure 1. Here, when $P(m)$ is fitted in equation (1), we have $\nu = 1$ and the exponent α is approximately given by equation (12) with $\lambda = \lambda_1$ and m_c depending on f and λ_1 . Note, for finite size N of the market, the distribution has a narrow initial growth upto

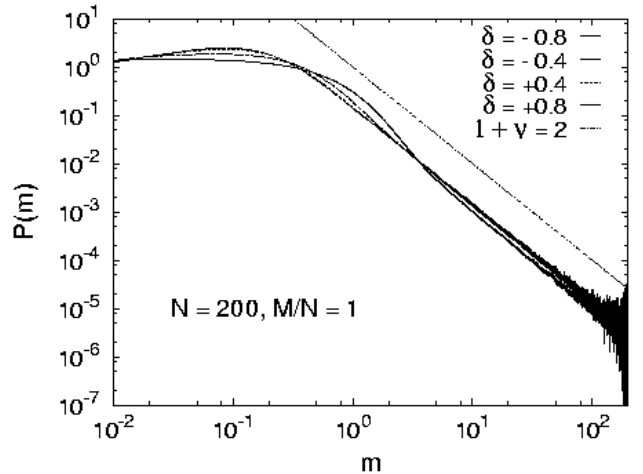


Fig. 6. Steady state money distribution $P(m)$ in the model for $N = 200$ agents with λ distributed as $\rho(\lambda) \sim \lambda^\delta$ with different values of δ . A guide to the power law with exponent $1 + \nu = 2$ is also provided. For all cases, the average money per agent $M/N = 1$.

a most-probable value m_p after which it falls off with a power-law tail for several decades. This Pareto law (with $\nu \simeq 1$) covers the entire range in m of the distribution $P(m)$ in the limit $N \rightarrow \infty$. We checked that this power law is extremely robust: apart from the uniform λ distribution used in the simulations in Figure 4, we also checked the results for a distribution

$$\rho(\lambda) \sim |\lambda_0 - \lambda|^\delta, \quad \lambda_0 \neq 1, \quad 0 < \lambda < 1, \quad (17)$$

of quenched λ values among the agents. The Pareto law with $\nu = 1$ is universal for all δ . The data in Figure 4 corresponds to $\lambda_0 = 0$, $\delta = 0$. For negative δ values, however, we get an initial (small m) Gibbs-like decay in $P(m)$ (see Fig. 6). Figure 7 shows that for $\lambda_0 = 1$, the resultant distribution is $P(m) \sim m^{-(1+\nu)}$, $\nu = 1 + \delta$.

In case of uniformly distributed saving propensity λ ($\rho(\lambda) = 1$, $0 \leq \lambda < 1$), the individual money distribution $P_\lambda(m)$ for agents with any particular λ value, although differs considerably, remains non-monotonic (see Fig. 8), similar to that for uniform λ market with $m_p(\lambda)$ shifting with λ (see Fig. 3). Few subtle points may be noted though: while for uniform λ the $m_p(\lambda)$ were all less than of the order of unity (average money per agent is fixed to $M/N = 1$; see Fig. 3), for distributed λ case $m_p(\lambda)$ can be considerably larger and can approach to the order of N for large λ (see Fig. 8). This is consistent with the empirically known fact that the large-income people usually have larger saving factors [53].

There is also a marked qualitative difference in fluctuations: while for fixed λ , the fluctuations in time (around the most-probable value) in the individuals' money $m_i(t)$ gradually decreases with increasing λ , for quenched distribution of λ , the trend gets reversed.

We investigated on the range of distribution of the saving propensities in a certain interval $a < \lambda_i < b$, where, $0 < a < b < 1$. For uniform distribution within the range,

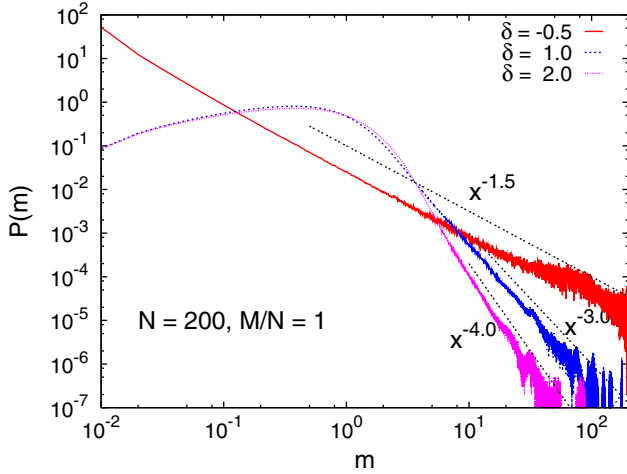


Fig. 7. Steady state money distribution $P(m)$ in the model for $N = 200$ agents with λ distributed as $\rho(\lambda) \sim |1 - \lambda|^\delta$ with different values of δ . The distributions $P(m)$ have power law tails $P(m) \sim m^{-(1+\nu)}$, where the power law exponents $1 + \nu$ approximately equal to $2 + \delta$ indicated by the dotted straight lines. For all cases, the average money per agent $M/N = 1$.

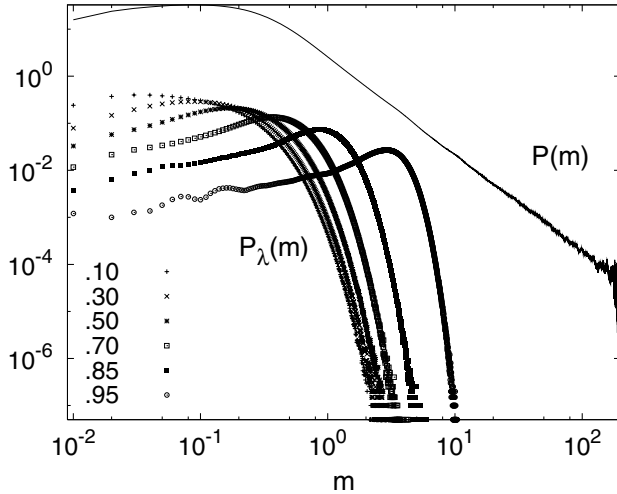


Fig. 8. Steady state money distribution $P_\lambda(m)$ for some typical values of λ ($=0.1, 0.3, 0.5, 0.7, 0.85, 0.95$) in the distributed λ model. The data is collected from the ensembles with $N = 200$ agents. The total distribution of money $P(m)$ is also plotted for comparison. For all cases, the average money per agent $M/N = 1$.

we observe the appearance of the same power law in the distribution but for a narrower region. As may be seen from Figure 9, as $a \rightarrow b$, the power-law behavior is seen for values a or b approaching more and more towards unity: for the same width of the interval $|b - a|$, one obtains power-law (with the same value of ν) when $b \rightarrow 1$. This indicates that for fixed λ , $\lambda = 0$ correspond to a Gibbs distribution, and one observes a power law in $P(m)$ when λ has got a non-zero width of its distribution extending up to $\lambda = 1$. It must be emphasized at this point that we are talking about the limit $\lambda \rightarrow 1$, since any agent having $\lambda = 1$ will result in condensation of money with that

