OSCILLATIONS OF A THREE-COMPONENT ASSEMBLY IN THE PRESENCE OF A MAGNETIC FIELD USING THE GENERALIZED BGK COLLISION MODEL

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(*Received October 4, 1962*)

The BGK collision model for one-component assembly of neutral particles has been extended to two-component assembly of charged particles by Gross and Krook (1956) and later on modified by Bhatnagar (1962). Following the lines of the latter, the model has been generalized to N-component assembly of both charged and neutral particles. This model is further applied to the study of small amplitude plasma oscillations in an assembly consisting of ions, electrons and neutral particles in the direction perpendicular to a uniform magnetic field. The dispersion relation splits up into two, one determining the transverse oscillations and the other longitudinal oscillations. In the transverse oscillations for small wave numbers k, it has been shown that apart from the Gross-gaps occurring at the multiples of gyro-frequencies of electrons and ions, if the magnetic energy density M is greater than one-third the kinetic energy density K of charged particles, and terms only up to k^2 are retained, five more forbidden ranges of frequencies occur. If $M < \frac{1}{3}K$, the number of additional gaps reduces to three. When $M = 0$, Oster's (1960) result is obtained as a particular case. The oscillations of neutral particles excited by collisions are strong at low frequencies, whereas for high frequencies they are mostly damped out. Exact analytical and graphical discussion of the transverse dispersion relations is given. Longitudinal propagation has been studied under very restricted circumstance numerically and it is shown that, unless the magnetic field is very high, propagation is possible for all frequencies. For a sufficiently high magnetic field, when the Alfvén velocity is comparable with the velocity of light, there is one forbidden range but, for the discussion of such high velocities, one should work with the relativistic equations.

In the present note, we have generalized the collision model proposed by Bhatnagar *et al.* (1954) to an N-component assembly and have used it for the study of small oscillations of a plasma consisting of electrons, neutral molecules and positive ions in the presence of a uniform magnetic field. It is interesting to find that this incorporates the salient characteristics of the medium-like as well as particle-like behaviour of the plasma.

1. In this section, we shall discuss the generalization of the collision model proposed by Bhatnagar *et al.* (1954) for one-component assembly and its subsequent modifications for two-component assembly by Gross and Krook (1956) and by Bhatnagar (1962), in a form suitable for the discussion of the properties of an N-component assembly. As emphasized in the previous investigations, though this model is approximate, it satisfies the requirements

of the conservation laws and preserves certain essential features of the collision mechanism such as 'persistence of velocity', and enables us to study a wide variety of problems over the whole range of the mean free path preserving their essential features.

Let $f_s(\vec{\xi}_s, \vec{r}, t)$ be the distribution function of the sth component and $\vec{\xi}_s$, the molecular velocity of the particles of the sth component at the point $\vec{r}$ and at time t. In the standard notations of Chapman and Cowling (1960) the set of the Boltzmann equations describing the behaviour of the assembly is

$$\frac{\partial f_s}{\partial t}+\xi_{si}\frac{\partial f_s}{\partial x_i}+\frac{F_{si}}{m_s}\frac{\partial f_s}{\partial \xi_{si}} = \sum_{r=1}^{N}\int\int\Big[f_r'\Big(\vec{\xi}_r', \vec{r}, t\Big)f_s'\Big(\vec{\xi}_s', \vec{r}, t\Big)$$

$$-f_r\Big(\vec{\xi}_r, \vec{r}, t\Big)f_s\Big(\vec{\xi}_s, \vec{r}, t\Big)\Big]g_{rs}bdbd\epsilon d\vec{\xi}_r, \quad \ldots \quad (1.1)$$

$$s = 1, 2, \ldots N,$$

where

$$g_{rs} = \left|\vec{\xi}_r-\vec{\xi}_s\right|,$$

m_s is the mass of a particle of sth type, $\vec{F}_s$ the force acting on it and the primes denote the value of the quantities after collision. We first interpret the two terms occurring in the Boltzmann collision integral, namely

$$(a) \qquad -f_s\Big(\vec{\xi}_s, \vec{r}, t\Big)\int\int f_r\Big(\vec{\xi}_r, \vec{r}, t\Big)g_{rs}bdbd\epsilon d\vec{\xi}_r, \quad \ldots \quad \ldots \quad (1.2)$$

and

$$(b) \qquad \int\int f_r'\Big(\vec{\xi}_r', \vec{r}, t\Big)f_s'\Big(\vec{\xi}_s', \vec{r}, t\Big)g_{rs}bdbd\epsilon d\vec{\xi}_r. \quad \ldots \quad \ldots \quad (1.3)$$

The term (a) represents the number of particles of the sth type absorbed from a given definite velocity range $(\vec{\xi}_s, d\vec{\xi}_s)$ due to collision with rth type of particles, while the term (b) represents the number of particles of the sth type brought into that range due to collisions with the particles of the rth type. Following Bhatnagar *et al.* (1954), Gross and Krook (1956) and Bhatnagar (1962), we approximate (a) by

$$(c) \quad -\frac{N_r(\vec{r}, t)}{\sigma_{rs}}f_s\Big(\vec{\xi}_s, \vec{r}, t\Big) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

and (b) by

$$(d) \quad \frac{N_r(\vec{r}, t)N_s(\vec{r}, t)}{\sigma_{rs}}\Phi_{rs}\Big(\vec{\xi}_s, \vec{r}, t\Big). \quad \ldots \quad \ldots \quad \ldots \quad (1.5)$$

We can understand the significance of the above approximations as follows:

(c) The number of particles absorbed from the range $(\vec{\xi}_s, d\vec{\xi}_s)$ due to collision with the particles of rth type is clearly proportional to the number of particles $f_s(\vec{\xi}_s, \vec{r}, t)$ of the sth type in that range and to the number of collisions N_r/σ_{rs} that a particle of sth type undergoes with particles of rth type per unit time and per unit volume. In general the number of collisions N_r/σ_{rs} would depend upon g_{rs}, but to make the model workable we assume it to be a constant and equal to some suitable average value of the collision frequency. We also note that $\sigma_{rs} = \sigma_{sr}$.

(*d*) The emission term (*b*) is the non-linear term and is the actual source of difficulty. The number of particles of the sth type emitted into that range is proportional to the number of collisions $\frac{N_s N_r}{\sigma_{rs}}$ taking place per unit time and per unit volume and to the probability $\Phi_{rs}(\vec{\xi}_s, \vec{r}, t)$ that a particle of sth type is brought by a collision into that velocity range. Evidently Φ_{rs} is also the relative distribution function of the scattered particles. In the absence of any knowledge of the actual collision mechanism, the best assumption that we can make about it is that the emitted particles are distributed randomly and hence Φ_{rs} is taken to be locally Maxwellian with mean velocity $\vec{u}_{rs}(\vec{r}, t)$ about the mean temperature $T_{rs}(\vec{r}, t)$. Hence we get

$$\Phi_{rs}\left(\vec{\xi}_s, \vec{r}, t\right) = \left(\frac{m_s}{2\pi K T_{rs}}\right)^{3/2} \exp\left[-\frac{m_s}{2KT_{rs}}\left(\vec{\xi}_s - \vec{u}_{rs}\right)^2\right], \quad \ldots \quad \ldots \quad (1.6)$$

where K denotes the Boltzmann constant.

The choice of $\vec{u}_{rs}$ and T_{rs}, besides satisfying the requirements of conservation laws, must give correct initial and asymptotic behaviour of the assembly.

With the proposed collision model, the kinetic equation for the distribution function f_s becomes

$$\frac{\partial f_s}{\partial t} + \xi_{st}\frac{\partial f_s}{\partial x_t} + \frac{F_{st}}{m_s}\frac{\partial f_s}{\partial \xi_{st}} = \sum_{r=1}^{N}\left[-\frac{N_r}{\sigma_{rs}} f_s + \frac{N_r N_s}{\sigma_{rs}}\Phi_{rs}\right], \quad \ldots \quad \ldots \quad (1.7)$$

$$s = 1, 2, \ldots N.$$

We note the following point:

The equations (1.7) look linear, partial differential equations in the distribution functions, but the definition of the N_r and the self-consistent electromagnetic field introduces non-linearity and the integral terms once again, thereby preserving those essential features of the exact Boltzmann equations (1.1).

Conservation laws: (i) *Conservation of the mass of the sth component.*—On multiplying the right-hand side of (1.7) by m_s and integrating with respect to $\vec{\xi}_s$, we get

$$\sum_{r=1}^{N}\left[\frac{-N_r N_s}{\sigma_{rs}} m_s + \frac{N_r N_s}{\sigma_{rs}} m_s\right],$$

since

$$N_s = \int f_s d\vec{\xi}_s,$$

and

$$1 = \int \Phi_{rs} d\vec{\xi}_s. \qquad (1.8)$$

Thus the model ensures the conservation of mass of each component separately, as it should, in the absence of inelastic collisions. Application of this law does not provide any equation between $\vec{u}_{rs}$ and T_{rs}.

(ii) *Conservation of momentum.*—Since we have assumed in the model that it is possible to distinguish and separate the contribution to the momentum of each component during collision with any other specified component, the amount of momentum change in the sth component due to collision with the particles of rth component should be equal to the amount of momentum change in the rth component due to collision with particles of sth component with the sign reversed. Multiplying the right-hand side of (1.7) by $m_s\vec{\xi}_s$ and integrating over $\vec{\xi}_s$, the change in momentum of the sth component due to collisions is given by

$$\sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}} m_s(\vec{u}_{rs} - \vec{u}_{ss}), \qquad (1.9)$$

where

$$\vec{u}_{ss} = \frac{1}{N_s}\int f_s \vec{\xi}_s d\vec{\xi}_s, \qquad (1.10)$$

is the mean velocity of sth component and the prime denotes that $r \neq s$.

