PROPAGATION OF A BORE PRODUCED BY THE SUDDEN BREAK OF A DAM

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[Received 12 September 1968]

SUMMARY

In this paper we study the progress of a bore, produced by the sudden break of a dam, when there is a flow of water ahead of the dam and the bed has a mild slope and offers resistance, employing Whitham's rule. We first derive certain interesting results from the general discussion of the differential equation, expressing the variation of the bore strength with the undisturbed Froude number, M_0, ratio of bed slope to bed resistance, $g\alpha/R \equiv a^2$ and the bore strength $M(x) \equiv \sqrt{\{h(x)/h_0(x)\}}$ where $h(x)$ and $h_0(x)$ are the bore height and the undisturbed height of the water immediately ahead of the bore, the horizontal distance x being measured from the dam. The parameters M_0 and a^2 combine to influence the bore strength in a very special way. We also examine the asymptotic cases when the bore strength $M \to \infty$ and $M \to 1$. The intermediate cases are investigated numerically to bring out the effects of the parameters, α, a^2, M_0 and the dam height on the strength of the bore, its velocity and the fluid velocity behind it.

1. Introduction

In this paper we consider the production and propagation of a bore when a dam suddenly breaks. The problem is essentially the same as was considered by Craya (1) and Re' (2) and which has also been reported by Stoker (3). It may be stated as follows: There is water of certain height behind the dam, while ahead of it there is an established steady flow of water. The bed ahead of the dam has a mild slope and also offers resistance to flow which varies empirically as the square of the flow velocity. No steady-state solution of the existing flow ahead of the dam was, however, considered in (1) or (2). As pointed out by Stoker (3), the numerical solution given by Re' through step-by-step integration is too approximate. Stoker (3) has also discussed the solution of this problem in the idealized form when there is no resistance of the bed and bed-slope is zero, so that the solution corresponds to the well-known 'simple wave' solution in gas-dynamics. Of course the entire work, referred to above, has been done in the framework of shallow water theory whose equations can be transformed in such a way that they become one-dimensional gas-dynamics equations (the well known gas-dynamic analogy) so that all the techniques employed in gas-dynamics can be readily made use of. The above stated problem with the difference that there is not water ahead of the dam has been considered by Dressler (4) and Whitham (5). Indeed, the problem then becomes entirely analogous to the expansion of a gas into vacuum in gas-dynamics as discussed by Greenspan and Butler (6). Incidentally the expansion for the solution used in (4) and (6) are in essence exactly the same, though there is no mention of this fact in (6).

Keller *et al.* (7) have discussed the climb of a bore above a sloping beach using an approximate method of Whitham (8). We have also made use of Whitham's technique (8) in the present paper. Its extreme simplicity and a good accuracy have led to its use in a large number of investigations, particularly in astrophysics (see, for example, (9), (10)). The rule consists in substituting the shock conditions on the forward characteristic if the shock or bore diverges and on the negative characteristic if they converge. The result is a first-order non-linear differential equation in one of the shock (bore) strength parameters and distance from a reference point and hence the entire course and strength of the shock (bore) can be found, using the initial condition from the physical situation. The approximation in this theory arises from the neglect of disturbances that come from behind the shock (bore), but generally they have been found to be negligible. For example, Whitham's rule and the exact numerical methods (7), (8) yield nearly the same results in the case of a bore climbing up a sloping beach referred to above.

In the present paper, we have first studied the established steady flow when the resistance coefficient is taken according to Chézy's formula assuming the bed-slope to be constant as has been done by Dressler (11). This equation expresses the relation between Froude number $M_0 = u_0/\sqrt{(gh_0)}$ and distance from the original site of the dam. Then the 'shallow water' equations with the Chézy resistance formula and the bed-slope are combined with the bore conditions according to Whitham's rule to obtain a differential equation in bore strength $M = \sqrt{(h/h_0)}$ and M_0. This equation and the equation for M_0 are simultaneously solved. The initial bore strength is obtained from the simple-wave solution of Stoker (3) without bed-resistance and bed-slope. Then we consider some general results from the equations concerning the effects of bed-slope, ratio of bed-slope to bed-resistance and the undisturbed Froude number on the bore-strength. We have also studied analytically the asymptotic forms of this equation when $M \to \infty$ and $M \to 1$. We have numerically integrated these equations to study the above effects. The results reveal a very intimate relation between the ratio of bed-slope and resistance coefficient and the undisturbed Froude number in influencing the propagation of the bore. We finally remark that our mathematical analysis is similar to that of Bird (10) in his investigation on the heating of the outer solar atmosphere.