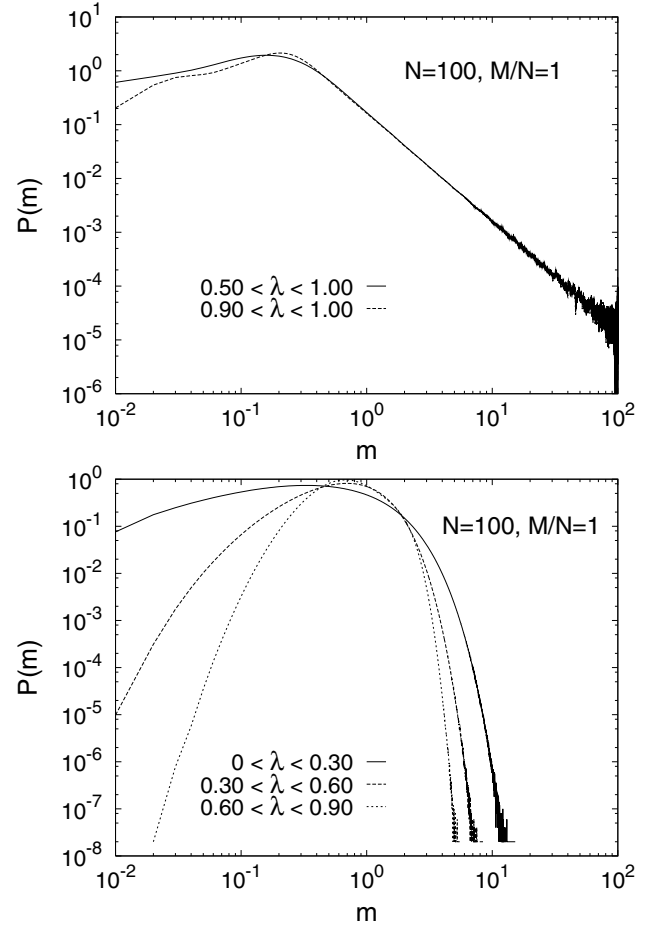


Fig. 9. Steady state money distribution in cases when the saving propensity λ is distributed uniformly within a range of values: (a) λ distribution extends up to 1, money distribution shows power law both for lower cut-offs 0.5 and 0.9; (a) width of λ distribution is 0.3, money distribution shows a power law in a narrow region only for $0.6 < \lambda < 0.9$. The power law exponent is $\nu \simeq 1$ in all cases. All data shown here are for $N = 100$, $M/N = 1$.

particular agent. The role of the agents with high saving propensity ($\lambda \rightarrow 1$) is crucial: the power law behavior is truly valid up to the asymptotic limit if $\lambda = 1$ is included. Indeed, had we assumed $\lambda_0 = 1$ in equation (17), the Pareto exponent ν immediately switches over to $\nu = 1 + \alpha$. Of course, $\lambda_0 \neq 1$ in equation (17) leads to the universality of the Pareto distribution with $\nu = 1$ (irrespective of λ_0 and α). Obviously, $P(m) \sim \int_0^1 P_\lambda(m) \rho(\lambda) d\lambda \sim m^{-2}$ for $\rho(\lambda)$ given by equation (17) and $P(m) \sim m^{-(2+\alpha)}$ if $\lambda_0 = 1$ in equation (17) (for large m values).

Another numerical study [44] analysed the average money of the agent with the maximum savings factor $\langle m(\lambda_{\max}) \rangle$. This study concludes on the time evolution of the money of this agent, and finds a scaling behavior

$$\langle m(\lambda_{\max}) \rangle / N (1 - \lambda_{\max})^{0.725} \sim \mathcal{G}[t(1 - \lambda_{\max})]. \quad (18)$$

This implies that the stationary state for the agent with the maximum value of λ is reached after a relaxation time

$$\tau \propto (1 - \lambda_{\max})^{-1}. \quad (19)$$

The average money $\langle m(\lambda_{\max}) \rangle$ of this agent is also found to scale as

$$[\langle m(\lambda_{\max}) \rangle / N] N^{-0.15} \sim \mathcal{F}[(1 - \lambda_{\max}) N^{1.5}]. \quad (20)$$

The scaling function $\mathcal{F}[x] \rightarrow x^{-\kappa}$ as $x \rightarrow 0$ with $\kappa \approx 0.76$. This means $\langle m(\lambda_{\max}) \rangle N^{-1.15} \sim (1 - \lambda_{\max})^{-0.76} N^{-1.14}$ or $\langle m(\lambda_{\max}) \rangle \sim (1 - \lambda_{\max})^{-0.76} N^{0.01}$. Since for a society of N traders $(1 - \lambda_{\max}) \sim 1/N$ this implies

$$\langle m(\lambda_{\max}) \rangle \sim N^{0.77}. \quad (21)$$

These model income distributions $P(m)$ compare very well with the wealth distributions of various countries: Data suggests Gibbs like distribution in the low-income range (more than 90% of the population) and Pareto-like in the high-income range [19,16,32] (less than 10% of the population) of various countries. In fact, we compared one model simulation of the market with saving propensity of the agents distributed following equation (17), with $\lambda_0 = 0$ and $\delta = -0.7$ [50]. The qualitative resemblance of the model income distribution with the real data for Japan and USA in recent years is quite intriguing. In fact, for negative δ values in equation (17), the density of traders with low saving propensity is higher and since $\lambda = 0$ ensemble yields Gibbs-like income distribution equation (6), we see an initial Gibbs-like distribution which crosses over to Pareto distribution equation (1) with $\nu = 1.0$ for large m values. The position of the crossover point depends on the value of α . It is important to note that any distribution of λ near $\lambda = 1$, of finite width, eventually gives Pareto law for large m limit. The same kind of crossover behavior (from Gibbs to Pareto) can also be reproduced in a model market of mixed agents where $\lambda = 0$ for a finite fraction of population and λ is distributed uniformly over a finite range near $\lambda = 1$ for the rest of the population.

5 Analytical studies

There have been a number of attempts to study the uniform savings model (Model B, Sect. 3.2) analytically (see e.g., [54]), but no closed form expression for the steady state distribution $P(m)$ has yet been arrived at. Kar Gupta [55] investigated the nature of the transition matrices from the equations (14) and (15) and concluded that the effect of introducing a saving propensity leads to a nonsingular transition matrix, and hence a time irreversible state.

We review now some of the investigations on the steady state distribution $P(m)$ of money resulting from the equations (14) and (15) representing the trading and money dynamics (Model C, Sect. 3.3) in the distributed savings case. The dynamics of money distribution is solved in two limiting cases. In one case, the evolution of the mutual

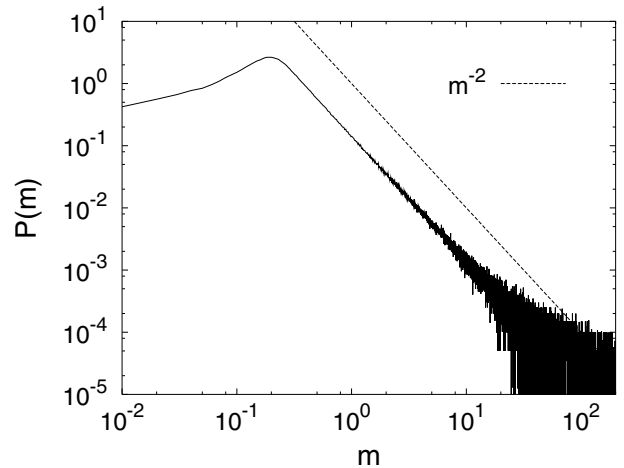


Fig. 10. Steady state money distribution $P(m)$ against m in a numerical simulation of a market with $N = 200$, following equations (14) and (15) with $\epsilon_{ij} = 1/2$. The dotted line corresponds to $m^{-(1+\nu)}$; $\nu = 1$. Here, the average money per agent $M/N = 1$.

money difference among the agents is investigated and one looks for a self-consistent equation for its steady state distribution. In the other case, a master equation for the money distribution function is developed [56,57].