The conservation of total momentum of the sth and the rth components in collisions between themselves separately yields

$$m_s(\vec{u}_{rs} - \vec{u}_{ss}) + m_r(\vec{u}_{sr} - \vec{u}_{rr}) = 0, \qquad (1.11)$$

$$r, s = 1, 2, \dots N.$$

The momentum change due to collision for the whole gas is

$$\sum_{s=1}^{N}\sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}} m_s(\vec{u}_{rs} - \vec{u}_{ss}) = \tfrac{1}{2}\sum_{s=1}^{N}\sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}}\{m_s(\vec{u}_{rs} - \vec{u}_{ss}) + m_r(\vec{u}_{sr} - \vec{u}_{rr})\}. \qquad (1.12)$$

Evidently in view of (1.11) there is no change in the total momentum for the whole assembly, though the momentum of each component is not conserved during collisions.

(iii) *Conservation of energy.*—Multiplying the right-hand side of (1.7) by $\frac{1}{2}\, m_s \vec{\xi}_s^{\,2}$ and integrating with respect to $\vec{\xi}_s$, we find that the expression for the change of energy of the sth component due to collision is

$$\sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}} \cdot \frac{m_s}{2}\left[\frac{3K}{m_s}(T_{rs}-T_{ss})+\left(\vec{u}_{rs}^{\,2}-\vec{u}_{ss}^{\,2}\right)\right], \quad \ldots \quad \ldots (1.13)$$

where

$$\frac{3KT_{ss}}{m_s} = \frac{1}{N_s}\int f_s\left(\vec{\xi}_s-\vec{u}_{ss}\right)^2 d\vec{\xi}_s. \quad \ldots \quad \ldots \quad \ldots (1.14)$$

Arguing as in case (ii) the energy conservation leads to the equation

$$m_s\left[\frac{3K}{m_s}(T_{rs}-T_{ss})+\vec{u}_{rs}^{\,2}-\vec{u}_{ss}^{\,2}\right]+m_r\left[\frac{3K}{m_r}(T_{sr}-T_{rr})+\vec{u}_{sr}^{\,2}-\vec{u}_{rr}^{\,2}\right] = 0, \ldots \quad (1.15)$$

$$r,\, s = 1, 2, \ldots N,$$

for each pair of components of sth and rth type. Again it is evident that the total change in energy of the whole assembly due to collisions vanishes identically in view of (1.15).

The equations (1.11) and (1.15) constitute $\frac{3N(N-1)}{2}+\frac{N(N-1)}{2} = 2N(N-1)$ equations between $3N(N-1)+N(N-1) = 4N(N-1)$ scalar unknowns $\vec{u}_{rs}$ and T_{rs}. Hence the number of equations is not sufficient to determine them uniquely. Consequently, we have to supply $2N(N-1)$ additional relations between them from other physical considerations.

The mean velocity $\vec{u}_{rs}$ of the sth type particles, emitted after collision with rth type of particles, depends on the mean velocities of the colliding particles, for the sake of simplicity we shall assume that $\vec{u}_{rs}$ is a linear combination of $\vec{u}_{rr}$ and $\vec{u}_{ss}$:

$$\vec{u}_{rs} = a_{rr}\vec{u}_{rr}+a_{rs}\vec{u}_{ss}. \quad \ldots \quad \ldots \quad \ldots (1.16)$$

The assumption of linearity is apparently very restrictive, but these terms may be treated as the first two terms of the expansion of $\vec{u}_{rs}$. Additional support in favour of these assumptions is provided by the form of the expression for the rate of transfer of momentum from one component to the other deduced later on the basis of (1.16).

Similarly, we shall assume that

$$T_{rs} = b_{rr}T_{rr}+b_{rs}T_{ss}+D_{rs}\vec{u}_{rr}^{\,2}+E_{rs}\vec{u}_{rr}\cdot\vec{u}_{ss}+F_{rs}\vec{u}_{ss}^{\,2}. \quad \ldots (1.17)$$

The phenomenological assumptions (1.16) and (1.17) introduce $7N(N-1)$ constants which have to be evaluated by considering some simple non-equilibrium phenomena like relaxation problem.

2. *Relaxation problem.*—The basic transfer equations are obtained by taking the successive moments of the kinetic equations (1.7):

$$U_s \equiv \frac{\partial N_s}{\partial t} + \frac{\partial}{\partial x_j}(N_s u_{ssj}) - \frac{N_s}{m_s} < \frac{\partial F_{sj}}{\partial \xi_{sj}} > = 0, \quad \text{(2.1)}$$

$$V_s \equiv \frac{\partial}{\partial t}(N_s u_{ssi}) + \frac{\partial}{\partial x_j}[N_s (P_s)_{ij}] - \frac{N_s}{m_s} < \frac{\partial}{\partial \xi_{sj}}(F_{sj}\xi_{si}) > = \sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}}(u_{rsi} - u_{ssi}), \quad \text{(2.2)}$$

$$W_s \equiv \frac{\partial}{\partial t}\left(\frac{3KN_s T_{ss}}{m_s}\right) + \frac{\partial}{\partial x_j}[N_s Q_{sj}] + 2N_s\left[(P_s)_{ij}\frac{\partial u_{ssj}}{\partial x_i} - u_{ssi}u_{ssj}\frac{\partial u_{ssj}}{\partial x_i}\right]$$

$$- \frac{N_s}{m_s} < \frac{\partial}{\partial \xi_{sj}}\left[F_{sj}\left(\vec{\xi}_s - \vec{u}_{ss}\right)^2\right] > = \sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}}\left[\frac{3K}{m_s}(T_{rs} - T_{ss}) + \left(\vec{u}_{rs} - \vec{u}_{ss}\right)^2\right], \quad \text{(2.3)}$$

where

$$(P_s)_{ij} = \frac{1}{N_s}\int f_s(\xi_{si} - u_{ssi})(\xi_{sj} - u_{ssj})d\vec{\xi}_s$$

and (2.4)

$$\vec{Q}_s = \frac{1}{N_s}\int f_s \vec{\xi}_s(\vec{\xi}_s - \vec{u}_{ss})^2 d\vec{\xi}_s .$$

and $< >$ stands for the operator

$$\frac{1}{N_s}\int d\vec{\xi}_s f_s.$$

For studying the relaxation problem we neglect the external forces, and so the only force acting on the particles is the self-consistent electric field given by

$$\frac{\partial}{\partial x_j}(E_j) = 4\pi \sum_s e_s N_s \,. \quad \text{(2.5)}$$

Further, neglecting the space dependence of the physical quantities, we get

$$\frac{\partial N_s}{\partial t} = 0, \quad \text{(2.6)}$$

$$\frac{\partial}{\partial t}(N_s \vec{u}_{ss}) = \sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}}(\vec{u}_{rs} - \vec{u}_{ss}), \quad \text{(2.7)}$$

$$\frac{\partial}{\partial t}\left(\frac{3KN_s T_{ss}}{m_s}\right) = \sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}}\left[\frac{3K}{m_s}(T_{rs} - T_{ss}) + \left(\vec{u}_{rs} - \vec{u}_{ss}\right)^2\right], \quad \text{(2.8)}$$

$$\sum_s e_s N_s = 0. \quad \text{(2.9)}$$

The last equation asserts that the assembly is electrostatically neutral at every stage, and so

$$\vec{E} = 0 \quad \ldots \quad (2.10)$$

for all time t.

This fact has been used in writing (2.7). The equation (2.6) gives $N_s =$ constant for all time t. Substituting (1.16) and (1.17) in (2.7) and (2.8), we get

$$\frac{\partial}{\partial t}(\vec{u}_{ss}) = \sum_{r=1}^{N}{}' \frac{N_r}{\sigma_{rs}}[a_{rr}\vec{u}_{rr} - (1-a_{rs})\vec{u}_{ss}], \quad \ldots \quad (2.11)$$

$$\frac{\partial}{\partial t}(T_{ss}) = \sum_{r=1}^{N}{}' \frac{N_r}{\sigma_{rs}}[b_r T_{rr} - (1-b_{rs})T_{ss} + D_{rs}\vec{u}_{rr}^{\,2} + E_{rs}\vec{u}_{rr}\cdot\vec{u}_{ss} + F_{rs}\vec{u}_{ss}^{\,2}$$

$$+\frac{m_s}{3K}\{a_{rr}\vec{u}_{rr} - (1-a_{rs})\vec{u}_{ss}\}^2], \quad \ldots \quad (2.12)$$

$$s = 1, 2, \ldots N.$$

We have to solve these equations under the following conditions:

Initial conditions: at $t = 0$

$$\vec{u}_{ss} = \vec{A}_s,\ T_{ss} = T_s.$$

Final conditions: as $t \to \infty$,

$$\vec{u}_{ss} \to \vec{u}_{\infty},\ T_{ss} \to T_{\infty},$$

$$\frac{\partial \vec{u}_{ss}}{\partial t} \to 0,\ \frac{\partial T_{ss}}{\partial t} \to 0,\ s = 1, 2, \ldots N. \quad \ldots \quad (2.13)$$

The explicit solution of the simultaneous differential equations (2.11) and (2.12) involves considerable analysis and hence we shall follow simpler physical considerations to evaluate the unknown constants.

The phenomenological relations (1.16) and (1.17) hold in any inertial frame of reference, so that changing over to a frame moving with an arbitrary uniform velocity $\vec{u}_0$, the equation (1.16) will be invariant to this transformation, if

$$a_{rr} = 1 - a_{rs}. \quad \ldots \quad (2.14)$$

Similarly, the invariance of (1.17) under the transformation yields

$$(D_{rs} + \tfrac{1}{2}E_{rs})\vec{u}_{rr} + (F_{rs} + \tfrac{1}{2}E_{rs})\vec{u}_{ss} = 0,$$

and

$$D_{rs} + E_{rs} + F_{rs} = 0. \quad \ldots \quad (2.15)$$

The last equation should hold good for all time t, and in particular for time $t = 0$, so that

$$(D_{rs} + \tfrac{1}{2}E_{rs})\vec{A}_r + (F_{rs} + \tfrac{1}{2}E_{rs})\vec{A}_s = 0.$$

Since $\vec{A}_r$ and $\vec{A}_s$ can be prescribed arbitrarily,

$$D_{rs} = -\tfrac{1}{2}E_{rs} = F_{rs}. \qquad (2.16)$$

Because of (2.16), the condition (2.15) is automatically satisfied.