2. Equations of motion and boundary conditions

We denote the depth of water by $h(x, t) = h_0(x)+\eta(x, t)$, $h_0(x)$ being the depth of water in the steady flow ahead of the dam and $\eta(x, t)$ being the height of water in the disturbed region behind the bore above the $h_0(x)$ level, where x and t are the distance from the dam site, and time, respectively. The equation of shallow water theory are:

$$\eta_t+[(\eta+h_0)u]_x = 0, \tag{2.1}$$

$$u_t+uu_x+g\eta_x+R\left(\frac{u}{c}\right)^2 = 0, \tag{2.2}$$

where c is the 'sound speed'

$$c = (gh)^{\frac{1}{2}} \tag{2.3}$$

and $R(u/c)^2$ is the Chézy resistance term, R having the dimensions of acceleration. In terms of c, equations (2.1) and (2.2) become

$$2c_t+2uc_x+cu_x = 0, \tag{2.4}$$

$$u_t+uu_x+2cc_x-g\frac{dh_0}{dx}+R\left(\frac{u}{c}\right)^2 = 0. \tag{2.5}$$

Along the positive characteristic

$$dx/dt = u+c, \tag{2.6}$$

with the corresponding compatibility condition

$$du+2dc-g\frac{dh_0}{u+c}+R\left(\frac{u}{c}\right)^2\frac{dx}{u+c} = 0. \tag{2.7}$$

In terms of the bore strength

$$M = c/c_0 = \sqrt{(h/h_0)}, \tag{2.8}$$

the bore conditions are given by

$$u = c_0\left[M_0+\left(\frac{M^2+1}{2}\right)^{\frac{1}{2}}\frac{M^2-1}{M}\right], \tag{2.9}$$

$$U = c_0\left[M_0+\left(\frac{M^2+1}{2}\right)^{\frac{1}{2}}M\right], \tag{2.10}$$

where u and U are the particle velocity just behind the bore and bore velocity respectively and M_0 is the Froude number of the undisturbed flow

$$M_0 = u_0/c_0. \tag{2.11}$$

The steady flow ahead of the bore is given by

$$h_0 u_0 = K, \tag{2.12}$$

$$\frac{dh_0}{dx} = \frac{\alpha-(R/g)M_0^2}{1-M_0^2}, \tag{2.13}$$

or

$$\frac{dM_0}{dx} = -\frac{3\alpha}{2a^2}\frac{M_0^{5/3}}{K'}\frac{a^2-M_0^2}{1-M_0^2} \tag{2.14}$$

where α is the constant bed slope, $a^2 = g\alpha/R$ and $K' = K^{\frac{2}{3}}g^{-\frac{1}{3}}$ has the dimension of a length and depends on the initial conditions only. Though equation (2.14) can be integrated in a closed form, it is more convenient in the present problem to use it in its differential form.

Following Whitham's rule (8), we substitute the bore conditions (2.9) and (2.10) along the positive characteristic (2.7), making use of the undisturbed steady flow equation (2.14) to obtain

$$\frac{dM}{dM_0} = \frac{f_1(M,M_0)+f_2(M,M_0,a^2)}{D}, \tag{2.15}$$

where

$$f_1(M,M_0) = M^2\left[\left\{M^2-MM_0+(M^2-1)\left(\frac{M^2+1}{8}\right)^{\frac{1}{2}}\right\}\times\right.$$
$$\left.\times\left\{MM_0+M^2+(M^2-1)\left(\frac{M^2+1}{2}\right)^{\frac{1}{2}}\right\}-M^2\right], \tag{2.16}$$

$$f_2(M,M_0,a^2) = \frac{(1-M_0^2)\left[MM_0+(M^2-1)\left(\frac{M^2+1}{2}\right)^{\frac{1}{2}}\right]^2}{a^2-M_0},$$

and

$$D = \frac{3}{2}\frac{MM_0}{\sqrt{\{2(M^2+1)\}}}\left[MM_0+M^2+(M^2-1)\left(\frac{M^2+1}{2}\right)^{\frac{1}{2}}\right]\times$$
$$\times[\sqrt{\{8(M^2+1)\}}+(M^2+1)^2+M^2(M^2-1)].$$

Equation (2.15) reveals some interesting results. The denominator D is always positive. The numerator consists of two terms: $f_1(M, M_0)$ gives the effect of bore strength and undisturbed Froude number M_0 on the change of bore strength, while $f_2(M, M_0, a^2)$ gives the interaction of bore strength, undisturbed Froude number and the ratio of bed-slope to resistance coefficient, $a^2 = g\alpha/R$. It is interesting to note that this equation contains these two effects only in the ratio $g\alpha/R$ and not α and R/g separately.