5.1 Distribution of money difference

Clearly in the process as considered (dynamics defined by Eqs. (14) and (15)), the total money $(m_i + m_j)$ of the pair of agents i and j remains constant, while the difference Δm_{ij} evolves as

$$\begin{aligned} (\Delta m_{ij})_{t+1} \equiv (m_i - m_j)_{t+1} &= \left(\frac{\lambda_i + \lambda_j}{2} \right) (\Delta m_{ij})_t \\ &+ \left(\frac{\lambda_i - \lambda_j}{2} \right) (m_i + m_j)_t \\ &+ (2\epsilon_{ij} - 1)[(1 - \lambda_i)m_i(t) + (1 - \lambda_j)m_j(t)]. \end{aligned} \quad (22)$$

Numerically, as shown in Figure 4, we observe that the steady state money distribution in the market becomes a power law, following such tradings when the saving factor λ_i of the agents remain constant over time but varies from agent to agent widely. As shown in the numerical simulation results for $P(m)$ in Figure 10, the law, as well as the exponent, remains unchanged even when $\epsilon_{ij} = 1/2$ for every trading. This can be justified by the earlier numerical observation [40,50] for fixed λ market ($\lambda_i = \lambda$ for all i) that in the steady state, criticality occurs as $\lambda \rightarrow 1$ where of course the dynamics becomes extremely slow. In other words, after the steady state is realized, the third term in equation (22) becomes unimportant for the critical behavior. For simplicity, we concentrate on this case, where the above evolution equation for Δm_{ij} can be written in a more simplified form as

$$(\Delta m_{ij})_{t+1} = \bar{\lambda}_{ij} (\Delta m_{ij})_t + \tilde{\lambda}_{ij} (m_i + m_j)_t, \quad (23)$$

where $\bar{\lambda}_{ij} = \frac{1}{2}(\lambda_i + \lambda_j)$ and $\tilde{\lambda}_{ij} = \frac{1}{2}(\lambda_i - \lambda_j)$. As such, $0 \leq \bar{\lambda} < 1$ and $-\frac{1}{2} < \tilde{\lambda} < \frac{1}{2}$.

The steady state probability distribution D for the modulus $\Delta = |\Delta m|$ of the mutual money difference between any two agents in the market can be obtained from equation (23) in the following way provided Δ is very much larger than the average money per agent $= M/N$. This is because, using equation (23), large Δ can appear at $t + 1$, say, from ‘scattering’ from any situation at t for which the right hand side of equation (23) is large. The possibilities are (at t) m_i large (rare) and m_j not large, where the right hand side of equation (23) becomes $\simeq (\bar{\lambda}_{ij} + \tilde{\lambda}_{ij})(\Delta_{ij})_t$; or m_j large (rare) and m_i not large (making the right hand side of Eq. (23) becomes $\simeq (\bar{\lambda}_{ij} - \tilde{\lambda}_{ij})(\Delta_{ij})_t$); or when m_i and m_j are both large, which is a much rarer situation than the first two and hence is negligible. Consequently for large Δ the distribution D satisfies

$$\begin{aligned} D(\Delta) &= \int d\Delta' D(\Delta') \\ &\quad \times \langle \delta(\Delta - (\bar{\lambda} + \tilde{\lambda})\Delta') + \delta(\Delta - (\bar{\lambda} - \tilde{\lambda})\Delta') \rangle \\ &= 2 \left\langle \left(\frac{1}{\bar{\lambda}} \right) D \left(\frac{\Delta}{\bar{\lambda}} \right) \right\rangle, \end{aligned} \quad (24)$$

where we have used the symmetry of the $\tilde{\lambda}$ distribution and the relation $\bar{\lambda}_{ij} + \tilde{\lambda}_{ij} = \lambda_i$, and have suppressed labels i, j . Here $\langle \dots \rangle$ denote average over λ distribution in the market, and δ denotes the δ -function. Taking now a uniform random distribution of the saving factor λ , $\rho(\lambda) = 1$ for $0 \leq \lambda < 1$, and assuming $D(\Delta) \sim \Delta^{-(1+\nu)}$ for large Δ , we get

$$1 = 2 \int_0^1 d\lambda \lambda^\nu = 2(1 + \nu)^{-1}, \quad (25)$$

giving $\nu = 1$. No other value fits the above equation. This also indicates that the money distribution $P(m)$ in the market also follows a similar power law variation, $P(m) \sim m^{-(1+\nu)}$ and $\nu = 1$. Distribution of Δ from numerical simulations also agree with this result.

A detailed analysis of the master equation for the kinetic exchange process and its solution for a special case can be seen in references [56, 57]. For a pioneering study of the kinetic equations for the two-body scattering process and a more general solution, see references [45, 46].

5.2 A mean field explanation

One can also derive the above results in a mean field limit, where the money redistribution equations for the individual agents participating in a trading process can be reduced to a stochastic map in m^2 [58]. The trick is to take the product of equations (14) and (15) and look for the time evolution of m^2 :

$$\begin{aligned} m_i(t+1)m_j(t+1) &= \alpha_i(\epsilon_t, \lambda_i)m_i^2(t) + \alpha_j(\epsilon_t, \lambda_j)m_j^2(t) \\ &\quad + \alpha_{ij}(\epsilon_t, \lambda_i, \lambda_j)m_i(t)m_j(t). \end{aligned} \quad (26)$$

Since ϵ_{ij} in equations (5), (10) and (16) keeps on changing (with time t) with the pairs of scatterer (i, j) , we use here ϵ_t to denote its explicit time dependence. We now introduce a mean-field-like approximation by replacing each of the quadratic quantities m_i^2 , m_j^2 and $m_i m_j$ by a mean quantity m^2 . Therefore equation (26) is replaced by its mean-field-like approximation

$$m^2(t+1) = \eta(t)m^2(t) \quad (27)$$

where $\eta(t)$ is an algebraic function of λ_i , λ_j and ϵ_t ; it has been observed in numerical simulations of the model that the value of ϵ_t , whether it is random or constant, has no effect on the steady state distribution [56] and the time dependence of $\eta(t)$ results from the different values of λ_i and λ_j encountered during the evolution of the market. Denoting $\log(m^2)$ by x , equation (27) can be written as

$$x(t+1) = x(t) + \delta(t), \quad (28)$$

where $\delta(t) = \log \eta(t)$ is a random number that changes with each time-step. The transformed map (Eq. (28)) depicts a random walk and therefore the ‘displacements’ x in the time interval $[0, t]$ follows the normal distribution

$$\mathcal{P}(x) \sim \exp\left(-\frac{x^2}{t}\right). \quad (29)$$

Now

$$\mathcal{P}(x)dx \equiv P(m)dm^2 \quad (30)$$

where $P(m)$ is the log-normal distribution of m^2 :

$$P(m) \sim \frac{1}{m^2} \exp\left[-\frac{(\log(m^2))^2}{t}\right]. \quad (31)$$

The normal distribution in equation (29) spreads with time (since its width is proportional to \sqrt{t}) and so does the normal factor in equation (31) which eventually becomes a very weak function of m and may be assumed to be a constant as $t \rightarrow \infty$. Consequently $P(m)$ assumes the form of a simple power law:

$$P(m) \sim \frac{1}{m^2} \text{ for } t \rightarrow \infty, \quad (32)$$

that is clearly the Pareto law for the model. Hence, the power law behavior obtained here agrees with the simulation results.