As each component of the system relaxes to the same mean velocity $\vec{u}_\infty$ and the mean temperature T_∞, taking limit as $t\to\infty$ in the equations (1.16), (1.17), (2.11) and (2.12), we have, in view of (2.13),

$$a_{rr}+a_{rs} = 1, \qquad (2.17a)$$

$$D_{rs}+E_{rs}+F_{rs} = 0. \qquad (2.17b)$$

$$b_{rr} = 1-b_{rs}. \qquad (2.17c)$$

The conditions (2.17*a*) and (2.17*b*) are the same as (2.14) and (2.15). Thus the relations (2.14), (2.16) and (2.17*c*) constitute the conditions imposed on these phenomenological constants by the considerations of the relaxation problem. Utilizing these relations, we can rewrite (1.16) and (1.17) as

$$\vec{u}_{rs}-\vec{u}_{ss} = a_{rr}\left(\vec{u}_{rr}-\vec{u}_{ss}\right), \qquad (2.18)$$

and

$$T_{rs}-T_{ss} = b_{rr}\,(T_{rr}-T_{ss})+D_{rs}\left(\vec{u}_{rr}-\vec{u}_{ss}\right)^2. \qquad (2.19)$$

Substituting (2.18) in the conservation laws (1.11), we get

$$(m_s a_{rr}-m_r a_{ss})\left(\vec{u}_{rr}-\vec{u}_{ss}\right) = 0,$$

or

$$\frac{a_{rr}}{m_r} = \frac{a_{ss}}{m_s} = \alpha_{rs} = \alpha_{sr} \text{ (say)}. \qquad (2.20)$$

Similarly, substitution of (2.19) in (2.15) yields

$$(b_{rr}-b_{ss})(T_{rr}-T_{ss})+\left\{D_{rs}+D_{sr}+\frac{m_s a_{rr}}{K}(a_{rr}+a_{ss}-2)\right\}\left(\vec{u}_{rr}-\vec{u}_{ss}\right)^2 = 0$$

where we have used (2.20). Since we can prescribe the initial mean temperatures and mean velocities arbitrarily, we have

$$b_{rr} = b_{ss} = \beta_{rs} = \beta_{sr} \text{ (say)} \qquad (2.21)$$

and

$$D_{rs}+D_{sr} = \frac{m_s a_{rr}}{3K}(2-a_{rr}-a_{ss}). \qquad (2.22)$$

Thus we get the additional relations (2.20), (2.21) and (2.22) from the considerations of conservation laws. Consequently, out of the $7N(N-1)$ constants which we have introduced, $\frac{3N(N-1)}{2}$ constants are yet to be determined. Neither the relaxation problem nor the conservation laws are adequate to determine these constants. This is as expected since so far we have completely ignored the actual mode of interaction that takes place between various particles in the assembly, during collisions. We shall, therefore, determine these constants approximately from the considerations of the molecular interactions.

Rewriting (2.18) as

$$m_s(\vec{u}_{rs}-\vec{u}_{ss}) = \left(\frac{m_r+m_s}{m_r}a_{rr}\right)\cdot\frac{m_r m_s}{m_r+m_s}\cdot(\vec{u}_{rr}-\vec{u}_{ss}),$$

we conclude that $\frac{m_r+m_s}{m_r}a_{rr}$ represents the fraction of the average relative momentum of the colliding particles that has been imparted to the scattered particle of the sth type in a collision with a rth type of particle. If the molecules are treated as rigid elastic spheres, Jeans' (1954a) relation gives

$$a_{rr} = \frac{m_r}{m_r+m_s}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.23)$$

so that

$$\alpha_{rs} = \alpha_{sr} = \frac{1}{m_r+m_s}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.24)$$

In general,

$$a_{rr} = \frac{m_r}{m_r+m_s}\left(2\sin\frac{\chi_{rs}}{2}\right)_{\text{average}}$$

where χ_{rs} denotes the angle through which the relative velocity of the rth type of particle with respect to sth type of particle turns due to collision between them. Following Jeans (1954a) we can show

$$a_{rr} \simeq 0{\cdot}113\frac{m_r}{m_r+m_s} \text{ for Coulomb law}$$

$$\simeq 0{\cdot}023\frac{m_r}{m_r+m_s} \text{ for Maxwellian law of inverse fifth power.}$$

In a similar fashion, from (2.19) we conclude that $\frac{6K(m_r+m_s)}{m_r m_s}D_{rs}$ represents the fraction of the average kinetic energy that has been utilized in raising the temperature of the scattered particle of sth type due to collision with a particle of rth type, after deducting the direct heat transfer given by $3K\beta_{rs}(T_{rr}-T_{ss})$. We have to assign appropriate values to β_{rs} using Newton's law of heat transfer between the rth and sth components from experimental considerations.

Simpler but approximate expressions for β_{rs} and D_{rs} can be obtained, following the suggestion of Prof. Cowling (*private communication*) in the following manner under the assumption that the velocities of the scattered particles are randomly oriented:

The velocities of the molecule m_s before and after collision with a molecule m_r are

$$\vec{G}_{rs}+\frac{m_r}{m_r+m_s}\vec{g}_{rs} \text{ and } \vec{G}_{rs}+\frac{m_r}{m_r+m_s}\vec{g}\,'_{rs}, \quad \ldots \quad \ldots \quad (2.25)$$

so that the mean velocity of the scattered sth type of particle is

$$\vec{u}_{rs}=\frac{m_r\vec{u}_{rr}+m_s\vec{u}_{ss}}{m_r+m_s}, \quad \ldots \quad (2.26)$$

agreeing with equations (2.18) and (2.23). Also the mean kinetic energy of the molecule m_s after collision with molecule m_r is

$$\left[\tfrac{1}{2}m_s\left(\vec{G}_{rs}+\frac{m_r}{m_r+m_s}\vec{g}'_{rs}\right)^2\right]_{\text{average}}$$

$$=\left[\tfrac{1}{2}m_s\left\{\vec{G}^2_{rs}+\frac{m_r^2}{(m_r+m_s)^2}\vec{g}^2_{rs}\right\}\right]_{\text{average}}$$

$$=\left[\tfrac{1}{2}\frac{m_s}{(m_r+m_s)^2}\left\{(m_r+m_s)^2\vec{G}^2_{rs}+m_s^2\vec{C}_s^2+m_r^2\vec{C}_r^2\right.\right.$$

$$\left.\left.+m_r^2(\vec{u}_{rr}-\vec{u}_{ss})^2+m_s^2(\vec{C}_s^2+\vec{C}_r^2)\right\}\right]_{\text{average}}$$

where $\vec{C}_s$ denotes the peculiar velocity of the sth type of particle, on using the fact that the averages of the peculiar velocities are zero.

Since

$$\left[\tfrac{1}{2}m_s\vec{C}_s^2\right]_{\text{average}}=\frac{3KT_{ss}}{2},$$

and the above average according to the model is also equal to

$$\left[\tfrac{1}{2}m_s\vec{G}^2_{rs}\right]_{\text{average}}+\frac{3KT_{rs}}{2},$$

we get

$$\frac{3KT_{rs}}{2}=\frac{m_sm_r^2}{2(m_r+m_s)^2}(\vec{u}_{rr}-\vec{u}_{ss})^2+\frac{3K}{2}\left\{\left(m_r^2+m_s^2\right)T_{rr}+2m_rm_sT_{ss}\right\}. \quad \ldots \quad (2.27)$$

Comparing (2.27) with (2.19), we have

$$b_{rr}=\frac{m_r^2+m_s^2}{(m_r+m_s)^2}, \quad \ldots \quad (2.28)$$

and

$$D_{rs}=\frac{m_sm_r^2}{3K(m_r+m_s)^2}. \quad \ldots \quad (2.29)$$

From symmetry we can write

$$D_{sr}=\frac{m_rm_s^2}{3K(m_r+m_s)^2}, \quad \ldots \quad (2.30)$$

so that

$$D_{rs}+D_{sr}=\frac{m_rm_s}{3K(m_r+m_s)},$$

which is the same as (2.22) on using Jeans' relation which holds under the present assumptions.

We have also attempted to evaluate D_{rs} taking into consideration the actual nature of the interaction between the particles. We have in the notation of Chapman and Cowling:

$$\frac{6K(m_r+m_s)}{m_r m_s} D_{rs} = \left[\frac{\frac{1}{2}m_s\left(\vec{\xi}_s^{\prime 2}-\vec{\xi}_s^{2}\right)}{\frac{1}{2}\frac{m_r m_s}{m_r+m_s}\vec{g}_{rs}^{2}}\right]_{\text{average}}$$

$$= \left[\frac{4(\vec{G}_{rs}\,.\,\vec{k})(\vec{g}_{rs}\,.\,\vec{k})}{\vec{g}_{rs}^{2}}\right]_{\text{average}}$$

Following the method of Jeans (1954*a*), we can easily show that

$$D_{rs} \cong 0{\cdot}10\,\frac{m_r m_s}{6K(m_r+m_s)} \text{ for Coulomb law}$$

$$\cong 0{\cdot}04\,\frac{m_r m_s}{6K(m_r+m_s)} \text{ for Maxwellian law.}$$

We note that the conditions (2.14), (2.16), (2.17*c*), (2.21) and (2.22) can also be obtained (Bhatnagar 1962) by the straightforward solution of the differential equations (2.11) and (2.12). This approach further gives the relaxation times $\tau(\vec{u}_{ss})$ and $\tau(T_{ss})$ for the mean velocities and the mean temperatures as

$$\tau(\vec{u}_{ss}) = 2\tau(T_{ss}) = \frac{1}{\sum_{r=1}^{N}\frac{N_r}{\sigma_{rs}}}, \qquad \ldots \quad \ldots \quad \ldots \quad (2.31)$$

so that the steady mean temperature is attained earlier than the steady mean velocity.