We note the following:

(i) $$f_1(M, M_0) > 0 \quad \text{if}$$

$$M_0 < \frac{1}{M}\left[\frac{M^2-1}{32}\left\{9M^4+32M^2-9+48M^2\left(\frac{M^2+1}{2}\right)^{\frac{1}{2}}\right\}\right]^{\frac{1}{2}}-\frac{M^2-1}{4M}\left(\frac{M^2+1}{2}\right)^{\frac{1}{2}} \tag{2.17}$$

which follows easily after some algebra. We have calculated bounds of M_0 as given by this inequality when $M = 1{\cdot}5$, 2, 3, 5:

$$M_0 < 1.75, 3.05, 5.98, 13.44, \quad \text{respectively.}$$

If, for a given value of M, M_0 exceeds the value given in the table, the bore strength decreases with M_0. However, since the undisturbed flow is usually sub-critical or mildly supercritical, the bore strength will be generally amplified, due to this term.

(ii) $$f_2(M, M_0, a^2) \to \infty \quad \text{if} \quad M_0 \to a^2 \quad \text{and} \quad a \neq 1$$

(we shall presently consider the case $a = 1$). This is the only singularity of the differential equation (2.15). The differential equation (2.15) shows that the bore strength increases beyond limit when the undisturbed Froude number $M_0 \to a^2$.

The following cases need special mention and discussion:

(a) When $a^2 < 1$, the resistance effect is more important than the gravitational acceleration due to bed slope. It can be easily seen from the expression for $f_2(M, M_0, a^2)$ that

$$\left.\begin{aligned} f_2(M, M_0, a^2) > 0 \quad \text{if} \quad M_0 < a^2 \quad \text{or} \quad M_0 > 1,\\ f_2(M, M_0, a^2) < 0 \quad \text{if} \quad a^2 < M_0 < 1.\end{aligned}\right\} \tag{2.18}$$

Thus, the bore strength increases with M_0 if the undisturbed flow is supercritical or the undisturbed Froude number is less than a^2, while it decreases with increasing M_0 if the undisturbed Froude number lies between a^2 and 1, implying that it is less than one so that the flow is subcritical.

(b) When $a^2 > 1$, the resistance effect is less important than the gravitational effect and

$$\left.\begin{aligned} f_2(M, M_0, a^2) > 0 \quad \text{if} \quad M_0 < 1 \quad \text{or} \quad M_0 > a^2,\\ f_2(M, M_0, a^2) < 0 \quad \text{if} \quad 1 < M_0 < a^2,\end{aligned}\right\} \tag{2.19}$$

showing that the bore strength increases with M_0 if the undisturbed flow is sub-critical or if the flow is supercritical such that the undisturbed Froude number is greater than $a^2 > 1$ and the bore strength diminishes with increasing M_0 if the undisturbed Froude number lies between 1 and a^2.

As pointed out earlier, the strengthening or attenuation of the bore strongly depends on the relative magnitude of a^2 and M_0 as is very neatly brought out by the above discussion.

Before considering some numerical results, we discuss the two asymptotic cases, that is, when the bore is very strong and when the bore is very weak. Whitham's rule is known to give extremely good results in these asymptotic cases (**8**).

3. Asymptotic cases

(A) When the height of the dam, h_1, is very large and, correspondingly, the initial bore height $h \gg h_0$ so that $M \to \infty$, we obtain from equation (2.15), after some simplification,

$$\frac{1}{h_0}\frac{dh_0}{dM} = -\frac{4}{M}$$

so that

or

$$\left.\begin{aligned} M &\propto h_0^{-\frac{1}{4}} \\ \eta &\propto h_0 \end{aligned}\right\} \tag{3.1}$$