5.3 Average money at any saving propensity and the distribution

Several numerical studies investigated [59, 60] the saving factor λ and the average money held by an agent whose savings factor is λ . This numerical study revealed that the product of this average money and the unsaved fraction remains constant, or in other words, the quantity

$$\langle m(\lambda) \rangle (1 - \lambda) = c \quad (33)$$

where c is a constant. This key result has been justified using a rigorous analysis by Mohanty [61]. We give below a simpler argument and proceed to derive the steady state distribution $P(m)$ in its general form.

In a mean field approach, one can calculate [61] the distribution for the ensemble average of money for the model with distributed savings. It is assumed that the distribution of money of a single agent over time is stationary, which means that the time averaged value of money of any agent remains unchanged independent of the initial value of money. Taking the ensemble average of all terms on both sides of equation (14), one can write

$$\langle m_i \rangle = \lambda_i \langle m_i \rangle + \langle \epsilon \rangle \left[(1 - \lambda_i) \langle m_i \rangle + \left\langle \frac{1}{N} \sum_{j=1}^N (1 - \lambda_j) m_j \right\rangle \right]. \quad (34)$$

It is assumed that any agent on the average, interacts with all others in the system. The last term on the right is replaced by the average over the agents. Writing

$$\overline{\langle (1 - \lambda) m \rangle} \equiv \left\langle \frac{1}{N} \sum_{j=1}^N (1 - \lambda_j) m_j \right\rangle \quad (35)$$

and since ϵ is assumed to be distributed randomly and uniformly in $[0, 1]$, so that $\langle \epsilon \rangle = 1/2$, equation (34) reduces to

$$(1 - \lambda_i) \langle m_i \rangle = \overline{\langle (1 - \lambda) m \rangle}.$$

Since the right side is free of any agent index, it suggests that this relation is true for any arbitrary agent, i.e., $\langle m_i \rangle (1 - \lambda_i) = \text{constant}$, where λ_i is the saving factor of the i th agent (as in Eq. (33)) and what follows is:

$$d\lambda \propto \frac{dm}{m^2}. \quad (36)$$

An agent with a particular saving propensity factor λ therefore ends up with a characteristic average wealth m given by equation (33) such that one can in general relate the distributions of the two:

$$P(m) dm = \rho(\lambda) d\lambda. \quad (37)$$

This, together with equations (33) and (34) gives [61]

$$P(m) = \rho(\lambda) \frac{d\lambda}{dm} \propto \frac{\rho(1 - \frac{c}{m})}{m^2}, \quad (38)$$

giving $P(m) \sim m^{-2}$ for large m for uniform distribution of savings factor λ , i.e., $\nu = 1$; and $\nu = 1 + \delta$ for $\rho(\lambda) = (1 - \lambda)^\delta$. This study therefore explains the origin of the universal ($\nu = 1$) as well as the non-universal ($\nu = 1 + \delta$) Pareto exponent values in the distributed savings model, as discussed in Section 4 and shown in Figures 6 and 7.

6 Other model studies

Sinha [62,63] considered an iterative map approach to distribution of wealth in an economy, along with models that employed yard-sale (YS) as well as theft and fraud (TF) [25] for asset exchange, yielding interesting results. A recent study [64] also considers combinations of these strategies, along with partial savings in a class of models. Recent detailed studies [65] of empirical data and analysis of the distribution functions present a strong case in favor of gas-like models for economic exchanges. Other studies calculated the holding time [66] of money, which indicated in turn the mobility of the money in a model under a given dynamics. Another similar study [67] calculated the velocity of money in a life-cycle model. Studies of gas-like or particle-exchange models have already been carried out on complex networks [68,69]. Similar models study the effect of risk aversion and subsequent emergence of Gibbs and power-law distributions in different cases [70], while another study tunes the rate of money transfer to obtain Boltzmann and Gibbs-like money distributions [71]. Similarly, one can introduce asymmetry in favor of either of the traders in a trade-investment framework and produce power law distributions in wealth distributions [72]. Preferential spending behavior can also lead to similar results [73]. Recently, Angle [74] has also proposed a macro-model for the inequality process to explain the upward surge of the Pareto tail in recent time for the US waged income data. Düring and Toscani [75] recently formulated hydrodynamic equations for such kinetic models of markets.

There are evidences of emerging income inequality arising as a consequence of resource flow in hierarchical organizations [76], and the resulting income distribution is power law distributed.

6.1 A model with ‘annealed’ savings

In a real trading process, the concept of ‘saving factor’ cannot be attributed to a quantity that is invariant with time. A saving factor always changes with time or trading. In some of the earlier works [50], we reported the case of annealed savings, where the savings factor λ_i changes with time in the interval $[0, 1]$, but does not give rise to a power law in $P(m)$ [50]. But, there are some special cases of annealed saving can give rise to a power law distribution of $P(m)$.

We proposed [77] a slightly different model of an annealed saving case. Let us associate a parameter μ_i ($0 < \mu_i < 1$) with each agent i such that the savings factor λ_i randomly assumes a value in the interval $[\mu_i, 1]$ at each time or trading. The trading rules are of course unaltered and governed by equations (14) and (15). Now, considering a suitable distribution $\zeta(\mu)$ of μ over the agents, one can produce money distributions with power-law tail. The only condition that needs to be satisfied is that $\zeta(\mu)$ should be non-vanishing as $\mu \rightarrow 1$. Figure 11 shows the case when $\zeta(\mu) = 1$. Numerical simulations suggest that the behavior of the wealth distribution is similar to the quenched

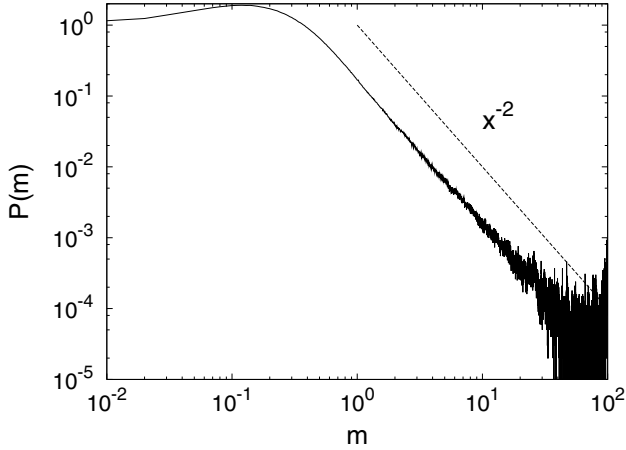


Fig. 11. Distribution $P(m)$ of money m in case of annealed savings λ varying randomly in $[\mu, 1)$. Here, $\zeta(\mu)$ has a uniform distribution. The distribution produces a power law tail with Pareto exponent $\nu = 1$. The simulation has been done for a system of $N = 100$ agents, with average money per agent $M/N = 1$. $P(m)$ is the steady state distribution after 4×10^4 Monte Carlo steps, and the data is averaged over an ensemble of 10^5 .

savings case. In other words, only if $\zeta(\mu) \propto |1 - \mu|^\delta$, it is reflected in the Pareto exponent as $\nu = 1 + \delta$ [77]. μ_i is interpreted as the lower bound of the saving distribution of the i th agent. Thus, while agents are allowed to randomly save any fraction of their money, the bound ensures that there is always a non-vanishing fraction of the population that assumes high saving fraction.