With the help of these relations, the final expressions for the transport equations are

$$U_s = 0,$$

$$V_s = \sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}} m_r \alpha_{rs}(u_{rri}-u_{ssi}),$$

$$W_s = \sum_{r=1}^{N}{}' \frac{N_r N_s}{\sigma_{rs}}\left[\frac{3K}{m_s}\beta_{rs}(T_{rr}-T_{ss})+\left(\frac{3K}{m_s}D_{rs}+m_r^2\alpha_{rs}^2\right)(\vec{u}_{rr}-\vec{u}_{ss})^2\right] \quad \ldots \quad (2.32)$$

We note that our phenomenological relations and the subsequent conditions lead to simple and physically meaningful expressions for momentum and heat transports. The momentum transferred to sth component due to collision with rth component is

$$\frac{N_r N_s}{\sigma_{rs}} m_r m_s \alpha_{rs} (\vec{u}_{rr} - \vec{u}_{ss}) \quad \ldots \quad (2.33)$$

whereas the energy transferred is

$$\frac{3K}{2} \beta_{rs}(T_{rr} - T_{ss}) + \tfrac{1}{2}(3KD_{rs} + m_s m_r^2 \alpha_{rs}^2)(\vec{u}_{rr} - \vec{u}_{ss})^2 + \tfrac{1}{2} m_s (\vec{u}_{rs}^2 - \vec{u}_{ss}^2) \quad \ldots \quad (2.34)$$

The expression (2.33) for momentum transfer is quite reasonable and similar expressions have been used on other physical considerations by Cowling (1959) and Spitzer (1959). Interpreting the three terms of (2.34) as the energy transferred due to difference in temperature of the two components, the amount of mean relative kinetic energy retained in the sth component itself and the amount of energy gained by the sth component due to the increase in the mean velocity of the scattered particles, we see that all the mechanisms that can change its energy content have been fully covered. These considerations lend support to the plausibility of the present model.

3. *Oscillations of the plasma transverse to a strong magnetic field.*—We shall now apply the above model to the study of the small amplitude wave propagation in a plasma composed of ions, electrons and neutral particles in equilibrium state. Without loss of generality, we can choose the z-axis along the direction of constant external magnetic field $\vec{H}_0$.

The kinetic equations are

$$\frac{\partial f_s}{\partial t} + \xi_{si} \frac{\partial f_s}{\partial x_i} + \frac{e_s}{m_s}\left(E_i + \frac{\epsilon_{ijk}\xi_{sj}H_k}{c}\right)\frac{\partial f_s}{\partial \xi_{si}} = -\left(\sum_{r=1}^{3} \frac{N_r}{\sigma_{rs}}\right) f_s + N_s \left(\sum_{r=1}^{3} \frac{N_r}{\sigma_{rs}} \Phi_{rs}\right), \quad \ldots \quad (3.1)$$

$$s = 1, 2$$

and

$$\frac{\partial f_3}{\partial t} + \xi_{3i} \frac{\partial f_3}{\partial x_i} = -\left(\sum_{r=1}^{3} \frac{N_r}{\sigma_{r3}}\right) f_3 + N_3 \left(\sum_{r=1}^{3} \frac{N_r}{\sigma_{r3}} \Phi_{r3}\right), \quad \ldots \quad (3.2)$$

where the suffixes 1, 2 and 3 denote respectively electrons, ions and neutral particles. In the sequel, s would always assume the values of 1 and 2 only unless stated otherwise and we shall treat the neutral particles separately.

Assuming that initially the assembly is in equilibrium with the mean temperature T_0 and with no mean motion and that there is cylindrical symmetry in velocity space about the z-axis which is also the direction of the initial magnetic field, we can show that the equilibrium distribution is given by

$$f_{s0} = N_{s0}(\Phi_{rs})_0 = N_{s0}\left(\frac{m_s}{2\pi K T_0}\right)^{\frac{3}{2}} \exp\left(-\frac{m_s}{2KT_0}\vec{\xi}_s^2\right). \quad \ldots \quad (3.3)$$

In the disturbed state, let the distribution function be given by $f_s = f_{s0} + g_s$, where g_s is small compared to f_{s0}. In a similar fashion, let $N_s = N_{s0}(1+n_s)$, $T_{rs} = T_0(1+t_{rs})$, $\vec{H} = \vec{H}_0+\vec{h}$, while $\vec{u}_{ss}$, $\vec{u}_{rs}$ and $\vec{E}$ are only the perturbations. Neglecting the squares and products of small perturbations and the induced magnetic field $\vec{h}$ in comparison with $\vec{H}_0$ the first order Boltzmann equations become

$$\frac{\partial g_s}{\partial t}+\xi_{si}\frac{\partial g_s}{\partial x_i}+\frac{e_s}{m_s}E_i\frac{\partial f_{s0}}{\partial \xi_{si}}-\omega_s\frac{\partial g_s}{\partial \theta_s} = -\frac{1}{\sigma_s}g_s+\frac{1}{\sigma_s}f_{s0}n_s+f_{s0}\sum_{r=1}^{3}\frac{N_{r0}}{\sigma_{rs}}\phi_{rs}, \quad \ldots \quad (3.4)$$

$$\frac{\partial g_3}{\partial t}+\xi_{3i}\frac{\partial g_3}{\partial x_i} = -\frac{1}{\sigma_3}g_3+\frac{1}{\sigma_3}f_{30}n_3+f_{30}\sum_{r=1}^{3}\frac{N_{r0}}{\sigma_{r3}}\phi_{r3}, \quad \ldots \quad \ldots \quad (3.5)$$

where

$$\omega_s = \frac{e_sH}{cm_s}, \frac{1}{\sigma_s} = \sum_{r=1}^{3}\frac{N_{r0}}{\sigma_{rs}}, \frac{1}{\sigma_3} = \sum_{r=1}^{3}\frac{N_{r0}}{\sigma_{r3}},$$

$$\phi_{rs} = \phi'_{rs}+\phi''_{rs},$$

$$\phi'_{rs} = -\tfrac{3}{2}t_{rs}+\frac{3}{2a_s^2}t_{rs}\vec{\xi}_s^{\,2}+\frac{3}{a_s^2}\xi_{sz}u_{rsz},$$

$$\phi''_{rs} = \frac{3}{a_s^2}(\xi_{sx}u_{rsx}+\xi_{sy}u_{rsy}),$$

$$\xi_{sx} = \rho_s\cos\theta_s,\ \xi_{sy} = \rho_s\sin\theta_s, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

and

$$a_s^2 = \frac{3KT_0}{m_s} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

Since we are interested in the propagation in the direction perpendicular to the magnetic field, we choose the direction of propagation as the direction of x-axis, so that all the quantities may be taken to vary as $\exp\,[i(kx-\omega t)]$. This assumption reduces (3.4) and (3.5) to

$$\frac{\partial g_s}{\partial \theta_s}-\frac{i\left(k\rho_s\cos\theta_s-\omega-\dfrac{i}{\sigma_s}\right)}{\omega_s}g_s = -\frac{1}{\sigma_s\omega_s}f_{s0}n_s$$

$$-\frac{3f_{s0}}{\omega_s a_s^2}[\rho_sG_{s-}e^{i\theta_s}+\rho_sG_{s+}e^{-i\theta_s}+\xi_{sz}G_{sz}]+\frac{3f_{s0}}{2\omega_s}T_s\left(1-\frac{\vec{\xi}_s^{\,2}}{a_s^2}\right), \quad \ldots \quad (3.8)$$

and

$$i\left(k\xi_{3x}-\omega-\frac{i}{\sigma_3}\right)g_3 = \frac{1}{\sigma_3}f_{30}n_3+f_{30}\sum_{r=1}^{3}\frac{N_{r0}}{\sigma_{r3}}\phi_{r3}, \quad \ldots \quad \ldots \quad (3.9)$$

where

$$\vec{G}_1 = \frac{e_1}{m_1}\vec{E} + \frac{1}{\sigma_1'}\vec{u}_{11} + \frac{N_{20}}{\sigma_{21}} a_{22}\vec{u}_{22} + \frac{N_{30}}{\sigma_{31}} a_{33}\vec{u}_{33},$$

$$\vec{G}_2 = \frac{e_2}{m_2}\vec{E} + \frac{N_{10}}{\sigma_{12}} a_{11}\vec{u}_{11} + \frac{1}{\sigma_2'}\vec{u}_{22} + \frac{N_{30}}{\sigma_{32}} a_{33}\vec{u}_{33},$$

$$\vec{G}_3 = \frac{N_{10}}{\sigma_{13}} a_{11}\vec{u}_{11} + \frac{N_{20}}{\sigma_{23}} a_{22}\vec{u}_{22} + \frac{1}{\sigma_3'}\vec{u}_{33},$$

$$G_{s+} = \frac{G_{sx}+iG_{sy}}{2}, \quad G_{s-} = \frac{G_{sx}-iG_{sy}}{2},$$

$$\frac{1}{\sigma_1'} = \frac{N_{10}}{\sigma_{11}} + \frac{N_{20}}{\sigma_{21}} a_{21} + \frac{N_{30}}{\sigma_{31}} a_{31}, \qquad \text{.. .. .. (3.10)}$$

and

$$T_1 = \left(\frac{N_{10}}{\sigma_{11}} + \frac{N_{20}}{\sigma_{21}} b_{21} + \frac{N_{30}}{\sigma_{31}} b_{31}\right) t_{11} + \frac{N_{20}}{\sigma_{21}} b_{22} t_{22} + \frac{N_{30}}{\sigma_{31}} b_{33} t_{33}. \qquad \text{.. (3.11)}$$

with similar expressions for

$$T_2, \; T_3, \; \frac{1}{\sigma_2'} \text{ and } \frac{1}{\sigma_3'}.$$

Integrating with respect to θ_s and setting $\Omega_s = \omega + \frac{i}{\sigma_s}$ we get