and we recover the formula obtained earlier by Keller *et al.* (**7**). The above result shows that when the strength of the bore is very large, the bed resistance has no effect on its strength. Also, as was concluded in (**7**), the bore height tends to zero as $h_0 \to 0$, while the bore strength $M \to \infty$. It is interesting to compare these remarks with those made by Greenspan and Butler (**6**) while discussing the expansion of gas into vacuum which is exactly analogous to the present problem when $h_0 \to 0$, i.e. the bed is dry and the dam breaks instantaneously. In fact, in this case, there is no genuine shock or bore. The bore speed becomes infinitely large, as can be easily seen from the bore conditions, and so does the particle velocity behind the bore, so that the water rushes with a great speed but the 'bore regime' is meaningless as there is no water ahead of this 'virtual bore'. Indeed, the head of the disturbance is a characteristic and not a bore. The solution of the problem is a centred simple wave (and a somewhat distorted form of it, if the resistance and slope are significant) (**4**).

(B) We also discuss the case when $M-1$ is small so that the bore is weak, though this case is not relevant to the dam-break problem. If we substitute

$$M = 1+z, \tag{3.2}$$

where z is small, in equation (2.15), then to first order in z, we obtain

$$\frac{dz}{dM_0} = \frac{1}{6}\frac{a^2(5+4M_0+2M_0^2)+M_0(2M_0^4-M_0^3-10M_0^2-6M_0+4)}{M_0(1+M_0)^2(a^2-M_0^2)}z+ + \frac{1}{6}\frac{(1-a^2)M_0}{(1+M_0)(a^2-M_0^2)}. \tag{3.3}$$

To study the behaviour of this equation in the neighbourhood of the singularity $M_0 = a$, we put

$$M_0 = a+\xi, \tag{3.4}$$

where ξ is small so that equation (3.3) becomes, to first order in ξ,

$$\frac{dz}{d\xi}+\left(\frac{2a^3-a^2-5a+4}{12a}\frac{1}{\xi}+\frac{6a^4+a^3-14a^2-37a-4}{24a^2(1+a)}\right)z = \frac{a-1}{12}\frac{1}{\xi}-\frac{(a-1)^2}{24a(1+a)}. \tag{3.5}$$

This equation can be integrated to give

$$z\xi^m e^{n\xi} = \text{constant}-\frac{(a-1)^2\xi^m}{24a(1+a)n}e^{n\xi}+ + \frac{a-1}{6}\frac{4a^4-a^3-9a^2-14a-4}{24a^2(1+a)n}\int \xi^{m-1}e^{n\xi}d\xi, \tag{3.6}$$

where

$$m = (2a^3-a^2-5a+4)/12a, \quad n = (6a^4+a^3-14a^2-37a-4)/24a^2(1+a).$$

We find that $6a^4+a^3-14a^2-37a-4 = 0$ has only one positive root, namely $a = 2{\cdot}211$, so that $z \to \infty$ when $a \to 2{\cdot}211$ and the bore strength increases beyond limit and, in fact, the approximation which assumes z to be small breaks down. It is interesting to note that this critical value of a (whose magnitude was however found to be exactly 2), appears in different contexts, for example in the formation of roll-waves (**11**) which cannot be produced unless $a^2 > 4$. This condition is necessary for the instability of the steady flow and the formation of roll waves. This also arises in the discussion of kinematic waves (**12**), which cannot describe the phenomenon of flood waves when $a^2 > 4$.

4. Numerical results and discussion

We have studied numerically the cases listed in Table 1.

Our choice of parameters was motivated by two considerations, namely, to study the effect of variation of α, a^2, M_0 and h_1 and to bring out numerically the significance of our conclusions that we arrived at in section 2.

Case (1) is the same as considered by Re' (**2**), so that h_0 and h_1 are in metres and $u_0 = 1{\cdot}6$ m/sec. To obtain the initial bore height, we consider

TABLE 1

Case	α	a^2	M_0	h_0 (*metres*)	h_1 (*metres*)
1	0·9	1·5	0·3808	1·8	10·8
2	9	1·5	0·3808	1·8	10·8
3	0·9	10	0·3808	1·8	10·8
4	0·9	0·25	0·3808	1·8	10·8
5	0·9	1·5	1·2494	1·8	10·8
6	0·9	1·5	0·3808	1·8	30·0
7	0·9	1·5	2·000	1·8	10·8

the simple wave solution of the problem (3):

$$u_0+u+[(u_0-u)^2+2g(h_0+h)]^{\frac{1}{2}} = \frac{2(hu-h_0u_0)}{h-h_0}, \tag{4.1}$$

$$u = 2\{\sqrt{(gh_1)}-\sqrt{(gh)}\}+u_0h_0/h_1. \tag{4.2}$$

These equations are easily deduced from the simple wave solution by eliminating the sound speed c behind the bore and the bore velocity U so that knowing the data h_1, h_0, u_0, we can solve equations (4.1) and (4.2) simultaneously for h and u. Thus, the initial height of the bore becomes known.