6.2 A model with a non-consumable commodity

Money is certainly not the only quantity that circulates in a trading market. Exchange of goods is the main entity for transactions. Different economic conditions give rise to the fluctuation of price of these commodities and this plays an important role in the behavior of the market as a whole. The determination of ‘price’ is a complex phenomena and is decided by the dynamics of supply and demand of the particular commodity.

In the trading markets discussed in previous two chapters, modifications due to exchange of a consumable commodity hardly affects the distribution, as the commodity once bought or sold need not be accounted for. Consumable commodities effectively have no ‘price’, as due to their short lifetime to contribute to the total wealth of an individual. It is interesting however, to study the role of non-consumable commodities in such market models.

For sake of simplicity, we consider a simplified version of a market with a single non-consumable commodity [78]. As before, we consider a fixed number of traders or agents N who trade in a market with total money $\sum_i m_i(t) = M$ and total commodity $\sum_i c_i(t) = C$, $m_i(t)$ and $c_i(t)$ being the money and commodity respectively of the i -th agent at time t and are both non-negative. Needless to mention, both $m_i(t)$ and $c_i(t)$ change with time or trading t .

The market, as in previous cases, is closed, i.e., N , M and C are constants. The wealth w_i of an individual i in that case is, the sum of the money and commodity it possesses, i.e., $w_i = m_i + p_0 c_i$; where p_0 is the ‘global’ price. In course of trading, total money and total commodity are locally conserved, and this automatically conserves the total wealth. In such a market, one can define a global average price parameter $p_0 = M/C$, which is set to unity in this case, giving $w_i = m_i + c_i$. It may be noted at this point that in order to avoid the complication of restricting the commodity-money exchange and their reversal between the same agents, the Fisher velocity of money circulation (see e.g., Ref. [79]) is renormalised to unity here. In order to accommodate the lack of proper information and the ability of the agents to bargain etc., we will allow fluctuations θ in the price of the commodities at any trading (time): $p(t) = p_0 \pm \theta = 1 \pm \theta$. We find, the nature of steady state to be unchanged and independent of θ , once it becomes non-vanishing.

In general, the dynamics of money in this market looks the same as equations (3), (5), (8)–(10) or (14)–(16) depending on whether $\lambda_i = 0$ for all, $\lambda_i \neq 0$ but uniform for all i or $\lambda_i \neq \lambda_j$ respectively. However, all Δm are not allowed here; only those, for which $\Delta m_i \equiv m_i(t+1) - m_i(t)$ or Δm_j are allowed by the corresponding changes Δc_i or Δc_j in their respective commodities ($\Delta m > 0, \Delta c > 0$) [78]:

$$c_i(t+1) = c_i(t) + \frac{m_i(t+1) - m_i(t)}{p(t)} \quad (39)$$

$$c_j(t+1) = c_j(t) - \frac{m_j(t+1) - m_j(t)}{p(t)} \quad (40)$$

where $p(t)$ is the local-time ‘price’ parameter, a stochastic variable:

$$p(t) = \begin{cases} 1 + \theta & \text{with probability } 0.5 \\ 1 - \theta & \text{with probability } 0.5. \end{cases} \quad (41)$$

The role of the stochasticity in $p(t)$ is to imitate the effect of bargaining in a trading process. θ parametrizes the amount of stochasticity. The role of θ is significant in the sense that it determines the (relaxation) time the whole system takes to reach a dynamically equilibrium state; the system reaches equilibrium sooner for larger θ , while its magnitude does not affect the steady state distribution. It may be noted that, in course of trading process, certain exchanges are not allowed (e.g., in cases when a particular pair of traders do not have enough commodity to exchange in favor of an agreed exchange of money). We then skip these steps and choose a new pair of agents for trading.

In an ideal gas market without savings, money is exponentially distributed in presence of any finite value of θ . Again, commodity has a small initial peak before decaying exponentially. However, the total wealth $w = m + c$ has a form of a Gamma distribution.

For $\theta = 0$, however, wealth of each agent remains invariant with time as only the proportion of money and commodity interchange within themselves, as the ‘price’

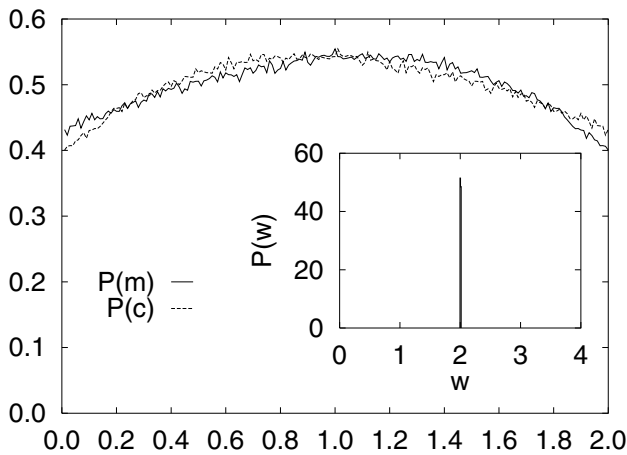


Fig. 12. Steady state distribution $P(m)$ of money m in a market with no savings (saving factor $\lambda = 0$) for no price fluctuations i.e. $\theta = 0$. The graphs show simulation results for a system of $N = 100$ agents, $M/N = 1$, $C/N = 1$; $m_i = c_i = 1$ at $t = 0$ for all agents i . The inset shows the distribution $P(w)$ of total wealth $w = m + c$. As $p = 1$, for $\theta = 0$, although m and c can change with tradings within the limit $(0 - 2)$ the sum is always maintained at 2.

factor remains constant. This of course happens irrespective of the savings factor being zero, uniform or distributed. For $\theta = 0$, the steady state distribution of money or commodity can take non-trivial forms: see Figure 12, but strictly a δ -function for total wealth, or at the value of wealth one starts with (see inset of Fig. 12 for the case $m_i = c_i = 1$ for all i) [78].

As mentioned already for $\theta \neq 0$, the steady state results are not dependent on the value of θ , the relaxation time of course decreases with increasing θ . In such a market with uniform savings, money distribution $P(m)$ has a form similar to a set (for $\lambda \neq 0$) of Gamma functions (see Fig. 13): a set of curves with a most-probable value shifting from 0 to 1 as saving factor λ changes from 0 to 1 (as in the case without commodity). The commodity distribution $P(c)$ has an initial peak and an exponential fall-off, without much systematics with varying λ (see inset of Fig. 13). The distribution $P(w)$ of total wealth $w = m + c$ behaves much like $P(m)$ (see Fig. 14). It is to be noted that since there is no precise correspondence with commodity and money for $\theta \neq 0$ (unlike when $\theta = 0$, when the sum is fixed), $P(w)$ cannot be derived directly from $P(m)$ and $P(c)$. However, there are further interesting features. Although they form a class of Gamma distributions, the set of curves for different values of saving factor λ seem to intersect at a common point, near $w = 1$. All the reported data are for a system of $N = 100$ agents, with $M/N = 1$ and $C/N = 1$ and for a case where the noise θ equals 0.5 [78].