$$g_s = \exp\left[\frac{i(k\rho_s \sin\theta_s - \Omega_s\theta_s)}{\omega_s}\right]\Bigg\{A_1(\rho_s, \xi_{sz})$$

$$+\left[-\frac{1}{\sigma_s\omega_s} f_{s0} n_s - \frac{3f_{s0}}{\omega_s a_s^2} G_{sz}\xi_{sz} + \frac{3f_{s0}}{2\omega_s} T_s\left(1 - \frac{\vec{\xi}_s^{\,2}}{a_s^2}\right)\right] \sum_{n=-\infty}^{\infty} J_n\left(-\frac{k\rho_s}{\omega_s}\right)\cdot \frac{e^{i\left(n+\frac{\Omega_s}{\omega_s}\right)\theta_s} - 1}{i\left(n+\frac{\Omega_s}{\omega_s}\right)}$$

$$-\frac{3f_{s0}}{\omega_s a_s^2}\rho_s G_{s-} \sum_{n=-\infty}^{\infty} J_n\left(-\frac{k\rho_s}{\omega_s}\right) \frac{e^{i\left(n+1+\frac{\Omega_s}{\omega_s}\right)\theta_s} - 1}{i\left(n+1+\frac{\Omega_s}{\omega_s}\right)}$$

$$-\frac{3f_{s0}}{\omega_s a_s^2}\rho_s G_{s+} \sum_{n=-\infty}^{\infty} J_n\left(-\frac{k\rho_s}{\omega_s}\right) \frac{e^{i\left(n-1+\frac{\Omega_s}{\omega_s}\right)} - 1}{i\left(n-1+\frac{\Omega_s}{\omega_s}\right)}\Bigg\},$$

where $A(\rho_s, \xi_{sz})$ is an arbitrary function. In order to determine it we make use of the physical fact that there is symmetry in the velocity space about the direction of the magnetic field so that g_s is periodic in θ_s with period 2π.

Thus we get

$$A(\rho_s, \xi_{sz}) = \left[-\frac{f_s}{\sigma_s \omega_s} n_s - \frac{3f_{s0}}{\omega_s a_s^2} G_{sz}\xi_{sz} + \frac{3f_{s0}}{2\omega_s} T_s \left(1 - \frac{\vec{\xi}_s^2}{a_s^2}\right)\right] \sum_{n=-\infty}^{\infty} \frac{J_n\left(-\frac{k\rho_s}{\omega_s}\right)}{i\left(n+\frac{\Omega_s}{\omega_s}\right)}$$

$$-\frac{3f_{s0}}{\omega_s a_s^2} \rho_s G_{s-} \sum_{n=-\infty}^{\infty} \frac{J_n\left(\frac{-k\rho_s}{\omega_s}\right)}{i\left(n+1+\frac{\Omega_s}{\omega_s}\right)} - \frac{3f_{s0}}{\omega_s a_s^2} \rho_s G_{s+} \sum_{n=-\infty}^{\infty} \frac{J_n\left(-\frac{k\rho_s}{\omega_s}\right)}{i\left(n-1+\frac{\Omega_s}{\omega_s}\right)},$$

so that finally we have

$$g_s = \frac{if_{s0}}{\omega_s} \exp\left(\frac{ik\rho_s \sin\theta_s}{\omega_s}\right) \left\{ \frac{3\rho_s G_{s-}}{a_s^2} \sum_{n=-\infty}^{\infty} J_{n-1}\left(-\frac{k\rho_s}{\omega_s}\right) \frac{e^{in\theta_s}}{n+\frac{\Omega_s}{\omega_s}} \right.$$

$$+\frac{3\rho_s G_{s+}}{a_s^2} \sum_{n=-\infty}^{\infty} J_{n+1}\left(-\frac{k\rho_s}{\omega_s}\right) \frac{e^{in\theta_s}}{n+\frac{\Omega_s}{\omega_s}}$$

$$\left. +\left[\frac{n_s}{\sigma_s} + \frac{3G_{sz}\xi_{sz}}{a_s^2} + \frac{3T_s}{2}\left(\frac{\vec{\xi}_s^2}{a_s^2} - 1\right)\right] \sum_{n=-\infty}^{\infty} J_n\left(-\frac{k\rho_s}{\omega_s}\right) \frac{e^{in\theta_s}}{n+\frac{\Omega_s}{\omega_s}} \right\}, \quad \ldots (3.12)$$

and

$$g_3 = \left[\frac{n_3}{\sigma_3} + \frac{3\vec{G}_3 \cdot \vec{\xi}_3}{a_3^2} + \frac{3T_3}{2}\left(\frac{\vec{\xi}_3^2}{a_3^2} - 1\right)\right] \frac{if_{30}}{\Omega_3 - k\xi_{3x}}. \quad \ldots \ldots (3.13)$$

The equations (3.12) and (3.13) express g_s and g_3 as linear combinations of the perturbations $\vec{E}$, $\vec{u}_{11}$, $\vec{u}_{22}$, $\vec{u}_{33}$, t_{11}, t_{22}, t_{33}, n_1, n_2 and n_3.

To find the dispersion relation we proceed as follows: Taking successive moments up to the third order of the equations (3.4) and (3.5) and utilizing the fact that all perturbations vary as $\exp[i(kx-\omega t)]$, we first obtain the following set of equations:

$$-\omega n_s + ku_{ssx} = 0, \quad \ldots \ldots \ldots \ldots (3.14)$$

$$-\omega n_3 + ku_{33x} = 0, \quad \ldots \ldots \ldots \ldots (3.15)$$

$$\mu_{r1}u_{11z} + \mu_{r2}u_{22z} + \mu_{r3}u_{33z} + \frac{e_r}{m_r}E_z = 0, \quad \ldots \ldots \ldots (3.16)$$

$$\left(i\Omega_r + \frac{1}{\sigma_r'} - \mu_{rr}\right)u_{rrx} + \mu_{r1}u_{11x} + \mu_{r2}u_{22x} + \mu_{r3}u_{33x} + \omega_r u_{rry} + \frac{e_r}{m_r}E_x + L_r = 0, \quad (3.17)$$

$$\left(i\Omega_r + \frac{1}{\sigma_r'} - \mu_{rr}\right)u_{rry} + \mu_{r1}u_{11y} + \mu_{r2}u_{22y} + \mu_{r3}u_{33y} - \omega_r u_{rrx} + \frac{e_r}{m_r}E_y + M_r = 0, \quad (3.18)$$

$$\left(i\Omega_r + \frac{1}{\sigma_r''} - \nu_{rr}\right)t_{rr} + \nu_{r1}t_{11} + \nu_{r2}t_{22} + \nu_{r3}t_{33} + iku_{rrx} + N_r = 0, \quad \ldots \ldots \ldots (3.19)$$

$$r = 1, 2, 3,$$

where

$$\mu_{rs} = \frac{N_{s0}}{\sigma_{rs}} a_{ss}, \quad \nu_{rs} = \frac{N_{s0}}{\sigma_{rs}} b_{ss}, \quad r \neq s, \qquad \text{(3.20)}$$

$$\mu_{ss} = \frac{i\Omega_s}{1 + \frac{\pi k}{\omega_s}[A_s(1, 0; 2) + A_s(1, 1; 2)]} + \frac{1}{\sigma_s}, \qquad \text{(3.21)}$$

$$\mu_{33} = \frac{i\Omega_3}{1 + kI_1} + \frac{1}{\sigma_3}, \qquad \text{(3.22)}$$

$$L_s = \frac{\pi k}{2\omega_s}\Big[\Big(\frac{n_s}{\sigma_s} - T_s\Big)\{2A_s(0, 0; 3) + A_s(2, 0; 3) + A_s(2, 2; 3)\}$$
$$+ \frac{3T_s}{a_s^2}\{2A_s(0, 0; 5) + A_s(2, 0; 5) + A_s(2, 2; 5)\}$$
$$+ \frac{3G_{s-}}{a_s^2}\{2A_s(1, 1; 4) + A_s(3, 1; 4) + A_s(1, 2; 4)\}$$
$$+ \frac{3G_{s+}}{a_s^2}\{2A_s(1, 0; 4) + A_s(1, -1; 4) + A_s(3, 2; 4)\}\Big] \qquad \text{(3.23)}$$

$$L_3 = k\left[\frac{n_3}{\sigma_3} I_2 + \frac{3}{a_3^2} G_{3x} I_3 - \frac{T_3}{2} I_2 + \frac{3T_3}{2a_3^2} I_4\right], \qquad \text{(3.24)}$$

$$M_s = \frac{\pi k}{2\omega_s}\left[\Big(\frac{n_s}{\sigma_s} - T_s\Big)\{A_s(2, 0; 3) - A_s(2, 2; 3)\} + \frac{3T_s}{a_s^2}\{A_s(2, 0; 5) - A_s(2, 2; 5)\}\right.$$
$$\left. + \frac{3G_{s-}}{a_s^2}\{A_s(3, 1; 4) - A_s(1, 2; 4)\} + \frac{3G_{s+}}{a_s^2}\{A_s(1, -1; 4) - A_s(3, 2; 4)\}\right]$$

$$M_3 = kG_{3y} I_1, \qquad \text{(3.25)}$$

$$N_s = \frac{\pi k}{\omega_s a_s^2}\left\{\frac{n_s}{\sigma_s}\left[A_s(1, 0; 4) + A_s(1, 1; 4) + \frac{a_s^2}{3}\{A_s(1, 0; 2) + A_s(1, 1; 2)\}\right]\right.$$
$$+ \frac{3T_s}{a_s^2}\left[A_s(1, 0; 6) + A_s(1, 1; 6) - \frac{a_s^2}{3}\{A_s(1, 0; 4) + A_s(1, 1; 4)\}\right]$$
$$+ \frac{3G_{s-}}{a_s^2}\left[A_s(2, 1; 5) + A_s(0, 1; 5) + \frac{a_s^2}{3}\{A_s(2, 1; 3) + A_s(0, 1; 3)\}\right]$$
$$\left. + \frac{3G_{s+}}{a_s^2}\left[A_s(0, -1; 5) + A_s(2, 1; 5) + \frac{a_s^2}{3}\{A_s(0, -1; 3) + A_s(2, 1; 3)\}\right]\right\} \qquad \text{(3.26)}$$