Before discussing the numerical results, we remark that we have carried out our integration from $x = 0$ to $x = 15$ for case (1) and to $x = 10$ for other cases except when the singularity appeared or the solution became a constant. From (2.14) we note that M_0 is, in fact, a function of $x\alpha$ and not of x and α separately so that our actual length scale is $x\alpha$. A given value of $x\alpha$ can be obtained by a suitable combination of values of x and α in infinitely many ways. While interpreting the solution we shall keep this fact in mind. Moreover, since K' has been measured in metres, $x\alpha$ has also been expressed in the same unit. In Figures 1 to 3, we have, for convenience, taken x in metres, while α has been taken to be either 0·9 or 9, representing two extreme cases.

We give in Table 2 below the sample results for one of the cases (1) which corresponds to the problem of Re' (2).

TABLE 2

Case (1): $\alpha = 0{\cdot}9$, $a^2 = 1{\cdot}25$, $M_0 = 0{\cdot}3808$, $h_0 = 1{\cdot}8m$, $h_1 = 10{\cdot}8m$

x	M_0	M	h_0	h	I	U (m/s)	u (m/s)
0·0	0·3808	1·618	1·8	4·713	1·618	10·75	7·253
1·0	0·2043	1·408	2·726	5·408	1·490	9·951	5·467
2·0	0·1327	1·299	3·635	6·137	1·390	9·788	4·460
3·0	0·0951	1·233	4·538	6·904	1·314	9·874	3·800
4·0	0·0725	1·190	5·440	7·698	1·254	10·08	3·330
5·0	0·0576	1·158	6·342	8·512	1·206	10·34	2·975
6·0	0·0472	1·136	7·243	9·339	1·165	10·64	2·697
7·0	0·0396	1·118	8·143	10·18	1·130	10·95	2·472
8·0	0·0338	1·104	9·044	11·02	1·100	11·27	2·286
9·0	0·0293	1·093	9·944	11·88	1·074	11·60	2·129
10·0	0·0257	1·084	10·84	12·73	1·050	11·92	1·995
11·0	0·0228	1·076	11·74	13·60	1·029	12·24	1·879
12·0	0·0205	1·069	12·64	14·46	1·009	12·56	1·777
13·0	0·0184	1·064	13·54	15·33	0·992	12·87	1·687
14·0	0·0167	1·059	14·45	16·20	0·975	13·18	1·607
15·0	0·0153	1·055	15·35	17·07	0·961	13·49	1·534

The variation of bore strength $I\ (\equiv (h - h_0)/h_{00})$, bore velocity U and particle velocity u, behind the bore are shown in Figures 1 to 3.

Cases 4, 5, and 7 merit separate discussion. Case (4) differs from case (1) in that $a^2 = 0{\cdot}25$ instead of 1·5, that is, bed resistance is more important. The bore strength increases continuously up to $x \simeq 0{\cdot}97$, where M_0 has diminished to 0·25 approximately from its initial value 0·3808 and the singularity in equation (2.15), $M_0 = a^2$, is approached. The strength of the bore increases beyond limit at this point and further progress of the bore cannot be followed.

In case (5), the initial undisturbed Froude number, M_{00}, is taken to be 1·249 instead of 0·3808 as in case (1), to study the effect of supercritical undisturbed flow on the bore propagation. The bore strength very gradually increases from its initial value 1·1098 to 1·1348 at $x \simeq 2{\cdot}5$ where $M_0^2 \to a^2$ (so that $dM_0/dx = 0$) and, in fact, the variable solution of the equation of undisturbed flow (2.14) merges with its constant solution $M_0 = a$. Therefore, after $x \simeq 2{\cdot}5$, M_0 remains constant and hence the bore strength (equation (2.15)) and bore velocity remain constant throughout the later course of the bore.