For λ distributed uniformly within the interval $0 \leq \lambda < 1$, the tails of both money and wealth distributions $P(m)$ and $P(w)$ have Pareto law behavior with a fitting exponent value $\nu = 1 \pm 0.02$ and $\nu = 1 \pm 0.05$ respectively (see Figs. 15 and 16 respectively), whereas

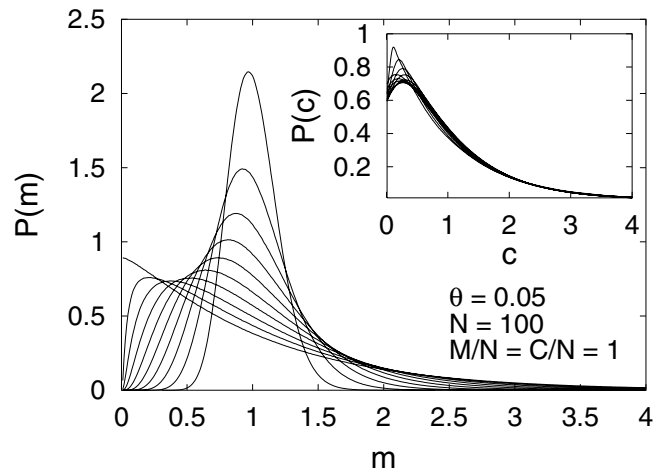


Fig. 13. Steady state distribution $P(m)$ of money m in the uniform savings commodity market for different values of saving factor λ (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 from left to right near the origin) for $\theta = 0.05$. The inset shows the distribution $P(c)$ of commodity c in the uniform savings commodity market for different values of saving factor λ . The graphs show simulation results for a system of $N = 100$ agents, $M/N = 1$, $C/N = 1$.

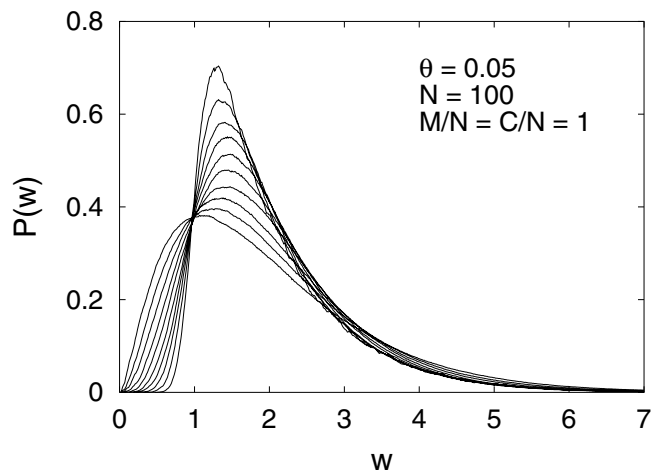


Fig. 14. Steady state distribution $P(w)$ of total wealth $w = m + c$ in the uniform savings commodity market for different values of saving factor λ (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 from left to right) for $\theta = 0.05$. The graphs show simulation results for a system of $N = 100$ agents, $M/N = 1$, $C/N = 1$.

the commodity distribution is still exponentially decaying (see inset of Fig. 15) [78].

A major limitation of these money-only exchange models considered earlier [3, 25, 37–40, 43, 45–52, 56, 61, 77, 80–83] is that they do not make any explicit reference to the commodities exchanged with the money and to the constraints they impose on the exchange process. Also, the wealth is not just the money is possession (unless the commodity exchanged with the money is strictly consumable). Here, we have studied the effect of a single non-consumable commodity on the money (and also wealth) distributions in the steady state, and allowing for local

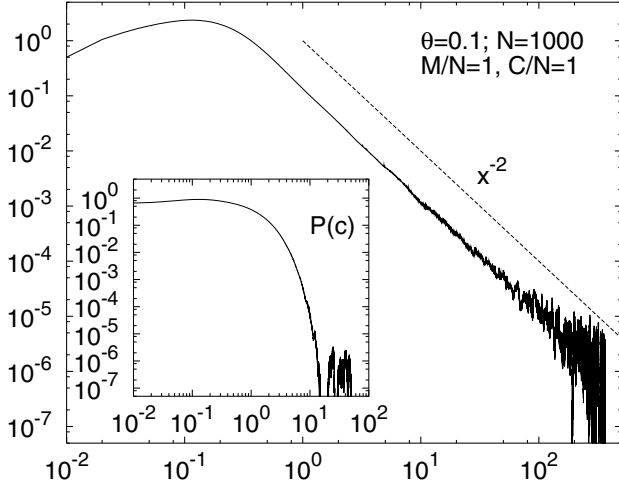


Fig. 15. Steady state distribution $P(m)$ of money m in the commodity market with distributed savings $0 \leq \lambda < 1$. $P(m)$ has a power-law tail with Pareto exponent $\nu = 1 \pm 0.02$ (a power law function x^{-2} is given for comparison). The inset shows the distribution $P(c)$ of commodity c in the same commodity market. The graphs show simulation results for a system of $N = 1000$ agents, $M/N = 1$, $C/N = 1$.

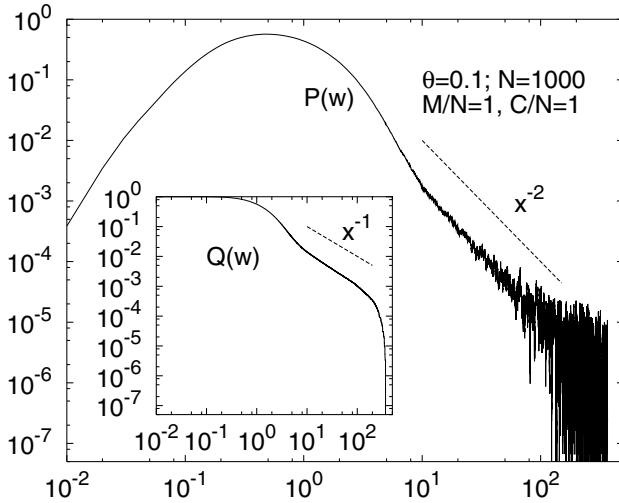


Fig. 16. Steady state distribution $P(w)$ of total wealth $w = m + c$ in the commodity market with distributed savings $0 \leq \lambda < 1$. $P(w)$ has a power-law tail with Pareto exponent $\nu = 1 \pm 0.05$ (a power law function x^{-2} is given for comparison). The inset shows the cumulative distribution $Q(w) \equiv \int_w^\infty P(w)dw$. The graphs show simulation results for a system of $N = 1000$ agents, $M/N = 1$, $C/N = 1$.

(in time) price fluctuation. Allowing for price fluctuation is very crucial for the model — it allows for the stochastic dynamics to play its proper role in the market. However, this model is quite different from that considered recently in reference [84], where p_0 is strictly unity and the stochasticity enters from other exogenous factors. In the sense that we also consider two exchangeable variables in the market, our model has some similarity with that in reference [85]. However, Silver et al. [85] consider

only random exchanges between agents (keeping the total conserved) while we consider random exchanges and also allowing for price fluctuations and savings. As such they only obtain the Gamma distribution in wealth, while our model produce both Gamma and Pareto distributions. In spite of many significant effects, the general feature of Gamma-like form of the money (and wealth) distributions (for uniform λ) and the power law tails for both money and wealth (for distributed λ) with identical exponents, are seen to remain unchanged. The precise studies (theories) for the money-only exchange models are therefore extremely useful and relevant.