$$N_3 = \frac{k}{a_3^2}\left[\frac{n_3}{\sigma_3}\left(I_3+\frac{2a_3^2}{3}I_1\right)+\frac{3}{2a_3^2}G_{3x}\left(I_4+\frac{2a_3^2}{3}I_2\right)\right.$$

$$\left.+\frac{3T_3}{2a_3^2}\left(I_5+\frac{a_3^2}{3}I_3+\frac{2}{9}I_1a_3^4\right)\right], \qquad (3.27)$$

$$A_s(p, \alpha\,;\, q) = \int_0^\infty d\rho_s \frac{3\rho_s^q}{2\pi a_s^2} e^{-\frac{3}{2a_s^2}\rho_s^2} \sum_{n=-\infty}^{\infty} \frac{J_{n+p}\left(-\frac{k\rho_s}{\omega_s}\right) J_n\left(-\frac{k\rho_s}{\omega_s}\right)}{n+\alpha+\frac{\Omega_s}{\omega_s}} \qquad (3.28)$$

$$I_p = \int_{-\infty}^{\infty} \left(\frac{3}{2\pi a_3^2}\right)^{\frac{1}{2}} \frac{e^{-\frac{3}{2a_3^2}\xi_{3x}^2}}{k\xi_{3x}-\Omega_3} \xi_{3x}^p \, d\xi_{3x}, \qquad (3.29)$$

and

$$\frac{1}{\sigma_1''} = \frac{N_{10}}{\sigma_{11}}+\frac{N_{20}}{\sigma_{21}}b_{21}+\frac{N_{30}}{\sigma_{31}}b_{31}, \qquad (3.30)$$

with similar expressions for $\frac{1}{\sigma_2''}$ and $\frac{1}{\sigma_3''}$.

In order to express $\vec{E}$ in terms of mean velocities, we utilize the Maxwell equations and the current equation.

$$\text{div}\,\vec{h} = 0$$

$$\text{curl}\,\vec{h} = \frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial \vec{E}}{\partial t},$$

$$\text{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{h}}{\partial t},$$

$$\text{div}\,\vec{E} = 4\pi(N_{10}e_1n_1+N_{20}e_2n_2),$$

and

$$\vec{J} = N_{10}e_1\vec{u}_{11}+N_{20}e_2\vec{u}_{22},$$

yielding

$$4\pi c^2k^2[N_{10}e_1u_{11x}+N_{20}e_2u_{22x}]\hat{x}+i\omega(\omega^2-c^2k^2)\vec{E} = 4\pi\omega^2[N_{10}e_1\vec{u}_{11}+N_{20}e_2\vec{u}_{22}], \qquad (3.31)$$

where $\hat{x}$ is the unit vector along x-axis.

(3.14) to (3.19) and (3.31) constitute a set of 18 homogeneous equations in 18 scalar unknowns and the condition for non-trivial solution leads to the dispersion relation.

4. *Dispersion relations.*—It is interesting to note that the set of equations (3.14) to (3.19) and (3.31) break up into two mutually exclusive sets (Bhatnagar 1960). The first set contains u_{11z}, u_{22z}, u_{33z} and E_z leading to the

dispersion relation corresponding to transverse wave and the other set defines the longitudinal wave. The dispersion relation for the transverse and longitudinal waves can be written as

$$\Delta_T = 0, \qquad (4.1)$$

and

$$\Delta_L = 0, \qquad (4.2)$$

where

$$\Delta_T = \left[\frac{i\Omega_3}{1+kI_1}+\frac{1}{\sigma_3'}\right]\left[\omega^2-c^2k^2-\omega_{p_1}^2x_1-\omega_{p_2}^2x_2-\frac{i}{\omega}\left\{\left(-\frac{1}{\sigma_1}+\frac{x_1}{\sigma_1'}\right)\omega_{p_1}^2x_1\right.\right.$$
$$\left.\left.+\left(-\frac{1}{\sigma_2}+\frac{x_2}{\sigma_2'}\right)\omega_{p_2}^2x_2+\frac{4\pi e_1e_2(e_1+e_2)N_{10}N_{20}\alpha_{12}}{\sigma_{12}c^2}x_1x_2\right\}\right], \qquad (4.3)$$

$$\Delta_L = \left\{i\Omega_3+\frac{1}{\sigma_3'}+\frac{k^2}{2\omega\sigma_3}I_2+\left(\frac{3k}{2a_3^2}I_3+\frac{k}{2}I_1\right)\frac{1}{\sigma_3'}\right\}\left\{-\frac{\omega^4}{c^2}(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)\right.$$
$$+2\omega^2k^2(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)+3\omega_{p_2}^2k^2\omega\omega_2(\omega_1^2-\omega^2)+3\omega_{p_1}^2k^2\omega\omega_1(\omega_2^2-\omega^2)$$
$$-2\omega_{p_2}^2\frac{\omega^2}{c^2}(\omega_1^2-\omega^2)\omega\omega_2-2\omega_{p_1}^2\frac{\omega^2}{c^2}(\omega_2^2-\omega^2)\omega\omega_1-\tfrac{2}{3}\omega_{p_2}^2\frac{\omega^2}{c^2}(\omega_1^2-\omega^2)$$
$$\times(\omega+\omega_2)\frac{k^2a_2^2}{\omega-2\omega_2}-\tfrac{2}{3}\omega_{p_1}^2\frac{\omega^2}{c^2}(\omega_2^2-\omega^2)(\omega+\omega_1)\frac{k^2a_1^2}{\omega-2\omega_1}+\omega_{p_1}^4\frac{\omega^2}{c^2}(\omega_2^2-\omega^2)$$
$$\times\left[1+\frac{4k^2a_1^2}{3(\omega^2-4\omega_1^2)}\right]-\omega_{p_1}^4k^2(\omega_2^2-\omega^2)+\omega_{p_2}^4\frac{\omega^2}{c^2}(\omega_1^2-\omega^2)$$
$$\times\left[1+\frac{4k^2a_2^2}{3(\omega^2-4\omega_2^2)}\right]-\omega_{p_2}^4k^2(\omega_1^2-\omega^2)-2\omega_{p_1}^2\omega_{p_2}^2\frac{\omega^2}{c^2}(\omega^2-\omega_1\omega_2)$$
$$+2\omega_{p_1}^2\omega_{p_2}^2k^2(\omega^2-\omega_1\omega_2)-\tfrac{4}{3}\omega_{p_1}^2\omega_{p_2}^2\frac{\omega^2}{c^2}\frac{k^2a_2^2(\omega^2+2\omega_2^2-3\omega_1\omega_2)}{\omega^2-4\omega_2^2}$$
$$\left.-\tfrac{4}{3}\omega_{p_1}^2\omega_{p_2}^2\frac{\omega^2}{c^2}\frac{k^2a_1^2(\omega^2+2\omega_1^2-3\omega_1\omega_2)}{\omega^2-4\omega_1^2}+\text{collisional terms}\right\}, \qquad (4.4)$$

where

$$\omega_{p_1}^2 = \frac{4\pi N_{10}e_1^2}{m_1}, \quad \omega_{p_2}^2 = \frac{4\pi N_{20}e_2^2}{m_2}, \qquad (4.5)$$

are the electron and ion plasma frequencies, and

$$x_1 = 1+\frac{\pi k}{\omega_1}[A_1(1, 0; 2)+A_1(1, 1; 2)],$$
$$x_2 = 1+\frac{\pi k}{\omega_2}[A_2(1, 0; 2)+A_2(1, 1; 2)], \qquad (4.6)$$

on retaining only the first order collisional effects.

Transverse oscillations.—The first factor of (4.3) represents the oscillations of the neutral particles damped due to the collisions among themselves and with ions and electrons. The form of I_1 shows that the integral would diverge if there are many particles moving in the x-direction with the wave velocity. However, if the particles move with velocities near the wave velocity, they will be accelerated to such an extent that they will run off the wave. Consequently only the waves having fairly large wave-length (small k) would be possible. Restricting $|k|$ to be sufficiently small, this factor reduces to

$$\frac{a_3^2k^2}{3\sigma_3'} = -i\omega^3\left[1-\frac{i}{\omega}\left(\frac{3}{\sigma_3}+\frac{1}{\sigma_3'}\right)\right], \qquad (4.7)$$

giving the wave velocity

$$V_w = \frac{\omega}{\text{Re}k} = a_3\sqrt{\frac{2}{3\sigma_3\omega}}\left[1+\frac{1}{2\omega}\left(\frac{3}{\sigma_3}+\frac{1}{\sigma_3'}\right)\right], \qquad (4.8)$$

and the damping distance,

$$Z = -\frac{1}{\text{Im}k} = a_3\sqrt{\frac{2}{3\sigma_3\omega^3}}\left[1-\frac{1}{2\omega}\left(\frac{3}{\sigma_3}+\frac{1}{\sigma_3'}\right)\right], \qquad (4.9)$$

provided the wave frequency is larger than the collisional frequencies which have been assumed small in the present investigation. This wave travels with modified sound speed and fairly good transmission takes place at low frequencies. All high-frequency oscillations are quickly damped out and their velocities tend to zero. Hence considerable energy transport by neutral particles can occur only in the wave of low frequencies. We can, therefore, avoid this loss of wave energy in inducing oscillations in the neutral component by considering only the waves with high frequencies.