In case (7), $M_{00} = 2{\cdot}00$ as against $M_{00} = 0{\cdot}3808$ for case (1), a bore of small strength, 0·701 ($I_0 = 1{\cdot}618$ for case (1)), is produced. We find that

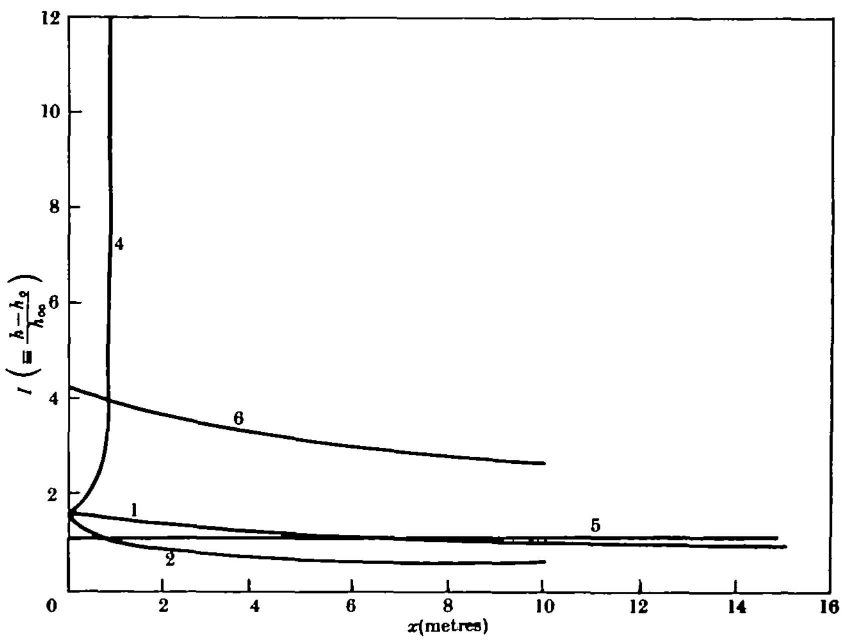

FIG. 1. Variation of bore strength with distance for different parameters (Table 1).

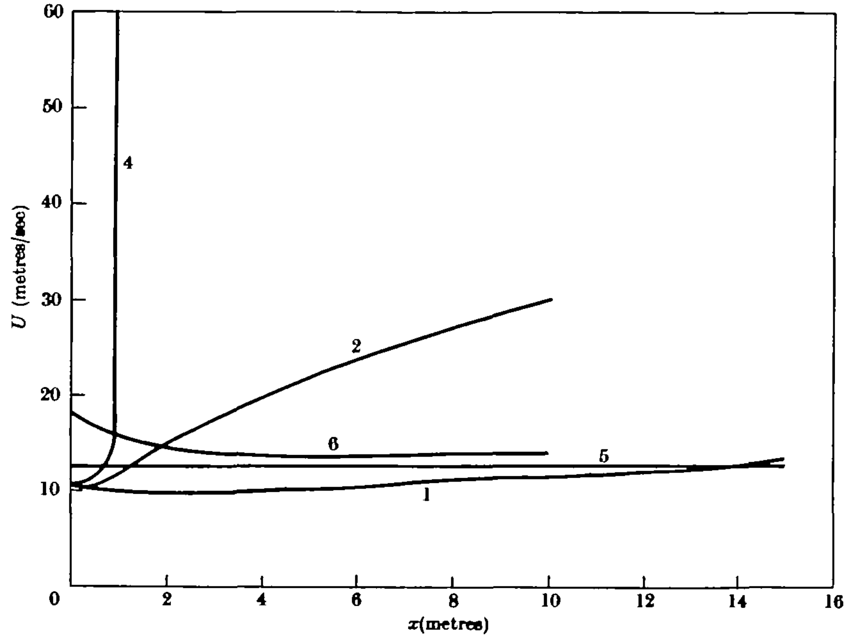

FIG. 2. Variation of bore velocity with distance for different parameters (Table 1).