7 Discussions

Empirical data for income and wealth distribution in many countries are now available, and they reflect a particular robust pattern (see Fig. 1). The bulk (about 90%) of the distribution resemble the century-old Gibbs distribution of energy for an ideal gas, while there are evidences of considerable deviation in the low income as well as high income ranges. The high income range data (for 5–10% of the population in any country) fits to a power law tail, known after Pareto, and the value of the (power law) exponent ranges between 1–3 and depends on the individual make-up of the economy of the society or country. There are also some reports of two distinct power law tails of such distributions (see e.g. [86]).

The analogy with a gas like many-body system has led to the formulation of the models of markets. The random scattering-like dynamics of money (and wealth) in a closed trading market, in analogy with energy conserved exchange models, reveals interesting features. The minimum modification required over such ideal gas-like kinetic exchange models seem to be the consideration of saving propensity of the traders. Self-organisation is a key emerging feature of these kinetic exchange models when saving factors are introduced. In the model with uniform savings (see Sect. 3.2), the Gamma-like distribution of wealth shows stable most-probable or peaked distribution with a most-probable value indicative of an economic scale dependent on the saving propensity or factor λ . Empirical observations in homogeneous groups of individuals as in waged income of factory laborers in UK and USA [7] and data from population survey in USA among students of different school and colleges produce similar distributions [49]. This is a relatively simpler case where a homogeneous population (say, characterised by a unique value of λ) could be identified.

In the model with distributed savings (see Sect. 3.3), the saving propensity is assumed to have a randomness and varies from agent to agent. One finds the emergence of a power law tail in money (and wealth) in cases where the saving factor is a quenched variable (does not change with tradings or time t) within different agents or traders. Several variants have been investigated for the basic model, including an ‘annealed’ version, some of which produce the Pareto-like power law (Eq. (1)). The money exchange

equations can be cast into a master equation, and the solution to the steady state money distribution giving the Pareto law with $\nu = 1$ have been derived using several approaches (see Sect. 5). The results of the mean field theory agree with the simulations. We have mostly used the terms ‘money’ and ‘wealth’ interchangeably, treating the models in terms of only one quantity, namely ‘money’ that is exchanged. Of course, wealth does not comprise of (paper) money only, and there have been studies distinguishing these two. We review one such model study in Sect. 6.2 where, in addition to money, a single non-consumable commodity, having local price fluctuations, was introduced. The steady state money and wealth (comprising of money and price weighted commodity) distributions were then investigated in the same market. Interestingly, the scaling behavior for high range of the money as well as the wealth are found to be similar (see Sect. 6.2), with identical Pareto exponent value for the distributed savings.

Study of such simple models here give some insight into the possible emergence of self organizations in such markets, evolution of the steady state distribution, emergence of Gamma-like distribution for the bulk and of the power law tail, as in the empirically observed distributions (Fig. 1). A study of these models in terms of quantities that parametrise the circulation of money [79] suggests that the model with distributed savings perform better. These studies bring some new insight into the some essential economic issues, including economic mobility.

These model studies also indicate the appearance of self-organization, and the self-organized criticality [87] in particular, in the simplest model so far; namely in the kinetic gas models, when the effect of random saving propensities [24] is incorporated. Our observations indicate that the Gibbs and the (self-organized critical) Pareto distributions fall in the same category and can appear naturally in the century-old and well-established kinetic theory of gas [88,89]: Gibbs distribution for no saving and Pareto distribution for agents with quenched random saving propensity. To some degree of approximation therefore, these studies indicate that the society or market behaves like an ideal gas, and the exchange of money and wealth looks similar as in the above models at a coarse-grained level. Statistical physics allows us to model and analyse such systems in analogy to a variety of many body systems studied traditionally within the framework of physics; see e.g., Yakovenko [90] for an alternative account on these developments.

These models have additional prospective future applications in other spheres of social as well as physical sciences. In social sciences, the knowledge of the mechanism by which such distributions of wealth emerge out of collective exchanges may find application in policy making and taxation [26]. In physical sciences, the corresponding particle exchange model can find important application in designing desired energy spectrum for different types of chemical reactions [91].

The authors are grateful to P. Bhattacharyya, A. Chakraborti, S.S. Manna, S. Marjit, S. Pradhan, S. Sinha and R.B. Stinchcombe for collaborations at various stages of the study. Useful discussions with J. Angle, A.S. Chakrabarti, A. Das, Y. Fujiwara, M. Gallegati, A. Kar Gupta, T. Lux, M. Marsili, P.K. Mohanty, L. Pietronero, P. Richmond, W. Souma, D. Stauffer, V.M. Yakovenko and S. Yarlagadda are also acknowledged.

References

1. V. Pareto, *Cours d'Économie Politique* (F. Rouge, Lausanne, 1897)
2. B.B. Mandelbrot, *Int. Econ. Rev.* **1**, 79 (1960)
3. *Econophysics of Wealth Distributions*, edited by A. Chatterjee, S. Yarlagadda, B.K. Chakrabarti (Springer Verlag, Milan, 2005)
4. *Econophysics and Sociophysics*, edited by B.K. Chakrabarti, A. Chakraborti, A. Chatterjee (Wiley-VCH, Berlin, 2006)
5. P. Richmond, S. Hutzler, R. Coelho, P. Repetowicz, in reference [4]
6. A.C. Silva, V.M. Yakovenko, in reference [3]
7. G. Willis, J. Mimkes, e-print [arXiv:cond-mat/0406694](https://arxiv.org/abs/cond-mat/0406694)
8. A.A. Drăgulescu, V.M. Yakovenko, *Eur. Phys. J. B* **20**, 585 (2001)
9. Y. Fujiwara, in reference [3]
10. C. Gini, *The Economic Journal* **31**, 124 (1921)
11. E.W. Montroll, M.F. Shlesinger, *Proc. Natl. Acad. Sci. USA* **79**, 3380 (1982)
12. H. Aoyama, W. Souma, Y. Nagahara, M.P. Okazaki, H. Takayasu, M. Takayasu, *Fractals* **8**, 293 (2000)
13. W. Souma, *Fractals* **9**, 463 (2000)
14. R. Gibrat, *Les Inégalités Économiques* (Librairie du Recueil Sirey, Paris, 1931)
15. Y. Fujiwara, W. Souma, H. Aoyama, T. Kaizoji, M. Aoki, *Physica A* **321**, 598 (2003)
16. A.A. Drăgulescu, V.M. Yakovenko, *Physica A* **299**, 213 (2001)
17. K. Okuyama, M. Takayasu, H. Takayasu, *Physica A* **269**, 125 (1999)
18. R.L. Axtell, *Science* **293**, 1818 (2001)
19. M. Levy, S. Solomon, *Physica A* **242**, 90 (1997)
20. R. Coelho, Z. Néda, J.J. Ramasco, M.A. Santos, *Physica A* **353**, 515 (2005)
21. A.Y. Abul-Magd, *Phys. Rev. E* **66**, 057104 (2002)
22. G. Hegyi, Z. Néda, M.A. Santos, *Physica A* **380**, 271 (2007)
23. S. Sinha, *Physica A* **359**, 555 (2006)
24. P.A. Samuelson, *Economics* (Mc-Graw Hill Int., Auckland, 1980)
25. B. Hayes, *American Scientist* **90**, 400 (2002)
26. J. Hogan, *New Scientist*, 6 (12 March 2005)
27. P. Ball, *Nature* **441**, 686 (2006); *Editorial*, *Nature* **441**, 667 (2006)
28. A. Carbone, G. Kaniadakis, A.M. Scarfone, *Editorial*, *Eur. Phys. J. B* **57**, 121 (2007)
29. M. Gallegati, S. Keen, T. Lux, P. Ormerod, *Physica A* **370**, 1 (2006)
30. L. Zyga, *World's economies show similarities in economic inequality*, April 6, 2007, <http://www.physorg.com/news95074548.html>