Retaining the terms up to k^2 only we find the second factor of (4.3) reduces to

$$\frac{c^2k^2}{\omega^2} = \frac{1}{\omega^2}\Bigg[(\omega^2-\omega_{p1}^2-\omega_{p2}^2)+\frac{i}{\omega}\left\{\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_1'}\right)\omega_{p1}^2+\left(\frac{1}{\sigma_2}-\frac{1}{\sigma_1'}\right)\omega_{p2}^2 - \frac{4\pi e_1e_2(e_1+e_2)N_{10}N_{20}\alpha_{12}}{\sigma_{12}c^2}\right\}\Bigg]$$
$$\times\left[1-\frac{a_1^2\omega_{p1}^2}{3c^2}\,\frac{1-\frac{i}{\omega}\left(\frac{1}{\sigma_1}-\frac{2}{\sigma_1'}\right)}{\omega_1^2-\Omega_1^2}-\frac{a_2^2\omega_{p2}^2}{3c^2}\cdot\frac{1-\frac{i}{\omega}\left(\frac{1}{\sigma_2}-\frac{2}{\sigma_2'}\right)}{\omega_2^2-\Omega_2^2}\right.$$
$$\left.-\frac{i}{\omega}\frac{4\pi e_1e_2(e_1+e_2)N_{10}N_{20}\alpha_{12}}{\sigma_{12}c^2}\left(\frac{a_1^2}{3(\omega_1^2-\Omega_1^2)}+\frac{a_2^2}{3(\omega_2^2-\Omega_2^2)}\right)\right]^{-1} \qquad (4.10)$$

It is clear that when $\omega = \omega_1$ or $\omega = \omega_2$, $k = 0$, so that there is no propagation when the frequency of the wave is equal to the gyro-frequency of the electron or the ion. If we retain the higher order terms in k^2, we can show that no wave propagation is possible at integral multiples of gyro-frequencies of the electrons or the ions. This fact was first pointed out by Gross (1951) for a single component collision-free assembly. Later on, Bernstein (1958), Kildal (1960) and Salpeter (1960) rigorously established the presence of Gross-gaps for the single component collision-free assemblies. In passing we may note that the present collision model preserves this essential microscopic feature of the multi-component plasma oscillations in the presence of a magnetic field and collisions.

We shall now draw attention to another interesting aspect, namely the existence of the forbidden bands of frequencies due to the medium-like behaviour of the assembly. Neglecting the collisional effects for the present, and considering only small k, the dispersion relation (4.10) reduces to

$$\frac{c^2k^2}{\omega^2} = \frac{1}{\omega^2}\left[\left(\omega^2-\omega_{p_1}^2-\omega_{p_2}^2\right)\left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_2^2\right)\right] \times\left[\omega^4-\left(\omega_1^2+\omega_2^2-\frac{a_1^2\omega_{p_1}^2}{3c^2}-\frac{a_2^2\omega_{p_2}^2}{3c^2}\right)\omega^2+\omega_1^2\omega_2^2 -\frac{a_1^2\omega_{p_1}^2\omega_2^2}{3c^2}-\frac{a_2^2\omega_{p_2}^2\omega_1^2}{3c^2}\right]^{-1}, \quad \ldots \quad (4.11)$$

$$= \frac{1}{\omega^2}\,\frac{\left(\omega^2-\omega_{p_1}^2-\omega_{p_2}^2\right)\left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_2^2\right)}{(\omega^2-\alpha)(\omega^2-\beta)} \ \text{(say)}, \quad \ldots \quad (4.12)$$

where

$$\alpha\beta = \frac{4\pi e_1^2e_2^2}{m_1^2m_2^2}\,\frac{H_0^2}{c^4}\left[\frac{H_0^2}{4\pi}-KT_0(N_{10}+N_{20})\right], \quad \ldots \quad (4.13)$$

and

$$\alpha+\beta = \frac{4\pi}{c^2}\,\frac{e_1^2}{m_1^2}\left(\frac{H_0^2}{4\pi}-KT_0N_{10}\right)+\frac{4\pi}{c^2}\,\frac{e_2^2}{m_2^2}\left(\frac{H_0^2}{4\pi}-KT_0N_{20}\right). \quad \ldots \quad (4.14)$$

(*a*) If $\frac{H_0^2}{4\pi} > KT_0(N_{10}+N_{20})$, that is the magnetic energy density is greater than one-third of the thermal energy density of electrons and ions, α and β are both positive. If we denote by x_1^2, x_2^2, x_3^2, x_4^2 and x_5^2 the quantities $\omega_{p_1}^2+\omega_{p_2}^2$, ω_1^2, ω_2^2, α and β in ascending order of magnitude, we find that

$$0<\omega<x_1, \quad x_2<\omega<x_3, \quad x_4<\omega<x_5, \quad \ldots \quad (4.15)$$

are the three forbidden ranges of frequency.

(*b*) If $\frac{H_0^2}{4\pi} = KT_0(N_{10}+N_{20})$ either α or β is zero. In this case

$$x_2 < \omega < x_3, \quad x_4 < \omega < x_5, \qquad \ldots \quad \ldots \quad \ldots \quad (4.16)$$

are the only forbidden ranges.

(*c*) If $\frac{H_0^2}{4\pi} < kT_0(N_{10}+N_{20})$ either α or β is negative and once again we have the forbidden ranges as in (*b*).

(*d*) If $H_0 = 0$, i.e. the magnetic field is absent, we have only one forbidden range, namely

$$0 < \omega < \sqrt{\omega_{p_1}^2+\omega_{p_2}^2}, \qquad \ldots \quad \ldots \quad \ldots \quad (4.17)$$

as given by Oster (1960) from continuum equations. This is an unexpected result as our treatment is not strictly valid when $H_0 = 0$.

Thus the assembly exhibits all the 'band-pass filter' characteristics of a medium in the presence of a magnetic field. When the magnetic field is absent, the entire character changes and it exhibits 'high-pass filter' characteristics, the lower critical frequency being $\sqrt{\omega_{p_1}^2+\omega_{p_2}^2}$. We observe that, while the continuum equations could predict a sort of filter properties of the medium, and the Gross (1951) method could give only the microscopic 'selector' properties of the assembly, the present treatment synthesizes both the characteristics. In addition, due to the simplicity of the model, collisional effects can be studied directly. Finally, it is interesting to note that for high-frequency transmission, energy transport by neutral particles is small, while it is fairly high for charged particles. For small frequencies, the neutral particles share a considerable portion of the energy transport, whereas the charged particles permit wave propagation through selected bands in the frequency range.

Longitudinal oscillations.—Because of the extreme complexity of the general treatment we consider only a special case in which the temperature fluctuations t_{11}, t_{22}, t_{33} are zero. The dispersion relation (4.2) reduces to two distinct equations. The first one is

$$i\Omega_3+\frac{1}{\sigma_3'}+\frac{k^2}{2\omega\sigma_3}I_2+\left(\frac{3k}{2a_3^2}I_3+kI_1\right)\frac{1}{\sigma_3'} = 0. \qquad \ldots \quad \ldots \quad (4.18)$$

This equation, as in the transverse case, gives an acoustic wave, modified by collisional effects. The second one is complicated and neglecting collisional effects and putting

$$\frac{\omega}{\omega_{p_1}} = \Omega, \frac{c}{a_1} = c', \frac{\omega_1}{\omega_{p_1}} = \lambda = \frac{V_A}{c}, \frac{a_1 k}{\omega_{p_1}} = Dk = k_1,$$

$$\frac{\omega_2}{\omega_{p_1}} = \frac{m_1}{m_2}\lambda = \delta\lambda \approx 0, \frac{\omega_{p_2}^2}{\omega_{p_1}^2} = \frac{a_2^2}{a_1^2} = \delta \approx 0, \qquad \ldots \quad \ldots \quad \ldots \quad (4.19)$$

where V_A = Alfvén velocity, D = Debye shielding distance, we have, after simplification,

$$\frac{c'^2k_1^2}{\Omega^2} = [\Omega^2(\Omega^2-\lambda^2)+2\lambda\Omega+1](\Omega^2-4\lambda^2)\Big[2\Omega^2(\Omega^2-\lambda^2)(\Omega^2-4\lambda^2)-3\lambda\Omega(\Omega^2-4\lambda^2)$$
$$+\frac{2}{3c'^2}\Omega^2(\Omega^2+3\lambda\Omega+2\lambda^4)-\frac{4\Omega^2}{3c'^2}+\Omega^2-4\lambda^2\Big]^{-1} \quad \ldots \quad (4.20)$$

(i) If $\lambda = 0$, i.e. the magnetic field is absent,

$$\frac{c'^2k_1^2}{\Omega^2} = \frac{\Omega^4+1}{2\Omega^4+\frac{2\Omega^2}{3c'^2}+1-\frac{4}{3c'^2}}, \quad \ldots \quad (4.21)$$

and, the right-hand side being positive always, transmission is possible for all frequencies.

(ii) If $\lambda = 0{\cdot}5$, i.e. the electron gyro-frequency is half the electron plasma frequency,

$$\frac{c'^2k_1^2}{\Omega^2} \simeq \frac{(\Omega+\frac{1}{2})(\Omega+1{\cdot}112)[(\Omega-0{\cdot}806)^2+1{\cdot}15]}{2[\Omega^4-0{\cdot}5\Omega^2-0{\cdot}75\Omega+0{\cdot}5]}, \quad \ldots \quad (4.22)$$

and there is transmission for all the range of frequencies.

(iii) If $\lambda = 1$, i.e. the electron gyro-frequency is equal to electron plasma frequency,

$$\frac{c'^2k_1^2}{\Omega^2} \simeq \frac{\Omega^2(\Omega^2-1)+2\Omega+1}{2(\Omega-0{\cdot}284)(\Omega-1{\cdot}356)[(\Omega+0{\cdot}8193)^2+0{\cdot}629]}, \quad \ldots \quad (4.23)$$

and there is one forbidden range of frequencies given by

$$0{\cdot}284\,\omega_{p_1} < \omega < 1{\cdot}356\,\omega_{p_1}. \quad \ldots \quad (4.24)$$

Thus, even this extremely crude treatment for the longitudinal oscillations predicts the gaps in the frequency range for a sufficiently large primitive magnetic field.