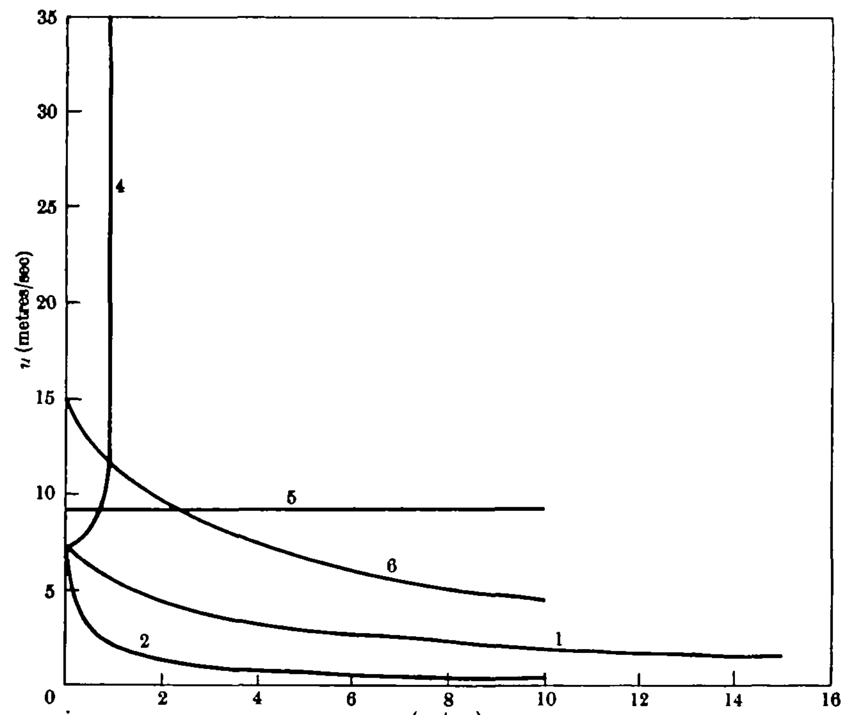

FIG. 3. Variation of particle velocity behind the bore with distance for different parameters (Table 1).

the bore strength rapidly decreases so that the bore is completely dissipated near $x \simeq 0{\cdot}6$.

Now we turn to other cases to study the effect of variation of other parameters, namely, dam height h_1, bed slope α, ratio of bed slope to bed resistance coefficient a^2. We find that the results in case (3) with $a^2 = 10$ are not very different from those of case (1) with $a^2 = 1{\cdot}5$, showing that if the slope of the bed and other parameters are kept constant while a^2 is changed to a large value (which means that resistance coefficient is much smaller than the bed slope), the bore propagation is hardly affected. In case (2), the bed slope, $\alpha = 9$, is 10 times that in case (1). One could, of course, also look at this case to be case (1) if α is taken to be $0{\cdot}9$ and the distance 10 times the corresponding distance of case (1). However, here we want to study the effect of the change of slope on the bore propagation. When the slope of the bed increases, the bore strength I decreases much more rapidly so that when $x = 10$, $I = 0{\cdot}612$ for $\alpha = 9$ and $I = 1{\cdot}05$ for $\alpha = 0{\cdot}9$, both showing, however, that the rate of decay of the bore becomes very small in the later stages of propagation. The bore velocity U first decreases up to $x \simeq 2{\cdot}2$ and then continuously increases for case (1) when slope is smaller while for case (2), this reversal takes place much earlier ($x \simeq 0{\cdot}75$). This trend is in conformity with that obtained by Keller *et al.* (7), if we allow for the change of the sign of the slope. Particle velocity behind the bore continuously decreases as the bore propagates, but the decrease is very large when the slope is large, for example at $x = 10{\cdot}00$, $u = 1{\cdot}995$ for $\alpha = 0{\cdot}9$ and $u = 0{\cdot}390$ for $\alpha = 9$ compared to the initial value $u = 7{\cdot}25$, at $x = 0$. This decrease is particularly significant in the early stages of propagation of the bore as is readily noted from Figure 3.

Finally, we discuss the effect of the change of the dam-height to 30 m, case (6), which gives the initial bore height as 8·924 m compared to 4·71 m for case (1). In this case the bore strength decreases from its initial value 4·203 to 2·711 at $x = 10$, while the bore velocity decreases (unlike case (1)) right up to $x = 7{\cdot}0$ where it begins to increase, though very gradually. The particle velocity behind the bore continuously decreases.

In all these cases, the bore settles down to nearly constant strength at $x \simeq 10$ and its decay afterwards becomes very slow. One would have to carry out the integration over a very large interval to locate the point where the bore finally decays completely. This was not possible due to lack of requisite computational facilities.

Stoker (3) has given the solution of Re' (2) to a distance of about 1·8 km, but perhaps it is difficult to make any useful comparison, considering the very approximate nature of Re's solution.

Acknowledgement

The authors are grateful to Mr. S. S. Krishnamoorthy for programming and running the problem on the Elliot 803 computer at the H.A.L. Bangalore.

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