31. S. Moss de Oliveira, P.M.C. de Oliveira, D. Stauffer, *Evolution, Money, War and Computers* (B.G. Tuebner, Stuttgart, Leipzig, 1999)
32. H. Aoyama, W. Souma, Y. Fujiwara, *Physica A* **324**, 352 (2003)
33. T. Di Matteo, T. Aste, S.T. Hyde, in *The Physics of Complex Systems (New Advances and Perspectives)*, edited by F. Mallamace, H.E. Stanley, *Proceedings of the International School of Physics "Enrico Fermi"* (IOS Press, Amsterdam 2004), p. 435
34. F. Clementi, M. Gallegati, *Physica A* **350**, 427 (2005)
35. F. Clementi, M. Gallegati, in [3]
36. N. Ding, Y. Wang, *Chinese Phys. Lett.* **24**, 2434 (2007)
37. B.K. Chakrabarti, S. Marjit, *Ind. J. Phys. B* **69**, 681 (1995)
38. S. Ispolatov, P.L. Krapivsky, S. Redner, *Eur. Phys. J. B* **2**, 267 (1998)
39. A.A. Drăgulescu, V.M. Yakovenko, *Eur. Phys. J. B* **17**, 723 (2000)
40. A. Chakraborti, B.K. Chakrabarti, *Eur. Phys. J. B* **17**, 167 (2000)
41. A. Chakraborti, *Int. J. Mod. Phys. C* **13**, 1315 (2002)
42. A. Chakraborti, S. Pradhan, B.K. Chakrabarti, *Physica A* **297**, 253 (2001)
43. M. Patriarca, A. Chakraborti, K. Kaski, *Phys. Rev. E* **70**, 016104 (2004)
44. K. Bhattacharya, G. Mukherjee, S.S. Manna, in reference [3]
45. P. Repetowicz, S. Hutzler, P. Richmond, *Physica A* **356**, 641 (2005)
46. P. Richmond, P. Repetowicz, S. Hutzler, in reference [3]
47. J. Angle, *Social Forces* **65**, 293 (1986)
48. T. Lux, in reference [3]
49. J. Angle, *Physica A* **367**, 388 (2006)
50. A. Chatterjee, B.K. Chakrabarti, S.S. Manna, *Physica A* **335**, 155 (2004)
51. A. Chatterjee, B.K. Chakrabarti, S.S. Manna, *Phys. Scripta T* **106**, 36 (2003)
52. B.K. Chakrabarti, A. Chatterjee, in *Application of Econophysics*, edited by H. Takayasu (Springer, Tokyo, 2004), pp. 280
53. K.E. Dynan, J. Skinner, S.P. Zeldes, *J. Pol. Econ.* **112**, 397 (2004)
54. A. Das, S. Yarlagadda, *Phys. Scripta T* **106**, 39 (2003)
55. A. Kar Gupta, *Physica A* **359**, 634 (2006)
56. A. Chatterjee, B.K. Chakrabarti, R.B. Stinchcombe, *Phys. Rev. E* **72**, 026126 (2005)
57. A. Chatterjee, B.K. Chakrabarti, R.B. Stinchcombe, in *Practical Fruits of Econophysics*, edited by H. Takayasu (Springer-Verlag, Tokyo, 2005), p. 333
58. P. Bhattacharyya, A. Chatterjee, B.K. Chakrabarti, *Physica A* **381**, 377 (2007)
59. M. Patriarca, A. Chakraborti, K. Kaski, G. Germano, in reference [3], p. 93
60. M. Patriarca, A. Chakraborti, G. Germano, *Physica A* **369**, 723 (2006)
61. P.K. Mohanty, *Phys. Rev. E* **74**, 011117 (2006)
62. S. Sinha, *Phys. Scripta T* **106**, 59 (2003)
63. S. Sinha, in reference [3], p. 177
64. M.A. Saif, P.M. Gade, *Physica A* **384**, 448 (2007)
65. H. Yuqing, *Physica A* **377**, 230 (2007)
66. Y. Wang, N. Ding, L. Zhang, *Physica A* **324**, 665 (2003)
67. Y. Wang, H. Qiu, *Physica A* **353**, 493 (2005)
68. M.-B. Hu, W.-X. Wang, R. Jiang, Q.-S. Wu, B.-H. Wang, Y.-H. Wu, *Eur. Phys. J. B* **53**, 273 (2006)
69. M.-B. Hu, R. Jiang, Q.-S. Wu, Y.-H. Wu, *Physica A* **381**, 476 (2007)
70. J.R. Iglesias, S. Gonçalves, G. Abramson, J.L. Vega, *Physica A* **342**, 186 (2004)
71. J.C. Ferrero, *Physica A* **341**, 575 (2004)
72. N. Scafetta, S. Picozzi, B.J. West, *Physica D* **193**, 338 (2004)
73. N. Ding, N. Xi, Y. Wang, *Eur. Phys. J. B* **36**, 149 (2003)
74. J. Angle, in *Econophysics of Markets and Business Networks*, edited by A. Chatterjee, B.K. Chakrabarti (Springer, Milan, 2007), p. 185
75. B. Düring, G. Toscani, *Physica A* **384**, 493 (2007)
76. S. Sinha, N. Srivastava, in *Econophysics of Markets and Business Networks*, edited by A. Chatterjee, B.K. Chakrabarti (Springer, Milan, 2007), pp. 215
77. A. Chatterjee, B.K. Chakrabarti, *Physica A* **382**, 36 (2007)
78. A. Chatterjee, B.K. Chakrabarti, *Eur. Phys. J. B* **54**, 399 (2006)
79. Y. Wang, N. Xi, N. Ding, in reference [4]
80. F. Slanina, *Phys. Rev. E* **69**, 046102 (2004)
81. A. Kar Gupta, in reference [4]
82. A. Chatterjee, B.K. Chakrabarti, in reference [3]
83. A. Chatterjee, S. Sinha, B.K. Chakrabarti, *Current Science* **92**, 1383 (2007); e-print [arXiv:physics/0703201](https://arxiv.org/abs/physics/0703201)
84. M. Ausloos, A. Pełalski, *Physica A* **373**, 560 (2007)
85. J. Silver, E. Slud, K. Takamoto, *J. Econ. Theory* **106**, 417 (2002)
86. P. Richmond, in *Econophysics of Markets and Business Networks*, edited by A. Chatterjee, B.K. Chakrabarti (Springer, Milan, 2007), p. 261
87. P. Bak, *How Nature Works* (Oxford University Press, Oxford, 1997)
88. See e.g. L.D. Landau, E.M. Lifshitz, *Statistical Physics* (Pergamon Press, Oxford, 1968)
89. P. Richmond, B.K. Chakrabarti, A. Chatterjee, J. Angle, in *Econophysics of Stock and other Markets*, edited by A. Chatterjee, B.K. Chakrabarti (Springer, Milan, 2006), 244
90. V.M. Yakovenko, e-print [arXiv:0709.3662v3](https://arxiv.org/abs/0709.3662v3)
91. N. Scafetta, B.J. West, *J. Phys: Cond. Matter* **19**, 065138 (2007)