5. In this section we shall consider the general properties of the dispersion relation for the transverse wave (4.3) neglecting the collisions. For this purpose, following Gross (1951), Dnestrovskii and Kostomarov (1961), we write (4.3) in the following forms, neglecting the neutral mode,

$$D(s) = s-1+\sum_i \frac{\beta_i^2}{2\alpha_i \sin \alpha_i\pi}\int_0^{2\pi} \exp[-s\alpha_i^2\gamma_i(1-\cos\gamma)]\cos(\gamma-\pi)\alpha_i\,d\gamma, \quad \ldots \quad (5.1)$$

and

$$D(s) = s-1+\sum_i \frac{\beta_i^2}{\alpha_i}\left\{\sum_{n=-\infty}^{\infty}\frac{I_n(s\alpha_i^2\gamma_i)}{\alpha_i-n}\right\}, \quad \ldots \quad (5.2)$$

where

$$s = \frac{c^2k^2}{\omega^2},\ \alpha_i = \frac{\omega}{\omega_i},\ \beta_i = \frac{\omega_{pi}}{\omega_i},\ \gamma_i = \frac{KT_0}{m_i c^2},$$

$$I_n(x) = i^{-n}e^{-x}J_n(ix). \qquad \ldots \qquad (5.3)$$

and the summation extends over $i = 1$ and $i = 2$.

A. *Investigation of real roots of $D = 0$:*

For cold plasma $\gamma_i = 0$, the dispersion relation (5.2) reduces to

$$D \equiv s-1+\frac{\beta_1^2}{\alpha_1^2}+\frac{\beta_2^2}{\alpha_2^2} = 0.$$

so that

(i) when $\alpha_1 < \beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}},\ s < 0,$

and the waves are exponentially decaying with x,

(ii) when $\alpha_1 = \beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}},\ s = 0.$

implying no actual propagation of disturbance, and

(iii) when $\alpha_1 > \beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}},\ s > 0,$

implying the existence of the undamped travelling wave.

For a non-relativistic thermal plasma, for which γ_1 is small, we infer the existence of a real root by comparing the signs of $D(s)$ when $s = \pm\infty$, 0. We have

$$D(0) = \frac{\beta_1^2}{\alpha_1^2}\left(1+\frac{m_1}{m_2}\right)-1, \qquad \ldots \qquad (5.4)$$

and from (5.1)

$$D(s) \simeq s+\sqrt{\frac{\pi}{8}}\sum_i \beta_i^2\frac{\exp\left(-2s\alpha_i^2\gamma_i\right)}{\alpha_i \sin\alpha_i\pi\left(-s\alpha_i^2\gamma_i\right)^{\frac{1}{2}}} \text{ as } |s| \to \infty, \qquad \ldots \qquad (5.5)$$

so that when $s = +\infty$, $D(s)$ is > 0. When $s \to -\infty$, the sign of D is the same as that of $\sin\alpha_1\pi$ as can be inferred on writing (5.5) as

$$D(s) \simeq s+\sqrt{\frac{\pi}{8}}+\frac{\beta_1^2\exp\left(-2s\alpha_1^2\gamma_1\right)}{\alpha_1\sin\alpha_1\pi\left(-s\alpha_1^2\gamma_1\right)^{\frac{1}{2}}}\left[1+\frac{\sin\alpha_1\pi}{\sin\alpha_2\pi}\exp\left\{2s\alpha_1^2\gamma_1\left(1-\frac{m_1^3}{m_2^3}\right)\right\}\right],$$

$$\simeq \frac{\beta_1^2\exp\left(-2s\alpha_1^2\gamma_1\right)}{\alpha_1\sin\alpha_1\pi\left(-s\alpha_1^2\gamma_1\right)^{\frac{1}{2}}}. \qquad \ldots \qquad (5.6)$$

Therefore when $\alpha_1 > \beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}}$, we have $D(0) < 0$, and $D(+\infty) > 0$, so that $D = 0$ can have an odd number of positive roots. If γ_1 is sufficiently small, there is one root close to the root of the dispersion relation for the cold plasma. Let $p<\beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}}<p+1$, where p is a positive integer. If p is even and α_1 varies from $\beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}}$ to p, $\sin\alpha_1\pi$ is >0, so that there can be an odd number of negative roots. When $\gamma_1\rightarrow 0$, there is only one root which tends to $-\infty$. When p is odd, $\sin\alpha_1\pi<0$, so that there can be an even number of negative roots. For sufficiently small γ_1, there is no negative root at all.

In the resonance regions $\alpha_1\simeq n$, we have from (5.2)

$$\frac{\beta_1^2}{n}I_n(sn^2\gamma_1)\simeq(\alpha_1-n)\left[1-s-\frac{\beta_1^2}{n}I_0(sn^2\gamma_1)+0\left(\frac{m_1}{m_2}\right)\right]. \quad .. \quad (5.7)$$

Let n be odd.—As $\alpha_1\rightarrow n-0$, $\sin\alpha_1\pi$ remains positive, so that there is one positive and one negative root. The equation (5.7) can be satisfied for positive values of s only when s is very large and for negative values of s when $s\rightarrow 0$. Thus the positive root tends to ∞ and the negative root to zero.

As $\alpha_1\rightarrow n+0$, $\sin\alpha_1\pi$ is <0, so that there is no negative root at least when $\gamma_1\rightarrow 0$, and the $+$ ve root $\rightarrow 0$.

When n is even.—As $\alpha_1\rightarrow n-0$, $\sin\alpha_1\pi<0$, so that there is no negative root and the positive root tends to ∞. As $\alpha_1\rightarrow n+0$, $\sin\alpha_1\pi>0$ and there is a positive as well as a negative root both tending to zero.

When $\alpha_1<\beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}}$, $D(0)>0$ and $D(+\infty)>0$. In general there can be an even number of positive roots, but for a sufficiently small γ_1, $D=0$ has no positive root. Also let $p<\beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}}<p+1$, and if p is even, as α_1 decreases from $\beta_1\left(1+\frac{m_1}{m_2}\right)^{\frac{1}{2}}$ to p, $\sin\alpha_1\pi$ is >0, so that there can be an even number of positive and negative roots. For sufficiently small γ_1, there is a pair of $-$ve roots only both of which are very large $\rightarrow-\infty$ as $\gamma_1\rightarrow 0$. If p is odd, $\sin\alpha_1\pi$ is <0, and there is one positive root and one negative root, for small γ_1. To investigate near resonance regions, we once again make use of (5.7).

When n is odd.—As $\alpha_1\rightarrow n-0$, $\sin\alpha_1\pi>0$, and so there is a pair of real roots for small γ_1, one $\rightarrow\infty$ and the other $\rightarrow+0$. As $\alpha_1\rightarrow n+0$, $\sin\alpha_1\pi<0$, so that there is a very large negative root.

When n is even.—As $\alpha_1\rightarrow n-0$, $\sin\alpha_1\pi<0$, and there is a large negative root, and as $\alpha_1\rightarrow n+0$, $\sin\alpha_1\pi>0$, and there are no real roots.

B. *Investigation of complex roots.*—In order to investigate the complex roots, we treat s as a complex variable. The dispersion relation (5.2) has no finite singularities so that the variation of the argument of $D(s)$ merely gives the number of zeros multiplied by 2π. We shall find the variation in the argument of $D(s)$ along the circle $C : |s| = R$, where R is sufficiently large. Taking $s = Re^{i\theta}$ on the circle, as $\cos\theta$ is positive in $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ from (5.6) we find that the argument of $D(s)$ changes by $\pi + 0\left(\frac{e^{-R}}{R^{3/2}}\right)$. As s goes round the other half of the circle C, the argument of $D(s)$ changes by a quantity of the order, $4\alpha_1^2\gamma_1 R + \frac{\pi}{2}$, as R is large, so that as $R \to \infty$, the change in the argument of $D(s)$ round $C \to \infty$. Therefore $D(s)$ must have infinitely many roots. Since for sufficiently small γ_1, only a finite number of real roots are possible, there are an infinity of pairs of complex conjugate roots.

The detailed behaviour of $D(s)$ can be seen from Fig. 1 which plots the values of $\sqrt{s}$ against α_1 on the upper half plane for positive s and $i\sqrt{-s}$ in the lower half plane for negative s as determined by (5.2), for particular values $\gamma_1 = 0{\cdot}1$ and $\beta_1 = 0{\cdot}5$ for a fully ionized hydrogen plasma. The plots in the upper half plane represent the undamped waves, while the plots in the lower half plane represent the purely damped waves. The points, where it crosses $\sqrt{s} = 0$ axis, determine the frequencies for which the plasma oscillates as a whole but there is no propagation.

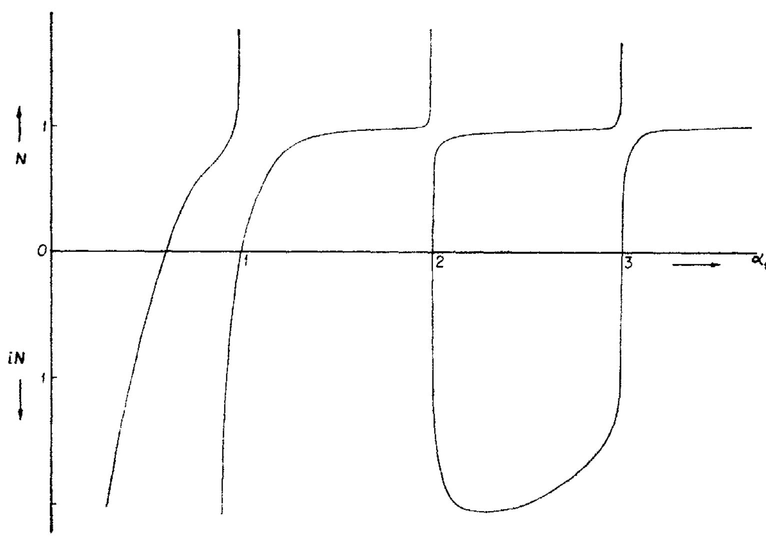

FIG. 1.

ACKNOWLEDGEMENT

We are extremely grateful to Prof. T. G. Cowling, F.R.S., for kindly going through the manuscript critically and for his valuable suggestions.

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